Strong Convergence Theorems under Hybrid Methods for Two Nonlinear Mappings in Banach Spaces (Study on Nonlinear Analysis and Convex Analysis)

Strong Convergence Theorems under Hybrid Methods for Two Nonlinear Mappings in Banach Spaces

慶應義塾大学研究教育センター, 中国医薬大学神経網際計算研究センター
高橋渉 (Wataru Takahashi)
Keio Research and Education Center for Natural Sciences, Keio University, Japan and
Research Center for Interneural Computing, China Medical University Hospital,
China Medical University, Taichung 40447, Taiwan
Email: wataru@is.titech.ac.jp; wataru@a00.itscom.net

Abstract. In this article, using the hybrid method defined by Nakajo and Takahashi [17], we first obtain a strong convergence theorem for two noncommutative nonlinear mappings in a Banach space. Next, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota [25], we prove another strong convergence theorem for the mappings in a Banach space. Using these results, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space and a Banach space.

2010 *Mathematics Subject Classification*: 47H05, 47H09
Keywords and phrases: Fixed point, skew-generalized nonspreading mapping, hybrid method, shrinking projection method.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. In 2010, Kocourek, Takahashi and Yao [11] defined a broad class of nonlinear mappings in a Hilbert space: A mapping $T : C \to H$ is called *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

Such a mapping T is called (α, β)-*generalized hybrid*. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is *nonspreading* [14, 15] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [22] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous [8]. Recently, by using the hybrid method of Nakajo and Takahashi [17], Hojo and Takahashi [2] obtained a strong convergence theorem for two noncommutative generalized hybrid mappings in a Hilbert space. Furthermore, by using the shrinking projection method of Takahashi, Takeuchi and Kubota [25], they proved another strong convergence theorem in a Hilbert space.

In this article, using the hybrid method defined by Nakajo and Takahashi [17], we first obtain a strong convergence theorem for two noncommutative nonlinear mappings in a Banach space. Next, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota [25], we prove another strong convergence theorem for the mappings in a Banach space. Using these results, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space and a Banach space.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}, \quad \forall \epsilon \in \mathbb{R} \ \text{ with } \ 0 \le \epsilon \le 2.$$

A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. If C is a nonempty, closed and convex subset of a strictly convex Banach space E and $T : C \to E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [9]. For a Banach space E,. the duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called *smooth*. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. For more details, see [19, 20, 21]. Let E be a smooth Banach space. The function $\phi\colon E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E,$$

where J is the duality mapping of E; see [1] and [10]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{2.2}$$

From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \quad \forall x, y, z, w \in E. \tag{2.3}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.4}$$

Let E be a smooth, strictly convex and reflexive Banach space. Let $\phi_*\colon E^* \times E^* \to \mathbb{R}$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2, \quad \forall x^*, y^* \in E^*,$$

where J is the duality mapping of E. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx), \quad \forall x, y \in E. \tag{2.5}$$

The following results are in Xu [28] and Kamimura and Takahashi [10].

Lemma 2.1 ([28])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.2 ([10])**.** *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.3 ([10])**.** *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called *generalized nonexpansive* [5] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y), \quad \forall x \in C,\ y \in F(T).$$

Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* [18] if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E,\ t \geq 0.$$

A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [4, 5] for more details. The following results are in Ibaraki and Takahashi [5].

Lemma 2.4 ([5]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([5]). *Let C be a nonempty and closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [13] proved the following results:

Lemma 2.6 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty and closed subset of E. Then the following are equivalent:*

(a) C is a sunny generalized nonexpansive retract of E;
(b) C is a generalized nonexpansive retract of E;
(c) JC is closed and convex.

Lemma 2.7 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y\in C}\phi(x, y)$.

Ibaraki and Takahashi [7] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.8 ([7]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.6 and 2.8.

Lemma 2.9 ([7]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be *monotone* if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [21]. Let E be a smooth, strictly convex and reflexive Banach space and let B be a maximal monotone operator of E^* into 2^E. For each $r > 0$ and $x \in E$, consider the set

$$J_r x = \{z \in E : x \in z + rBJz\}.$$

Then $J_r x$ consists of one point. Such J_r is called the *sunny generalized resolvent* of B and is denoted by $J_r = (I + BJ)^{-1}$. It follows that for any $x, y \in E$ and $r > 0$,

$$\langle x - J_r x - (y - J_r y), JJ_r x - JJ_r y\rangle \geq 0. \tag{2.6}$$

See [5] for more details.

