CONVERGENCE OF SOME ITERATIVE METHODS FOR MONOTONE INCLUSION, VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS (Study on Nonlinear Analysis and Convex Analysis)

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ABSTRACT. In this paper, we introduce two iterative methods (one implicit method and one explicit method) for finding a common element of the zero point set of a set-valued maximal monotone operator, the solution set of the variational inequality problem for a continuous monotone mapping, and the fixed point set of a continuous pseudocontractive mapping in a Hilbert space. Then we establish strong convergence of the proposed iterative methods to a common point of three sets, which is a solution of a certain variational inequality. Further, we find the minimum-norm element in common set of three sets. The main theorems develop and complement some well-known results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let $T : C \to C$ be a self-mapping on C. We denote by $Fix(T)$ the set of fixed points of S.

The monotone inclusion problem plays an essential role in the theory of nonlinear analysis and optimization. Let $B : H \to 2^H$ be a maximal monotone operator. The monotone inclusion problem consists of finding a zero element of B, that is, a solution of the inclusion problem:

$$0 \in Bx. \tag{1.1}$$

The solution set of the problem (1.1) is denoted by $B^{-1}0$. A classical method for solving the problem is proximal point algorithm, proposed by Martinet [9] and generalized by Rockafellar [10]. In some concrete cases including variational inequalities, the monotone inclusion problem requires to find a zero of the sum of two monotone operator. That is, in the case of $F = A + B$, where A and B are monotone operators, the problem is reduced to as follows:

$$\text{find } z \in C \text{ such that } 0 \in (A + B)z.$$

The solution set of this problem is denoted by $(A + B)^{-1}0$.

Let $A : C \to H$ be a nonlinear mapping. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

This problem is called Hartmann-Stampacchia variational inequality ([12]). We denote the set of solutions of the variational inequality problem (1.2) by $VI(C, A)$. As we also know, variational inequality theory has emerged as an important tool in studying a wide class of numerous problem in physics, optimization, variational inequalities, minimax problem, Nash equilibrium problem in noncooperative games and others.

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A fixed point problem is to find a fixed point z of a nonlinear mapping T with property:

$$z \in C,\ Tz = z. \tag{1.3}$$

In order to study the variational inequality problem (1.2) coupled with the fixed point problem (1.3), many researchers have invented some iterative methods for finding an element of $VI(C,A) \cap Fix(T)$, where A and T are nonlinear mappings. For instance, in case that $A : C \to H$ is an inverse-strongly monotone mapping and $T : C \to C$ is a nonexpansive mapping, see [4, 5] and the references therein, and in case that $A : C \to H$ is a continuous monotone mapping and $T : C \to C$ is a continuous pseudocontractive mapping, see [3, 15, 19].

In 2016, Jung [7] proposed an iterative method based on Yamada's hybrid steepest descent method [17] for finding an element of $Fix(T) \cap VI(C,A) \cap B^{-1}0$, where $T : C \to C$ is a continuous pseudocontractive mapping, $A : C \to H$ is continuous monotone mapping, and $B : H \to 2^H$ is a maximal monotone operator.

Some iterative methods for finding an element of $Fix(T) \cap (A+B)^{-1}0$ have been provided by several authors. For instance, in case that $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an inverse-strongly monotone mapping andand $B : H \to 2^H$ is a maximal monotone operator, see [14].

In this paper, as a continuation of study in this direction, we introduce new implicit and explicit iterative methods for finding a common element of the set $\Omega := Fix(T) \cap VI(C,A) \cap B^{-1}0$, where $T : C \to C$ is a continuous pseudocontractive mapping, $A : C \to H$ is a continuous monotone mapping and $B : H \to 2^H$ is a maximal monotone operator. Then we establish strong convergence of the sequences generated by the proposed iterative methods to a common point of three sets, which is a solution of a certain variational inequality. As a direct consequence, we find the unique minimum-norm element of Ω. The main theorems develop and complement some well-known results in the literature.

2. Preliminaries and Lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

A mapping A of C into H is called *monotone* if

$$\langle x - y, Ax - Ay \rangle \geq 0, \quad \forall x,\ y \in C.$$

A mapping A of C into H is called *α-inverse-strongly monotone* (see [4]) if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x,\ y \in C.$$

Clearly, the class of monotone mappings includes the class of α-inverse-strongly monotone mappings.

