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# On THE AVERAGE JOINT CYCLE INDEX AND THE AVERAGE JOINT WEIGHT ENUMERATOR (Research on finite groups, algebraic combinatorics, and vertex algebras)

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# ON THE AVERAGE JOINT CYCLE INDEX AND THE AVERAGE JOINT WEIGHT ENUMERATOR

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ABSTRACT. In this paper, we introduce the concept of the complete joint cycle index and the average complete joint cycle index, and discuss a relation with the complete joint weight enumerator and the average complete joint weight enumerator respectively in coding theory.

**Key Words and Phrases.** Cycle index, Complete weight enumerator.

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## 1. INTRODUCTION

Let $G$ be a permutation group on a set $\Omega$, where $|\Omega| = n$. For each element $h \in G$, we can decompose the permutation $h$ into a product of disjoint cycles; let $c(h, i)$ be the number of $i$-cycles occurring by the action of $h$.

**Definition 1.1** (Cameron [2])**.** The *cycle index* of $G$ is the polynomial $Z(G; s_1, \ldots, s_n)$ in indeterminates $s_1, \ldots, s_n$ defined as

$$Z(G; s_1, \ldots, s_n) = \sum_{g \in G} s_1^{c(g,1)} \cdots s_n^{c(g,n)}.$$

**Example 1.1.** Let $G$ be the symmetric group of degree 4. Each partition of 4 is the cycle type of some element of $G$. We have the following number of elements of $G$ corresponding to each partition:

| Partition | 4 | 31 | 22 | 211 | 1111 |
|---|---|---|---|---|---|
| # of elements | 6 | 8 | 3 | 6 | 1 |

TABLE 1. Partitions and element numbers in $G$.

Therefore, the cycle index of $G$ is

$$Z(G; s_1, s_2, s_3, s_4) = 6s_4 + 8s_1 s_3 + 3s_2^2 + 6s_2 s_1^2 + s_1^4.$$

**Definition 1.2** (Miezaki-Oura [6])**.** The *complete cycle index* of $G$ is the polynomial $Z(G; s(h,i) : h \in G, i \in \mathbb{N})$ in indeterminates $\{s(h,i) \mid h \in G, i \in \mathbb{N}\}$ given by

$$Z(G; s(h,i) : h \in G, i \in \mathbb{N}) = \sum_{h \in G} \prod_{i \in \mathbb{N}} s(h,i)^{c(h,i)},$$

where $\mathbb{N} := \{x \in \mathbb{Z} \mid x > 0\}$.

Let $\mathbb{F}_q$ be the finite field of order $q$, where $q$ is a prime power. An $\mathbb{F}_q$-linear code $C$ is a vector subspace of $\mathbb{F}_q^n$. The *dual code* of a code $C$ is

$$C^\perp := \{\mathbf{v} \in \mathbb{F}_q^n \mid \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{u} \in C\},$$

where $\mathbf{u} \cdot \mathbf{v} := \sum_{i=1}^n u_i v_i$ denotes the *inner product* of $\mathbf{u}$ and $\mathbf{v}$. If $C = C^\perp$, then $C$ is called *self-dual*. The *weight* of $\mathbf{u} \in C$ is denoted by $\text{wt}(\mathbf{u}) := \#\{i \mid u_i \neq 0\}$. For $\mathbf{u} \in C$, we denote the *composition* of $\mathbf{u}$ by $s(\mathbf{u}) := (s_a(\mathbf{u}) : a \in \mathbb{F}_q)$, where $s_a(\mathbf{u}) := \#\{i \mid u_i = a\}$.

**Definition 1.3.** Let $C$ be an $\mathbb{F}_q$-linear code of length $n$. Then the *weight enumerator* of $C$ is the homogeneous polynomial

$$w_C(x,y) := \sum_{\mathbf{u} \in C} x^{n-\text{wt}(\mathbf{u})} y^{\text{wt}(\mathbf{u})} \in \mathbb{C}[x,y],$$

and the *complete weight enumerator* of $C$ is defined as:

$$\mathbf{cwe}_C(x_a : a \in \mathbb{F}_q) := \sum_{\mathbf{u} \in C} \prod_{a \in \mathbb{F}_q} x_a^{s_a(\mathbf{u})} \in \mathbb{C}[x_a : a \in \mathbb{F}_q].$$

**Definition 1.4.** Let $C$ be an $[n,k]$ *linear code* over $\mathbb{F}_q$. We construct a *permutation group* $G(C)$ from $C$ whose *cycle index* is the *weight enumerator*. The group we construct is the *additive* group of $C$. We let it act on the set

$$\{1, \ldots, n\} \times \mathbb{F}_q \quad \text{(a set of cardinality } nq)$$

in the following way: the codeword $(u_1, \ldots, u_n)$ acts as the permutation

$$(i, x) \mapsto (i, x + u_i)$$

of the set $\{1, \ldots, n\} \times \mathbb{F}_q$. The group $G(C)$ is an elementary *abelian group* of order $q^k$. We call the cycle index

$$Z(G(C); s_1, \ldots, s_n)$$

the *cycle index for a code* $C$. We call the complete cycle index

$$Z(G(C); s(g,i) : g \in G(C), i \in \mathbb{N})$$

the *complete cycle index for a code $C$.*

**Example 1.2.** Let $C := \{(0,0), (0,1), (1,0), (1,1)\} = \mathbb{F}_2^2$ be a code. Again let $G(C)$ be the permutation groups on

$$\{1,2\} \times \mathbb{F}_2 = \{(1,0), (1,1), (2,0), (2,1)\}.$$

For $\mathbf{u} = (u_1, u_2) \in C$ acts as a permutation on $\{1,2\} \times \mathbb{F}_2$ as follows:

$$(i, x) \mapsto (i, x + u_i).$$

Now let $\mathbf{u} = (0,1) \in C$. Then

$$(1,0) \mapsto (1, 0 + 0) = (1,0) \Leftarrow \text{1-cycle},$$
$$(1,1) \mapsto (1, 1 + 0) = (1,1) \Leftarrow \text{1-cycle},$$
$$(2,0) \mapsto (2, 0 + 1) = (2,1) \mapsto (2, 1 + 1) = (2,0) \Leftarrow \text{2-cycle}.$$

