Identical Duals : Gap Function (Study on Nonlinear Analysis and Convex Analysis)

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Identical Duals

— Gap Function —

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Abstract

We consider *identical duals* of two pairs of minimization (primal) problems and maximization (dual) problems from a view point of *gap function*. The identical dual means that both optimum points of a primal problem and its dual one are identical. An identity

$$\text{(CI)}\quad \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n \;=\; x_0\mu_1$$

is called *complementary* [17]. The complementary identity leads to a gap function. We show that the complementary identity and the gap function play a fundamental part in analyzing an identical duality between primal and dual.

1 Identical Dual 1

As a pair of primal problem and dual problem, we take n-variable optimization problems:

$$(\mathrm{P}_1)\quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + x_n^2 \\ \text{subject to} & \text{(i)}\ \ x \in R^n, \quad \text{(ii)}\ \ x_0 = c \end{array}$$

$$(\mathrm{D}_1)\quad \begin{array}{ll} \text{Maximize} & \displaystyle 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k-\mu_{k+1})^2\right] - \mu_n^2 - \mu_n^2 \\ \text{subject to} & \text{(i)}\ \ \mu \in R^n. \end{array}$$

First we present an identity, which plays a fundamental role in analyzing the pair. Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$(\mathrm{C}_1)\quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n$$

holds true. This identity is called *complementary*. Furthermore the complementary identity implies that

$$
\begin{aligned}
(\mathrm{QI}_1)\quad & \sum_{k=1}^{n}\left[(x_{k-1}-x_k)^2+x_k^2\right]+\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]+2\mu_n^2-2c\mu_1 \\
&= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k-\mu_k)^2+(x_k-\mu_k+\mu_{k+1})^2\right]+(x_{n-1}-x_n-\mu_n)^2+(x_n-\mu_n)^2.
\end{aligned}
$$

This is an identity on $R^n \times R^n$, which is called *quadratic*.

Now we define three functions $f,\ g : R^n \to R^1,\ h : R^n \times R^n \to R^1$ by

$$
\begin{aligned}
f(x) &= \sum_{k=1}^{n}\left[(x_{k-1}-x_k)^2+x_k^2\right] \\
g(\mu) &= 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-2\mu_n^2 \\
h(x,\mu) &= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k-\mu_k)^2+(x_k-\mu_k+\mu_{k+1})^2\right]+(x_{n-1}-x_n-\mu_n)^2+(x_n-\mu_n)^2.
\end{aligned}
$$

They are called *primal*, *dual* and *gap* functions, respectively. Then (QI_1) is summarized as follows.

Lemma 1 *It holds that*

$$(\mathrm{QI}_1) \qquad f(x)-g(\mu) \;=\; h(x,\mu).$$

We consider a linear system of $2n$-equation on $2n$-variable $(x,\ \mu)$:

$$
(\mathrm{EC}_1) \qquad
\begin{aligned}
c-x_1 &= \mu_1, & x_1 &= \mu_1-\mu_2 & & \\
x_{k-1}-x_k &= \mu_k, & x_k &= \mu_k-\mu_{k+1} & & 2 \le k \le n-1 \\
x_{n-1}-x_n &= \mu_n, & x_n &= \mu_n. & &
\end{aligned}
$$

Lemma 2 *It holds that*

(i) $h(x,\mu) \ge 0 \quad \forall (x,\ \mu) \in R^n \times R^n$

(ii) $h(x,\mu) = 0 \Longleftrightarrow (x,\mu)$ *satisfies* (EC_1).

Corollary 1 *It holds that*

(i) $f(x) \ge g(\mu) \quad \forall (x,\ \mu) \in R^n \times R^n$

(ii) $f(x) = g(\mu) \Longleftrightarrow (x,\mu)$ *satisfies* (EC_1).

Definition 1 We say that that (P_1) and (D_1) are *dual to each other* and (EC_1) is an *equality condition* (EC) if Corollary 1 (i), (ii) hold. Then we say that one is *dual* of the other. This definition applies for any triplet such as $\{(\mathrm{P}_1),\ (\mathrm{D}_1),\ (\mathrm{EC}_1)\}$.