A mapping T of C into H is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

and T is said to be *k-strictly pseudocontractive* (see [2]) if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is *nonexpansive* (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive.

Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings and the class of nonexpansive mappings as a subclass. Moreover, this inclusion is strict (see Example 5.7.1 and Example 5.7.2 in [1]).

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in H : Bx \neq \emptyset\}$. A set-valued mapping B is said to be a *monotone operator* on H if $\langle x-y, u-v\rangle \geq 0$ for all $x,\ y \in dom(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator $J_\lambda^B = (I + \lambda B)^{-1} : H \to dom(B)$, which is called the *resolvent* of B.

Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is well-known that $B^{-1}0 = Fix(J_\lambda^B)$ for all $\lambda > 0$ is closed and convex and the resolvent J_λ^B is firmly nonexpansive, that is,

$$\|J_\lambda^B x - J_\lambda^B y\|^2 \leq \langle x - y, J_\lambda^B x - J_\lambda^B y\rangle, \quad \forall x,\ y \in H,$$

and that the resolvent identity

$$J_\lambda^B x = J_\mu^B \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^B x\right)$$

holds for all $\lambda,\ \mu > 0$ and $x \in H$.

In a real Hilbert space H, the following hold:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle,$$

and

$$\|\alpha x + \beta y\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta\|x - y\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2,$$

for all $x,\ y \in H$ and $\alpha,\ \beta \in (0,1)$ with $\alpha + \beta = 1$.

We recall that

(i) a mapping $V : C \to H$ is said to be *l-Lipschitzian* if there exists a constant $l \geq 0$ such that
$$\|Vx - Vy\| \leq l\|x - y\|, \quad \forall x,\ y \in C;$$

(ii) a mapping $G : C \to H$ is said to be *η-strongly monotone* if there exists a constant $\eta > 0$ such that
$$\langle Gx - Gy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x,\ y \in C.$$

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([1]). *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.2 ([13]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a real Banach space E, and let $\{\gamma_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n$ for all $n \geq 1$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.3 ([16]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \xi_n)s_n + \xi_n\delta_n, \quad \forall n \geq 1,$$

where $\{\xi_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\xi_n\} \subset [0,1]$ *and* $\sum_{n=1}^\infty \xi_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \xi_n|\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

The following lemmas are Lemma 2.3 and Lemma 2.4 of Zegeye [18], respectively.

Lemma 2.4 ([18])**.** *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping. Then, for $\nu > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Az\rangle + \frac{1}{\nu}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

For $\nu > 0$ and $x \in H$, define $A_\nu : H \to C$ by

$$A_\nu x = \left\{z \in C : \langle y - z, Az\rangle + \frac{1}{\nu}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}.$$

Then the following hold:

(i) *A_ν is single-valued;*

(ii) *A_ν is firmly nonexpansive, that is,*

$$\|A_\nu x - A_\nu y\|^2 \leq \langle x - y, A_\nu x - A_\nu y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(A_\nu) = VI(C, A)$;*

(iv) *$VI(C, A)$ is a closed convex subset of C.*

Lemma 2.5 ([18])**.** *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C\right\}.$$

Then the following hold:

(i) *T_r is single-valued*;

(ii) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(T_r) = Fix(T)$*;

(iv) *$Fix(T)$ is a closed convex subset of C.*

The following lemma is a variant of a Minty lemma (see [9]).

Lemma 2.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $G : C \to H$ is monotone and weakly continuous along segments, that is, $G(x + ty) \to G(x)$ weakly as $t \to 0$. Then the variational inequality*

$$\widetilde{x} \in C, \quad \langle G\widetilde{x}, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C,$$

is equivalent to the dual variational inequality

$$\widetilde{x} \in C, \quad \langle Gp, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C.$$

The following lemmas can be easily proven (see [17]), and therefore, we omit their proof.