Therefore the partition is 211.

| $\mathbf{u} \in C$ | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
|---|---|---|---|---|
| **Partitions** | 1111 | 211 | 211 | 22 |

TABLE 2. Elements and Partitions in $G(C)$.

Therefore, the cycle index, $Z(G(C), s_1, s_2) = s_1^4 + 2s_1^2 s_2 + s_2^2$. Then the complete cycle index,

$$Z(G(C); s(h,i) : h \in G(C), i \in \mathbb{N})$$
$$= s((0,0), 1)^2 s((0,0), 1)^2 + s((0,1), 1)^2 s((0,1), 2)^1$$
$$+ s((1,0), 2)^1 s((1,0), 1)^2 + s((1,1), 2)^1 s((1,1), 2)^1.$$

**Theorem 1.1** ([2, 6]). *Let $C$ be a linear code over $\mathbb{F}_q$ of length $n$, where $q$ is a power of the prime number $p$. Then we have the following results.*

(i) $W_C(x, y) = Z(G(C); s_1 \leftarrow x^{1/q}, s_p \leftarrow y^{p/q})$.

(ii) *Let $T$ be a map defined as: for each $g = (u_1, \ldots, u_n) \in C$ and $i \in \{1, \ldots, n\}$, if $u_i = 0$, then $s(g, 1) \mapsto x_{u_i}^{1/q}$; if $u_i \neq 0$, then $s(g, p) \mapsto x_{u_i}^{p/q}$. Then $\mathbf{cwe}_C(x_a : a \in \mathbb{F}_q) = T(Z(G(C); s(g, i) : g \in G(C), i \in \mathbb{N}))$.*

The notion of the joint weight enumerator of two $\mathbb{F}_q$-linear codes was introduced in [5]. Further, the notion of the $g$-fold complete joint weight enumerator of $g$ linear codes over $\mathbb{F}_q$ was given in [7].

**Definition 1.5** ([7]). Let $C$ and $D$ be two linear codes of length $n$ over $\mathbb{F}_q$. The *complete joint weight enumerator* of codes $C$ and $D$ is defined as follows:

$$\mathcal{J}_{C,D}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^2) := \sum_{\mathbf{u} \in C, \mathbf{v} \in D} \prod_{\mathbf{a} \in \mathbb{F}_q^2} x_{\mathbf{a}}^{n_{\mathbf{a}}(\mathbf{u}, \mathbf{v})},$$

where $n_{\mathbf{a}}(\mathbf{u}, \mathbf{v})$ denotes the number of $i$ such that $\mathbf{a} = (u_i, v_i)$.

Let $G$ and $H$ be two permutation groups on a set $\Omega$, where $|\Omega| = n$. Again let $\mathcal{G}_{G,H} := G \times H$ be the direct product of $G$ and $H$. For each element $(g, h) \in \mathcal{G}_{G,H}$, where $g \in G$ and $h \in H$, we can decompose each permutation of the pair $(g, h)$ into a product of disjoint cycles. Let $c(gh, i)$ be the number of $i$-cycles occurring by the action of $gh$, where $gh$ denotes the *product of permutations* $g$ and $h$ which acts on $\Omega$ as $(gh)(\alpha) = h(g(\alpha))$ for any $\alpha \in \Omega$.

**Definition 1.6.** The *complete joint cycle index* of permutation groups $G$ and $H$ is the polynomial

$$\mathcal{Z}_{G,H}(s((g, h), i)) := \mathcal{Z}(\mathcal{G}_{G,H}; s((g, h), i) : (g, h) \in \mathcal{G}_{G,H}, i \in \mathbb{N})$$

in indeterminates $s((g, h), i)$, where $(g, h) \in \mathcal{G}_{G,H}$ and $i \in \mathbb{N}$, given by

$$\mathcal{Z}_{G,H}(s((g, h), i)) := \sum_{(g,h) \in \mathcal{G}_{G,H}} \prod_{i \in \mathbb{N}} s((g, h), i)^{c(gh, i)}.$$

The concept of the complete joint cycle index is used in Theorem 2.1. Theorem 2.1 gives a relation between complete joint cycle index and complete joint weight enumerator. This generalizes the earlier work Theorem 1.1. Further, we give the notion of the $r$-fold complete joint cycle index and the $(\ell, r)$-fold complete joint weight enumerator. In this paper we also give a link between the $r$-fold complete joint cycle index and the $(\ell, r)$-fold complete joint weight enumerator. The link is a generalization of Theorem 2.1. This result presents us a new application of constructing the average $r$-fold complete joint cycle index and a motivation to establish a relation with the average $(\ell, r)$-fold complete joint weight enumerator.