From Corollary 1, it turns out that both are *dual to each other*, and (EC_1) is an *equality condition*.

Lemma 3 (EC_1) *has a unique solution:*

$$x = (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n)$$
$$= \frac{c}{F_{2n+1}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1), \tag{1}$$
$$\mu = (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n)$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-2},\ \ldots, F_{2n-2k},\ \ldots,\ F_4,\ F_2). \tag{2}$$

Here $\{F_n\}$ is the *Fibonacci sequence*. This is defined as the solution to the second-order linear difference equation

$$x_{n+2} - x_{n+1} - x_n = 0, \qquad x_1 = 1,\ x_0 = 0. \tag{3}$$

n	$\cdots$	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
F_n	$\cdots$	-1	1	0	1	1	2	3	5	8	13	21	34	55	89

Table 1 Fibonacci sequence $\{F_n\}$

Proof. From (EC_1), we have a pair of linear systems of n-variable on n-equation:

$$(\mathrm{EQ}_1)\qquad \begin{aligned} c &= 3x_1 - x_2 \\ x_1 &= 3x_2 - x_3 \\ &\ \vdots \\ x_{n-2} &= 3x_{n-1} - x_n \\ x_{n-1} &= 2x_n \end{aligned} \qquad \begin{aligned} c &= 2\mu_1 - \mu_2 \\ \mu_1 &= 3\mu_2 - \mu_3 \\ &\ \vdots \\ \mu_{n-2} &= 3\mu_{n-1} - \mu_n \\ \mu_{n-1} &= 3\mu_n. \end{aligned}$$

The left system has a solution x in (1), while the right has a solution μ in (2). □

Theorem 1 *The primal* (P_1) *has a minimum value* $m = c(c - \hat{x}_1) = \dfrac{F_{2n}}{F_{2n+1}}c^2$ *at a path*

$$\hat{x} = (\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n)$$
$$= \frac{c}{F_{2n+1}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1).$$

The dual (D_1) *has a maximum value* $M = c\mu_1^* = \dfrac{F_{2n}}{F_{2n+1}}c^2$ *at a path*

$$\mu^* = (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*,\ \mu_n^*)$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-2},\ \ldots, F_{2n-2k},\ \ldots,\ F_4,\ F_2).$$

Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then a complementary identity

$$(\mathrm{C}_1) \quad c\mu_1 \; = \; \sum_{k=1}^{n-1}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + x_n\mu_n$$

holds true.

Let us define two sequences $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ from $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ through

$$\begin{aligned} &y_1 = c - x_1,\ y_2 = x_1,\ y_3 = x_1 - x_2,\ y_4 = x_2,\ y_5 = x_2 - x_3 \\ &\quad \ldots,\ y_{2n-2} = x_{n-1},\ y_{2n-1} = x_{n-1} - x_n,\ y_{2n} = x_n \\ &\nu_1 = \mu_1,\ \nu_2 = \mu_1 - \mu_2,\ \nu_3 = \mu_2,\ \nu_4 = \mu_2 - \mu_3,\ \nu_5 = \mu_3 \\ &\quad \ldots,\ \nu_{2n-2} = \mu_{n-1} - \mu_n,\ \nu_{2n-1} = \mu_n,\ \nu_{2n} = \mu_n \end{aligned} \tag{4}$$

, respectively. Then an identity

$$(\mathrm{C}_1^*) \quad c\nu_1 \; = \; \sum_{k=1}^{2n} y_k\nu_k$$

holds *under a constraint* – a linear system of $4n$-variables (y, ν) on $2n$-equations – :

$$(\mathrm{C}^{y\nu}) \qquad \begin{array}{rcl@{\qquad}rcl} c &=& y_1 + y_2 & \nu_1 &=& \nu_2 + \nu_3 \\ y_2 &=& y_3 + y_4 & \nu_3 &=& \nu_4 + \nu_5 \\ &\vdots& & &\vdots& \\ y_{2n-4} &=& y_{2n-3} + y_{2n-2} & \nu_{2n-3} &=& \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} &=& y_{2n-1} + y_{2n} & \nu_{2n-1} &=& \nu_{2n}. \end{array}$$

An equality (C_1^*) with constraint $(\mathrm{C}^{y\nu})$ is called a $2n$-variable *conditional complementarity.* This is simply written as (C_1^*) under $(\mathrm{C}^{y\nu})$.