Lemma 2.7. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$, and let $G : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa,\ \eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x,\ y \in H.$$

That is, $\mu G - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

Lemma 2.8. *Let H be a real Hilbert space H. Let $G : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < t < 1$. Then $I - t\mu G : H \to H$ is a contractive mapping with a constant $1 - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.*

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$;
- C is a nonempty closed convex of H;
- $B : H \to 2^H$ is a maximal monotone operator with $dom(B) \subset C$;
- $B^{-1}0$ is the set of zero points of B, that is, $B^{-1}0 = \{z \in H : 0 \in Bz\}$;
- $J^B_{\lambda_t} : H \to \mathrm{dom}(B)$ is the resolvent of B for $\lambda_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t \to 0} \lambda_t > 0$;
- $J^B_{\lambda_n} : H \to \mathrm{dom}(B)$ is the resolvent of B for $\lambda_n \in (0, \infty)$ and $\liminf_{n\to\infty} \lambda_n > 0$;
- $G : C \to C$ is a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa,\ \eta > 0$;
- $V : C \to C$ is an l-Lipschitzian mapping with constant $l \in [0, \infty)$;
- Constants $\mu > 0$ and $\gamma \geq 0$ satisfy $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $A : C \to H$ is a continuous monotone mapping;
- $VI(C, A)$ is the solution set of the variational inequality problem (1.2) for A;
- $T : C \to C$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;
- $A_{\nu_t} : H \to C$ is a mapping defined by
$$A_\nu x = \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{\nu} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}$$
for $x \in H$ and $\nu_t \in (0, \infty), t \in (0, 1)$, $\liminf_{t\to 0} \nu_t > 0$;
- $A_{\nu_n} : H \to C$ is a mapping defined by
$$A_{\nu_n} x = \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{\nu_n} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}$$
for $x \in H$ and $\nu_n \in (0, \infty)$, $\liminf_{n\to\infty} \nu_n > 0$;
- $T_{r_t} : H \to C$ is a mapping defined by
$$T_{r_t} x = \left\{ z \in C : \langle Tz, y - z \rangle - \frac{1}{r_t} \langle y - z, (1 + r_t)z - x \rangle \leq 0, \quad \forall y \in C \right\}$$
for $x \in H$ and $r_t \in (0, \infty), t \in (0, 1)$, and $\liminf_{t\to 0} r_t > 0$;
- $T_{r_n} : H \to C$ is a mapping defined by
$$T_{r_n} x = \left\{ z \in C : \langle Tz, y - z \rangle - \frac{1}{r_n} \langle y - z, (1 + r_n)z - x \rangle \leq 0, \quad \forall y \in C \right\}$$
for $x \in H$ and $r_n \in (0, \infty)$, and $\liminf_{n\to\infty} r_n > 0$;
- $\Omega := Fix(T) \cap VI(C, A) \cap B^{-1}0 \neq \emptyset$.

By Lemma 2.4 and Lemma 2.5, we note that A_{ν_t}, A_{ν_n}, T_{r_t} and T_{r_n} are nonexpansive, $VI(C, A) = Fix(A_{\nu_t}) = Fix(A_{\nu_n})$ and $Fix(T_{r_t}) = Fix(T_{r_n}) = Fix(T)$.

Now, we introduce the following iterative method that generates a net $\{x_t\}$ in an implicit way:

$$x_t = T_{r_t}(t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t),\ t \in (0, 1). \tag{3.1}$$

For $t \in (0, 1)$, consider the following mapping Q_t on C defined by

$$Q_t x = T_{r_t}(t\gamma V x + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x),\ \forall x \in C.$$

Then, since T_{r_t}, $J^B_{\lambda_t}$ and A_{ν_t} are nonexpansive, for $x,\ y \in C$, we have

$$\begin{aligned}
&\|Q_t x - Q_t y\| \\
&= \|T_{r_t}(t\gamma V x + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x) - (T_{r_t}(t\gamma V y + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} y))\| \\
&\le \|T_{r_t}(t\gamma V x + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x) - T_{r_t}(t\gamma V y + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} y)\| \\
&\le \|t\gamma V x + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x - (t\gamma V y + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} y)\| \\
&\le t\|\gamma V x - \gamma V y\| + \|(I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x - (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} y\| \\
&\le t\gamma l\|x - y\| + (1 - t\tau)\|x - y\| \\
&= (1 - (\tau - \gamma l)t)\|x - y\|.
\end{aligned}$$

Since $0 < 1-(\tau - \gamma l)t < 1$, Q_t is a contractive mapping. By Banach contraction principle, Q_t has a unique fixed point $x_t \in C$, which uniquely solves the fixed point equation

$$x_t = T_{r_t}(t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t),\ t \in (0,1).$$

We summarize the basic property of $\{x_t\}$ and $\{y_t\}$, where $y_t = t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t$.