## 2. THE RELATION

In this section, from any two $\mathbb{F}_q$-linear codes, we construct two permutation groups, whose complete joint cycle index is essentially the complete joint weight enumerator of codes.

**Definition 2.1.** Let $C$ and $D$ be two linear codes of length $n$ over $\mathbb{F}_q$. We construct from $C$ and $D$ two permutation groups $G(C)$ and $H(D)$

respectively. The groups $G(C)$ and $H(D)$ are the additive group of $C$ and $D$ respectively. We let each group act on the set $\{1, \ldots, n\} \times \mathbb{F}_q$ in the following way: the codeword $(u_1, \ldots, u_n)$ acts as the permutation

$$(i, x) \mapsto (i, x + u_i)$$

of the set $\{1, \ldots, n\} \times \mathbb{F}_q$. We define the *product* of two permutations $(u_1, \ldots, u_n) \in C$ and $(v_1, \ldots, v_n) \in D$ as follows:

$$(i, x) \mapsto (i, x + u_i + v_i)$$

of a set $\{1, \ldots, n\} \times \mathbb{F}_q$. Let $\mathcal{G}_{C,D} := G(C) \times H(D)$. We call the complete joint cycle index

$$\mathcal{Z}_{C,D}(s((g,h),i)) := \mathcal{Z}(\mathcal{G}_{C,D}; s((g,h),i) : (g,h) \in \mathcal{G}_{C,D}, i \in \mathbb{N})$$

the complete joint cycle index for codes $C$ and $D$.

**Example 2.1.** Let

$$C := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, D := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Then the complete joint weight enumerator is

$$x_{00}^2 + x_{01}^2 + x_{00}x_{10} + x_{01}x_{11} + x_{10}x_{00} + x_{11}x_{01} + x_{10}^2 + x_{11}^2.$$

Let $G(C)$ and $H(D)$ are the permutation groups on $\{1, 2\} \times \mathbb{F}_2$. In the following calculation, for $g \in G(C)$ and $h \in H(D)$, we prefer to write the indeterminates $s((g, h), i)$ as

$$s\left(\begin{pmatrix} g \\ h \end{pmatrix}, i\right).$$

Then the joint cycle index is

$$s\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1\right)^2 s\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1\right)^2 + s\left(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, 2\right)^1 s\left(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, 2\right)^1$$

$$+ s\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1\right)^2 s\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 2\right)^1 + s\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 2\right)^1 s\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 1\right)^2$$

$$+ s\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 2\right)^1 s\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 1\right)^2 + s\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 1\right)^2 s\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 2\right)^1$$

$$+ s\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, 2\right)^1 s\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, 1\right)^2 + s\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, 1\right)^2 s\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, 1\right)^2.$$

Now we have the following result.

**Theorem 2.1.** *Let $C$ and $D$ be two codes over $\mathbb{F}_q$ of length $n$, where $q$ is a power of the prime number $p$. Let $\mathcal{J}_{C,D}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^2)$ be the complete joint weight enumerator and $\mathcal{Z}(\mathcal{G}_{C,D}; s((g,h),i) : (g,h) \in \mathcal{G}_{C,D}, i \in \mathbb{N})$ be the complete joint cycle index.*

*Let $T$ be a map defined as follows: for each $g = (u_1, \ldots, u_n) \in C$ and $h = (v_1, \ldots, v_n) \in D$, and for $i \in \{1, \ldots, n\}$, if $u_i + v_i = 0$, then*

$$s((g,h),1) \mapsto x_{u_i v_i}^{1/q};$$

*if $u_i + v_i \neq 0$, then*

$$s((g,h),p) \mapsto x_{u_i v_i}^{p/q}.$$

*Then we have*

$$\mathcal{J}_{C,D}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^2) = T(\mathcal{Z}(\mathcal{G}_{C,D}; s((g,h),i) : (g,h) \in \mathcal{G}_{C,D}, i \in \mathbb{N})).$$

## 3. $r$-FOLD COMPLETE JOINT CYCLE INDEX

Let $G_1, G_2, \ldots, G_r$ be $r$ permutation groups on a set $\Omega$, where $|\Omega| = n$. Again let $\mathcal{G}_{G_1, \ldots, G_r} := G_1 \times \cdots \times G_r$ be the direct product of $G_1, G_2, \ldots, G_r$. For any element $(g_1, g_2, \ldots, g_r) \in \mathcal{G}_{G_1, \ldots, G_r}$, where $g_k \in G_k$ for $k \in \{1, 2, \ldots, r\}$, we can decompose each permutation $g_k$ into a product of disjoint cycles. Let $c(g_k, i)$ be the number of $i$-cycles occurring by the action of $g_k$. Now the $r$-*fold complete joint cycle index* of $G_1, G_2, \ldots, G_r$ is the polynomial

$$\mathcal{Z}_{G_1, \ldots, G_r}(s((g_1, \ldots, g_r), i))$$
$$:= \mathcal{Z}(\mathcal{G}_{G_1, \ldots, G_r}; s((g_1, \ldots, g_r), i) : (g_1, \ldots, g_r) \in \mathcal{G}_{G_1, \ldots, G_r}, i \in \mathbb{N})$$

in indeterminates $s((g_1, \ldots, g_r), i)$, where $(g_1, \ldots, g_r) \in \mathcal{G}_{G_1, \ldots, G_r}$ and $i \in \mathbb{N}$, given by