3 Strong convergence theorems by hybrid methods

In this section, using the hybrid method by Nakajo and Takahashi [17], we first prove a strong convergence theorem for two noncommutative generic skew 2-generalized nonspreading mappings in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to C$ is called *generic 2-generalized nonspreading* [23] if there exist $\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0, \gamma_2, \gamma_1, \gamma_0, \delta_2, \delta_1, \delta_0 \in \mathbb{R}$ such that
$\alpha_2 + \alpha_1 + \alpha_0 + \beta_2 + \beta_1 + \beta_0 \geq 0, \quad \alpha_2 + \alpha_1 + \alpha_0 > 0 \quad$ and

$$\begin{aligned}\alpha_2\phi(T^2x,Ty) &+ \alpha_1\phi(Tx,Ty) + \alpha_0\phi(x,Ty)\\ &+ \beta_2\phi(T^2x,y) + \beta_1\phi(Tx,y) + \beta_0\phi(x,y)\\ &\leq \gamma_2\{\phi(Ty,T^2x) - \phi(Ty,Tx)\} + \gamma_1\{\phi(Ty,Tx) - \phi(Ty,x)\}\\ &\quad + \gamma_0\{\phi(Ty,x) - \phi(Ty,T^2x)\} + \delta_2\{\phi(y,T^2x) - \phi(y,Tx)\}\\ &\quad + \delta_1\{\phi(y,Tx) - \phi(y,x)\} + \delta_0\{\phi(y,x) - \phi(y,T^2x), \quad \forall x, y \in C.\end{aligned} \tag{3.1}$$

A mapping $T : C \to E$ is called *generic generalized nonspreading* [26] if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \beta > 0$ and

$$\begin{aligned}\alpha\phi(Tx,Ty)&+\beta\phi(x,Ty) + \gamma\phi(Tx,y) + \delta\phi(x,y)\\ &\leq \varepsilon\{\phi(Ty,Tx) - \phi(Ty,x)\} + \zeta\{\phi(y,Tx) - \phi(y,x)\}, \quad \forall x, y \in C.\end{aligned} \tag{3.2}$$

We call such a mapping a *generic* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$*-generalized nonspreading mapping.* A generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading mapping $T : C \to E$ is *generalized nonspreading* in the sense of Kocourek, Takahashi and Yao [12] if $\alpha + \beta = -\gamma - \delta = 1$ in (3.2). In particular, putting $\alpha = 1$, $\beta = \delta = 0$, $\gamma = \varepsilon = -1$ and $\zeta = 0$ in (3.2), we obtain that

$$\phi(Tx,Ty) + \phi(Ty,Tx) \leq \phi(Tx,y) + \phi(Ty,x), \quad \forall x, y \in C.$$

Such a mapping is *nonspreading* in the sense of Kohsaka and Takahashi [15]. A nonspreading mapping is obtained from a resolvent of a maximal monotone operator in a Banach space; see [15]. A mapping $T : C \to C$ is called *generic skew 2-generalized nonspreading* [23] if there exist $\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0, \gamma_2, \gamma_1, \gamma_0, \delta_2, \delta_1, \delta_0 \in \mathbb{R}$ such that

$\alpha_2 + \alpha_1 + \alpha_0 + \beta_2 + \beta_1 + \beta_0 \geq 0, \quad \alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\begin{aligned}\alpha_2\phi(Ty,T^2x) &+ \alpha_1\phi(Ty,Tx) + \alpha_0\phi(Ty,x)\\ &+ \beta_2\phi(y,T^2x) + \beta_1\phi(y,Tx) + \beta_0\phi(y,x)\\ &\leq \gamma_2\{\phi(T^2x,Ty) - \phi(Tx,Ty)\} + \gamma_1\{\phi(Tx,Ty) - \phi(x,Ty)\}\\ &\quad + \gamma_0\{\phi(x,Ty) - \phi(T^2x,Ty)\} + \delta_2\{\phi(T^2x,y) - \phi(Tx,y)\}\\ &\quad + \delta_1\{\phi(Tx,y) - \phi(x,y)\} + \delta_0\{\phi(x,y) - \phi(T^2x,y)\}, \quad \forall x,y \in C.\end{aligned} \tag{3.3}$$

