On $\tilde{D}$-separable polynomials in skew polynomial rings of derivation type

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Abstract

The notion of $(\tilde{\rho}, \tilde{D})$-separable polynomials in skew polynomial rings was introduced by S. Ikehata, and X. Lou gave a characterization of $\tilde{\rho}$-separable polynomials in skew polynomial rings of automorphism type. In this paper, we shall give a new characterization of $\tilde{D}$-separable polynomials in skew polynomial rings of derivation type.

1 Introduction and Preliminaries

Let A/B be a ring extension with common identity. A/B is said to be *separable* if the A-A-homomorphism of $A\otimes_B A$ onto A defined by $z\otimes w \mapsto zw$ $(z, w \in A)$ splits. It is well known that A/B is separable if and only if there exists $\sum_i z_i \otimes w_i \in (A\otimes_B A)^A$ such that $\sum_i z_i w_i = 1$, where $(A\otimes_B A)^A = \{\theta \in A\otimes_B A \,|\, u\theta = \theta u \ (\forall\, u \in A)\}$.

Throughout this paper, let B be an associative ring with identity element 1, and D a derivation of B. By $B[X, D]$ we denote the skew polynomial ring in which the multiplication is given by $\alpha X = X\alpha + D(\alpha)$ for any $\alpha \in B$. Moreover, by $B[X; D]_{(0)}$, we denote the set of all monic polynomials f in $B[X; D]$ such that $fB[X; D] = B[X; D]f$. From now on, let $f = \sum_{i=0}^{m} X^i a_i \in B[X; D]_{(0)}$ $(m \geq 1, a_m = 1)$, $A = B[X; D]/fB[X; D]$, and $x = X + fB[X; D]$. As was shown in [3, Lemma 1.6], we see that f is in $B[X; D]_{(0)}$ iff $a_i \in B^D$ $(0 \leq i \leq m-1)$ and

$$a_i\alpha = \sum_{j=i}^{m} \binom{j}{i} D^{j-i}(\alpha)a_j \quad (\forall\, \alpha \in B, 0 \leq i \leq m-1). \tag{1.1}$$

Since $f \in B^D[X]$, there is a derivation $\tilde{D}$ of A which is naturally induced by D (that is, $\tilde{D}$ is defined by $\tilde{D}\left(\sum_{j=0}^{m-1} x^j c_j\right) = \sum_{j=0}^{m-1} x^j D(c_j)$ $(c_j \in B)$). Now we consider the following A-A-homomorphisms:

$$\begin{cases} \mu : A\otimes_B A \to A, \quad \mu(z\otimes w) = zw \\ \xi : A\otimes_B A \to A\otimes_B A, \quad \xi(z\otimes w) = \tilde{D}(z)\otimes\tilde{\rho}(w) + z\otimes\tilde{D}(w) \end{cases} \quad (z, w \in A)$$

We say that f is a *separable polynomial* in $B[X; D]$ if A is a separable extension of B, namely, there exists an A-A-homomorphism $\nu : A \to A\otimes_B A$ such that $\mu\nu = 1_A$ (the identity map of A). Moreover, f is called a *$\tilde{D}$-separable polynomial* in $B[X; D]$

if there exists an A-A-homomorphism $\nu : A \to A \otimes_B A$ such that $\mu\nu = 1_A$ and $\xi\nu = \nu\tilde{D}$. The notion of $\tilde{D}$-separable polynomials was introduced by S. Ikehata (cf. [3]). By [7, Theorem 2.1] and [3, Theorem 2.1], the following lemma is already known.

Lemma 1.1. *Let $f = \sum_{i=0}^{m} X^i a_i$ $(m \geq 1, a_m = 1)$ be in $B[X; D]_{(0)}$. The following are equivalent.*

(1) *f is $\tilde{D}$-separable in $B[X; D]$.*

(2) *f is separable in $C(B^D)[X]$, where $C(B^D)$ is the center of $B^D = \{b \in B \mid D(b) = 0\}$.*

(3) *$\delta(f)$ is invertible in $C(B^D)$, where $\delta(f)$ is the discriminant of f.*

In this paper, we shall characterize $\tilde{D}$-separable polynomials in $B[X; D]$. In section 2, we define a D-matrix over B, and we shall mention briefly on it. In section 3, we shall give a new characterization of $\tilde{D}$-separable polynomial by making use of the trace map. Moreover, we shall show the "distance" between separability and $\tilde{D}$-separability.

