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Kazama-Suzuki Coset Vertex Superalgebras at Admissible Levels

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1 Introduction

1.1 Verlinde Formula for Irrational Theories

One of the most fundamental problems in the study of two-dimensional conformal field theories (CFTs) is to compute *fusion rules* among primary fields. Each model of CFT specifies a vertex operator superalgebra as its chiral symmetry algebra. Mathematically, primary fields in a given model are identified with highest weight modules of the vertex operator superalgebra and their fusion rules are defined by the corresponding 3-point conformal blocks on the genus zero curve $\mathbf{P}^1(\mathbb{C})$ (see e.g. [Zhu94]). When the vertex operator superalgebra satisfies certain good conditions (*rationality* and *C_2 -cofiniteness*), fusion rules can be computed in terms of the *modular S -matrix* via the *Verlinde formula*, which is originally proposed by E. Verlinde in [Ver88] and proved by Y.-Z. Huang in [Hua08]. Such models are referred to as *rational* CFTs and have been extensively studied in various contexts.

Recently, a certain class of irrational chiral CFTs turns out to play an important role in the conjectural 2d/4d correspondence firstly investigated by C. Beem et. al. in [BLL⁺15]. One of the best studied irrational chiral CFTs is the Wess–Zumino–Novikov–Witten (WZNW) model associated with the simple affine vertex operator algebra $L_k(\mathfrak{sl}_2)$, where k is a special rational number known as a Kac–Wakimoto admissible level (see [KW88] for detail). D. Adamović and A. Milas proved in [AM95] that irreducible highest weight $L_k(\mathfrak{sl}_2)$ -modules coincide with the Kac–Wakimoto admissible highest weight modules of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ of level k . Based on the celebrated modular invariance property of the Kac–Wakimoto characters, one can define the corresponding modular S -matrix. However, there are no known general results about relationships between fusion rules and the modular S -matrix. In fact, a naive application of the Verlinde formula computes negative fusion rules from the modular S -matrix. To fix this failure, T. Creutzig and D. Ridout in [CR13] computed an extended *modular S -data* and proposed a conjectural Verlinde formula. We note that this conjectural Verlinde formula is verified only for ordinary (= \mathfrak{sl}_2 -integrable) highest weight $L_k(\mathfrak{sl}_2)$ -modules by T. Creutzig, Y.-Z. Huang, and J. Yang in [CHY18].

1.2 Our Result

In this note we consider a super-analog of Creutzig–Ridout’s conjecture. More precisely, we construct the $\mathcal{N} = 2$ superconformal vertex operator superalgebra $\mathbb{L}_{c(k)}$ from the affine vertex operator algebra $L_k(\mathfrak{sl}_2)$ by the Kazama–Suzuki supersymmetric coset construction and formulate an analogous Verlinde formula for $\mathbb{L}_{c(k)}$. Based on the modular S -data associated with irreducible highest weight $\mathbb{L}_{c(k)}$ -modules at a general Kac–Wakimoto admissible level

$$k = -2 + \frac{p}{p'} \quad ((p, p') \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 2}: \text{coprime})$$

computed by the author in [Sat19], we obtain the following result:

Theorem 1.1. Fusion rules among irreducible highest weight $\mathbb{L}_{c(-\frac{1}{2})}$ -modules are correctly computed from the modular S -data in [Sat19] via the conjectural Verlinde formula.

We divide our computation of fusion rules into two steps. First, we estimate the fusion rules from above by using the *Frenkel–Zhu theory*. Roughly speaking, the Zhu algebra associated to a vertex operator superalgebra V is an associative superalgebra $\mathbf{Zhu}(V)$ generated by the degree preserving adjoint actions of V . More generally, in [FZ92], I. Frenkel and Y. Zhu constructed $\mathbf{Zhu}(V)$ -bimodules from highest weight V -modules. Frenkel–Zhu’s bimodules have nice applications to the computation of fusion rules. For more details of the Frenkel–Zhu theory, we refer the reader to [KW94, DLM98a, DLM98b, Li99, DZ06].