$$\mathcal{Z}_{G_1, \ldots, G_r}(s((g_1, \ldots, g_r), i))$$
$$:= \sum_{(g_1, \ldots, g_r) \in \mathcal{G}_{G_1, \ldots, G_r}} \prod_{i \in \mathbb{N}} s((g_1, \ldots, g_r), i)^{c(g_1 \cdots g_r, i)}.$$

where $g_1 \cdots g_r$ denotes the *product of permutations* $g_1, \ldots, g_r$ which acts on $\Omega$ as $(g_1 \cdots g_r)(\alpha) = g_r(\cdots g_1(\alpha) \cdots)$ for any $\alpha \in \Omega$. If $G_1 = \cdots = G_r = G$, then we call $\mathcal{Z}(\mathcal{G}_{G, \ldots, G}; s((g_1, \ldots, g_r), i) : (g_1, \ldots, g_r) \in \mathcal{G}_{G, \ldots, G}, i \in \mathbb{N})$ the $r$-*fold complete multi-joint cycle index* of $G$.

**Definition 3.1.** We denote, $\Pi^\ell := C_1 \times \cdots \times C_\ell$, where $C_1, \ldots, C_\ell$ be the linear codes of length $n$ over $\mathbb{F}_q$. We call $\Pi^\ell$ as the $\ell$-*fold joint code*

of $C_1, \ldots, C_\ell$. We denote an element of $\Pi^\ell$ by

$$\tilde{\mathbf{c}} := (\mathbf{c}_1, \ldots, \mathbf{c}_n) := \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & \ldots & \vdots \\ a_{\ell 1} & \ldots & a_{\ell n} \end{pmatrix},$$

where $\mathbf{c}_i := {}^t(a_{1i}, \ldots, a_{\ell i}) \in \mathbb{F}_q^\ell$ and $\mu_j(\tilde{\mathbf{c}}) := (a_{j1}, \ldots, a_{jn}) \in C_j$.

Now let $\Pi_1^\ell, \ldots, \Pi_r^\ell$ be the $\ell$-fold joint codes (not necessarily the same) over $\mathbb{F}_q$. For $k \in \{1, \ldots, r\}$, we denote, $\Pi_k^\ell := C_{k1} \times \cdots \times C_{k\ell}$, where $C_{k1}, \ldots, C_{k\ell}$ be the linear codes of length $n$ over $\mathbb{F}_q$. An element of $\Pi_k^\ell$ is denoted by

$$\tilde{\mathbf{c}}_k := (\mathbf{c}_{k1}, \ldots, \mathbf{c}_{kn}) := \begin{pmatrix} a_{11}^{(k)} & \ldots & a_{1n}^{(k)} \\ a_{21}^{(k)} & \ldots & a_{2n}^{(k)} \\ \vdots & \ldots & \vdots \\ a_{\ell 1}^{(k)} & \ldots & a_{\ell n}^{(k)} \end{pmatrix},$$

where $\mathbf{c}_{ki} := {}^t(a_{1i}^{(k)}, \ldots, a_{\ell i}^{(k)}) \in \mathbb{F}_q^\ell$. and $\mu_j(\tilde{\mathbf{c}}_k) := (a_{j1}^{(k)}, \ldots, a_{jn}^{(k)}) \in C_{kj}$. Then the $(\ell, r)$-*fold complete joint weight enumerator* of $\Pi_1^\ell, \ldots, \Pi_r^\ell$ is defined as follows:

$$\mathcal{J}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r}) := \sum_{\tilde{\mathbf{c}}_1 \in \Pi_1^\ell, \ldots, \tilde{\mathbf{c}}_r \in \Pi_r^\ell} \prod x_{\mathbf{a}}^{n_{\mathbf{a}}(\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_r)},$$

where $n_{\mathbf{a}}(\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_r)$ denotes the number of $i$ such that $\mathbf{a} = (\mathbf{c}_{1i}, \ldots, \mathbf{c}_{ri})$. For $r = 2$ and $\ell = 1$ the complete $(\ell, r)$-fold joint weight enumerator coincide with complete joint weight enumerator (Definition 1.5).

**Definition 3.2.** We construct from $\Pi^\ell$ a permutation group $G(\Pi^\ell)$. The group we construct is the additive group of $\Pi^\ell$. We let it act on the set $\{1, \ldots, n\} \times \mathbb{F}_q^\ell$ in the following way: $(\mathbf{c}_1, \ldots, \mathbf{c}_n)$ acts as the permutation

$$\left( i, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{pmatrix} \right) \mapsto \left( i, \begin{pmatrix} x_1 + a_{1i} \\ x_2 + a_{2i} \\ \vdots \\ x_\ell + a_{\ell i} \end{pmatrix} \right)$$

of the set $\{1, \ldots, n\} \times \mathbb{F}_q^\ell$. Now let $G_1(\Pi_1^\ell), \ldots, G_r(\Pi_r^\ell)$ be $r$ permutation groups. We define the *product* of $r$ permutations $(\mathbf{c}_{11}, \ldots, \mathbf{c}_{1n}) \in$

$\Pi_1^\ell, \ldots, (\mathbf{c}_{r1}, \ldots, \mathbf{c}_{rn}) \in \Pi_r^\ell$ as follows:

$$\left( i, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{pmatrix} \right) \mapsto \left( i, \begin{pmatrix} x_1 + \sum_{k=1}^r a_{1i}^{(k)} \\ x_2 + \sum_{k=1}^r a_{2i}^{(k)} \\ \vdots \\ x_\ell + \sum_{k=1}^r a_{\ell i}^{(k)} \end{pmatrix} \right)$$

of a set $\{1, \ldots, n\} \times \mathbb{F}_q^\ell$. Let $\mathcal{G}_{\Pi_1^\ell, \ldots, \Pi_r^\ell} := G_1(\Pi_1^\ell) \times \cdots \times G_r(\Pi_r^\ell)$. We call the $r$-fold complete joint cycle index

$$\mathcal{Z}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}(s((g_1, \ldots, g_r), i))$$
$$:= \mathcal{Z}(\mathcal{G}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}; s((g_1, \ldots, g_r), i) : (g_1, \ldots, g_r) \in \mathcal{G}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}, i \in \mathbb{N})$$

the $r$-fold complete joint cycle index for $\Pi_1^\ell, \ldots, \Pi_r^\ell$.