2 D-matrix

In this section, let $B^D = \{b \in B \mid D(b) = 0\}$, and $C(B^D)$ the center of B^D. For any $b \in B$, I_b will represent the inner derivation effected by b (i.e. $I_b(\alpha) = \alpha b - b\alpha$ for any $\alpha \in B$). In [12], X. Lou defined the ρ-matrix, and he characterized $\tilde{\rho}$-separable polynomials in $B[X; \rho]$ by making use of it. In constant, we shall define the D-matrix as follows:

Definition 2.1. (1) An element b in B is called a *D-element* if $I_b(B) \subset D(B)$ and $B^D \subset \operatorname{Ker} I_b$, where $\operatorname{Ker} I_b$ is the kernel of I_b.

(2) A matrix P over B is called a *D-matrix* if every entry of P is a D-element.

Remark 1. In Definition 2.1 (1), if a D-element b is in B^D then the condition $B^D \subset \operatorname{Ker} I_b$ implies that $b \in C(B^D)$.

Lemma 2.2. *Let b and c be D-elements in B.*

(1) *$b + c$ is also a D-element.*

(2) *If either b or c is in B^D, then bc is also a D-element.*

(3) *If $b \in B^D$ and b is invertible in B, then b^{-1} is also a D-element.*

Proof. Let α and β be arbitrary elements in B and B^D, respectively. Assume that b and c are D-elements. Then there exist $b', c' \in B$ such that

$$I_b(\alpha) = \alpha b - b\alpha = D(b'), \quad I_c(\alpha) = \alpha c - c\alpha = D(c'). \tag{2.1}$$

Moreover, we see that $\beta b = b\beta$ and $\beta c = c\beta$.

(1) Since the equation (2.1), we have $I_{b+c}(\alpha) = \alpha(b+c) - (b+c)\alpha = (\alpha b - b\alpha) + (\alpha c - c\alpha) = D(b') + D(c') = D(b' + c')$. Therefore $I_{b+c}(B) \subset D(B)$. It is obvious that $\beta(b+c) - (b+c)\beta = 0$, and hence $B^D \subset \operatorname{Ker} I_{b+c}$.

(2) Assume that b is in B^D. Note that $bc = cb$ and $I_b(\alpha c) = D(b'')$ for some $b'' \in B$. So it follows from the equation (2.1) that $I_{bc}(\alpha) = \alpha bc - bc\alpha = \alpha cb - b\alpha c + b\alpha c - bc\alpha = I_b(\alpha c) + bI_c(\alpha) = D(b'') + bD(c') = D(b'' + bc')$. Thus $I_{bc}(B) \subset D(B)$. Clearly, we see that $\beta bc - bc\beta = 0$. Therefore $B^D \subset \operatorname{Ker} I_{bc}$.

(3) Assume that $b \in B^D$ and b is invertible. Since $bb^{-1} = 1$ and $b \in B^D$, we obtain

$$0 = D(1) = D(bb^{-1}) = bD(b^{-1}).$$

This implies that $b^{-1} \in B^D$. Since the equation (2.1), we have $b^{-1}\alpha - \alpha b^{-1} = b^{-1}D(b')b^{-1} = D(b^{-1}b'b^{-1})$. Hence $I_{b^{-1}}(B) \subset D(B)$. In addition, $\beta b = b\beta$ means that $\beta b^{-1} = b^{-1}\beta$. Thus $B^D \subset \operatorname{Ker} I_{b^{-1}}$. □

Lemma 2.3. *Let P be a D-matrix over B^D.*

(1) $\det(P)$ *is a D-element in $C(B^D)$.*

(2) *If P is an invertible matrix, then the inverse matrix of P is also a D-matrix.*

(3) *Assume that $P = P^T$ (the transpose of P). If P has a left (or right) inverse matrix which is a D-matrix, then* $\det(P)$ *is invertible in $C(B^D)$.*

Proof. Let $P = [p_{ij}]_{n\times n}$ be a D-matrix over B^D for some positive integer n. In particular, P is a matrix over $C(B^D)$ by Remark 1.

(1) It is obvious by Lemma 2.2 (1) (2).

(2) Assume that P is an invertible matrix, and P^* be the cofactor matrix of P. It follow from Lemma 2.2 (1) (2) that every entry of P^* is a D-element (i.e. P^* is a D-matrix). By Lemma 2.2 (3) and the assertion (1), moreover, $\det(P)^{-1}$ is also a D-element. Therefore $P^{-1} = \det(P)^{-1}P^*$ is a D-matrix.