Second, we estimate the fusion rules from below by using a certain *free field realization*. By using the oscillator representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2 = \widehat{\mathfrak{sp}}_2$ of level $k = -\frac{1}{2}$, we identify the simple vertex operator algebra $L_{-\frac{1}{2}}(\mathfrak{sl}_2)$ with a \mathbb{Z}_2 -orbifold of the free bosonic $\beta\gamma$ -system. Then, by the Kazama–Suzuki coset construction, we realize the simple vertex operator superalgebra $\mathbb{L}_{c(-\frac{1}{2})}$ as a Heisenberg coset of a \mathbb{Z}_2 -orbifold of the tensor product of the $\beta\gamma$ -system and the free fermionic bc -system. This construction enables us to find non-trivial intertwining operators (cf. [Zhu94, Proposition 7.4]) and completes our computation.

The detail of the proof will appear soon in a substantially revised version of our earlier draft [KS18].

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2 Superconformal Vertex Algebra

2.1 Notation

We first recall the vertex operator superalgebra associated with the two-dimensional $\mathcal{N} = 2$ superconformal algebra of central charge $c \in \mathbb{C}$.

Proposition 2.1 ([Ada99]). There exists a unique vertex superalgebra \mathbb{V}^c which is strongly generated by \mathbb{Z}_2 -homogeneous fields $\mathbf{L}(z)$, $\mathbf{J}(z)$, $\mathbf{G}^+(z)$, $\mathbf{G}^-(z)$ with parity

$$\deg(\mathbf{L}(z)) = \deg(\mathbf{J}(z)) = \bar{0}, \quad \deg(\mathbf{G}^\pm(z)) = \bar{1}$$

only subject to the following operator product expansions:

$$\begin{aligned} \mathbf{L}(z)\mathbf{L}(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2\mathbf{L}(w)}{(z-w)^2} + \frac{\partial_w \mathbf{L}(w)}{z-w}, \\ \mathbf{L}(z)\mathbf{G}^\pm(w) &\sim \frac{\frac{3}{2}\mathbf{G}^\pm(w)}{(z-w)^2} + \frac{\partial_w \mathbf{G}^\pm(w)}{z-w}, \quad \mathbf{L}(z)\mathbf{J}(w) \sim \frac{\mathbf{J}(w)}{(z-w)^2} + \frac{\partial_w \mathbf{J}(w)}{z-w}, \\ \mathbf{J}(z)\mathbf{J}(w) &\sim \frac{c/3}{(z-w)^2}, \quad \mathbf{J}(z)\mathbf{G}^\pm(w) \sim \frac{\pm \mathbf{G}^\pm(w)}{z-w}, \\ \mathbf{G}^\pm(z)\mathbf{G}^\pm(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2\mathbf{L}(w) + 2\mathbf{J}(w)}{(z-w)^2} + \frac{\partial_w \mathbf{J}(w)}{z-w}. \end{aligned}$$

In addition, the Virasoro field $\mathbf{L}(z)$ gives rise to a vertex operator superalgebra structure of central charge c on \mathbb{V}^c .

Remark 2.2. Every \mathbb{V}^c -module admits a natural action of the Neveu-Schwarz sector of the $\mathcal{N} = 2$ superconformal algebra generated by

$$L_n := \mathbf{L}_{(n+1)}, \quad G_r^\pm := \mathbf{G}_{(r+\frac{1}{2})}^\pm, \quad J_n := \mathbf{J}_{(n)},$$

where

$$A_{(n)} := \oint z^n A(z) \frac{dz}{2\pi\sqrt{-1}}.$$

The Verma module $\mathcal{M}_{h,j,c}$ of highest weight $(h, j, c) \in \mathbb{C}^3$ is a \mathbb{Z}_2 -graded \mathbb{V}^c -module freely generated by an even vector $|h, j, c\rangle$ subject to the relations

$$\begin{aligned} L_n |h, j, c\rangle &= G_r^\pm |h, j, c\rangle = J_n |h, j, c\rangle := 0 \quad (n, r > 0), \\ L_0 |h, j, c\rangle &:= h |h, j, c\rangle, \quad J_0 |h, j, c\rangle := j |h, j, c\rangle, \quad C |h, j, c\rangle := c |h, j, c\rangle. \end{aligned}$$

We write $\mathcal{L}_{h,j,c}$ for the irreducible quotient \mathbb{V}^c -module of $\mathcal{M}_{h,j,c}$.