Now let $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ satisfy $(\mathrm{C}_1^{y\nu})$. Then an elementary inequality with equality

$$2xy \; \leq \; x^2 + y^2 \quad \text{on} \;\; R^2 \; ; \;\; x = y \tag{5}$$

yields

$$2c\nu_1 \; \leq \; \sum_{k=1}^{2n}(y_k^2 + \nu_k^2).$$

Thus we have an inequality

$$2c\nu_1 - \sum_{k=1}^{2n}\nu_k^2 \; \leq \; \sum_{k=1}^{2n}y_k^2.$$

The sign of equality holds iff

$$(\mathrm{EC}_1) \quad y_k = \nu_k \quad 1 \le k \le 2n. \tag{6}$$

Hence we have a pair of conditional optimization problems:

$$(\mathrm{P}_1^*) \quad \begin{array}{rl} \text{minimize} & y_1^2 + y_2^2 + \cdots + y_{2n-1}^2 + y_{2n}^2 \\ \text{subject to} & (1) \;\; y_1 + y_2 = c \\ & (2) \;\; y_3 + y_4 = y_2 \\ & \vdots \\ & (n-1) \;\; y_{2n-3} + y_{2n-2} = y_{2n-4} \\ & (n) \;\; y_{2n-1} + y_{2n} = y_{2n-2} \\ & (n+1) \;\; y \in R^{2n} \end{array}$$

$$(\mathrm{D}_1^*) \quad \begin{array}{rl} \text{Maximize} & 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2 + \nu_{2n}^2) \\ \text{subject to} & [1] \;\; \nu_2 + \nu_3 = \nu_1 \\ & [2] \;\; \nu_4 + \nu_5 = \nu_3 \\ & \vdots \\ & [n-1] \;\; \nu_{2n-2} + \nu_{2n-1} = \nu_{2n-3} \\ & [n] \;\; \nu_{2n} = \nu_{2n-1} \\ & [n+1] \;\; \nu \in R^{2n}. \end{array}$$

Let (AC_1) be an *augmentation* of the system $(\mathrm{C}_1^{y\nu})$ with the additional equality condition (EC_1):

$$(\mathrm{AC}_1) \quad \begin{array}{rclrcl} c &=& y_1 + y_2 & \nu_1 &=& \nu_2 + \nu_3 \\ y_2 &=& y_3 + y_4 & \nu_3 &=& \nu_4 + \nu_5 \\ &\vdots& & &\vdots& \\ y_{2n-4} &=& y_{2n-3} + y_{2n-2} & \nu_{2n-3} &=& \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} &=& y_{2n-1} + y_{2n} & \nu_{2n-1} &=& \nu_{2n} \end{array}$$
$$y_k = \nu_k \quad 1 \le k \le 2n.$$

The linear system (AC_1) is of $4n$-variables on $4n$-equations. Let (y, ν) satisfy (AC_1). Then both sides become a common value with five expressions:

$$(5\mathrm{V}_1) \quad \begin{aligned} & y_1^2 + y_2^2 + \cdots + y_{2n}^2 \\ &= cy_1 \\ &= 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n}^2) \\ &= \nu_1^2 + \nu_2^2 + \cdots + \nu_{2n}^2 \\ &= c\nu_1. \end{aligned}$$

The system (AC_1) has indeed a unique common solution:

$$y = (y_1,\ y_2,\ \ldots,\ y_k,\ \ldots,\ y_{2n-1},\ y_{2n})$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1),$$
$$\nu = (\nu_1,\ \nu_2,\ \ldots,\ \nu_k,\ \ldots,\ \nu_{2n-1},\ \nu_{2n})$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1).$$

