Seiberg-Witten Floer homotopy and contact structures (Intelligence of Low-dimensional Topology)

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1 Backgrounds

We develop homotopy theoretical aspects of Seiberg-Witten theory and give two kinds of applications to low dimensional topology. This paper is a survey of [IMT21].

1.1 Several open questions to low dimensional topology

In [IMT21], we consider the following two problems:

(i) sliceness problem of knots in general 4-manifolds, and

(ii) giving constraints on Betti numbers of symplectic caps.

We mainly review background of (i) in this subsection. For (ii), see Subsection 1.4.

Definition 1.1. A knot K in S^3 is *smoothly (resp. topologically) slice* if K bounds a smooth (resp. locally flat) properly embedded 2-disk in D^4.

It is known that a topologically slice but not smoothly slice knot provides an exotic $\mathbb{R}^4$. Thus sliceness is closely related to smooth structures of 4-manifolds. Also, the concordance relation $K_1 \sim K_2$ is defined as the sliceness of $K_1\#(-\overline{K}_2)$, where $\overline{K}$ means the mirror image and $-K$ is K with the opposite sign. The quotient set

$$\mathcal{C} := \{ \text{ all oriented knots } \}/\sim \text{:concordance}$$

is called the *knot concordance group* and it admits an abelian group structure via the connected sum. The group $\mathcal{C}$ has been studied via various techniques. There are several effective tools to study the subgroup

$$\mathcal{T} := \{\text{topologically slice knots }\} \subset \mathcal{C}$$

in various theories including Heegaard Floer theory [OS03, MO07, OS08, OS11, Hom14, HW16, OSS17, HM17, DHST19, AKS20], Khovanov homology [Ras10] and gauge theory [KM13, DS19, KM19]. For example, there are several studies finding $\mathbb{Z}^\infty$-subgroups or summands in $\mathcal{T}$ in Heegaard Floer theory [OSS17, KP18, Hom19, AKS20].

Instead of D^4, we consider sliceness for general 4-manifolds with S^3-boundary. Let X be a closed, oriented, connected, smooth 4-manifold and K a knot in S^3

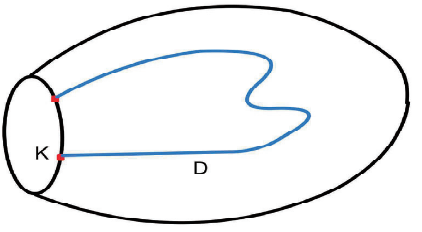

Figure 1: Slice disk in $X \setminus \operatorname{int} D^4$

Definition 1.2. [MMSW19, Definition 6.2] K is *smoothly (resp. topologically) H-slice* in X if K bounds a properly embedded smooth (resp. locally flat) null-homologous disc D in $X^\circ = X - \operatorname{Int} B^4$.

In 4-dimensional topology, the following quantity is important:

$b^+(X) := \dim$ (a maximal positive definite subspace of the intersection form of X).

The following are known typical obstructions to H-sliceness:

(i) obstructions to topological H-sliceness : signature and Rochlin type theorem [Gil81, KR20, MMP20a, Rob65]

(ii) obstructions to smooth H-sliceness in definite 4-manifolds($b^+ = 0$) : Heegaard Floer τ-invariant[OS03], Rasmussen invariant[Ras10], Thurston-Bennequin number[Pla04], other Heegaard Floer and gauge theoretic obstructions [KM13, MMSW19]

(iii) obstructions to smooth H-sliceness in indefinite 4-manifolds($b^+ > 0$): generalized Thurston-Bennequin inequality [MR06], 10/8-type constraints [MMP20a], an obstruction from Bauer-Furuta invariant [MMP20a]

It is proved in [Schneiderman'10] that, for a knot K in S^3, there exists $N > 0$ such that K is smoothly H-slice in $\#_N S^2 \times S^2$. Thus H-sliceness depends on 4-manifolds. Also, recently, in [MMP20b], the existence of an exotic pair (X, X') of closed 4-manifolds admitting smoothly H-slice knot K in X but not in X' is proved. As above, H-sliceness is also related to smooth structures of 4-manifolds. As a main result, we give an obstruction to H-sliceness in a certain class of 4-manifolds including a certain class of symplectic 4-manifolds. Our main tool is the author's invariant [Iid19] defined via Seiberg-Witten theory.

1.2 Spin^c structures and Dirac operators

In this subsection, we introduce Spin^c structures and Dirac operators as preliminaries of Seiberg-Witten theory. For more details, see [KM07]. Spin^c structures give a nice 1st order elliptic operator called the Dirac operator. Dirac operators originate in Dirac's approach to give a physical description of electrons based on the theory of special relativity and quantum mechanics.