**Remark 3.1.** Let $\Pi_1^\ell = \cdots = \Pi_r^\ell = \Pi^\ell$, where

$$\Pi^\ell := C_1 \times \cdots \times C_\ell,$$

for the $\mathbb{F}_q$-linear codes $C_1, \ldots, C_\ell$ of length $n$. Then we call

$$\mathcal{Z}(\mathcal{G}_{\Pi^\ell, \ldots, \Pi^\ell}; s((g_1, \ldots, g_r), i) : (g_1, \ldots, g_r) \in \mathcal{G}_{\Pi^\ell, \ldots, \Pi^\ell}, i \in \mathbb{N})$$

the *$r$-fold complete multi-joint cycle index for $\Pi^\ell$*.

Again let $C_1 = \cdots = C_\ell = C$, for some $\mathbb{F}_q$-linear code $C$ of length $n$. Then we denote $\Pi^\ell$ by $C^\ell$, that is,

$$C^\ell := \underbrace{C \times \cdots \times C}_{\ell}.$$

We call $\mathcal{Z}(\mathcal{G}_{C^\ell, \ldots, C^\ell}; s((g_1, \ldots, g_r), i) : (g_1, \ldots, g_r) \in \mathcal{G}_{C^\ell, \ldots, C^\ell}, i \in \mathbb{N})$ the *$r$-fold complete multi-joint cycle index for $C^\ell$*. Note that if $r = 1$, the $r$-fold complete multi-joint cycle index for $C^\ell$ coincide with the *complete cycle index of genus $\ell$* for code $C$ in the sense of Miezaki-Oura [6].

Now we give a generalization of Theorem 2.1 as follows.

**Theorem 3.1.** *For $k \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, \ell\}$, let $C_{kj}$ be an $\mathbb{F}_q$-linear code of length $n$, where $q$ is a power of the prime number $p$. Again let $\Pi_k^\ell$ be the $\ell$-fold joint code of $C_{k1}, \ldots, C_{k\ell}$. Let $\mathcal{J}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}(x_\mathbf{a} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r})$ be the $(\ell, r)$-fold complete joint weight enumerator of $\Pi_1^\ell, \ldots, \Pi_r^\ell$, and*

$$\mathcal{Z}(\mathcal{G}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}; s((g_1, \ldots, g_r), i) : (g_1, \ldots, g_r) \in \mathcal{G}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}, i \in \mathbb{N})$$

*be the $r$-fold complete joint cycle index for $\Pi_1^\ell, \ldots, \Pi_r^\ell$.*

*Let $T$ be a map defined as follows: for each $g_1 = (\mathbf{c}_{11}, \ldots, \mathbf{c}_{1n}) \in \Pi_1^{\ell}, \ldots, g_r = (\mathbf{c}_{r1}, \ldots, \mathbf{c}_{rn}) \in \Pi_r^{\ell}$, and for $i \in \{1, \ldots, n\}$, if $\sum_{k=1}^{r} \mathbf{c}_{ki} = \mathbf{0}$, then*

$$s((g_1, \ldots, g_r), 1) \mapsto x_{\mathbf{c}_{1i} \ldots \mathbf{c}_{ri}}^{1/q^{\ell}};$$

*if $\sum_{k=1}^{r} \mathbf{c}_{ki} \neq \mathbf{0}$, then*

$$s((g_1, \ldots, g_r), p) \mapsto x_{\mathbf{c}_{1i} \ldots \mathbf{c}_{ri}}^{p/q^{\ell}}.$$

*Then we have*

$$\mathcal{J}_{\Pi_1^{\ell}, \ldots, \Pi_r^{\ell}}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r}) =$$
$$T(\mathcal{Z}(\mathcal{G}_{\Pi_1^{\ell}, \ldots, \Pi_r^{\ell}}; s((g_1, \ldots, g_r), i) : (g_1, \ldots, g_r) \in \mathcal{G}_{\Pi_1^{\ell}, \ldots, \Pi_r^{\ell}}, i \in \mathbb{N})).$$

## 4. Main Results

In [8], the notion of the average joint weight enumerators was given. Further, the notion of the average $r$-fold complete joint weight enumerators was given in [4]. In this section, we give the concept of the average complete joint cycle index and provide a relation with average complete joint weight enumerator of codes. We also give an analogy of Theorem 3.1 for the average complete joint cycle index. For two permutation groups $G$ and $G'$ on $\Omega$, where $|\Omega| = n$, we write $G' \cong G$ if $G$ and $G'$ are isomorphic as permutation groups.