Theorem 2 *The primal* (P_1) *has a minimum value* $m = \frac{F_{2n}}{F_{2n+1}}c^2$ *at a path*

$$\hat{y} = (\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_k,\ \ldots,\ \hat{y}_{2n-1},\ \hat{y}_{2n})$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1).$$

The dual (D_1) *has a maximum value* $M = \frac{F_{2n}}{F_{2n+1}}c^2$ *at a path*

$$\nu^* = (\nu_1^*,\ \nu_2^*,\ \ldots,\ \nu_k^*,\ \ldots,\ \nu_{2n-1}^*,\ \nu_{2n}^*)$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1).$$

Both optimal solutions (point and value) are identical:

$$\hat{x} = \mu^*, \quad m = M.$$

Further both are Fibonacci.

Thus *Fibonacci Identical Duality* (FID) holds between (P_1) and (D_1) [15–17].

We remark that the $2n$-*variable* pair is a transliteration from n-*variable* one (P_1), (D_1).

2 Identical Dual 2

Next we consider the following pair

$$(P_m)\quad \begin{array}{ll} \text{minimize} & \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + \frac{F_{m+1}}{F_m}x_n^2 \\ \text{subject to} & \text{(i)}\ \ x \in R^n, \quad \text{(ii)}\ \ x_0 = c \end{array}$$

$$(D_m)\quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 - \frac{F_m}{F_{m+1}}\mu_n^2 \\ \text{subject to} & \text{(i)}\ \ \mu \in R^n, \end{array}$$

where $\{F_n\}$ is the *Fibonacci sequence*. The identity (C_1) is enhanced to

$$(\mathrm{C}_m)\ c\mu_1 = \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + \sqrt{\frac{F_{m+1}}{F_m}}\,x_n\sqrt{\frac{F_m}{F_{m+1}}}\,\mu_n$$

where $m \ge 1$. This identity is called F_m-*complementary*.

Furthermore the complementary identity implies that

$$(\mathrm{QI}_m)\quad \begin{aligned} &\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + \frac{F_{m+1}}{F_m}x_n^2 \\ &\qquad + \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k-\mu_{k+1})^2\right] + \mu_n^2 + \frac{F_m}{F_{m+1}}\mu_n^2 - 2c\mu_1 \\ &= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k-\mu_k)^2 + (x_k-\mu_k+\mu_{k+1})^2\right] \\ &\qquad + (x_{n-1}-x_n-\mu_n)^2 + \left(\sqrt{\frac{F_{m+1}}{F_m}}\,x_n - \sqrt{\frac{F_m}{F_{m+1}}}\,\mu_n\right)^2. \end{aligned}$$

This is an identity on $R^n \times R^n$, which is called *quadratic*.

Now we define three functions $f,\ g : R^n \to R^1,\ h : R^n \times R^n \to R^1$ by

$$\begin{aligned} f(x) &= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + \frac{F_{m+1}}{F_m}x_n^2 \\ g(\mu) &= 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k-\mu_{k+1})^2\right] - \mu_n^2 - \frac{F_m}{F_{m+1}}\mu_n^2 \\ h(x,\mu) &= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k-\mu_k)^2 + (x_k-\mu_k+\mu_{k+1})^2\right] \\ &\qquad + (x_{n-1}-x_n-\mu_n)^2 + \left(\sqrt{\frac{F_{m+1}}{F_m}}\,x_n - \sqrt{\frac{F_m}{F_{m+1}}}\,\mu_n\right)^2. \end{aligned}$$

They are called *primal*, *dual* and *gap* functions, respectively. Then (QI_m) is summarized as follows.

Lemma 4 *It holds that*

$$(\mathrm{QI}_m)\qquad f(x) - g(\mu) = h(x,\mu).$$

We consider a linear system of $2n$-equation on $2n$-variable $(x,\ \mu)$:

$$(\mathrm{EC}_m)\qquad \begin{aligned} c - x_1 &= \mu_1, & x_1 &= \mu_1 - \mu_2 & & \\ x_{k-1} - x_k &= \mu_k, & x_k &= \mu_k - \mu_{k+1} & & 2 \le k \le n-1 \\ x_{n-1} - x_n &= \mu_n, & \frac{F_{m+1}}{F_m}x_n &= \mu_n. & & \end{aligned}$$

Lemma 5 *It holds that*

(i) $h(x,\mu) \geq 0 \quad \forall(x,\ \mu) \in R^n \times R^n$

(ii) $h(x,\mu) = 0 \Longleftrightarrow (x,\mu)$ *satisfies* (EC_m).