Definition 1.3. *A Spin^c structure* on a Riemannian 4-manifold (X, g) is a pair

$$\mathfrak{s} = (S = S^+ \oplus S^-, \rho : \Lambda^* T^* X \otimes \mathbb{C} \to \mathrm{Hom}(S, S)),$$

where S^+ and S^- are hermite vector bundles with rank 2 called positive and negative spinor bundles and ρ is a bundle map called the *Clifford multiplication*. Relations

$$\rho(e^i)\rho(e^j) + \rho(e^j)\rho(e^i) = -2g(e^i, e^j)$$

and

$$\rho(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \rho(e^{i_1}) \cdots \rho(e^{i_k}), \quad i_1 < \cdots < i_k$$

are imposed for any local orthonormal frame e^0, e^1, e^2, e^3. The bundles S^+ and S^- are -1 and $+1$ eigenspaces for

$$\rho(e^0 \wedge e^1 \wedge e^2 \wedge e^3) : S \to S.$$

respectively.

Two Spin^c structures (S, ρ) and (S', ρ') are *isomorphic* if there exists unitary isomorphism $S \to S'$ which interwines with Clifford multiplications.

Definition 1.4. A unitary connection $\nabla^A = A$ on S is called a *Spin^c structure* if it satisfies

$$\nabla^A_V(\rho(W)\Phi) = \rho(\nabla^A_V W)\Phi + \rho(W)\nabla^A_V \Phi$$

for any vector fields V, W and any section Φ of S.

Let us denote by $\mathcal{A}(\mathfrak{s})$ the space of Spin^c connections. This is an affine space modeled on the space of imaginary valued 1-forms $i\Omega^1_X$.

Definition 1.5. The *Dirac operator* is given by

$$D_A = \rho \circ \nabla_A = \sum_{i=0}^{3} \rho(e_i)\nabla^A_{e_i} : \Gamma(S) \to \Gamma(S).$$

The operator D_A is decomposed as

$$D_A = \begin{bmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{bmatrix}$$

with respect to the decomposition $S = S^+ \oplus S^-$, where

$$D_A^+ : \Gamma(S^+) \to \Gamma(S^-) \text{ and } D_A^- : \Gamma(S^-) \to \Gamma(S^+).$$

The operators $D_A^\pm$ are first order elliptic operators. Although a spin structure does not always exist on 4-manifolds, a spin^c structure exists on any 4-manifold. Let us write

$$\mathrm{Spin}^c_X := \{\text{all spin}^c \text{ structures}\}/\text{isomorphism}.$$

There is a free transitive action of $H^2(X; \mathbb{Z})$ on Spin^c_X for any closed 4-manifolds. Thus there a one to one correspondence (non-canonical) between $H^2(X; \mathbb{Z})$ and Spin^c_X.

1.3 Seiberg-Witten theory

In this subsection, we review Seiberg-Witten theory. For more details, see [KM07]. Seiberg-Witten theory has been an effective tool in the studies of 3 and 4-manifolds. A typical example of such studies is finding an exotic pair of 4-manifolds. This theory is based on a non-linear PDE called the Seiberg-Witten equation.

Let X be a closed oriented 4-manifold. For simplicity we assume $b_1(X) = 0$. Fix a Riemannian metric and a Spin^c structure $\mathfrak{s}$ on X.

Definition 1.6. The *Seiberg-Witten equation* is a non-linear 1-st order PDE for a pair $(A, \Phi) \in \mathcal{A}(\mathfrak{s}) \times \Gamma(S^+)$ defined by

$$\begin{cases} \frac{1}{2}\rho(F^+_{A^t}) = (\Phi\Phi^*)_0 \\ D^+_A\Phi = 0, \end{cases} \tag{1}$$

where F_{A^t} is the curvature 2-form of the $U(1)$ connection induced from A on the $U(1)$ bundle $\det S^+$, $+$ in $F^+_{A^t}$ means the self dual component with respect to the Hodge star operator $* : \Omega^2_X \to \Omega^2_X$, and $(\Phi\Phi^*)_0$ is the traceless part of $\Phi\Phi^* : S^+ \to S^+$.

Roughly speaking, the Seiberg-Witten invariant of $(X, \mathfrak{s})$ is obtained by counting the number of points in the quotient space

$$\mathcal{M}(X, \mathfrak{s}) := \{ \text{ all solutions } (A, \Phi) \text{ to } (1)\}/\mathcal{G}_X,$$

where

$$\mathcal{G}_X := \mathrm{Map}(X, U(1))$$

and the action of $u \in \mathcal{G}_X$ on (A, Φ) is given by

$$u \cdot (A, \Phi) := (A - u^{-1}du, u\Phi).$$

However, in general, the moduli space $\mathcal{M}(X, \mathfrak{s})$ may have quotient singularities. The quantity $b^+(X)$ control the existence of singularities. When $b^+ > 0$, for a generic perturbation, $\mathcal{M}(X, \mathfrak{s})$ is an orientable compact smooth manifold of dimension

$$d(\mathfrak{s}) := \frac{1}{4}(c_1^2(S^+)[X] - 2\chi(X) - 3\sigma(X)) = c_2(S^+)[X].$$

Fix an orientation of a line

$$\Lambda(\mathfrak{s}) := \Lambda^{max}(H^0(X; \mathbb{R}) \oplus H^1(X; \mathbb{R}) \oplus H^+(X; \mathbb{R})).$$