A mapping $T : C \to E$ is called *generic skew generalized nonspreading* [26] if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \beta > 0$ and

$$\begin{aligned}\alpha\phi(Ty,Tx) &+ \beta\phi(Ty,x) + \gamma\phi(y,Tx) + \delta\phi(y,x)\\ &\leq \varepsilon\{\phi(Tx,Ty) - \phi(x,Ty)\} + \zeta\{\phi(Tx,y) - \phi(x,y)\}, \quad \forall x,y \in C.\end{aligned} \tag{3.4}$$

We call such a mapping a *generic* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$*-skew generalized nonspreading mapping.* For example, a generic $(1, 0, -1, 0, -1, 0)$-skew generalized nonspreading mapping is a *skew nonspreading* mapping in the sense of Ibaraki and Takahashi [6], i.e.,

$$\phi(Tx,Ty) + \phi(Ty,Tx) \leq \phi(x,Ty) + \phi(y,Tx), \quad \forall x,y \in C.$$

A skew nonspreading mapping is obtained from a sunny generalized resolvent of a maximal monotone operator in a Banach space; see [15]. Let $T : C \to E$ be a generic skew generalized nonspreading mapping satisfying (3.4). Putting $x = u \in F(T)$ in (3.4), we have that

$$\phi(Ty,u) \leq \phi(y,u), \quad \forall y \in C,\ u \in F(T). \tag{3.5}$$

This implies that T is generalized nonexpansive [5]. The following proposition was proved by Takahashi [23].

Proposition 3.1 ([23])**.** *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty, closed and convex subset of E and let T be a generic 2-generalized nonspreading mapping of C into C. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup z$, $x_n - Tx_n \to 0$ and $x_n - T^2x_n \to 0$, then $z \in F(T)$.*

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the *duality mapping* of T; see also [27] and [3]. It is easy to show that if T is a mapping of C into itself, then T^* is a mapping of JC into itself. In fact, for any $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$ from the property of T. So we have

$$T^*x^* = JTJ^{-1}x^* \in JC.$$

Then T^* is a mapping of JC into itself.

Lemma 3.2 ([24])**.** *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let T be a generic skew 2-generalized nonspreading mapping of C into itself such that $F(T) \neq \emptyset$. Then, for any bounded sequence $\{z_n\}$ of C such that $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ and $\lim_{n\to\infty} \|z_n - T^2z_n\| = 0$, every weak cluster point of $\{Jz_n\}$ belongs to $JF(T)$.*

Theorem 3.3 ([24]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let S and T be generic skew 2-generalized nonspreading mappings of C into C such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n)T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n)T^2 x_n\big), \\ C_n = \{z \in C : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, Jz - Jx_n \rangle \leq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_n \cap Q_n}$ is the sunny generalized nonexpansive retraction of E onto $C_n \cap Q_n$, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = R_{F(S)\cap F(T)} x_1$, where $R_{F(S)\cap F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$.

Next, we prove a strong convergence theorem by the shrinking projection method [25] for two noncommutative generic skew-generalized nonspreading mappings in a Banach space.

Theorem 3.4 ([24]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let S and T be generic skew 2-generalized nonspreading mappings of C into C such that $F(S) \cap F(T) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n)T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n)T^2 x_n\big), \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to $z_0 = R_{F(S)\cap F(T)} x_1$, where $R_{F(S)\cap F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$.

4 Applications

In this section, using Theorems 3.3 and 3.4, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space and a Banach space. As a direct result of Theorem 3.3, we have the following theorem for generic skew 2-generalized nonspreading mappings in a Banach space.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let S be a generic skew 2-generalized nonspreading mapping of C into E such that $F(S) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n S x_n + c_n S^2 x_n, \\ C_n = \{z \in C : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, Jz - Jx_n \rangle \leq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_n \cap Q_n}$ is the sunny generalized nonexpansive retraction of E onto $C_n \cap Q_n$, $c, d \in \mathbb{R}$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ satisfy the following:

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = R_{F(S)} x_1$, where $R_{F(S)}$ is the sunny generalized nonexpansive retraction of E onto $F(S)$.