Corollary 2 *It holds that*

(i) $f(x) \geq g(\mu) \quad \forall(x,\ \mu) \in R^n \times R^n$

(ii) $f(x) = g(\mu) \Longleftrightarrow (x,\mu)$ *satisfies* (EC_m).

From Corollary 2, it turns out that (P_m) and (D_m) are dual to each other, and (EC_m) is an equality condition. The equality condition (EC_m) is a linear system of $2n$-equations on $2n$-variables (x,μ).

Lemma 6 *Let* (x,μ) *satisfy* (EC_m). *Then both sides become a common value with five expressions:*

$$(5\mathrm{V}_m) \qquad \begin{aligned} f(x) &= c(c-x_1) = g(\mu) \\ &= \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 + \frac{F_m}{F_{m+1}}\mu_n^2 = c\mu_1. \end{aligned}$$

The primal (P_m) *has a minimum value*

$$m = f(x) = c(c-x_1)$$

at x, *while the dual* (D_m) *has a maximum value*

$$M = g(\mu) = \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 + \frac{F_m}{F_{m+1}}\mu_n^2 = c\mu_1$$

at μ.

Lemma 7 (EC_m) *has indeed a unique solution:*

$$\begin{aligned} x &= (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-2},\ F_{m+2n-4},\ \ldots, F_{m+2n-2k},\ \ldots,\ F_{m+2},\ F_m), \end{aligned} \tag{7}$$

$$\begin{aligned} \mu &= (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-1},\ F_{m+2n-3},\ \ldots, F_{m+2n-2k+1},\ \ldots,\ F_{m+3},\ F_{m+1}). \end{aligned} \tag{8}$$

Proof. From (EC_m), we have a pair of linear systems of n-variable on n-equation:

$$(\mathrm{EQ}_m) \qquad \begin{aligned} c &= 3x_1 - x_2 & \qquad c &= 2\mu_1 - \mu_2 \\ x_1 &= 3x_2 - x_3 & \mu_1 &= 3\mu_2 - \mu_3 \\ &\ \vdots & &\ \vdots \\ x_{n-2} &= 3x_{n-1} - x_n & \mu_{n-2} &= 3\mu_{n-1} - \mu_n \\ x_{n-1} &= \frac{F_{m+2}}{F_m}x_n & \mu_{n-1} &= \frac{F_{m+3}}{F_{m+1}}\mu_n. \end{aligned}$$

The left system has a solution x in (7), while the right has a solution μ in (8). $\square$

Let us define two sequences $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ from $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ through

$$\begin{aligned} & y_1 = c - x_1,\ y_2 = x_1,\ y_3 = x_1 - x_2,\ y_4 = x_2,\ y_5 = x_2 - x_3 \\ & \quad \ldots,\ y_{2n-2} = x_{n-1},\ y_{2n-1} = x_{n-1} - x_n,\ y_{2n} = x_n \\ & \nu_1 = \mu_1,\ \nu_2 = \mu_1 - \mu_2,\ \nu_3 = \mu_2,\ \nu_4 = \mu_2 - \mu_3,\ \nu_5 = \mu_3 \\ & \quad \ldots,\ \nu_{2n-2} = \mu_{n-1} - \mu_n,\ \nu_{2n-1} = \mu_n,\ \nu_{2n} = \mu_n \end{aligned} \tag{9}$$