Then an orientation of $\mathcal{M}(X, \mathfrak{s})$ is induced from it. Then the Seiberg-Witten invariant is defined to be

$$SW(X, \mathfrak{s}) := \int_{\mathcal{M}(X,\mathfrak{s})} U^{\frac{1}{2}d(\mathfrak{s})},$$

where U is a cohomology class coming from an isomorphism

$$H^*((\mathcal{A}(\mathfrak{s}) \times (\Gamma(S^+) \setminus \{0\}))/\mathcal{G}_X : \mathbb{Z}) \cong H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[U].$$

When $b^+(X) > 1$, $SW(X, \mathfrak{s})$ is independent on the choces of perturbations. Thus $SW(X, \mathfrak{s})$ is an invariant of $(X, \mathfrak{s})$. This construction gives a map

$$SW_X : \mathrm{Spin}^c_X \to \mathbb{Z}.$$

The map SW gives a strong tool to study smooth structures of 4-manifolds.

Example 1.7. When $X = 3\mathbb{C}P^2 \# 20\overline{\mathbb{C}P^2}$, $SW_X = 0$. But for $X' = K3\#\overline{\mathbb{C}P^2}$, $SW_{X'} \neq 0$. Since X and X' have the same intersection form, by Freedman theory [Fre82], X and X' are homeomorphic. Thus we can conclude that X and X' give an exotic pair.

In 2004, Bauer-Furuta [BF04] introduced a refinement of the Seiberg-Witten invariant:

$$BF_X : \mathrm{Spin}^c_X \to \bigoplus_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} \lim_{M,N\to\infty} [(\mathbb{R}^{m+M} \oplus \mathbb{C}^{n+N})^+, (\mathbb{R}^M \oplus \mathbb{C}^N)^+]_{S^1}.$$

The invariant BF_X is defined by using a method called *finite dimensional approximation* of the Seiberg-Witten equation.

It is proved in [BF04] that BF_X recovers SW_X when we fix an orientation of $\Lambda_{\mathfrak{s}}$ for any $\mathfrak{s}$.

The following example gives a pair of 4-manifolds detected by the Bauer-Furuta invariant but not by the Seiberg-Witten invariant.

Example 1.8. When $X = 6\mathbb{C}P^2 \# 39\overline{\mathbb{C}P^2}$, $BF_X =$ constant. But, when $X' = K3\#K3\#\overline{\mathbb{C}P^2}$, then we still have nontrivial result $BF_{X'} \neq$ constant. Since X and X' have the same intersection form, by Freedman theory [Fre82], X and X' are homeomorphic. Thus X and X' give an exotic pair.

1.4 Symplectic and contact structures

In order to obtain obstruction to H-sliceness of knots, we use symplectic fillings and find a solution to Seiberg-Witten equation. For more details on contact structures in 3-dimension and symplectic structures in 4-dimension , see [OS04, GS99].

Symplectic structures are structure on even dimensional manifolds and originate from the classical mechanics. Let X be an even dimensional oriented manifold. Denote its dimension by $2n$.

Definition 1.9. A 2-form ω on X is called a *symplectic form* if it is a closed form and $\omega^n > 0$.

Liouville vector fields and contact type hypersurfaces serve as convenient tools to conduct cut-and-paste operation respecting symplectic strutures. Contact type hypersurfaces are also typical examples of contact manifolds. First we give the definition of the contact structure.

Definition 1.10. Let Y be an oriented odd dimensional manifold. Denote its dimension by $2n+1$. A codimension 1 distribution $\xi \subset TY$ is called a contact structure if there exists a 1-form θ on Y such that

$$\mathrm{Ker}\ \theta = \xi \text{ and } \theta \wedge (d\theta)^n > 0.$$

θ is called a contact 1-form. The following is a natural situation in which contact structures appear.

Definition 1.11. A vector field v defined on a symplectic manifold (X, ω) is called a *Liouville vector field* if

$$\mathcal{L}_v \omega = \omega$$

holds. A codimension 1 submanifold $Y \subset X$ is called a contact type hypersurface if there exists a Liouville vector field defined near Y and transverse to Y. Then $\theta := \iota_v \omega$ is a contact form on Y.

Let $(Y_0, \xi_0), (Y_1, \xi_1)$ be contact manifolds with the same dimensions.

Definition 1.12. A cobordism equipped with a symplectic structure (W, ω) from Y_0 to Y_1 is called a *strong symplectic cobordsim* if there exists a Liouville vector field v defined near collor neighborhoods on ∂W such that

1. v is transverse to ∂W
2. v is inward on Y_0 and outward on Y_1, and
3. for $i = 0, 1$, $\xi_i = \text{Ker } (\iota_v \omega)$ on Y_i.

The contact manifold (Y_0, ξ_0) is called the *concave boundary* of (W, ω) and (Y_1, ξ_1) called the *convex boundary* of (W, ω).

