Inverse problems and theory of reproducing kernels – theory

Saburou Saitoh
Institute of Reproducing Kernels
saburou.saitoh@gmail.com

March 4, 2021

1 Introduction

At least, until about 20 years ago, we had very difficult inverse problems that are important in many practical problems (fundamentals) as follows:

1): Inverse source problem; that is in the Poisson equation

$$\triangle u = -\rho,$$

from the observation of the potential u for the out side of the support ρ, look for the source ρ.

2): The problem in the heat conduction; that is, from some heat $u(x,t)$ observation at a time t, look for the initial heat $u(x,0)$.

3): Real inversion formulas for the Laplace transform.

These problems were indeed difficult in both mathematics and numerical realizations of the solutions and so, they are called ill-posed problems and very famous difficult problems.

We were able to solve these problems in both senses of mathematics and numerical problems by using the theory of reproducing kernels applying the Tikhonov regularization. However, for the real inversion formula of the Laplace transform, we needed the great power of computers by H. Fujiwara.

For any practical numerical analysis, the important problem is on the discretization procedure for analytical inverse problem solutions. At this very important point, we will see that the theory of reproducing kernels is very good mathematics. These global theories were published in the book [23] and our method is applicable in some general problems in the viewpoint of practical problems. Here, we state their essential parts. Here, we will state its theoritical parts and Professor T. Matsuura will give their numerical examples, by using computer graphics.

In order to fix our background in this paper, following [18, 19, 21], we first recall a general theory for linear mappings in the framework of Hilbert spaces using the general theory of reproducing kernels.

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space, and consider E to be an abstract set and $\mathbf{h}$ a Hilbert $\mathcal{H}$-valued function on E. Then, a very general linear transform from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued functions on E will be given by

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{1}$$

in the framework of Hilbert spaces.

In general, a complex-valued function is called a *positive definite quadratic form function* on the set E, or shortly, *positive definite function*, when it satisfies the property that, for an arbitrary function $X : E \to \mathbf{C}$ and for any finite subset F of E,

$$\sum_{p,q \in F} \overline{X(p)} X(q) k(p,q) \geq 0.$$

In order to investigate the linear mapping (1), we form a positive definite quadratic form function $K(p,q)$ on $E \times E$ defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \quad \text{on} \quad E \times E. \tag{2}$$

Then, the followings are fundamental ([18]):

Proposition 1.1 *(I) The range of the linear mapping (1) by $\mathcal{H}$ is characterized as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(p,q)$ whose characterization is given by the two properties: (i) $K(\cdot, q) \in H_K(E)$ for any $q \in E$ and, (ii) for any $f \in H_K(E)$ and for any $p \in E$, $(f(\cdot), K(\cdot . p))_{H_K(E)} = f(p)$.*

(II) In general, the inequality

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}$$

holds. Here, for any member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathcal{H}$ satisfying

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \quad \text{on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}. \tag{3}$$

(III) In general, the inversion formula in (1) in the form

$$f \mapsto \mathbf{f}^* \tag{4}$$

in (II) holds, by using the reproducing kernel Hilbert space $H_K(E)$.

The typical ill-posed problem (1) becomes a well-posed problem, because the image space of (1) is characterized as the reproducing kernel Hilbert space $H_K(E)$ with the isometric identity (3), which may be considered as a generalization of the *Pythagorean* theorem.

However, this viewpoint is a mathematical one and is not a numerical one and not easy to deal with analytical and numerical problems.

2 Inversion Formulas

Consider the inversion in (1) formally, however, this idea will be very important for the general inversions and for discretization method.

Following the above general situation, let $\{\mathbf{v_j}\}$ be a complete orthonormal basis for $\mathcal{H}$. Then, for

$$v_j(p) = (\mathbf{v_j}, \mathbf{h}(\mathbf{p}))_{\mathcal{H}},$$

$$\mathbf{h}(p) = \sum_j (\mathbf{h}(p), \mathbf{v_j})_{\mathcal{H}} \mathbf{v_j} = \sum_{\mathbf{j}} \overline{\mathbf{v_j}(\mathbf{p})} \mathbf{v_j}.$$

Hence, by setting

$$\overline{\mathbf{h}}(p) = \sum_j v_j(p) \mathbf{v_j},$$

$$\overline{\mathbf{h}}(\cdot) = \sum_j v_j(\cdot) \mathbf{v_j}.$$

Thus, define

$$(f, \overline{\mathbf{h}}(p))_{H_K} = \sum_j (f, v_j)_{H_K} \mathbf{v_j}.$$

For simplicity, write as follows:

$$H_K = H_K(E).$$

Then, formally, we obtain:

Proposition 2.1 *Assume that for $f \in H_K$*

$$(f, \overline{\mathbf{h}})_{H_K} \in \mathcal{H}$$

and for all $p \in E$,

$$(f, (\mathbf{h}(\mathbf{p}), \mathbf{h}(\cdot))_{\mathcal{H}})_{\mathbf{H_K}} = ((\mathbf{f}, \overline{\mathbf{h}})_{\mathbf{H_K}}, \mathbf{h}(\mathbf{p}))_{\mathcal{H}}.$$

Then,

$$\|f\|_{H_K} \leq \|(f, \overline{\mathbf{h}})_{H_K}\|_{\mathcal{H}}.$$

If $\{\mathcal{F}h(p); p \in E\}$ is complete in $\mathcal{H}$, then equality always holds.