**Definition 4.1.** Let $G_1, \ldots, G_r$ be $r$ permutation groups on a set $\Omega$, where $|\Omega| = n$. Then the $(G_1, \ldots, G_r)$-*average $r$-fold complete joint cycle index* of $G_1, \ldots, G_r$ is the polynomial

$$\mathcal{Z}_{G_1, \ldots, G_r}^{av}(s((g_1', \ldots, g_r'), i)) := \mathcal{Z}^{av}(\mathcal{G}_{G_1', \ldots, G_r'}; s((g_1', \ldots, g_r'), i) :$$
$$G_1' \cong G_1, \ldots, G_r' \cong G_r, (g_1', \ldots, g_r') \in \mathcal{G}_{G_1', \ldots, G_r'}, i \in \mathbb{N}),$$

in indeterminates $s((g', \ldots, g_r'), i)$ where $g_1' \in G_1', \ldots, g_r' \in G_r'$, and $i \in \mathbb{N}$ defined by

$$\mathcal{Z}_{G_1, \ldots, G_r}^{av}(s((g_1', \ldots, g_r'), i))$$
$$:= \frac{1}{\prod_{k=1}^{r} N_{\cong}(G_k)} \sum_{G_1' \cong G_1} \cdots \sum_{G_r' \cong G_r} \mathcal{Z}_{G_1', \ldots, G_r'}(s((g_1', \ldots, g_r'), i)),$$

where $N_{\cong}(G_k) := \sharp\{G_k' \mid G_k' \cong G_k\}$.

In this paper we only consider the case $G_1$-average complete joint cycle index. The $G_1$-*average $r$-fold complete joint cycle index* of $G_1, \ldots, G_r$ is the polynomial

$$\mathcal{Z}_{G_1, \ldots, G_r}^{av}(s((g_1', \ldots, g_r'), i)) := \mathcal{Z}^{av}(\mathcal{G}_{G_1', \ldots, G_r'}; s((g_1', \ldots, g_r'), i) :$$

$$G'_1 \cong G_1, (g'_1, \ldots, g_r) \in \mathcal{G}_{G'_1, \ldots, G_r}, i \in \mathbb{N}),$$

in indeterminates $s((g', \ldots, g_r), i)$ where $g'_1 \in G'_1$, $g_2 \in G_2, \ldots, g_r \in G_r$, and $i \in \mathbb{N}$ defined by

$$\mathcal{Z}^{av}_{G_1, \ldots, G_r}(s((g'_1, \ldots, g_r), i)) := \frac{1}{N_\cong(G_1)} \sum_{G'_1 \cong G_1} \mathcal{Z}_{G'_1, \ldots, G_r}(s((g'_1, \ldots, g_r), i)),$$

where $N_\cong(G_1) := \sharp\{G'_1 \mid G'_1 \cong G_1\}$.

**Example 4.1.** Let $S_3$ be the symmetric group on $\{1, 2, 3\}$. Again let $G_1$ and $G_2$ be two subgroup of $S_3$ such that $G_1 = \langle(1, 2)\rangle$ and $G_2 = \langle(1, 3, 2)\rangle$. Then the subgroups of $S_3$ that are isomorphic as permutation group to $G_1$ are $\langle(1, 2)\rangle, \langle(1, 3)\rangle, \langle(2, 3)\rangle$. That is $N_\cong(G_1) = 3$. Therefore

$$\mathcal{Z}^{av}_{G_1, G_2}(s((g'_1, g_2), i))$$

$$= \frac{1}{3}(\mathcal{Z}_{\langle(1,2)\rangle, G_2}(s((g'_1, g_2), i)) + \mathcal{Z}_{\langle(1,3)\rangle, G_2}(s((g'_1, g_2), i))$$

$$+ \mathcal{Z}_{\langle(2,3)\rangle, G_2}(s((g'_1, g_2), i)))$$

$$= \frac{1}{3}(s(((1), (1)), 1)^3 + s(((1), (1, 2, 3)), 3)^1 + s(((1), (1, 3, 2)), 3)^1$$

$$+ s(((1, 2), (1)), 1)^1 s(((1, 2), (1)), 2)^1$$

$$+ s(((1, 2), (1, 2, 3)), 1)^1 s(((1, 2), (1, 2, 3)), 2)^1$$

$$+ s(((1, 2), (1, 3, 2)), 1)^1 s(((1, 2), (1, 3, 2)), 2)^1$$

$$+ s(((1), (1)), 1)^3 + s(((1), (1, 2, 3)), 3)^1 + s(((1), (1, 3, 2)), 3)^1$$

$$+ s(((1, 3), (1)), 1)^1 s(((1, 3), (1)), 2)^1$$

$$+ s(((1, 3), (1, 2, 3)), 1)^1 s(((1, 3), (1, 2, 3)), 2)^1$$

$$+ s(((1, 2), (1, 3, 2)), 1)^1 s(((1, 2), (1, 3, 2)), 2)^1$$

$$+ s(((1), (1)), 1)^3 + s(((1), (1, 2, 3)), 3)^1 + s(((1), (1, 3, 2)), 3)^1$$

$$+ s(((2, 3), (1)), 1)^1 s(((2, 3), (1)), 2)^1$$

$$+ s(((2, 3), (1, 2, 3)), 1)^1 s(((2, 3), (1, 2, 3)), 2)^1$$

$$+ s(((2, 3), (1, 3, 2)), 1)^1 s(((2, 3), (1, 3, 2)), 2)^1)$$

**Definition 4.2.** We write $S_n$ for the symmetric group acting on the set $\{1, 2, \ldots, n\}$. Let $C$ be any linear code of length $n$ over $\mathbb{F}_q$, and $\mathbf{u} = (u_1, \ldots, u_n) \in C$. Then $\sigma(\mathbf{u}) := (u_{\sigma(1)}, \ldots, u_{\sigma(n)})$ for a permutation $\sigma \in S_n$. Now the code $C' := \sigma(C) := \{\sigma(\mathbf{u}) \mid \mathbf{u} \in C\}$ for $\sigma \in S_n$ is called *permutationally equivalent* to $C$, and denoted by $C \sim C'$. Then the *average r-fold complete joint weight enumerator* of codes $C_1, \ldots, C_r$