Definition 1.13. A symplectic cobordism (W, ω) is called a *symplectic filling* of (Y_1, ξ_1) when Y_0 is empty and called a *symplectic cap* of (Y_0, ξ_0) when Y_1 is empty.

If $(W_0, \omega_0) : (Y_0, \xi_0) \to (Y_1, \xi_1)$ and $(W_1, \omega_1) : (Y_1, \xi_1) \to (Y_2, \xi_2)$ are strong symplectic cobordisms, we can glue two symplectic structures and construct another strong symplectic cobordism $(W_0 \cup W_1, \omega_0 \cup \omega_1) : (Y_0, \xi_0) \to (Y_2, \xi_2)$. An important remark is that there is asymmetry between convex and concave boundaries.

Definition 1.14. A contact structure on a closed manifold is called *symplectically fillable* if it has a symplectic filling.

In this paper, we focus on dimension 3 and 4. It is shown by Etnyre-Honda [EH02] independently that any closed contact 3-manifold has a symplectic cap. On the other hand, a closed contact manifold does not have a symplectic filling in general. For example, it is proved in [Lis98] that $-\Sigma(2, 3, 5)$ does not admit any symplectic fillings. There are several known constraints on the topology of symplectic fillings of a contact 3-manifold (Y, ξ), but not much is known about the topology of symplectic caps.

We give a topological constraint on symplectic caps for certain spherical 3-manifolds.

We will also use a knot invariant coming from contact geometry called *maximal Thurston–Bennequin invariant*. Let Y be an oriented homology 3-sphere and ξ a contact structure.

Definition 1.15. A *Legendrian knot* K is a knot in (Y, ξ) such that $T_p K \subset \xi_p$ for any $p \in K$.

A Legendrian knot has a diagramatic representation called a *front projection*. A Legendrian knot K admits a framing coming from $\xi|_K$, which is called the *contact framing*.

Definition 1.16. For any knot $K \subset S^3$, *maximal Thurston–Bennequin invariant* for K is defined to be

$$TB(K) := \max_{\forall \text{Legendrian rep of } K \in (S^3, \xi_{std})} \{ \text{ (contact framing) } - \text{ (Seifert framing) } \},$$

where ξ_{std} is the standard contact structure on S^3.

One can check that $TB(K)$ is a $\mathbb{Z}$-valued isotopy invariant of knots. The important property of $TB(K)$ is following Thurston-Bennequin type inequality proved by Plamenevskaya [Pla04]:

$$TB(K) \leq 2\tau(K) - 1,$$

where $\tau(K)$ is a concordance invariant called the *tau-invariant* introduced in [OS03]. Since the tau invariant gives a lower bound for the smooth 4-ball genus, $TB(K)$ obstructs H-sliceness in D^4 and, also in negative definite 4-manifold with S^3-boundary satisfying $b_1 = 0$.

1.5 Seiberg-Witten theory and symplectic/contact structures

Taubes's non-vanishing result of the Seiberg-Witten invariant for symplectic manifolds is one of the most fundamental results in the relation between Seiberg-Witten theory and symplectic/contact structures. Before stating this result, note that a symplectic structure determines a canonical Spin^c structure. We explain it in the case of dimension 4. Let (X, ω) be a symplectic 4-manifold. There exits a compatible almost complex structure J unique up to homotopy. A Riemannien metric on X is determined by

$$g_J = \omega(\cdot, J\cdot).$$

The triple (ω, J, g_J) consist so-called almost Kähler structure.

Then

$$\begin{cases} S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2} \\ S^- = \Lambda^{0,1} \\ \rho = \sqrt{2}\,\mathrm{Symb}(\bar{\partial} + \bar{\partial}^*) : T^*X \to \mathrm{Hom}(S^+, S^-) \end{cases}$$

gives a Spin^c structure on X. Here $\mathrm{Symb}(\bar{\partial} + \bar{\partial}^*)$ is the principal symbol of

$$\bar{\partial} + \bar{\partial}^* : \Omega_X^{0,0} \oplus \Omega_X^{0,2} \to \Omega_X^{0,1}.$$

This construction gives a well-defined isomorphism class of Spin^c structure. We denote it by $\mathfrak{s}_\omega$.

Theorem 1.17 ([Tau94])**.** *Let (X, ω) be a closed symplectic 4-manifold with $b^+(X) \geq 2$. Then*

$$SW_X(\mathfrak{s}_\omega) = \pm 1.$$

Actually, we can give a solution under a certain perturbation: Fix (ω, J, g_J) as before. Define $(A_0, \Phi_0) \in \mathcal{A}(\mathfrak{s}_\omega) \times \Gamma(S^+)$ as follows.