Furthermore, if:

$$(\mathbf{f_0}, (\mathbf{f}, \overline{\mathbf{h}})_{\mathbf{H_K}})_{\mathcal{H}} = ((\mathbf{f_0}, \mathbf{h})_{\mathcal{H}}, \mathbf{f})_{\mathbf{H_K}} \quad \mathbf{for} \quad \mathbf{f_0} \in \mathbf{N}(\mathbf{L}).$$

Then, for $\mathbf{f}^*$ *in (II) and (III)*

$$\mathbf{f}^* = (\mathbf{f}, \overline{\mathbf{h}})_{\mathbf{H_K}}.$$

In particular, note that the basic assumption $(f, \overline{\mathbf{h}})_{H_K} \in \mathcal{H}$ in Proposition 2.1, is, in general, not valid and very delicate for many analytical problems and we need some delicate treatment for the inversion.

In order to derive a general inversion formula that is widely applicable in analysis, assume that the both Hilbert spaces $\mathcal{H}$ and H_K are given as $\mathcal{H} = L_2(T, dm), H_K \subset L_2(E, d\mu)$, on the sets T and E, respectively (assume that for $dm, d\mu$ measurable sets T, E, they are the Hilbert spaces comprising $dm, d\mu-$ L_2 integrable complex-valued functions, respectively.) Therefore, consider the integral transform

$$f(p) = \int_T F(t)\overline{h(t,p)}dm(t). \tag{5}$$

Here, $h(t,p)$ is a function on $T \times E$, $h(\cdot, p) \in L_2(T, dm)$, and $F \in L_2(T, dm)$. The corresponding reproducing kernel for (2) is given by

$$K(p,q) = \int_T h(t,q)\overline{h(t,p)}dm(t) \quad \text{on} \quad E \times E.$$

The norm of the reproducing kernel Hilbert space H_K is represented as $L_2(E, d\mu)$.

Under these situations:

Proposition 2.2 *Assume that an approximating sequece* $\{E_N\}_{N=1}^{\infty}$ *of* E *satisfies (a)* $E_1 \subset E_2 \subset \cdots \subset \cdots$, *(b)* $\bigcup_{N=1}^{\infty} E_N = E$, *(c)* $\int_{E_N} K(p,p)d\mu(p) < \infty$, $(N = 1, 2, ...)$.

Then, for $f \in H_K$ *satisfying* $\int_{E_N} f(p)h(t,p)d\mu(p) \in L_2(T, dm)$ *for any* N, *the sequence*

$$\left\{ \int_{E_N} f(p)h(t,p)d\mu(p) \right\}_{N=1}^{\infty} \tag{6}$$

converges to F^* *in (4) in Proposition 1.1 in the sense of* $L_2(T, dm)$ *norm.*

Practically for many cases, the assumptions in Proposition 2.2, will be satisfied automatically, and so Proposition 2.2 may be applied in many cases. Proposition 2.2 will give a new viewpoint and method for the Fredholm integral equation (5) of the first kind that is a fundamental integral equation. The method and solution has the following properties:

1) The use of the naturally determined reproducing kernel Hilbert space H_K which is determined by the integral kernel.

2) The solution is given in the sense of $\mathcal{H}$ norm convergence.

3) The solution (inverse) is given by $\boldsymbol{f}^*$ in Proposition 1.1.

4) For the ill-posed problem in (5), the solution is given as a well-posed solution.

This viewpoint is, however, a mathematical and theoritical one. In practical and physical linear systems, the observation data will be *a finite number of data containing error or noises,* and so we will meet to various delicate problems numerically.

The basic assumption here for the integral kernels is to belong to some Hilbert spaces. However, as a very typical integral transform, in the case of Fourier integral transform, the integral kernel does not belong to $L_2(\mathbf{R})$ and, however, we can establish the isometric identity and inversion formula in the space $L_2(\mathbf{R})$.

We can develop some general integral transform theory containing the Fourier integral transform case that the integral kernel does not belong to any Hilbert space, based on the general concept of generalized reproducing kernels in [22, 14].

3 Best Approximations, as a connection

For numerical treatments and practical constructions of the analytical solutions, we will need some approximate solutions and the Tikhonov regularizations.