, respectively. Then an identity

$$(\mathrm{C}_m^*) \qquad c\nu_1 = \sum_{k=1}^{2n-1} y_k \nu_k + \sqrt{\frac{F_{m+1}}{F_m}}\, y_{2n} \sqrt{\frac{F_m}{F_{m+1}}}\, \nu_{2n}$$

holds *under a constraint* – a linear system of $4n$-variables (y, ν) on $2n$-equations – :

$$(\mathrm{C}^{y\nu}) \qquad \begin{aligned} c &= y_1 + y_2 & \nu_1 &= \nu_2 + \nu_3 \\ y_2 &= y_3 + y_4 & \nu_3 &= \nu_4 + \nu_5 \\ &\ \vdots & &\ \vdots \\ y_{2n-4} &= y_{2n-3} + y_{2n-2} & \nu_{2n-3} &= \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} &= y_{2n-1} + y_{2n} & \nu_{2n-1} &= \nu_{2n}. \end{aligned}$$

An equality (C_m^*) with constraint $(\mathrm{C}^{y\nu})$ is called a $2n$-variable *conditional complementarity*. This is simply written as (C_m^*) under $(\mathrm{C}^{y\nu})$.

Now let $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ satisfy $(\mathrm{C}^{y\nu})$. Then the elementary inequality with equality yields

$$2c\nu_1 \le \sum_{k=1}^{2n-1} (y_k^2 + \nu_k^2) + \frac{F_{m+1}}{F_m} y_{2n}^2 + \frac{F_m}{F_{m+1}} \nu_{2n}^2.$$

Thus we have an inequality

$$2c\nu_1 - \sum_{k=1}^{2n-1} \nu_k^2 - \frac{F_m}{F_{m+1}} \nu_{2n}^2 \le \sum_{k=1}^{2n-1} y_k^2 + \frac{F_{m+1}}{F_m} y_{2n}^2.$$

The sign of equality holds iff

$$(\mathrm{EC}_m) \qquad y_k = \nu_k \quad 1 \le k \le 2n-1, \quad F_{m+1} y_{2n} = F_m \nu_{2n}. \tag{10}$$

We remark that an equivalence

$$\sqrt{\frac{F_{m+1}}{F_m}}\, y_{2n} = \sqrt{\frac{F_m}{F_{m+1}}}\, \nu_{2n} \Longleftrightarrow \frac{F_{m+1}}{F_m} y_{2n} = \nu_{2n}$$

yields the last equality.

Hence we have a pair of conditional optimization problems:

$$(\mathrm{P}_m^*)\quad \begin{array}{ll} \text{minimize} & y_1^2 + y_2^2 + \cdots + y_{2n-1}^2 + \dfrac{F_{m+1}}{F_m} y_{2n}^2 \\ \text{subject to} & (1)\ \ y_1 + y_2 = c \\ & (2)\ \ y_3 + y_4 = y_2 \\ & \quad \vdots \\ & (n-1)\ \ y_{2n-3} + y_{2n-2} = y_{2n-4} \\ & (n)\ \ y_{2n-1} + y_{2n} = y_{2n-2} \\ & (n+1)\ \ y \in R^{2n} \end{array}$$

$$(\mathrm{D}_m^*)\quad \begin{array}{ll} \text{Maximize} & 2c\nu_1 - \left(\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2 + \dfrac{F_m}{F_{m+1}} \nu_{2n}^2\right) \\ \text{subject to} & [1]\ \ \nu_2 + \nu_3 = \nu_1 \\ & [2]\ \ \nu_4 + \nu_5 = \nu_3 \\ & \quad \vdots \\ & [n-1]\ \ \nu_{2n-2} + \nu_{2n-1} = \nu_{2n-3} \\ & [n]\ \ \nu_{2n} = \nu_{2n-1} \\ & [n+1]\ \ \nu \in R^{2n}. \end{array}$$

Let (AC_m) be an *augmentation* of the system ($\mathrm{C}_m^{y\nu}$) with the additional equality condition (EC_m):

$$(\mathrm{AC}_m)\quad \begin{array}{rlrl} c = & y_1 + y_2 & \nu_1 = & \nu_2 + \nu_3 \\ y_2 = & y_3 + y_4 & \nu_3 = & \nu_4 + \nu_5 \\ \vdots & & \vdots & \\ y_{2n-4} = & y_{2n-3} + y_{2n-2} & \nu_{2n-3} = & \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} = & y_{2n-1} + y_{2n} & \nu_{2n-1} = & \nu_{2n} \end{array}$$

$$y_k = \nu_k \quad 1 \le k \le 2n-1, \quad F_{m+1} y_{2n} = F_m \nu_{2n}.$$

The linear system (AC_m) is of $4n$-variables on $4n$-equations. Let (y, ν) satisfy (AC_m).