$$\Phi_0 = (1, 0) \in \Omega_X^{0,0} \oplus \Omega_X^{0,2} = \Gamma(S^+).$$

Note that the virtual dimension is

$$d(\mathfrak{s}_\omega) = c_2(S^+) = 0$$

since Φ_0 is a nowhere-vanishing section of S^+. The map

$$\begin{aligned}\mathcal{A}(\mathfrak{s}_\omega) &\to \Omega^1(i\underline{\mathbb{R}} \oplus \Lambda^{0,2})\\ A &\mapsto \nabla_A \Phi_0\end{aligned}$$

is well defined since a Spin^c connection is unitary. Since this is a $i\Omega^1_X$-affine map, there exists unique $A_0 \in \mathcal{A}(\mathfrak{s}_\omega)$ such that $\Omega^1(i\underline{\mathbb{R}})$ part of $\nabla_{A_0}\Phi_0$ is zero. Then we can check that

$$D^+_{A_0}\Phi_0 = 0.$$

See for example [HT99]. Obviously, (A_0, Φ_0) is a solution to

$$\frac{1}{2}\rho(F^+_A) - (\Phi\Phi^*)_0 = \frac{1}{2}\rho(F^+_{A_0}) - (\Phi_0\Phi_0^*)_0.$$

Thus (A_0, Φ_0) is a solution to the Seiberg-Witten equation perturbed by $\frac{1}{2}\rho^{-1}(F^+_{A_0}) - (\Phi_0\Phi_0^*)_0$. Furthermore, we can show that for a large constant r, $(A_0, r\Phi_0)$ is the unique solution to the perturbed equation

$$\begin{cases}\frac{1}{2}\rho(F^+_A) - (\Phi\Phi^*)_0 &= \frac{1}{2}\rho(F^+_{A_0}) - r(\Phi_0\Phi_0^*)_0\\ D^+_A\Phi &= 0.\end{cases}$$

and actually contributes to ± 1 to the Seiberg-Witten invariant (i.e. transversality is automatically satisfied. See Lemma 3.11 of [KM97]).

Inspired by Taubes's work on Seiberg-Witten theory on symplectic manifolds, Kronheimer-Mrowka ([KM97]) constructed an integer valued invariant (defined up to sign) for 4-manifold with contact boundary. Note that an oriented 2-plane field ξ on a oriented 3-manifold Y defines a Spin^c structure $\mathfrak{s}_\xi$ as follows: Let W be a compact oriented 4-manifold with non-empty boundary and $\xi = \text{Ker}\ \theta$ be a contact structure on the boundary. Fix a complex structure J on ξ. Then

$$g_1 = \theta \otimes \theta + \frac{1}{2}d\theta(\cdot, J\cdot)$$

is a Riemannian metric on ∂W. Consider a manifold

$$C = \mathbb{R}^{\geq 1}_t \times \partial W$$

equipped with a conical metric

$$g_0 = dt^2 + t^2 g_1$$

and symplectic structure

$$\omega_0 = \frac{1}{2}d(t^2\theta).$$

These define an almost Kähler structure on C and in turn determines a Spin^c structure $\mathfrak{s}_0$. Let

$$W^+ = W \cup_{\partial W} C$$

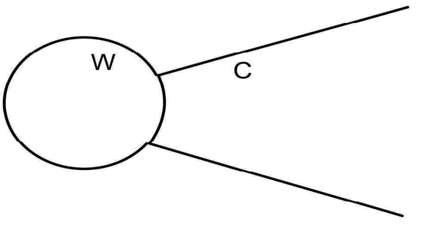

Figure 2: Conical-end 4-manifold

be a 4-manifold with a conical end obtained by gluing W and C along boundaries and $\mathrm{Spin}^c(W,\xi)$ be the set of isomorphis class of pairs $(\mathfrak{s},h)$ where $\mathfrak{s}$ is a Spin^c structure on W^+ and $h:\mathfrak{s}|_W\to\mathfrak{s}_0$ is an isomorphism. This is a $H^2(W,\partial W;\mathbb{Z})$ torsor. Kronheimer-Mrowka defnied an invariant

$$KM_{X,\xi}:\mathrm{Spin}^c(W,\xi)\to\mathbb{Z}$$

up to overall sign. This is constructed from the perturbed Seiberg-Witten equation on W^+.

Using Bauer-Furuta's method of finite dimensional approximation, the first author [Iid19] defined a refinement of Kronheimer-Mrowka's invariant

$$BF(W,\xi,\mathfrak{s})\in\pi^{st}_{d(\mathfrak{s})}/\pm 1$$

for $\mathfrak{s}\in\mathrm{Spin}^c(W,\xi)$ when $b_3(W)=0$. Here

$$d(\mathfrak{s})=\langle e(S^+,\Phi_0),[W,\partial W]\rangle$$

is the relative Euler number relative to the section Φ_0 on C defined from the almost Kähler structure. The first author proved that the mapping degree of $BF(W,\xi,\mathfrak{s})$ is equal to Kronheimer-Mrowka's invariant when $d(\mathfrak{s})=0$.

Using the connected sum formula below, we can prove a certain non-vanishing result.