Let L be any bounded linear operator from a reproducing kernel Hilbert space H_K into a Hilbert space $\mathcal{H}$. Then, the following problem is a classical and fundamental problem known as the best approximate mean square norm problem: For any member $\mathbf{d}$ of $\mathcal{H}$, we would like to find

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}}.$$

It is clear that we are considering *operator equations*, generalized solutions and corresponding generalized inverses within the framework of $f \in H_K$ and $\mathbf{d} \in \mathcal{H}$, having in mind

$$Lf = \mathbf{d}. \tag{7}$$

However, this problem has a complicated structure, specially in the infinite dimension Hilbert spaces case, leading in fact to the consideration of generalized inverses (in the Moore-Penrose sense). Following the reproducing kernel theory, we can realize its complicated structure. Anyway, the problem turns to be well-posed within the reproducing kernels theory framework in the following proposition:

Proposition 3.1 *For any member* $\mathbf{d}$ *of* $\mathcal{H}$*, there exists a function* $\tilde{f}$ *in* H_K *satisfying*

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} = \|L\tilde{f} - \mathbf{d}\|_{\mathcal{H}} \tag{8}$$

if and only if, for the reproducing kernel Hilbert space H_k *admitting the kernel defined by* $k(p, q) = (L^*LK(\cdot, q), L^*LK(\cdot, p))_{H_K}$

$$L^*\mathbf{d} \in H_k. \tag{9}$$

Furthermore, when there exists a function $\tilde{f}$ *satisfying (8), there exists a uniquely determined function that minimizes the norms in* H_K *among the functions satisfying the equality, and its function* $f_{\mathbf{d}}$ *is represented as follows:*

$$f_{\mathbf{d}}(p) = (L^*\mathbf{d}, L^*LK(\cdot,p))_{H_k} \quad on \quad E. \tag{10}$$

Here, the adjoint operator L^* of L, as we see, from

$$(L^*\mathbf{d})(p) = (L^*\mathbf{d}, K(\cdot,p))_{H_K} = (\mathbf{d}, LK(\cdot,p))_{\mathcal{H}}$$

is represented by known $\mathbf{d}, L, K(p,q)$, and $\mathcal{H}$. From this Proposition 3.1, we see that the problem is well-established by the theory of reproducing kernels, that is the existence, uniqueness and representation of the solutions in the problem are well-formulated. In particular, note that the adjoint operator is represented in a good way; this fact will be very important. The extremal function $f_{\mathbf{d}}$ is the *Moore-Penrose generalized inverse* $L^{\dagger}\mathbf{d}$ of the equation $Lf = \mathbf{d}$. The criteria (9) is involved and so the Moore- Penrose generalized inverse $f_{\mathbf{d}}$ is not good, when the data contain error or noises in some practical cases.

4 The Tikhonov Regularization

We shall consider some practical and more concrete representation in the extremal functions involved in the Tikhonov regularization by using the theory of reproducing kernels. Recall that for compact operators the singular values and singular functions representations are popular and in a sense, the representation may be considered complicated.

Furthermore, when $\mathbf{d}$ contains error or noises, error estimates are important. For this fundamental problem, we have the following results:

At first, we need

Proposition 4.1 *Let $L : H_K \to \mathcal{H}$ be a bounded linear operator, and define the inner product,* **with a small positive** α

$$\langle f_1, f_2\rangle_{H_{K_\alpha}} = \alpha\,\langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_{\mathcal{H}}$$

for $f_1, f_2 \in H_K$. Then $(H_K, \langle\cdot,\cdot\rangle_{H_{K_\alpha}})$ is a reproducing kernel Hilbert space whose reproducing kernel is given by

$$K_\alpha(p,q) = [(\alpha + L^*L)^{-1}K_q](p).$$

Here, $K_\alpha(p,q)$ is the solution $\tilde{K}_\alpha(p,q)$ of the functional equation

$$\tilde{K}_\alpha(p,q) + \frac{1}{\alpha}(L\tilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\alpha}K(p,q), \tag{11}$$

that is corresponding to the Fredholm integral equation of the second kind for many concrete cases. Here,

$$\tilde{K}_q = \tilde{K}_\alpha(\cdot,q) \in H_K \quad for \quad q \in E, \qquad K_p = K(\cdot,p) \quad for \quad p \in E.$$

Proposition 4.2 *In the Tikhonov functional*

$$f \in H_K \mapsto \left\{\alpha \|f \,:\, H_K\|^2 + \|Lf - \mathbf{d} \,:\, \mathcal{H}\|^2\right\} \in \mathbf{R}$$

attains the minimum and the minimum is attained only at $f_{\mathbf{d},\alpha} \in H_K$ *such that*

$$(f_{\mathbf{d},\alpha})(p) = \langle \mathbf{d}, LK_\alpha(\cdot, p)\rangle_{\mathcal{H}}.$$

Furthermore, $(f_{\mathbf{d},\alpha})(p)$ *satisfies*

$$|(f_{\mathbf{d},\alpha})(p)| \leq \sqrt{\frac{K(p,p)}{2\alpha}}\|\mathbf{d}\|_{\mathcal{H}}. \tag{12}$$

This proposition means that in order to obtain good approximate solutions, we must take a sufficiently small α, however, here we have restrictions for them, as we see, when $\mathbf{d}$ moves to $\mathbf{d}'$, by considering $f_{\mathbf{d},\alpha}(p) - f_{\mathbf{d}',\alpha}(p)$ in connection with the relation of the difference $\|\mathbf{d} - \mathbf{d}'\|_{\mathcal{H}}$. This fact is a very natural one, because we cannot obtain good solutions from the data containing errors. Here we wish to know how to take a small α a priori and what is the bound for it. These problems are very important practically and delicate ones, and we have many methods.