2.2 Oscillator Realization

Let $V^k(\mathfrak{sl}_2)$ be the universal affine vertex algebra generated by $\mathbf{E}, \mathbf{H}, \mathbf{F}$ and π_b^k its Heisenberg vertex subalgebra generated by \mathbf{H} . Let $\langle \beta\gamma \rangle$ and $\langle bc \rangle$ denote the free bosonic $\beta\gamma$ -system and the free fermionic bc -system, respectively. By using the Kazama-Suzuki coset construction (see e.g. [Ada99])

$$\mathbb{V}_{\frac{3k}{k+2}}^{\frac{3k}{k+2}} \otimes \pi_b^{k+2} \rightarrow V^k(\mathfrak{sl}_2) \otimes \langle bc \rangle,$$

which is uniquely determined by

$$\mathbf{G}^+ \otimes \mathbf{1} \mapsto \sqrt{\frac{2}{k+2}} \mathbf{E} \otimes b, \quad \mathbf{G}^- \otimes \mathbf{1} \mapsto \sqrt{\frac{2}{k+2}} \mathbf{F} \otimes c, \quad \mathbf{1} \otimes \mathbf{H} \mapsto \mathbf{H} \otimes \mathbf{1} + \mathbf{1} \otimes 2b_{(-1)}c,$$

and the oscillator representation (see e.g. [FF85])

$$V^{-\frac{1}{2}}(\mathfrak{sl}_2) \rightarrow \langle \beta\gamma \rangle; \quad \mathbf{E} \mapsto -\beta_{(-1)}\beta, \quad \mathbf{H} \mapsto -\beta_{(-1)}\gamma, \quad \mathbf{F} \mapsto \gamma_{(-1)}\gamma,$$

we construct a free field realization of \mathbb{V}^{-1} as follows.

Lemma 2.3. There exist a unique vertex algebra homomorphism

$$\mathbb{V}^{-1} \otimes \pi_b^{\frac{3}{2}} \rightarrow \langle \beta\gamma \rangle \otimes \langle bc \rangle \quad (2.1)$$

such that

$$\begin{aligned} \mathbf{G}^+ \otimes \mathbf{1} &\mapsto -\frac{1}{\sqrt{3}}\beta_{(-1)}\beta \otimes b, \\ \mathbf{G}^- \otimes \mathbf{1} &\mapsto \frac{1}{\sqrt{3}}\gamma_{(-1)}\gamma \otimes c, \\ \mathbf{1} \otimes \mathbf{H} &\mapsto \phi^- \otimes \mathbf{1} + \mathbf{1} \otimes 2\phi^+, \end{aligned}$$

where $\phi^- := -\beta_{(-1)}\gamma$ and $\phi^+ := b_{(-1)}c$.

Let \mathbb{L}_c denote the simple quotient vertex operator superalgebra of \mathbb{V}^c .

Corollary 2.4. The map (2.1) induces a vertex superalgebra isomorphism

$$\mathbb{L}_{-1} \simeq C := \text{Com}(\pi_b^{\frac{3}{2}}, \langle \beta\gamma \rangle \otimes \langle bc \rangle).$$

Proof. One can verify that the induced homomorphism $\mathbb{V}^{-1} \rightarrow C$ is surjective by comparing their graded dimensions. Since $\pi_b^{\frac{3}{2}}$ and $\langle \beta\gamma \rangle \otimes \langle bc \rangle$ are simple, so is C (see e.g. [ACKL17, Lemma 2.1]). This completes the proof. \square

2.3 Spectral Flow Twist

In this subsection V denotes $\langle \beta\gamma \rangle$ or \mathbb{L}_c and \mathbf{A} denotes ϕ^- or \mathbf{J} , respectively. Since $A_0 := \mathbf{A}_{(0)}$ acts diagonally on V with integer eigenvalues and fixes the conformal vector, $\sigma := \exp(\pi\sqrt{-1}A_0)$ gives a vertex operator superalgebra involution of V . In addition, one can consider the notion of strongly (\mathbb{C} -)graded V -modules in the sense of [HLZ14, Definition 2.25] with respect to the operator A_0 . A strongly graded V -module M admits A_0 -eigenspace decomposition

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

and each component M_λ further decomposes into finite-dimensional L_0 -eigenspaces with the lower truncation condition. Such modules play an important role in a non- C_2 -cofinite situation. In fact, neither $\langle \beta\gamma \rangle$ nor \mathbb{L}_{-1} is C_2 -cofinite.