Theorem 1.18 ([Iid19]). *Let $(W,\mathfrak{s}_W)$ be an oriented Spin^c compact 4-manifold whose boundary is a contact 3-manifold (Y,ξ) with $b_3(W)=0$, $\mathfrak{s}_W|_Y=\mathfrak{s}_\xi$ and let $(X,\mathfrak{s}_X)$ be a closed Spin^c 4-manifold with $b_1(X)=0$. Then, we have*

$$\Psi(W\#X,\xi,\mathfrak{s}_W\#\mathfrak{s}_X)=\Psi(W,\xi,\mathfrak{s}_W)\wedge BF(X,\mathfrak{s}_X) \quad (2)$$

in the stable homotopy group up to sign. Here we forget the S^1 action of $BF(X,\mathfrak{s}_X)$.

Since Iida's invariant is $\pm\mathrm{Id}$ up to sign and stable homotopy for any weak symplectic filling with $b_3=0$ [Iid19, Corollary 4.3], thus by Theorem 1.18, we obtain the following non-vanishing results.

Theorem 1.19 ([Iid19]). *Let (W,ω) be a weak symplectic filling of a contact 3-manifold (Y,ξ) with $b_3(W)=0$. We consider a closed sympectic 4-manifold (X_1,ω_1) with $b_1(X_i)=0$ and $b_2^+(X_i)\equiv 3\bmod 4$. Then, one has*

$$\Psi(W\#X_1,\mathfrak{s}_\omega\#\mathfrak{s}\#\mathfrak{s}_{\omega_1})\neq 0.$$

2 Statements of results

2.1 Obstructions to H-sliceness

Using tecniques in contact/ symplectic topology and a certain adjunction type inequality for Iida's invariant, we prove the following non-H-sliceness results for symplectic 4-manifolds.

Theorem 2.1 ([IMT21]). *For a knot $K \subset S^3$ satisfying $TB(K) > 0$, K is not smoothly H-slice in any closed symplectic 4-manifold X with $b_1 = 0$ and $b^+ \equiv 3 \bmod 4$.*

A sequence of closed symplectic 4-manifolds with $b_1 = 0$ and $b^+ \equiv 3 \bmod 4$ is given by elliptic surfaces $\{E(2n)\}_{n \in \mathbb{Z}_{>0}}$. Let $T(p, q)$ be the (p, q)-torus knot. As a consequence, we can prove the following result:

Corollary 2.2. *The Whitehead doubles $\{Wh_0^+(T_{2,2^n-1})\}$ of torus knots are not smoothly H-slice in $K3$.*

Proof. It is proven in [EH01] that

$$TB(T(p, q)) = pq - p - q \text{ if } p, q > 0.$$

It is not difficult to see the following fact using front projection of Legendrian knots:

Lemma 2.3. *For any knot K in S^3 satisfying $TB(K) > 0$, one has $TB(Wh_0^+(K)) > 0$.*

Thus $\{Wh_0^+(T_{2,2^n-1})\}$ and $K3$ satisfy assumptions of Theorem 2.1. □

Moreover, our sequence $\{Wh_0^+(T_{2,2^n-1})\}$ is interesting in the following sense:

(i) $\{Wh_0^+(T_{2,2^n-1})\}$ bounds topologically disk in D^4, hence topologically disk in $K3$ ([Fre82]),

(ii) $Wh_0^+(T_{2,2^n-1})$ is smoothly slice in $K3$ ([MMP20b]), and

(iii) $\{Wh_0^+(T_{2,2^n-1})\}$ are linearly independent in the knot concordance group $\mathcal{C}$ ([HK12]).

Let us give a sketch of proof. The main gauge theoretic ingredient is the following vanishing result:

Theorem 2.4 ([IMT21]). *Let $(W, \mathfrak{s})$ be an oriented Spin^c compact 4-manifold whose boundary is a contact 3-manifold (Y, ξ) with $b_3(W) = 0$ and $\mathfrak{s}|_Y = \mathfrak{s}_\xi$.*

If a non-torsion homology class in $H_2(W, \partial W; \mathbb{Z})$ is realized by a smoothly embedded 2-sphere whose self-intersection number is non-negative, then

$$\Psi(W, \xi, \mathfrak{s}_W) = 0.$$

This is a special case of the adjunction inequality given in [IMT21]. The proof of Theorem 2.4 is a standard neck stretching argument as in the original proof of Kronheimer-Mrowka's adjunction inequality [KM94].

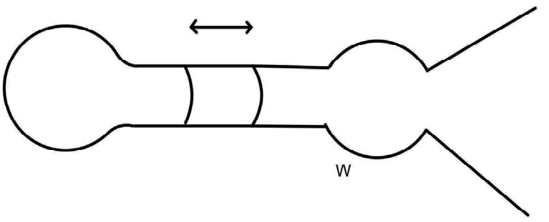

Figure 3: Neck stretching argument in the proof of Theorem 2.4

Sketch of proof of Theorem 2.1. A tubular neighborhood of a H-slice disk in $X \setminus \operatorname{int} D^4$ is diffeomorphic to the trace $W_0(\overline{K})$ of 0-surgery along the mirror image of K. We consider the trace $W_0(K)$ of 0-surgery along K, which admits a symplectic filling structure by the assumption $TB(K) > 0$. We take a symplectic cobordism W from $\partial(W_0(K))$ to a some contact rational homology 3-sphere (Y, ξ). Set