The basic idea may be given as follows. We examine for various α tending to zero, the corresponding extremal functions. By examining the sequence of the approximate extremal functions, when it converges to some function numerically and after then when the sequence diverges numerically, it will give the bound for α numerically. See (Fujiwara et al 2008; Fujiwara et al 2009; Fujiwara 2010).

For this important problem and the method of L-curve, see (Lawson et al 1972; Hansen 1992), for example.

The Tikhonov regularization is very popular and widely applicable in numerical analysis for its practical power. The application of the theory of reproducing kernels will give more concrete representations of the extremal functions in the Tikhonov regularization.

Indeed, we were able to give many good numerical solutions containing the typical difficult inverse problems in the Poisson equation and heat equation. We will refer to the most difficult problem in the real inversion formula of the Laplace transform.

For an up-to-date information on the Laplace inversion formula and its applications, see [15].

4.1 Real and Numerical Inversions of the Laplace Transform

Consider the inversion formula of the Laplace transform

$$(\mathcal{L}F)(p) = f(p) = \int_0^\infty e^{-pt}F(t)dt, \quad p > 0$$

for some natural function spaces.

On the positive real line $\mathbf{R}^+$, consider the norm

$$\left\{\int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt\right\}^{1/2}$$

for absolutely continuous functions F satisfying $F(0) = 0$. This space H_K admits the reproducing kernel

$$K(t,t') = \int_0^{\min(t,t')} \xi e^{-\xi} d\xi. \tag{13}$$

Then,

$$\int_0^\infty |(\mathcal{L}F)(p)p|^2 dp \leq \frac{1}{2}\|F\|_{H_K}^2; \tag{14}$$

that is, $(\mathcal{L}F)(p)p$ is a bounded linear operator from H_K into $L_2(\mathbf{R}^+, dp) = L_2(\mathbf{R}^+)$. So, the following result holds:

Proposition 4.3 *For any $g \in L_2(\mathbf{R}^+)$ and for any $\alpha > 0$, in the sense*

$$\inf_{F \in H_K}\left\{\alpha \int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt + \|(\mathcal{L}F)(p)p - g\|_{L_2(\mathbf{R}^+)}^2\right\} \tag{15}$$

$$= \alpha \int_0^\infty |F_{\alpha,g}^{*\prime}(t)|^2 \frac{1}{t} e^t dt + \|(\mathcal{L}F_{\alpha,g}^*)(p)p - g\|_{L_2(\mathbf{R}^+)}^2$$

there exists a uniquely determined best approximate function $F_{\alpha,g}^*$ and it is represented by

$$F_{\alpha,g}^*(t) = \int_0^\infty g(\xi)\left(\mathcal{L}K_\alpha(\cdot,t)\right)(\xi)\xi d\xi. \tag{16}$$

Here, $K_\alpha(\cdot,t)$ is determined by the functional equation for $K_{\alpha,t'} = K_\alpha(\cdot,t')$, $\quad K_t = K(\cdot,t)$,

$$K_\alpha(t,t') = \frac{1}{\alpha}K(t,t') - \frac{1}{\alpha}((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}K_t)(p)p)_{L_2(\mathbf{R}^+)}. \tag{17}$$

We calculate the approximate inverse $F_{\alpha,g}^*(t)$ by using (16). By taking the Laplace transform of (17) with respect to t, by changing the variables t and t'

$$(\mathcal{L}K_\alpha(\cdot,t))(\xi) = \frac{1}{\alpha}(\mathcal{L}K(\cdot,t))(\xi) - \frac{1}{\alpha}((\mathcal{L}K_{\alpha,t})(p)p, (\mathcal{L}(\mathcal{L}K_\cdot)(p)p))(\xi))_{L_2(\mathbf{R}^+)}. \tag{18}$$

Here,

$$K(t,t') = \begin{cases} -te^{-t} - e^{-t} + 1 & \text{for} \quad t \leq t' \\ -t'e^{-t'} - e^{-t'} + 1 & \text{for} \quad t \geq t'. \end{cases}$$

$$(\mathcal{L}K(\cdot,t'))(p) = e^{-t'p}e^{-t'}\left[\frac{-t'}{p(p+1)} + \frac{-1}{p(p+1)^2}\right] + \frac{1}{p(p+1)^2}.$$

$$\int_0^\infty e^{-qt'}(\mathcal{L}K(\cdot,t'))(p)dt' = \frac{1}{pq(p+q+1)^2}.$$

