Mathematics of Image Reconstruction in Sparse-View CT and Interior CT

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1. Introduction

Since 2000, it has been widely recognized that radiation dose in CT examinations increases cancer risk. To overcome this drawback, new designs of CT scanners called sparse-view CT and interior CT have been actively investigated in CT community. As shown in Fig. 1(a), the sparse-view CT refers to CT in which the number of projection data is reduced to decrease patient dose as well as to accelerate data acquisition. As shown in Fig. 1(b), the interior CT refers to CT in which X-rays are radiated only to a small region of interest (ROI) such as heart or breast to decrease patient dose. A key in these scanners is how to reconstruct images with sufficient quality from the limited projection data. In this paper, we introduce research activities on this topic for unfamiliar readers based on my talk in RIMS workshop "Recent developments on inverse problems for partial differential equations and their applications".

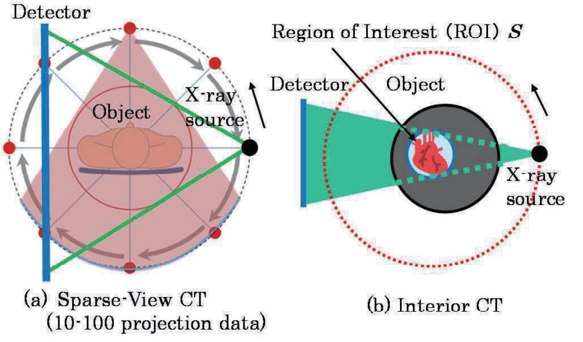

Fig. 1 Principles of sparse-view CT and interior CT.

2. Image Reconstruction in Sparse-View CT

This section is concerned with image reconstruction in the sparse-view CT using Compressed Sensing (CS). It is well-known that the solution to image reconstruction in the sparse-view CT is not unique such that the associated null-space of Radon transform operator is rather complicated. Up to the middle of 2000, this problem had been tackled with a variety of image reconstruction methods. However, the conclusion was that it is impossible to reconstruct sufficient images from a small number of projection data. In 2006, however, Candes *et al.* [1] and Donoho [2] discovered a new class of solution methods for inverse problems called Compressed Sensing (CS). CS is a promising technique, which is able to reconstruct high-quality images even from a variety of limited projection data. Since then, many researches have demonstrated that CS is very powerful for the image reconstruction in sparse-view CT. Here, we explain two CS-based image reconstruction methods developed for the sparse-view CT. The first one is the standard method all over the world called Total Variation (TV) regularization, which has been already used in commercial CT scanners. The second one is our original CS called second-generation CS which improves CS in terms of image quality.

(1) Total Variation (TV) Regularization [3]

We denote an image to be reconstructed by $\vec{x}$, denote measured projection data by $\vec{b}$, and denote a system matrix relating $\vec{x}$ and $\vec{b}$ by A. In the TV regularization approach, image reconstruction is performed by minimizing the following cost function.

$$f(\vec{x}) = \beta \|\vec{x}\|_{\mathrm{TV}} + \left\|A\vec{x} - \vec{b}\right\|^2, \quad (1)$$

where the first term is TV norm of image, the second term is the least-squares data fidelity term, and β is the hyper-parameter to control the strength of regularization.

(2) Second-Generation Compressed Sensing [4]

The major drawback of TV regularization is that smooth intensity changes and image textures are lost when measurement condition is not very good, because it is based on the mathematical model that the image is piecewise constant. To further improve image quality, several improved regularization terms such as non-local TV and higher-order TV have been proposed. Second-generation CS is one of them which we have developed around 2015 [4]. In 2-nd generation CS, image reconstruction is performed by minimizing the following cost function.

$$f(\vec{x}) = \beta \|\vec{x} - M\vec{x}\|_1^1 + \left\|A\vec{x} - \vec{b}\right\|^2, \quad (2)$$

where M is a non-linear smoothing filter. A key in this approach is the regularization term (first term) in Eq. (2), which can be interpreted as follows. First, we sparsify the image $\vec{x}$ by computing a difference between the image $\vec{x}$ and the filtered (smoothed) image $M\vec{x}$. Then, we evaluate sparsity of the sparsified vector $\vec{x} - M\vec{x}$ by computing L^1 norm. The choice of non-linear filter has a significant effect on image quality. If we use median filter, achieved image quality is similar to that by the TV regularization. However, by using Non-Local Means (NLM) filter or bilateral filter having a strong power in the preservation of smooth intensity changes and textures, image quality can be significantly improved compared to the standard TV. We are using NLM filter.