Proposition 3.1. *Let the net $\{x_t\}$ be defined via (3.1) and let the net $\{y_t\}$ be defined by $y_t = t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t$ for $t \in (0,1)$. Let $w_t = A_{\nu_t} x_t$ for $t \in (0,1)$. Then*

(1) *$\{x_t\}$ and $\{y_t\}$ are bounded for $t \in (0,1)$;*
(2) *x_t defines a continuous path from $(0,1)$ into C and so does y_t provided $r_t,\ \lambda_t,\ \nu_t : (0,1) \to (0,\infty)$ are continuous and $0 < a \le \min\{r_t,\ \lambda_t,\ \nu_t\}$ for $t \in (0,1)$;*
(3) $\lim_{t\to 0} \|A_{\nu_t} x_t - J^B_{\lambda_t} A_{\nu_t} x_t\| = \lim_{t\to 0} \|w_t - J^B_{\lambda_t} w_t\| = 0$;
(4) $\lim_{t\to 0} \|x_t - w_t\| = 0$;
(5) $\lim_{t\to 0} \|x_t - y_t\| = 0$;
(6) $\lim_{t\to 0} \|x_t - J^B_{\lambda_t} A_{\nu_t} x_t\| = \lim_{t\to 0} \|x_t - J^B_{\lambda_t} w_t\| = 0$;
(7) $\lim_{t\to 0} \|x_t - T_{r_t} x_t\| = 0$;
(8) $\lim_{t\to 0} \|y_t - T_{r_t} y_t\| = 0$.

By using Proposition 3.1, we establish strong convergence of the path x_t, which guarantees the existence of solutions of the variational inequality (3.2) below.

Theorem 3.2. *Let the net $\{x_t\}$ be defined by (3.1). Let $r_t,\ \lambda_t,\ \nu_t : (0,1) \to (0,\infty)$ be continuous and $0 < a \le \min\{r_t,\ \lambda_t,\ \nu_t\}$ for $t \in (0,1)$. Then x_t converges strongly, as $t \to 0$, to a point $q \in \Omega$, which is the unique solution of the variational inequality:*

$$\langle(\mu G - \gamma V)q, p - q\rangle \ge 0, \quad \forall p \in \Omega. \tag{3.2}$$

By taking $V \equiv 0$, $G \equiv I$, $\mu = 1$ in Theorem 3.2, we obtain the following result.

Corollary 3.3. *Let the net $\{x_t\}$ be defined by*

$$x_t = T_{r_t}((1 - t)J^B_{\lambda_t} A_{\nu_t} x_t),\ t \in (0,1).$$

Let $r_t,\ \lambda_t,\ \nu_t : (0,1) \to (0,\infty)$ be continuous and $0 < a \le \min\{r_t,\ \lambda_t,\ \nu_t\}$ for $t \in (0,1)$. Then x_t converges strongly, as $t \to 0$, to q, which solves the following minimum-norm problem : find $q \in \Omega$ such that

$$\|q\| = \min_{x\in\Omega} \|x\|.$$

Now, we propose a new iterative algorithm which generates a sequence $\{x_n\}$ in an explicit way: for an arbitrarily chosen $x_0 \in C$,

$$n \ge 0, \qquad x_{n+1} = \beta_n x_n + (1 - \beta_n)T_{r_n}(\alpha_n \gamma V x_n + (I - \alpha_n \mu G)J^B_{\lambda_n} A_{\nu_n} x_n), \tag{3.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, and $\{r_n\},\ \{\lambda_n\},\ \{\nu_n\} \subset (0,\infty)$, and establish strong convergence of this sequence to a common element of Ω.

Theorem 3.4. *Let the sequence* $\{x_n\}$ *be generated iteratively by the explicit algorithm* (3.3)*. Let* $\{\alpha_n\}$*,* $\{\beta_n\} \subset (0,1)$ *and* $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C4) $0 < a \leq r_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(C5) $0 < a \leq \lambda_n < \infty$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;
(C6) $0 < a \leq \nu_n < \infty$ *and* $\lim_{n\to\infty} |\nu_{n+1} - \mu_n| = 0$.

Then $\{x_n\}$ *converges strongly to a point* $q \in \Omega$*, which is the unique solution of the variational inequality* (3.2)*.*

By taking $V \equiv 0$, $G \equiv I$, $\mu = 1$ in Theorem 3.4, we obtain the following result.