Therefore, by setting as $(\mathcal{L}K_\alpha(\cdot,t))(\xi)\xi = H_\alpha(\xi,t)$, we obtain the *Fredholm integral equation of the second kind*:

$$\alpha H_\alpha(\xi,t) + \int_0^\infty \frac{H_\alpha(p,t)}{(p+\xi+1)^2}dp = -\frac{e^{-t\xi}e^{-t}}{\xi+1}\left(t + \frac{1}{\xi+1}\right) + \frac{1}{(\xi+1)^2}, \quad (19)$$

which is corresponding to (11). By solving this integral equation, H. Fujiwara derived a very reasonable numerical inversion formula for the integral transform and he expanded very good algorithms for numerical and real inversion formulas of the Laplace transform. For more detailed references and comments for this equation, see (Fujiwara et al 2008; Fujiwara et al 2009; Fujiwara 2010).

In particular, H. Fujiwara solved the integral equation (11) with 6000 points discretization with *600 digits precision* based on the concept of *infinite precision* which is in turn based on *multiple-precision arithmetic*. Then, the regularization parameters were $\alpha = 10^{-100}, 10^{-400}$ surprisingly. For the integral equation, he used the *DE formula* by H. Takahashi and M. Mori, using double exponential transforms. H. Fujiwara was successful in deriving numerically the inversion for the Laplace transform of the distribution delta which was proposed by V. V. Kryzhniy as a difficult case. This fact will mean that the above results valid for very general functions approximated by the functions of the reproducing kernel Hilbert space $H_K(\mathbf{R}^+)$.

We showed many Figures for the numerical experiments in the complete version [8] by Professor H. Fujiwara. For the heat conduction problem, by [13].

The general theory in this section was extended to the Hilbert space framework by using the generalized reproducing kernels in [22] with Professor Y. Sawano.

5 The Aveiro Discretization Method

Meanwhile, in general, the reproducing kernel Hilbert space H_K has a complicated structure, and so we have to consider the approximate realization of the abstract Hilbert space H_K by taking a finite number of points of E. A finite number of data will be lead to a discretization principle and practical method, because computers can deal with a finite number of data.

By taking a finite number of points $\{p_j\}_{j=1}^n$, we set

$$K(p_j, p_{j'}) := a_{jj'}. \quad (20)$$

Then, if the matrix $A :=\| a_{jj'} \|$ is positive definite, then, the corresponding norm in H_A comprising the vectors $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ is determined by

$$\|\mathbf{x}\|_{H_A}^2 = \mathbf{x}^* \widetilde{A} \mathbf{x},$$

where $\widetilde{A} = \overline{A^{-1}} = \|\widetilde{a_{jj'}}\|$.

When we approximate the reproducing kernel Hilbert space H_K by the vector space H_A, then from Proposition 4.1, the following is directly derived:

Proposition 5.1 *In the linear mapping*

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H} \tag{21}$$

for

$$\{p_1, p_2, ..., p_n\},$$

the minimum norm inverse $\mathbf{f}_{A_n}^*$ *satisfying*

$$f(p_j) = (\mathbf{f}, \mathbf{h}(p_j))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H} \tag{22}$$

is given by

$$\mathbf{f}_{A_n}^* = \sum_{j=1}^{n} \sum_{j'=1}^{n} f(p_j) \widetilde{a_{jj'}} \mathbf{h}(p_{j'}), \tag{23}$$

where $\widetilde{a_{jj'}}$ *are assumed the elements of the complex conjugate inverse of the positive definite Hermitian matrix* A_n *constituted by the elements*

$$a_{jj'} = (\mathbf{h}(p_{j'}), \mathbf{h}(p_j))_{\mathcal{H}} = K(p_j, p_{j'}).$$

Here, the positive definiteness of A_n *is a basic assumption.*

The following proposition shows the convergence of the approximate inverses in Proposition 5.1.

Proposition 5.2 *Let* $\{p_j\}_{j=1}^{\infty}$ *be a sequence of distinct points on* E*, that is the positive definiteness in Proposition 5.1 for any* n *and a uniqueness set for the reproducing kernel Hilbert space* H_K*; that is, for any* $f \in H_K$*, if all* $f(p_j) = 0$*, then* $f \equiv 0$*. Then, in the space* $\mathcal{H}$

$$\lim_{n \to \infty} \mathbf{f}_{A_n}^* = \mathbf{f}^*. \tag{24}$$

From the result, we can obtain directly the ultimate realization of the reproducing kernel Hilbert spaces and the ultimate sampling theory:

Proposition 5.3 *(Ultimate realization of reproducing kernel Hilbert spaces). In the general situation and for a uniqueness set* $\{p_j\}$ *of the set* E *satisfying the linearly independence in Proposition 5.1,*

$$\|f\|_{H_K}^2 = \|\mathbf{f}^*\|_{\mathcal{H}}^2 = \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{j'=1}^{n} f(p_j) \widetilde{a_{jj'}} \overline{f(p_{j'})}. \tag{25}$$