over $\mathbb{F}_q$ are defined in [4] as:

$$\mathcal{J}^{av}_{C_1,\ldots,C_r}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^r) := \frac{1}{N_\sim(C_1')} \sum_{C_1' \sim C_1} \mathcal{J}_{C_1',C_2,\ldots,C_r}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^r),$$

where $N_\sim(C_1') := \sharp\{C_1' \mid C_1' \sim C_1\}$.

We call the $G_1$-average $r$-fold complete joint cycle index

$$\mathcal{Z}^{av}_{C_1,\ldots,C_r}(s((g_1', g_2, \ldots, g_r), i)) := \mathcal{Z}^{av}(\mathcal{G}_{C_1',C_2,\ldots,C_r};$$
$$s((g_1', g_2, \ldots, g_r), i) : C_1' \sim C_1, (g_1', g_2, \ldots, g_r) \in \mathcal{G}_{C_1',C_2,\ldots,C_r}, i \in \mathbb{N})$$

the $G_1$-average $r$-fold complete joint cycle index for codes $C_1, \ldots, C_r$.

The following theorem gives a connection between the $G_1$-average of $r$-fold complete joint cycle index and the average of $r$-fold complete joint weight enumerator.

**Theorem 4.1.** *Let $C_1, \ldots, C_r$ be the linear codes of length $n$ over $\mathbb{F}_q$, where $q$ is a power of the prime number $p$. Let $\mathcal{J}^{av}_{C_1,\ldots,C_r}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^r)$ be the average $r$-fold complete joint weight enumerator and*

$$\mathcal{Z}^{av}(\mathcal{G}_{C_1',C_2,\ldots,C_r}; s((g_1', g_2, \ldots, g_r), i) : C_1' \sim C_1, (g_1', g_2, \ldots, g_r) \in$$
$$\mathcal{G}_{C_1',C_2,\ldots,C_r}, i \in \mathbb{N})$$

*be the $G_1$-average complete joint cycle index for $C_1, \ldots, C_r$.*

*Let $T$ be a map defined as follows: for $\sigma \in S_n$, and $g_1 = (u_{11}, \ldots, u_{1n}) \in C_1, g_2 = (u_{21}, \ldots, u_{2n}) \in C_2, \ldots, g_r = (u_{r1}, \ldots, u_{rn}) \in C_r$, and for $i \in \{1, \ldots, n\}$, if $u_{1\sigma(i)} + u_{2i} + \cdots + u_{ri} = 0$, then*

$$s((g_1', g_2, \ldots, g_r), 1) \mapsto x^{1/q}_{u_{1\sigma(i)}u_{2i}\ldots u_{ri}};$$

*if $u_{1\sigma(i)} + u_{2i} + \cdots + u_{ri} \neq 0$, then*

$$s((g_1', g_2, \ldots, g_r), p) \mapsto x^{p/q}_{u_{1\sigma(i)}u_{2i}\ldots u_{ri}}.$$

*Then we have*

$$\mathcal{J}^{av}_{C_1,\ldots,C_r}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^r) = T(\mathcal{Z}^{av}(\mathcal{G}_{C_1',C_2,\ldots,C_r}; s((g_1', g_2, \ldots, g_r), i) :$$
$$C_1' \sim C_1, (g_1', g_2, \ldots, g_r) \in \mathcal{G}_{C_1',C_2,\ldots,C_r}, i \in \mathbb{N})).$$

**Definition 4.3.** For $S_n^\ell := \underbrace{S_n \times \cdots \times S_n}_{\ell}$, we define the semidirect product of $S_\ell$ and $S_n^\ell$ as

$$S_\ell \rtimes S_n^\ell := \{\iota := (\pi; \sigma_1, \ldots, \sigma_\ell) \mid \pi \in S_\ell \text{ and } \sigma_1, \ldots, \sigma_\ell \in S_n\}.$$

We recall the $\ell$-fold joint code, $\Pi^\ell$ and for $\tilde{\mathbf{c}} = (\mathbf{c}_1, \ldots, \mathbf{c}_n) \in \Pi^\ell$, the group $S_\ell \ltimes S_n^\ell$ acts on $\Pi^\ell$ as

$$\iota(\tilde{\mathbf{c}}) := (\iota(\mathbf{c}_1), \ldots, \iota(\mathbf{c}_n)) := \begin{pmatrix} a_{\pi(1)\sigma_1(1)} & \cdots & a_{\pi(1)\sigma_1(n)} \\ a_{\pi(2)\sigma_2(1)} & \cdots & a_{\pi(2)\sigma_2(n)} \\ \vdots & \cdots & \vdots \\ a_{\pi(\ell)\sigma_\ell(1)} & \cdots & a_{\pi(\ell)\sigma_\ell(n)} \end{pmatrix},$$