In a Hilbert space H, we have that $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. Using this and from (3.3), we obtain that T is a normally 2-generalized hybrid mapping in the sense of Kondo and Takahashi [16], i.e., there exist $\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0 \in \mathbb{R}$ such that $\alpha_2 + \alpha_1 + \alpha_0 + \beta_2 + \beta_1 + \beta_0 \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\alpha_2 \|T^2 x - Ty\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 + \beta_2 \|T^2 x - y\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \leq 0, \quad \forall x, y \in C.$$

Theorem 4.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be normally 2-generalized hybrid mappings with $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1 - \lambda_n) T x_n\big) + c_n\big(\mu_n S^2 x_n + (1 - \mu_n) T^2 x_n\big), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S) \cap F(T)} x_1$, where $P_{F(S) \cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Theorem 4.3. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty and closed subset of E such that JC is closed and convex. Let S and T be generic skew generalized nonspreading mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$.*

Let $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n) T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n) T^2 x_n\big), \\ C_n = \{z \in C : \phi(y_n, z) \le \phi(x_n, z)\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, Jz - Jx_n \rangle \le 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_n \cap Q_n}$ *is the sunny generalized nonexpansive retraction of* E *onto* $C_n \cap Q_n$, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \le \lambda_n, \mu_n \le b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \le a_n, b_n, c_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{F(S)\cap F(T)} x_1$, *where* $R_{F(S)\cap F(T)}$ *is the sunny generalized nonexpansive retraction of* E *onto* $F(S) \cap F(T)$.

Using Theorem 3.3, we have the following strong convergence theorem for finding a common null point of two maximal monotone operators in a Banach space.

Theorem 4.4. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* B *and* G *be maximal monotone operators of* E^* *into* 2^E *and let* J_r *and* Q_s *be sunny generalized resolvents for* $r > 0$ *and* $s > 0$ *of* B *and* G, *respectively. Suppose that* $B^{-1}0 \cap G^{-1}0 \neq \emptyset$. *Let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n J_r x_n + (1-\lambda_n) Q_s x_n\big) + c_n\big(\mu_n (J_r)^2 x_n + (1-\mu_n)(Q_s)^2 x_n\big), \\ C_n = \{z \in C : \phi(y_n, z) \le \phi(x_n, z)\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, Jz - Jx_n \rangle \le 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_n \cap Q_n}$ *is the sunny generalized nonexpansive retraction of* E *onto* $C_n \cap Q_n$, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \le \lambda_n, \mu_n \le b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \le a_n, b_n, c_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{(BJ)^{-1}0 \cap (GJ)^{-1}0} x_1$, *where* $R_{(BJ)^{-1}0 \cap (GJ)^{-1}0}$ *is the sunny generalized nonexpansive retraction of* E *onto* $(BJ)^{-1}0 \cap (GJ)^{-1}0$.

Similarly, using Theorem 3.4, we have the following results.

Theorem 4.5. *Let* E *be a uniformly convex and uniformly smooth Banach space and let* C *be a nonempty and closed subset of* E *such that* JC *is closed and convex. Let* S *be a generic skew-generalized nonspreading mapping of* C *into* E *such that* $F(S) \neq \emptyset$. *Let* $C_1 = C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n S x_n + c_n S^2 x_n, \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ *is the sunny generalized nonexpansive retraction of* E *onto* C_{n+1}*,* $c, d \in \mathbb{R}$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{F(S)}x_1$*, where* $R_{F(S)}$ *is the sunny generalized nonexpansive retraction of* E *onto* $F(S)$*.*

Theorem 4.6. *Let* H *be a real Hilbert space and let* C *be a nonempty, closed and convex subset of* H*. Let* S *and* T *be normally 2-generalized hybrid mappings such that* $F(S) \cap F(T) \neq \emptyset$*. Let* $C_1 = C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n) T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n) T^2 x_n\big), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ *is the metric projection of* H *onto* C_{n+1}*,* $a, b, c, d \in \mathbb{R}$*,* $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ *converges strongly to* $z_0 = P_{F(S)\cap F(T)}x_1$*, where* $P_{F(S)\cap F(T)}$ *is the metric projection of* H *onto* $F(S) \cap F(T)$*.*

Theorem 4.7. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* C *be a nonempty and closed subset of* E *such that* JC *is closed and convex. Let* S *and* T *be generic skew generalized nonspreading mappings of* C *into itself such that* $F(S) \cap F(T) \neq \emptyset$*. Let* $C_1 = C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n) T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n) T^2 x_n\big), \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ *is the sunny generalized nonexpansive retraction of* E *onto* C_{n+1}*,* $a, b, c, d \in \mathbb{R}$*,* $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{F(S)\cap F(T)}x_1$*, where* $R_{F(S)\cap F(T)}$ *is the sunny generalized nonexpansive retraction of* E *onto* $F(S) \cap F(T)$*.*