Proposition 5.4 *(Ultimate sampling theory). In the general situation and for a uniqueness set $\{p_j\}$ of the set E satisfying the linearly independence in Proposition 5.1,*

$$f(p) = \lim_{n\to\infty} (\mathbf{f}^*_{A_n}, \mathbf{h}(p))_{\mathcal{H}} = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}}\mathbf{h}(p_{j'}), \mathbf{h}(p) \right)_{\mathcal{H}} \tag{26}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}} K(p, p_{j'}).$$

In Proposition 5.1, for any given finite number $f(p_j)$, $j = 1, 2, ..., n$, the result in Proposition 5.1 is valid. Meanwhile, Proposition 5.2 and Proposition 5.4 are valid when we consider the sequence $f(p_j)$, $j = 1, 2, ...$, for any member f of H_K. The sequence $f(p_j)$, $j = 1, 2, ...$, for any member f of H_K is characterized by the convergence of (25) in Proposition 5.3. Then, any member f of H_K is represented by (26) in terms of the sequence $f(p_j)$, $j = 1, 2, ...$, in Proposition 5.4.

In the general setting in Proposition 5.1, assume that we observed the values $f(p_j) = \alpha_j, j = 1, 2, ..., n$, for a finite number of points $\{p_j\}$, then for the whole value $f(p)$ of the set E, how can we consider it?

One idea is to consider the function $f_1(p)$: among the functions satisfying the conditions $f(p_j) = \alpha_j, j = 1, 2, ..., n$, we consider the minimum norm member $f_1(p)$ in $H_K(E)$. This function $f_1(p)$ is determined by the formula,

$$f_1(p) = \sum_{j=1}^{n} C_j K(p, p_j)$$

where, the constants $\{C_j\}$ are determined by the formula:

$$\sum_{j=1}^{n} C_j K(p_{j'}, p_j) = \alpha_{j'}, j' = 1, 2, ..., n.$$

(of course, we assume that $\|K(p_{j'}, p_j)\|$ is positive definite).

For this problem, see, Mo, Y. and Qian, T. (2014) : Support vector machine adapted Tikhonov regularization method to solve Dirichlet problem ([16]), as a new numerical approrach by a usual computer system level, we use a special powerful computer system by H. Fujiwara. In particular, they can deal with errorness data.

Meanwhile, by Proposition 1.4, we can consider the function $f_2(p)$ defined by

$$f_2(p) = (\mathbf{f}^*_{A_n}, \mathbf{h}(p))_{\mathcal{H}}$$

in terms of $\mathbf{f}^*_{A_n}$ in Proposition 3.1. This interpolation formula is depending on the linear system.

For analytical problems, we need discretization and using a finite number of data in order to obtain approximate solutions by using computers, the typical methods are finite element method and difference method, however, their practical algorithms will be complicated depending on case by case, depending on the domains and depending on the dimensions, however, the above methods are essentially simple and uniform method in principle, called the *Aveiro discretization method*. However, the hard work part is to solve the linear simultaneous equations, computer powers requested are increasing and so, in future, the above simple method may be expected to become a standard method. For the general information and numerical results, see (Castro et al 2014; Castro et al 2014).

Many numerical experiments for the sampling theory by Proposition 5.4 were given by [9]. In particular:

We showed a general sampling theorem and the concrete numerical experiments for the simplest and typical examples. We gave the sampling theorem in the Sobolev Hilbert spaces with numerical experimences. For the Sobolev Hilbert spaces, sampling theorems seem to be a new concept.

For the typical Paley-Wiener spaces, the sampling points are automatically determined as the common sense, however, in our general sampling theorem, we can select the sampling points freely and so, case by case, following some *a priori* information of a considering function, we can take the effective sampling points. We showed these properties by the concrete examples, by many Figures.

5.1 A Typical Example of the Aveiro Discretization Method With ODE

Consider a prototype differential operator

$$Ly := \alpha y'' + \beta y' + \gamma y. \tag{27}$$

Here, consider a very general situation that the coefficients are *arbitrary* functions on their nature and on a general interval I.

For some practical construction of some natural solution of

$$Ly = g \tag{28}$$

for a very general function g on a general interval I,

Proposition 5.5 *(Castro et all 2014; Castro et al 2014) Let us fix a positive number h and take a finite number of points $\{t_j\}_{j=1}^n$ of I such that*

$$(\alpha(t_j), \beta(t_j), \gamma(t_j)) \neq \mathbf{0}$$

for each j. Then, an optimal solution y_h^A of the equation (28) is given by

$$y_h^A(t) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} F_h^A(\xi) e^{-it\xi} d\xi$$

in terms of the function $F_h^A \in L_2(-\pi/h, +\pi/h)$ in the sense that F_h^A has the minimum norm in $L_2(-\pi/h, +\pi/h)$ among the functions $F \in L_2(-\pi/h, +\pi/h)$ satisfying, for the characteristic function $\chi_h(t)$ of the interval $(-\pi/h, +\pi/h)$:

$$\frac{1}{2\pi}\int_{\mathbf{R}} F(\xi)[\alpha(t)(-\xi^2)+\beta(t)(-i\xi)+\gamma(t)]\chi_h(\xi)\exp(-it\xi)d\xi = g(t) \qquad (29)$$

for all $t = t_j$ and for the function space $L_2(-\pi/h, +\pi/h)$.