The system (AC_m) has indeed a unique solution:

$$\begin{aligned} y &= (y_1,\ y_2,\ \ldots,\ y_k,\ \ldots,\ y_{2n-2},\ y_{2n-1},\ y_{2n}) \\ &= \frac{c}{F_{m+2n}} (F_{m+2n-1},\ F_{m+2n-2},\ \ldots, F_{m+2n-k},\ \ldots,\ F_{m+2},\ F_{m+1},\ \underline{F_m}), \end{aligned}$$

$$\begin{aligned} \nu &= (\nu_1,\ \nu_2,\ \ldots,\ \nu_k,\ \ldots,\ \nu_{2n-2},\ \nu_{2n-1},\ \nu_{2n}) \\ &= \frac{c}{F_{m+2n}} (F_{m+2n-1},\ F_{m+2n-2},\ \ldots, F_{m+2n-k},\ \ldots,\ F_{m+2},\ F_{m+1},\ \underline{F_{m+1}}). \end{aligned}$$

Note that only the last elements are different, as underlined. However, in Case $m = 1$, both solutions are identical:

$$\begin{aligned} y &= (y_1,\ y_2,\ \ldots,\ y_k,\ \ldots,\ y_{2n-2},\ y_{2n-1},\ y_{2n}) \\ &= \nu = (\nu_1,\ \nu_2,\ \ldots,\ \nu_k,\ \ldots,\ \nu_{2n-2},\ \nu_{2n-1},\ \nu_{2n}) \\ &= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_3,\ F_2,\ \underline{F_1}). \end{aligned}$$

We note that $F_2 = F_1 = 1$.

Theorem 3 *The primal* (P_m) *has a minimum value* $m = \dfrac{F_{m+2n-1}}{F_{m+2n}}c^2$ *at a path*

$$\begin{aligned} \hat{y} &= (\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_k,\ \ldots,\ \hat{y}_{2n-2},\ \hat{y}_{2n-1},\ \hat{y}_{2n}) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-1},\ F_{m+2n-2},\ \ldots, F_{m+2n-k},\ \ldots,\ F_{m+2},\ F_{m+1},\ \underline{F_m}). \end{aligned}$$

The dual (D_m) *has a maximum value* $M = \dfrac{F_{m+2n-1}}{F_{m+2n}}c^2$ *at a path*

$$\begin{aligned} \nu^* &= (\nu_1^*,\ \nu_2^*,\ \ldots,\ \nu_k^*,\ \ldots,\ \nu_{2n-2}^*,\ \nu_{2n-1}^*,\ \nu_{2n}^*) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-1},\ F_{m+2n-2},\ \ldots, F_{m+2n-k},\ \ldots,\ F_{m+2},\ F_{m+1},\ \underline{F_{m+1}}). \end{aligned}$$

Both optimal solutions (point and value) are identical except for the last element:

$$\hat{y}_k = \nu_k^* \quad 1 \le k \le 2n-1, \quad m = M.$$

Further both are Fibonacci:

$$\hat{y}_k = \nu_k^* = \frac{F_{m+2n-k}}{F_{m+2n}}c \quad 1 \le k \le 2n-1, \quad \hat{y}_{2n} = \frac{F_m}{F_{m+2n}}c,\ \nu_{2n}^* = \frac{F_{m+1}}{F_{m+2n}}c$$

$$m = M = \frac{F_{m+2n-1}}{F_{m+2n}}c^2.$$

Thus *Fibonacci Identical* [1] *Duality* (FID) holds between (P_m) and (D_m) [15–17].

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[1] *Identical* means *identical except for the last element.*

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