Theorem 4.8. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* B *and* G *be maximal monotone operators of* E^* *into* 2^E *and let* J_r *and* Q_s *be sunny generalized resolvents for* $r > 0$ *and* $s > 0$ *of* B *and* G*, respectively. Suppose that* $B^{-1}0 \cap G^{-1}0 \neq \emptyset$*. Let* $C_1 = C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n J_r x_n + (1-\lambda_n) Q_s x_n\big) + c_n\big(\mu_n (J_r)^2 x_n + (1-\mu_n)(Q_s)^2 x_n\big), \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ *is the sunny generalized nonexpansive retraction of* E *onto* C_{n+1}*,* $a, b, c, d \in \mathbb{R}$*,* $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{(BJ)^{-1}0 \cap (GJ)^{-1}0} x_1$*, where* $R_{(BJ)^{-1}0 \cap (GJ)^{-1}0}$ *is the sunny generalized nonexpansive retraction of* E *onto* $(BJ)^{-1}0 \cap (GJ)^{-1}0$*.*

References

[1] Y. I. Alber, *Metric and generalized projections in Banach spaces: Properties and applications*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 15–50.

[2] M. Hojo, and W. Takahashi, *Strong convergence theorems by hybrid methods for noncommutative two nonlinear mappings in Hilbert spaces*, in Nonlinear Analysis and Convex Analtsis (M. Hojo, M. Hoshino and W. Takahashi Eds.), Yokohama Publishers, Yokohama, 2019, pp. 69–82.

[3] T. Honda, T. Ibaraki and W. Takahashi, *Duality theorems and convergence theorems for nonlineaqr mappings in Banach spaces*, Int. J. Math. Statis. **6** (2010), 46–64.

[4] T. Ibaraki and W. Takahashi, *Mosco convergence of sequences of retracts of four nonlinear projections in Banach spaces*, in Nonlinear Analysis and Convex Analtsis (W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2007, pp. 139–147.

[5] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[6] T. Ibaraki and W. Takahashi, *Fixed point theorems for new nonlinear mappings of nonexpansive type in Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), 21–32.

[7] T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces*, Contemp. Math., **513**, Amer. Math. Soc., Providence, RI, 2010, pp. 169–180.

[8] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2008, pp. 75–85.

[9] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79** (1978), 493–508.

[10] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach apace*, SIAM J. Optim. **13** (2002), 938–945.

[11] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[12] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Banach spaces*, Adv. Math. Econ. **15** (2011), 67–88.

[13] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 197-209.

[14] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.

[15] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.

[16] A. Kondo and W. Takahashi, *Attractive point and weak convergence theorems for normally N-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **3** (2017), 297-310.

[17] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372-378.

[18] S. Reich, *Asymptotic behavior of contractions in Banach spaces*, J. Math. Anal. Appl. **44** (1973), 57–70.

[19] S. Reich, *On the asymptotic behavior of nonlinear semigroups and the range of accretive operators*, J. Math. Anal. Appl. **79** (1981), 113–126.

[20] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[21] W. Takahashi, *Convex Analysis and Approximation of Fixed Points (Japanese)*, Yokohama Publishers, Yokohama, 2000.

[22] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[23] W. Takahashi, *Fixed point and weak convergence theorems for new generic generalized nonspreading mappings in Banach Spaces*, J. Nonlinear Convex Anal. **20** (2019), 603–623.

[24] W. Takahashi, *Strong convergence theorems by hybrid methods for generic skew 2-generalized nonspreading mappings in Banach Spaces*, J. Nonlinear Convex Anal. **20** (2019), 2425–2446.

[25] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.

[26] W. Takahashi, N.-C. Wong and J. C. Yao, *Attractive point and mean convergence theorems for new generalized nonspreading mappings in Banach Spaces*, Contemp. Math., vol. 636, Amer. Math. Soc., Providence, RI, 2015, pp. 225–248.

[27] W. Takahashi and J. C. Yao, *Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces*, Taiwanese J. Math. **15** (2011), 787–818.

[28] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1981), 1127–1138.