where $\iota(\mathbf{c}_i) := {}^t(a_{\pi(1)\sigma_1(i)}, \ldots, a_{\pi(\ell)\sigma_\ell(i)}) \in \mathbb{F}_q^\ell$. Then we call $\Pi^{\ell'} := \iota(\Pi^\ell) := \{\iota(\tilde{\mathbf{c}}) \mid \tilde{\mathbf{c}} \in \Pi^\ell\}$ an *equivalent $\ell$-fold joint code* to $\Pi^\ell$, and denoted by $\Pi^{\ell'} \sim \Pi^\ell$. Now the *average $(\ell, r)$-fold complete joint weight enumerator* of $\Pi_1^\ell, \ldots, \Pi_r^\ell$ is defined by

$$\mathcal{J}^{av}_{\Pi_1^\ell, \Pi_2^\ell, \ldots, \Pi_r^\ell}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r}) := \frac{1}{N_\sim(\Pi_1^{\ell'})} \sum_{\Pi_1^{\ell'} \sim \Pi_1^\ell} \mathcal{J}_{\Pi_1^{\ell'}, \Pi_2^\ell, \ldots, \Pi_r^\ell}(x_{\mathbf{a}}),$$

where $N_\sim(\Pi_1^{\ell'}) := \sharp\{\Pi_1^{\ell'} \mid \Pi_1^{\ell'} \sim \Pi_1^\ell\}$. We call the $G_1$-average $r$-fold complete joint cycle index

$$\mathcal{Z}^{av}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}(s((g_1', g_2, \ldots, g_r), i)) := \mathcal{Z}^{av}(\mathcal{G}_{\Pi_1^{\ell'}, \Pi_2^\ell, \ldots, \Pi_r^\ell};$$

$$s((g_1', g_2, \ldots, g_r), i) : \Pi_1^{\ell'} \sim \Pi_1^\ell, (g_1', g_2, \ldots, g_r) \in \mathcal{G}_{\Pi_1^{\ell'}, \Pi_2^\ell, \ldots, \Pi_r^\ell}, i \in \mathbb{N})$$

the $G_1$-average $r$-fold complete joint cycle index for $\ell$-fold joint codes $\Pi_1^\ell, \ldots, \Pi_r^\ell$.

In the following theorem, we give a relationship between the average $(\ell, r)$-fold complete joint weigh enumerator and the $G_1$-average $r$-fold complete joint cycle index for $\ell$-fold joint codes as a generalization of Theorem 4.1.

**Theorem 4.2.** *For $k \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, \ell\}$, let $C_{kj}$ be an $\mathbb{F}_q$-linear code of length $n$, where $q$ is a power of the prime number $p$. Again let $\Pi_k^\ell$ be an $\ell$-fold joint code of $C_{k1}, \ldots, C_{k\ell}$. Let $\mathcal{J}^{av}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r})$ be the average $(\ell, r)$-fold complete joint weight enumerator of $\Pi_1^\ell, \ldots, \Pi_r^\ell$, and*

$$\mathcal{Z}^{av}(\mathcal{G}_{\iota\Pi_1^\ell, \ldots, \Pi_r^\ell}; s((g_1', \ldots, g_r), i) : \Pi_1^{\ell'} \sim \Pi_1^\ell, (g_1', \ldots, g_r) \in \mathcal{G}_{\Pi_1^{\ell'}, \ldots, \Pi_r^\ell},$$
$$i \in \mathbb{N})$$

*be the $G_1$-average $r$-fold complete joint cycle index for $\Pi_1^\ell, \ldots, \Pi_r^\ell$.*

*Let $T$ be a map defined as follows: for $\iota = (\pi; \sigma_1, \ldots, \sigma_\ell) \in S_\ell \ltimes S_n^\ell$, and $g_1 = (\mathbf{c}_{11}, \ldots, \mathbf{c}_{1n}) \in \Pi_1^\ell, \ldots, g_r = (\mathbf{c}_{r1}, \ldots, \mathbf{c}_{rn}) \in \Pi_r^\ell$, and for $i \in \{1, \ldots, n\}$, if $\iota(\mathbf{c}_{1i}) + \mathbf{c}_{2i} + \cdots + \mathbf{c}_{ri} = \mathbf{0}$, then*

$$s((g_1', \ldots, g_r), 1) \mapsto x^{1/q^\ell}_{\iota(\mathbf{c}_{1i})\mathbf{c}_{2i}\ldots\mathbf{c}_{ri}};$$

*if $\iota(\mathbf{c}_{1i}) + \mathbf{c}_{2i} + \cdots + \mathbf{c}_{ri} \neq \mathbf{0}$, then*

$$s((g_1', \ldots, g_r), p) \mapsto x^{p/q^\ell}_{\iota(\mathbf{c}_{1i})\mathbf{c}_{2i}\ldots\mathbf{c}_{ri}}.$$

*Then we have*

$$\mathcal{J}^{av}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}(x_\mathbf{a} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r}) = T(\mathcal{Z}^{av}(\mathcal{G}_{\Pi_1^{\ell'}, \ldots, \Pi_r^\ell}; s((g_1', \ldots, g_r), i) :$$
$$\Pi_1^{\ell'} \sim \Pi_1^\ell, (g_1', \ldots, g_r) \in \mathcal{G}_{\Pi_1^{\ell'}, \ldots, \Pi_r^\ell}, i \in \mathbb{N})).$$

## FUTURE RESEARCH

We would like to study with the concept of the joint Jacobi polynomial for codes over $\mathbb{F}_q$. We are also interested in studying average joint Jacobi polynomials of codes over $\mathbb{F}_q$. Further we would like to investigate the average Jacobi intersection number of codes.

## ACKNOWLEDGMENTS

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