(3) Assume that $P = P^T$ and there exists a D-matrix $Q = [q_{ij}]_{n\times n}$ such that $QP = E$ (the identity matrix). Since q_{ij} is a D-element and $p_{ij} \in B^D$, we have $q_{k\ell}p_{ij} = p_{ij}q_{k\ell}$ $(1 \le i, j, k, \ell \le n)$. Then we see that $QP = E$ iff $\sum_{j=1}^n q_{ij}p_{jk} = \delta_{ik}$ (the Kronecker's delta) iff $\sum_{j=1}^n p_{jk}q_{ij} = \delta_{ik}$ iff $P^TQ^T = E$ iff $PQ^T = E$. Hence $Q = Q^T$ is the inverse matrix of P. We put here $D(Q) = [D(q_{ij})]_{n\times n}$. Since $p_{ij} \in B^D$, we obtain

$$0 = D(\delta_{ik}) = D\left(\sum_{j=1}^n q_{ij}p_{jk}\right) = \sum_{j=1}^n D(q_{ij}p_{jk}) = \sum_{j=1}^n D(q_{ij})p_{jk}.$$

This implies that $D(Q)P = O$ (the zero matrix), and hence $D(Q) = O$ (i.e. Q is a matrix over B^D). Therefore Q is a matrix over $C(B^D)$ by Remark 1. Since P and Q are matrices over $C(B^D)$ such that $PQ = QP = E$, we see that $\det(P)$ is invertible in $C(B^D)$. □

3 $\tilde{D}$-separability in $B[X;D]$

The conventions and notations employed in the preceding section will be used in this section. We shall use the following conventions:

- $A^{\tilde{D}} = \{z \in A \mid \tilde{D}(z) = 0\}$.
- $C(A^{\tilde{D}})$ is the center of $A^{\tilde{D}}$.
- $\pi_i : A \to A$ is the projection map defined by

$$\pi_i\left(\sum_{j=0}^{m-1} x^j c_j\right) = c_i \quad (c_i \in B, 0 \leq i \leq m-1).$$

- $\tau : A \to B$ is the trace map defined by

$$\tau\,(z) = \sum_{i=0}^{m-1} \pi_i(x^i z) \quad (z \in A).$$

- $T_f = [\tau(x^i x^j)]_{m \times m}$, where $m = \deg f$ (i.e. T_f is a $m \times m$ symmetric matrix whose $(i+1, j+1)$ element is $\tau(x^i x^j)$).
- $\delta(f) = \det(T_f)$ (the discriminant of f).

Remark 2. (1) Clearly, π_i $(0 \leq i \leq m-1)$ and τ are left $C(B^D)$- right B-homomorphisms.

(2) It is easy to see that

$$C(A^{\tilde{D}}) = \left\{C(B^D)[X] + fB[X;D]\right\}/f[B[X;D] \cong C(B^D)[X]/fC(B^D)[X].$$

First we shall show the following.

Lemma 3.1. (1) *Every a_i $(0 \leq i \leq m-1)$ is a D-element in $C(B^D)$.*

(2) *$T_f = [\tau(x^i x^j)]_{m \times m}$ is a D-matrix over $C(B^D)$.*

Proof. (1) Since the equation (1.1), we have

$$\begin{aligned} a_i\alpha &= \alpha a_i + \sum_{j=i+1}^{m} \binom{j}{i} D^{j-i}(\alpha)a_j \\ &= \alpha a_i + D\left(\sum_{j=i+1}^{m} \binom{j}{i} D^{j-i-1}(\alpha)a_j\right). \end{aligned}$$

This implies that $I_{a_i}(B) \subset D(B)$ and $B^D \subset \operatorname{Ker} I_{a_i}$ $(0 \le i \le m-1)$. Hence every a_i is a D-element.

(2) It is easy to see that every $\tau(x^i x^j)$ $(0 \le i, j \le m-1)$ is generated by a_k $(0 \le k \le m-1)$. Then, by Lemma 2.2 and the assertion (1), every $\tau(x^i x^j)$ is a D-element. Therefore T_f is a D-matrix. □

So we shall show the following lemma which gives a new equivalent condition of $\tilde{D}$-separability.