$$Z := (W_0(K) \cup_{\partial W_0(K)} W)\#X.$$

Then by the non-vanishing result Theorem 1.19, one has

$$\Psi(Z, \xi, \mathfrak{s}_Z) \neq 0.$$

where Z is the Spin^c structure induced by the symplectic structures on X, W, and $W_0(K)$. On the other hand,

$$S := (\text{core of } W_0(K)) \cup K \times I \cup (\text{core of } W_0(\overline{K}))$$

gives a smoothly embedded 2-sphere in Z such that

- $S \cdot S = 0$ and
- S is not a torsion in $H_2(Z, \partial Z)$.

Thus, by Theorem 2.4, we see

$$\Psi(Z, \xi, \mathfrak{s}_Z) = 0.$$

This gives a contradiction. □

2.2 Topology of symplectic caps

We give a constraint of topology of symplectic caps. In particular, we give a constraint on b^+.

Let X be a compact 4-manifold with connected boundary Y. The proof uses vanishing results of Ψ, which follows from the following geometric setting.

Definition 2.5 ([IMT21]). We say X has a *geometrically isolated 2-handle* if there is a 2-handle h in a handle decomposition of X such that h does not intersect any 1-handles in that handle decompositon, and the core of h is not a torsion in $H_2(X; \mathbb{Z})$.

Any 4-manifolds obtained obtained as 2-handle surgery have geometrically isolated 2-handle.

Our result on symplectic caps is as follows.

Theorem 2.6 ([IMT21]). *The following results hold:*

(i) The contact 3-manifold (S^3, ξ_{std}) does not admit any positive definite symplectic cap having a geometrically isolated 2-handle such that $b_1 = 0$ and $b_2^+ \geq 2$.

(ii) The contact 3-manifold $\Sigma(2,3,5)$ with the unique tight contact structure does not admit any positive definite symplectic cap having a geometrically isolated 2-handle such that $b_1 = 0$ and $b_2^+ \geq 2$, and there is no 2-torsion on its homology.

Remark 2.7. We have three remarks:

1. Since $\mathbb{C}P^2 \setminus \operatorname{int} D^4$ gives a symplectic cap of (S^3, ξ_{std}), $b_2^+(X) > 1$ is necessary.
2. We can recover the result by using Theorem 2.6(i) that any positive definite geometrically simply connected closed 4-manifold with $b_2^+ > 1$ does not admit a symplectic structure proved in [HL19, Theorem 1.1], [Yas19, Corollary 1.6].
3. The proof uses classification results for intersection forms of negative definite 4-manifolds bounded by S^3 and $\Sigma(2,3,5)$ proven by Donaldson [Don83] and Scaduto [Sca18].

Since there are few studies of topology of symplectic caps, Theorem 2.6 gives an interesting constraint.

Let us give a sketch of proof of Theorem 2.6. First, we prove the following constraint coming from Bauer-Furuta type invariants:

Theorem 2.8 ([IMT21]). *Let (Y, ξ) be a contact 3-manifold with a symplectic filling that has $b_1 = 0$. If $b_1(Y) = 0$ and Y admits a positive scalar curvature metric, then (Y, ξ) does not have positive definite symplectic cap X with $b_1 = 0$ and $b_2^+ \geq 2$ having a geometrically isolated 2-handle and a Spin^c structure $\mathfrak{s}_X$ such that*

$$\frac{-c_1^2(\mathfrak{s}_X) + b_2(X)}{8} = \delta(Y, \mathfrak{s}_\xi),$$

where $\delta(Y, \mathfrak{s})$ is Frøyshov invariant of $(Y, \mathfrak{s})$ with the convention $\delta(\Sigma(2,3,5)) = 1$.

Sketch of proof of Theorem 2.8. Let W be a symplectic filling of (Y, ξ). We consider

$$Z := (X \cup W) \# (-X).$$

We use the relative Bauer-Furuta invariant ([Man03, Kha15]) BF for the 4-manifold Z with boundary Y. Note that $X \cup W$ admits a symplectic structure. The assumptions for X imply

$$BF_{-X}(\mathfrak{s}_X) = id.$$

Since $X \cup W$ has a symplectic structure, the Spin^c structure $\mathfrak{s}_Z$ obtained by the connected sum of the Spin^c structure coming from the symplectic structure on $X \cup W$ and $\mathfrak{s}_X$ satisfies

$$BF_Z(\mathfrak{s}_Z) \neq 0.$$

By sliding a geometrically isolated 2-handle on X to the corresponding geometrically isolated 2-handle of $-X$, we again find an embedded 2-sphere with self intersection number 0. The standard neck stretching argument shows

$$BF_Z(\mathfrak{s}) = 0.$$

for all Spin^c structure $\mathfrak{s}$. This gives a contradiction. □

Sketch of proof of Theorem 2.6. Using Theorem 2.8 and Donaldson's theorem A [Don83] or a classification result of intersection forms of negative definite 4-manifolds bounded by $\Sigma(2,3,5)$ [Sca18], we can obtain a contradiction.