Figure 2 shows example reconstructed images in the sparse-view CT. We used real projection data of X-ray phase CT, where the sample object was a small piece of blended-polymer material (PS-rich region and PMMA-rich region are mixed in a complicated way). The image reconstruction was performed from only 23 projection data (less than 1/10 of the ordinary scan case). We compared reconstructed images by Filtered BackProjection (FBP) method, TV regularization, and 2-nd generation CS. It can be observed that CS succeeds in eliminating the streak artifact occurred in FBP method, and 2-nd generation CS significantly outperforms the TV regularization in terms of preserving smooth intensity changes.

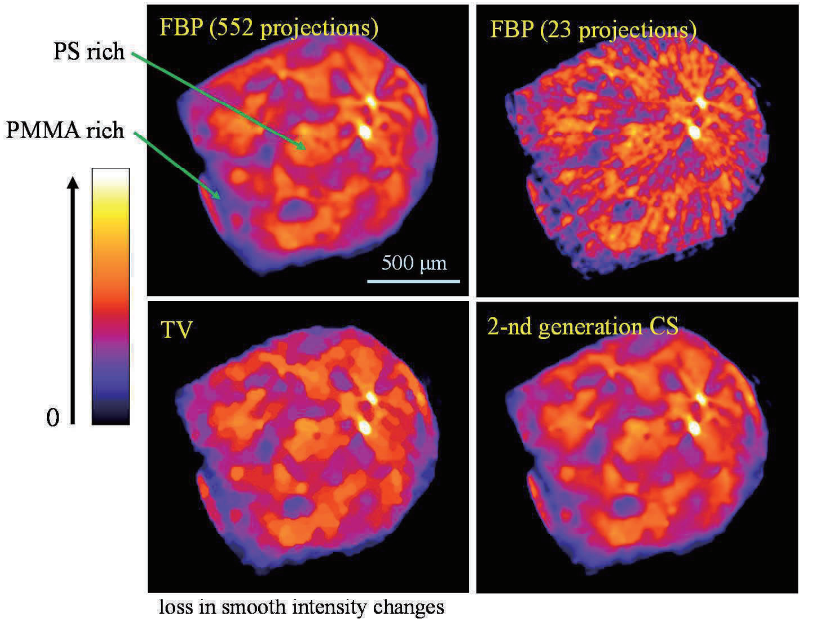

Fig. 2 Reconstructed images from only 23 projection data in the X-ray phase CT. The sample object was a small piece of blended-polymer material.

3. Image Reconstruction in Interior CT

This section is concerned with image reconstruction in the interior CT. For a long time up to 2007, it had been believed that exact image reconstruction in the interior CT is impossible, because Radon transform operator corresponding to the interior CT possesses a complicated null space. Since 2007, however, several exact solution methods have been discovered in CT community. To guarantee the solution uniqueness, some of them use small *a prior* knowledge on the object, and others use small additional measurement of projection data. Here, we explain the four existing exact approaches to the interior CT reconstruction centered on our research activities.

We begin by describing the definition of image reconstruction in the interior CT. We denote an object to be reconstructed by $f(x,y)$ and denote measured parallel-beam projection data by $p(r,\theta)$. As shown in Fig. 1(b), let consider the imaging situation in which it is enough to reconstruct $f(x,y)$ on a small ROI S located inside the object. In this case, we intuitively imagine that it is NOT necessary to measure $p(r,\theta)$ which does

NOT pass through the ROI S, because these rays do NOT have information of ROI. So, the interior CT refers to CT in which $p(r,\theta)$ is measured only for a set of straight lines passing through the ROI S and $f(x,y)$ is reconstructed only on the small ROI S. In 1986, Natterer [5] proved that the solution to image reconstruction in the interior CT is not unique. Since then, this problem had been an unsolved problem for a long time in CT fields. Since 2007, novel exact approaches to this problem has been discovered. Here, I briefly review four exact approaches. The existing exact approaches can be classified as follows.

(1) Method 1: Using *a Priori* Knowledge inside ROI [6],[7]

Ye *et al.* [6] and Kudo *et al.* [7] discovered the following uniqueness result. If $f(x,y)$ is known on an arbitrary small region B (having non-zero measure) located inside the ROI S as *a priori* knowledge, $f(x,y)$ is uniquely determined over the whole ROI S.

(2) Method 2: Using Piecewise Constancy over the Whole ROI [8]

In 2009, Yu and Wang [8] discovered the following uniqueness result. If it is known that $f(x,y)$ is piecewise constant over the whole ROI S, $f(x,y)$ is uniquely determined over the ROI S.