The best extremal function F_h^A is given by

$$F_h^A(\xi) = \sum_{j,j'=1}^{n} g(t_j)\widetilde{a_{jj'}}\overline{(\alpha(t_{j'})(-\xi^2)+\beta(t_{j'})(-i\xi)+\gamma(t_{j'}))}\exp(it_{j'}\xi). \qquad (30)$$

Here, the matrix $A = \{a_{jj'}\}_{j,j'=1}^{n}$ formed by the elements

$$a_{jj'} = K_{hh}(t_j, t_{j'})$$

with

$$\begin{aligned} K_{hh}(t,t') &= \frac{1}{2\pi}\int_{\mathbf{R}} [\alpha(t)(-\xi^2)+\beta(t)(-i\xi)+\gamma(t)]\overline{[\alpha(t')(-\xi^2)+\beta(t')(-i\xi)+\gamma(t')]} \\ &\quad \cdot\chi_h(\xi)\exp(-i(t-t')\xi)d\xi \qquad (31) \end{aligned}$$

is positive definite and the $\widetilde{a_{jj'}}$ are the elements of the inverse of $\overline{A}$ (the complex conjugate of A).

Therefore, the optimal solution y_h^A of the equation (28) is given by

$$\begin{aligned} y_h^A(t) = \sum_{j,j'=1}^{n} g(t_j)\widetilde{a_{jj'}}\frac{1}{2\pi}[-\overline{\alpha(t_{j'})}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \xi^2 e^{-i(t-t_{j'})\xi}d\xi \\ +i\overline{\beta(t_{j'})}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \xi e^{-i(t-t_{j'})\xi}d\xi + \overline{\gamma(t_{j'})}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i(t-t_{j'})\xi}d\xi]. \end{aligned}$$

At first, we are considering approximate solutions of the differential equation (28) and at this point, we are considering the Paley-Wiener function spaces with parameter h as approximating function spaces; the function spaces are formed by analytic functions of the entire functions of exponential type that are decreasing to zero exponential order. Next, by using the Fourier inversion, the differential equation (28) may be transformed to (29). However, to solve the integral equation (29) is very difficult for the generality of the coefficient functions. So, we assume (29) is valid on some finite number of points t_j. This assumption will be very reasonable for the discretization of the integral equation. By this assumption we can obtain an optimal approximate solutions in a very simple way.

Here, we assume that equation (28) is valid on I and so, as some practical case we would like to consider the equation in (28) on I satisfying some boundary conditions. In the present case, the boundary conditions are given as zero at infinity for $I = \mathbf{R}$.

However, our result gives the approximate general solutions satisfying boundary values. For example, for a finite interval (a, b), we consider $t_1 = a$ and $t_n = b$ and $\alpha(t_1) = \beta(t_1) = \alpha(t_n) = \beta(t_n) = 0$. Then, we can obtain the approximate solution having the arbitrary given boundary values $y_h^A(t_1)$ and $y_h^A(t_n)$. In addition, by a simple modification we may give the general approximate solutions satisfying the corresponding boundary values.

For a finite interval case I, following the boundary conditions, we can consider the corresponding reproducing kernels by the Sobolev Hilbert spaces. However, the concrete representations of the reproducing kernels are involved depending on the boundary conditions. However, we can still consider them and we can use them.

Of course, for a smaller h we can obtain a better approximate solution.

For the representation (31) of the reproducing kernel $K_{hh}(t, t')$, we can calculate it easily.

The very surprising facts are: for variable coefficients linear differential equations, we can represent their approximate solutions satisfying their boundary conditions without integrals. Approximate function spaces may be considered with the Paley-Wiener spaces and the Sobolev spaces. For many concrete examples and numerical examples, see (Castro et al 2014; Castro et al 2014). We showed Figures of the numerical experiments. See also (Rocha 2014) for some applications to nonlinear partial differential equations.

Acknowledgments

This paper and the meeting were, of course, supported by the RIMS, Kyoto University. The author wishes to express his sincere thanks to Professor Takashi OHE for his great contributions to the author and to the meeting.

References

[1] N. Aronszajn, Theory of reproducing kernels. Trans. Amer. Math. Soc., **68**(1950), 337–404.