Lemma 3.2. *The following are equivalent.*

(1) *f is $\tilde{D}$-separable in $B[X; D]$.*

(2) *T_f has a left (or right) inverse matrix which is a D-matrix.*

Proof. (1) $\Longrightarrow$ (2) Let f be $\tilde{D}$-separable in $B[X; D]$. So, by Lemma 1.1, $\delta(f) = \det(T_f)$ is invertible in $C(B^D)$, and hence T_f has an inverse matrix T_f^{-1}. Noting that T_f is a D-matrix, T_f^{-1} is also a D-matrix by Lemma 2.3 (2).

(2) $\Longrightarrow$ (1) Assume that T_f has a left inverse matrix which is a D-matrix. Then, by Lemma 2.3 (3), $\delta(f) = \det(T_f)$ is invertible in $C(B^D)$. Thus f is $\tilde{D}$-separable by Lemma 1.1. □

The following theorem at some extent shows the "distance" between separability and $\tilde{D}$-separability in $B[X; D]$.

Theorem 3.3. *The following are equivalent.*

(1) *f is $\tilde{D}$-separable in $B[X; D]$.*

(2) *f is separable in $B[X; D]$ with a separable set $\{z_i, w_i\}$ of A/B such that $\sum_i z_i\tau(w_i) = 1$.*

Proof. (1) $\Longrightarrow$ (2) Let f be $\tilde{D}$-separable in $B[X; D]$. So, by Lemma 1.1, f is separable in $C(B^D)[X]$. As was shown in [1, chapter III, Theorem 2.1], there exists separable system $\{z_i, w_i\}$ of $C(A^{\tilde{D}})/C(B^D)$ such that $\sum_i z_i\tau(w_i u) = u$ for any $u \in C(A^{\tilde{D}})$. So we can see that $\{z_i, w_i\}$ is still a separable system of A/B, and $\sum_i z_i\tau(w_i) = 1$.

(2) $\Longrightarrow$ (1) Assume that f is separable in $B[X;D]$ with a separable set $\{z_i, w_i\}$ of A/B such that $\sum_i z_i\tau(w_i) = 1$. We put here $z_i = \sum_{j=0}^{m-1} x^j c_{ij}$ and $w_i = \sum_{k=0}^{m-1} d_{ik}x^k$. We obtain then

$$\begin{aligned}\sum_i z_i \otimes w_i &= \sum_i \left(\sum_{j=0}^{m-1} x^j c_{ij} \otimes \sum_{k=0}^{m-1} d_{ik}x^k\right)\\ &= \sum_{j=0}^{m-1} x^j \otimes \left(\sum_{k=0}^{m-1}\sum_i c_{ij}d_{ik}x^k\right).\end{aligned}$$

We set $e_{jk} = \sum_i c_{ij}d_{ik}$ and $u_j = \sum_{k=0}^{m-1} e_{jk}x^k$. Clearly, $\{x^j, u_j\}$ is a still separable system of A/B such that $\sum_{j=0}^{m-1} x^j\tau(u_j) = 1$. Let $\widehat{\tau}$ be a map from $A \otimes_B A$ to A defined by $\widehat{\tau}(r_1 \otimes r_2) = r_1\tau(r_2)$ $(r_1, r_2 \in A)$. So we have

$$\begin{aligned}x^\ell &= x^\ell \sum_{j=0}^{m-1} x^j\tau(u_j)\\ &= \widehat{\tau}\left(x^\ell \sum_{j=0}^{m-1} x^j \otimes u_j\right)\\ &= \widehat{\tau}\left(\sum_{j=0}^{m-1} x^j \otimes u_j x^\ell\right)\\ &= \sum_{j=0}^{m-1} x^j\tau\left(\sum_{k=0}^{m-1} e_{jk}x^{k+\ell}\right)\\ &= \sum_{j=0}^{m-1} x^j\left(\sum_{k=0}^{m-1} e_{jk}\tau(x^{k+\ell})\right)\end{aligned}$$

This implies that $\sum_{k=0}^{m-1} e_{jk}\tau(x^{k+\ell}) = \delta_{j\ell}$. By setting $P = [e_{j+1,k+1}]_{m\times m}$, we have $PT_f = E$. Noting that $\alpha\sum_{j=0}^{m-1} x^j \otimes u_j = \sum_{j=0}^{m-1} x^j \otimes u_j\alpha$ for any $\alpha \in B$, we obtain $\sum_{j=\ell}^{m-1}\binom{j}{\ell}D^{j-\ell}(\alpha)u_j = u_j\alpha$. This implies that e_{jk} is a D-element, and hence P is a D-matrix. Therefore f is $\tilde{D}$-separable by Lemma 3.2. $\square$

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