On lattice points which become vertices of Heronian triangles II *

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1 Introduction

Heronian triangles are triangles whose side lengths and area are all integers. For example, the triangles with side lengths $3, 4, 5$, area 6, and with side lengths $5, 5, 6$, area 12, and others. Specially the triangle whose side lengths $3, 4$ and 5 is *Pythagorean triangle*, that is, a right triangle whose side lengths are all integers. We discuss the lattice points which become vertices of Heronian triangles.

In section 2, we state some results previously clarified in [1]. In section 3, we will describe the contents of the subsequent research. In this article, we will explain our result that every Heronian triangle can be realized as a lattice triangle. P.Yiu's result[2] is another approach to a similar problem. We will consider some properties about lattice points which become vertices of Heronian triangles.

2 Some results previously clarified by us

We get the following theorems.

Theorem 2.1 [1, Theorem 3.1] *There exist circles which contain n lattice points such that the distance between any pair of them is an integer. Specially, in case $n = 3$, there are three lattice points on circles which become vertices of Heronian triangles.*

Theorem 2.2 [1, Theorem 3.3] *Not on the line three points which belong to the following set consists of lattice points*

$$\{(0, \pm y), (0, 0), (\pm x_1, 0), \cdots, (\pm x_n, 0)\}$$

*This paper is a preliminary version and a final version will be submitted to elsewhere.

become vertices of Heronian triangles , where let p_i $(i = 1, \cdots, n)$ be odd prime number satisfying $p_i > p_j$ (for $i < j$), and $p_1 > 2p_2 \cdots p_n$, we can write $x_k = (p_1 \cdots p_k)^2 - 4(p_{k+1} \cdots p_n)^2$ $(k = 1, \cdots, n)$, $y = 4p_1 \cdots p_n$.

3 About the realization of a lattice triangle from a Heronian triangle

Lattice triangles(lattice Heronian triangles) are Heronian triangles whose all vertices are on lattice points. In this section, we consider a lattice triangle that is realized from a Heronian triangle.

3.1 Main claim

We describe one way to solve this problem. And we reveal the following proposition.

Proposition 3.1 *All Heronian triangles can be realized as lattice triangles.*

Outline of the proof. Every Heronian triangles can be put the coordinate plane by performing parallel translation and rotation.

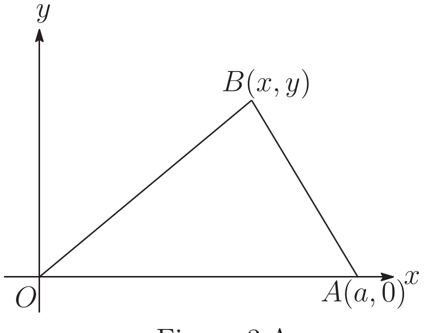

Figure 3.A

In this situation, x, y are both rational numbers. Therefore, by performing an appropriate similar transformation, we can x, y are both integers, that is, a point (x, y) is a lattice point. Therefore three points $(0, 0), (x, y), (a, 0)$ are all lattice points, and a Heronian $\triangle OAB$ is a lattice triangle. ■

3.2 Related results

Another approach to the same problem is P.Yiu's result[2]. P.Yiu apply elementary number theory such as integer solution of quartic equations.

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References

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