[2] L. P. Castro, H. Fujiwara, S, Saitoh, Y. Sawano, A. Yamada and M. Yamada, Fundamental error estimates inequalities for the Tikhonov regularization using reproducing kernels. International Series of Numerical Mathematics **161**(2010), Inequalities and Applications 2010, Springer, Basel: 87-101

[3] L. P. Castro, H. Fujiwara, M. M. Rodrigues, S. Saitoh and V. K. Tuan, Aveiro Discretization Method in Mathematics: A New Discretization Principle, MATHEMATICS WITHOUT BOUNDARIES: SURVEYS IN PURE

MATHEMATICS, Edited by Panos Pardalos and Themistocles M. Rassias 37-92 Springer (2014).

[4] L. P. Castro, H. Fujiwara, T. Qian and S. Saitoh, How to catch smoothing properties and analyticity of functions by computers? MATHEMATICS WITHOUT BOUNDARIES: SURVEYS IN INTERDISCIPLINARY RESEARCH, Edited by Panos Pardalos and Themistocles M. Rassias 101–116 Springer (2014).

[5] H. Fujiwara, Applications of reproducing kernel spaces to real inversions of the Laplace transform. RIMS Koukyuuroku 1618 (2008), 188–209.

[6] H. Fujiwara, T. Matsuura, S. Saitoh and Y. Sawano, (2009) Numerical real inversion of the Laplace transform by using a high-accuracy numerical method. Further Progress in Analysis: 574–583 World Sci. Publ., Hackensack, NJ (2009).

[7] H. Fujiwara, (2010) Numerical real inversion of the Laplace transform by reproducing kernel and multiple-precision arithmetric. Progress in Analysis and its Applications, Proceedings of the 7th International ISAAC Congress: 289–295 World Scientific (2010).

[8] H. Fujiwara and N. Higashimori, Numerical inversion of the Laplace transform by using multiple-precision arithmetic. Libertas Mathematica (new series), **34**(2014), No. 2, 5–21.

[9] H. Fujiwara and S.Saitoh, The general sampling theory by using reproducing kernels. CONTRIBUTIONS IN MATHEMATICS AND ENGINEERING In Honor of Constantin Caratheodory eds. Panos Pardalos and Th. M. Rassias, Springer (2010).

[10] P. C. Hansen, Analysis of discrete ill-posed problems by means of the L-curve. SIAM REVIEW, 34 (1992), 561-580.

[11] C. L. Lawson and R. J. Hanson, Solving least squares problems. Prentice-Hall, Englewood Cliffs. (1974).

[12] A. N. Kolmogoroff, Stationary sequences in Hilbert's space. Bolletin Moskovskogo Gosudarstvenogo Universiteta, Matematika (1941) 2: 40pp. (in Russian)

[13] T. Matsuura and S. Saitoh, Analytical and numerical inversion formulas in the Gaussian convolution by using the Paley-Wiener spaces. Applicable Analysis, **85** (2006), 901–915.

[14] T. Matsuura and S. Saitoh, General integral transforms by the concept of generalized reproducing kernels, P. Dan et al (eds.), New Trends in Analysis and Interdisciplinary Applications. Trends in Mathematics, Birkhäuser (2017), 379–386.

[15] V. Mishra and D. Rani, Laplace transform inversion using Bernstein operational matrix of integration and its application to differential and integral equations, Proc. Indian Acad. Sci. (Math. Sci.) (2020) 130:60 https://doi.org/10.1007/s12044-020-0573-x.

[16] Y. Mo and T. Qianm, Support vector machine adapted Tikhonov regularization method to solve Dirichlet problem. Appl. Math. Comput. **245**(2014), 509–519.

[17] E. M. Rocha, A reproducing kernel Hilbert discretization method for linear PDEs with nonlinear right-hand side. Libertas Mathematica (new series), **34**(2014), no. 2, 91-104.

[18] S. Saitoh, *Hilbert spaces induced by Hilbert space valued functions,* Proc. Amer. Math. Soc., **89** (1983), 74–78.

[19] S. Saitoh, Integral Transforms, Reproducing Kernels and their Applications. Pitman Res. Notes in Math. Series 369 (1997), Addison Wesley Longman, Harlow, CRC Press, Taylor & Francis Group, Boca Raton London, New York (in hard cover).

[20] S. Saitoh, Various operators in Hilbert space induced by transforms. International J. of Applied Math. 1 (1999), 111–126.

[21] S. Saitoh, Theory of reproducing kernels: Applications to approximate solutions of bounded linear operator equations on Hilbert spaces. Amer. Math. Soc. Transl. Ser. 2: 230 (2010), 107–137.

[22] S. Saitoh and Y, Sawano, Generalized reproducing kernels and generalized delta functions, P. Dan et al (eds.), New Trends in Analysis and Interdisciplinary Applications. Trends in Mathematics, Birkhäuser (2017), 395–400.

[23] S. Saitoh and Y. Sawano, Theory of Reproducing Kernels and Applications, Developments in Mathematics **44**, Springer (2016).