Corollary 3.5. *Let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) T_{r_n}((1-\alpha_n) J^B_{r_n} A_{r_n} x_n), \quad n \geq 0.$$

Let $\{\alpha_n\}$*,* $\{\beta_n\} \subset (0,1)$ *and* $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ *satisfy the conditions* (C1), (C2), (C3), (C4), (C5) *and* (C6) *in Theorem 3.4. Then* $\{x_n\}$ *converges strongly to a point* $q \in \Omega$*, which is the minimum-norm element of* Ω*.*

If in Theorem 3.4, we take $T \equiv I$, identity mapping on C, then we obtain the following result.

Corollary 3.6. *Suppose that* $\Omega_1 = VI(C, A) \cap B^{-1}0 \neq \emptyset$*. Let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n \gamma V x_n + (I - \alpha_n \mu G) J^B_{\lambda_n} A_{\nu_n} x_n), \quad n \geq 0.$$

Let $\{\alpha_n\}$*,* $\{\beta_n\} \subset (0,1)$ *and* $\{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ *satisfy the conditions* (C1), (C2), (C3), (C5) *and* (C6) *in Theorem 3.4. Then* $\{x_n\}$ *converges strongly to a point* $q \in \Omega_1$*, which is the unique solution of the following variational inequality:*

$$\langle (\mu G - \gamma V) q, p - q \rangle \geq 0, \quad \forall p \in \Omega_1.$$

If in Theorem 3.4, we have $C \equiv H$, then we have the following corollary.

Corollary 3.7. *Suppose that* $\Omega_2 = Fix(T) \cap A^{-1}0 \cap B^{-1}0 \neq \emptyset$*. Let* $T : H \to H$ *be a continuous pseudocontractive mapping and let* $A : H \to H$ *be a continuous monotone mapping. Let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) T_{r_n}(\alpha_n \gamma V x_n + (I - \alpha_n \mu G) J^B_{\lambda_n} A_{\nu_n} x_n), \quad n \geq 0.$$

Let $\{\alpha_n\}$*,* $\{\beta_n\} \subset (0,1)$ *and* $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ *satisfy the conditions* (C1), (C2), (C3), (C4), (C5) *and* (C6) *in Theorem 3.4. Then* $\{x_n\}$ *converges strongly to a point* $q \in \Omega_2$*, which is the unique solution of the following variational inequality:*

$$\langle (\mu G - \gamma V) q, p - q \rangle \geq 0, \quad \forall p \in \Omega_2.$$

.

Proof. Since $D(A) = H$, we note that $VI(H, A) = A^{-1}0$. So the result follows from Theorem 3.4. □

Remark 3.8. 1) It is worth pointing out that implicit and explicit iterative algorithms are new ones different from those announced by several authors; see, for instance, [6, 7, 14] and the references therein. In particular, we use the variable parameters r_t, λ_t, ν_t and r_n, λ_n, ν_n in comparison with the corresponding iterative algorithms in [6, 7, 14] and the references therein.

2) We know that $Fix(T) \cap VI(C, A) \cap B^{-1}0 \subset Fix(T) \cap (A + B)^{-1}0$ (see [7]). Thus, as results for finding a common element of the fixed point set of continuous pseudocontractive mapping more general than nonexpansive mapping and strictly pseudocontractive mapping and the zero point set of sum of maximal monotone operator and continuous monotone mapping more general than α-inverse strongly monotone mapping, Theorem 3.2 and Theorem 3.4 are new results, which develop and improve the corresponding results in [6, 11, 14] and the references therein.
3) Corollary 3.3 and Corollary 3.5 are also new results for finding a minimum norm point of $Fix(T) \cap VI(C, A) \cap B^{-1}0$, where T is a continuous pseudocontractive mapping, A is a continuous monotone mapping and B is a maximal monotone operator.
4) By taking $V \equiv 0$, $G \equiv I$ and $\mu = 1$ in Corollary 3.6 and Corollary 3.7, we can obtain new results for finding the minimum-norm point of $VI(C, A) \cap B^{-1}0$ and $Fix(T) \cap A^{-1}0 \cap B^{-1}0$, respectively.
5) As applications in [14], by using Theorem 3.2 and Theorem 3.4, we can propose implicit and explicit iterative algorithms for the equilibrium problems coupled with fixed point problem for continuous pseudocontractive mapping.

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