□

3 Open problems

Theorem 2.8 suggests the following conjecture:

Conjecture 3.1 ([IMT21])**.** A contact manifold (Y,ξ), where Y is an L-space and ξ is symplectically fillable, does not have a simply connected positive definite symplectic cap with $b_2^+ > 1$.

As a related question, for lens spaces, we have:

Question 3.2. Can we prove an anlogue of Theorem 2.6 for lens spaces?

In the proof of Theorem 2.6, the classification of negative definite lattices of 4-manifolds bounding a given 3-manifold is important. Thus we relate Question 3.2 with the follwoing problem:

Question 3.3. Classify all isomorphism classes of negative definite lattices of 4-manifolds bounding the lens space $L(p,q)$.

This problem is related to $\mathcal{I}(L(p,q))$ defined in [CP18].

We did not consider general 3-manifolds in Theorem 2.1. However, the generalized Thurston-Bennequin inequality is still true for general 3-manifolds [MR06].

Question 3.4. Can we prove Theorem 2.1 for general contact 3-manifolds?

Also, we have another direction to prove the generalized Thurston-Bennequin inequality given in [MR06] for Baur-Furuta type invariant. Originally, the generalized Thurston-Bennequin inequality is proved when Kronheimer-Mrowka's invariant [KM97] is non-trivial. Since Iida's invariant refines Kronheimer-Mrowka's invariant, it is natural to ask:

Question 3.5. Can we prove the generalized Thurston-Bennequin inequality for Iida's invariant?

In [IT20], the relative version of Iida's invariant [Iid19]

$$\Psi(Y,\xi) : S^0 \to \Sigma^* \, SWF(-Y, \mathfrak{s}_\xi)$$

is defined for any contact rational homology 3-sphere (Y,ξ), where $SWF(Y,\mathfrak{s}_\xi)$ is Manolescu's Seiberg-Witten-Floer stable homotopy type given in [Man03] and $*$ is a rational number determined by (Y,ξ). The invariant $\Psi(Y,\xi)$ is called *Seiberg-Witten Floer homotopy contact invariant*. On this invariant, we have the following conjecture:

Conjecture 3.6 ([IT20]). The invariant $\Psi(Y,\xi) : S^0 \to \Sigma^* \, SWF(-Y,\mathfrak{s})$ recovers Kronheimer-Mrowka-Ozváth-Szabó ([KMOS07])'s contact invariant

$$\psi(Y,\xi) \in \check{HM}_\bullet(-Y), \tag{3}$$

where $\check{HM}_\bullet(-Y)$ is the monopole Floer homology of Y.

See table 1 for relations between several "Seiberg-Witten type" invariants.

Also, there are not many calculations of $\Psi(Y,\xi)$. For example, the following is open:

Conjecture 3.7. For any overtwisted contact structure, the invariant Ψ given in [IT20] vanishes.

We list related invariants as follows:

	Counting	Finite dimensional approximation
closed 4-manifolds	SW-invariant $\in \mathbb{Z}$	BF-invariant $BF(X) : (\mathbb{R}^m \oplus \mathbb{C}^n)^+ \to (\mathbb{R}^{m'} \oplus \mathbb{C}^{n'})^+$
4-manifolds with contact boundary	KM-invariant $\in \mathbb{Z}/\{\pm 1\}$	Iida's invariant $\Psi(W,\xi) : (\mathbb{R}^M)^+ \to (\mathbb{R}^{M'})^+$
closed 3-manifolds	monopole Floer homology group "$HM_\bullet(Y)$"	SW Floer homotopy type $SWF(Y)$
4-manifolds with boundary	relative SW invariant "$\psi(X) \in HM_\bullet(\partial X)$"	relative BF invariant $BF(X) : (\mathbb{R}^m \oplus \mathbb{C}^n)^+ \to SWF(\partial X)$
contact 3-manifolds	contact invariant $\psi(Y,\xi) \in \check{HM}_\bullet(-Y)$	homotopy contact invariant $\Psi(Y,\xi) : (\mathbb{R}^M)^+ \to SWF(-Y)$

Table 1: Invariants

Also, it is natural to ask:

Question 3.8. Is there a contact rational homology 3-sphere (Y,ξ) such that

(1) Floer homotopy contact invariant $\Psi(Y,\xi) : S^0 \to \Sigma^* \, SWF(-Y,\mathfrak{s}_\xi)$([IT20]) does not vanish, but

(2) Kronheimer-Mrowka-Ozváth-Szabó ([KMOS07])'s contact invariant $\psi(Y,\xi) \in \check{HM}_\bullet(-Y)$ vanishes?

Since the double branched cover of transverse links admits a natural contact structure, it is interesting to ask:

Question 3.9. Can we define an invariant of transverse links from $\Psi(Y,\xi) : S^0 \to \Sigma^* \, SWF(-Y,\mathfrak{s}_\xi)$?

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