(3) Method 3: Using Piecewise Constancy over a Small Region inside ROI [9]

The major drawback of Yu and Wang's approach is that the *a priori* knowledge that $f(x,y)$ is piecewise constant over the whole ROI S is not correct for actual CT images having smooth intensity changes and textures. Consequently, there is a danger that this approach eliminates the smooth intensity changes and textures (as shown in Fig. 3). To overcome this drawback, Kudo [9] relaxed the necessary *a priori* knowledge as follows. If it is known that $f(x,y)$ is piecewise constant on an arbitrary small region B (having non-zero measure) located inside the ROI S, $f(x,y)$ is uniquely determined over the whole ROI S.

(4) Method 4: Using Minimum Additional Complete Projection Data [10]

All the previous exact approaches to the interior CT reconstruction, *i.e.* Method 1, Method 2, and Method 3, use *a priori* knowledge on the object to guarantee the solution uniqueness. In 2018, Kudo [10] discovered the following new approach, which is based on measuring additional minimum complete projection data covering the whole object (not ROI only). His result is summarized as follows. If we measure complete non-truncated projection data $p(r,\theta)$ (covering the whole object) over an arbitrary small angular range (having non-zero measure) $\theta \in E$ in addition to the interior CT projection data, $f(x,y)$ is uniquely determined over the whole ROI S. In addition, Kudo [10] proved that the inversion in this case is stable.

Figure 3 shows example reconstructed images in the interior CT. We used a CT image of human brain as a numerical phantom. The image reconstruction was performed from numerically computed projection data (with no noise) by using the four exact approaches. We also implemented the standard local FBP method used in commercial CT scanners to handle the interior CT case. In this method, the missing part of interior CT projection data is extrapolated with a smooth function and the resulting completed projection data is reconstructed by using FBP method. In Fig. 3, local FBP method suffered from the severe low-frequency shading artifact and the DC shift. As is worried, Yu and Wang's approach (Method 2) suffered from the loss in smooth intensity changes and textures. Other three approaches, *i.e.* Method 1, Method 3, and Method 4, succeeded in providing nice images. In this experiment, the most surprising discovery was the following. In implementing Method 4, we used only one (minimum) additional complete projection data in addition to the interior CT data, but the artifact could be almost completely eliminated leading to an almost perfect reconstruction.

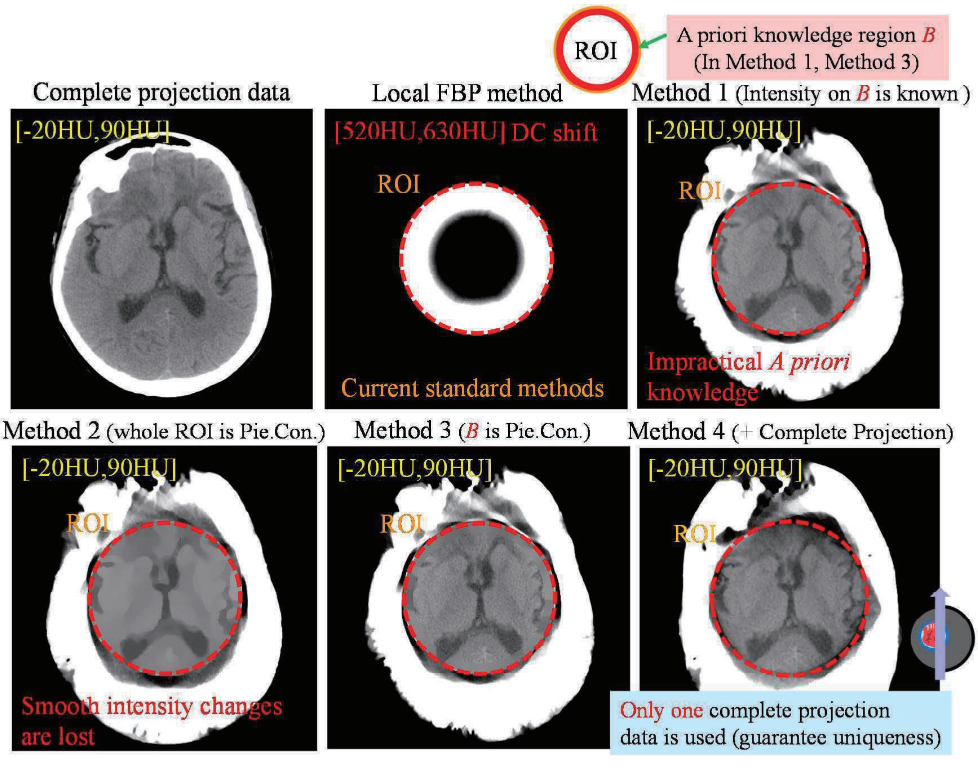

Fig. 3 Reconstructed images in the simulation studies of interior CT. In Method 1 and Method 3, *a priori* knowledge region *B* was set to the ring-shaped region shown on the top. In Method 4, we used only one complete (non-truncated) projection data corresponding to zero degree as an additional measurement.

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