Now we recall the following construction known as ‘‘spectral flow twist’’.

Lemma 2.5 ([Li97, Proposition 2,1]). Let $\varepsilon, \varepsilon' \in \{0, \frac{1}{2}\}$ and $\theta \in \mathbb{Z} + \varepsilon'$. For a $\sigma^{1-2\varepsilon}$ -twisted strongly graded V -module (M, Y_M) , we define

$$\Delta(\theta\mathbf{A}; z) := z^{\theta A_0} \exp\left(\sum_{\ell=1}^{\infty} \frac{\theta A_\ell}{-\ell} (-z)^{-\ell}\right) \in \text{End}(V)[[z^{\pm(\frac{1}{2}-\varepsilon')}]].$$

Then the even mapping

$$Y_{M^\theta}(\cdot; z) := Y_M(\Delta(\theta\mathbf{A}; z)\cdot; z) : V \rightarrow \text{End}(M)[[z^{\pm(\frac{1}{2}+|\varepsilon-\varepsilon'|)}]]$$

gives rise to a new $\sigma^{1-2|\varepsilon-\varepsilon'|}$ -twisted strongly graded V -module $M^\theta := (M, Y_{M^\theta})$, called a spectral flow twisted module of M .

Fusion rules are compatible with the spectral flow twist in the following sense.

Lemma 2.6 ([Li97, Proposition 2.4]). There exists a natural isomorphism

$$\mathrm{IO} \begin{pmatrix} N \\ L \ M \end{pmatrix} \simeq \mathrm{IO} \begin{pmatrix} N^{\theta+\theta'} \\ L^\theta \ M^{\theta'} \end{pmatrix}$$

for any \mathbb{Z}_2 -graded strongly graded V -modules L, M, N , and $\theta \in \frac{1}{2}\mathbb{Z}$, where $\mathrm{IO} \begin{pmatrix} N \\ L \ M \end{pmatrix}$ denotes the space of \mathbb{Z}_2 -graded intertwining operators of type $\begin{pmatrix} N \\ L \ M \end{pmatrix}$.

Since all the irreducible strongly graded \mathbb{L}_c -modules are highest weight modules and they are closed under taking the spectral flow twist, we only need to consider a complete system of representatives with respect to the twist.

3 Fusion Rules at Central Charge $c = -1$

In this section, we determine all the fusion rules among (\mathbb{Z}_2 -graded) irreducible highest weight \mathbb{L}_{-1} -modules. Our computation is divided into two steps:

1. estimation from above by using the Frenkel–Zhu theory;
2. estimation from below by using the free field realization.

The Zhu algebra $\mathrm{Zhu}(V)$ for a vertex operator superalgebra V is firstly introduced in [KW94] as a natural generalization of the original one in [Zhu96]. When $V = \mathbb{V}^c$ or \mathbb{L}_c , the following isomorphism is well-known (e.g. [Ada99, Remark 1.1]) and is proved in the same way as [KW94, Lemma 3.1].

Lemma 3.1. For $c \in \mathbb{C}$, there exists a unique algebra isomomorphism

$$\mathbb{C}[\mathfrak{h}, \mathfrak{q}] \xrightarrow{\simeq} \mathrm{Zhu}(\mathbb{V}_c); \quad \mathfrak{h} \mapsto [\mathbf{L}], \quad \mathfrak{q} \mapsto [\mathbf{J}],$$

where $[A]$ denotes the natural image of $A \in \mathbb{V}_c$ in the quotient space $\mathrm{Zhu}(\mathbb{V}_c)$. Moreover, $\mathrm{Zhu}(\mathbb{L}_c)$ is obtained as a quotient algebra of $\mathrm{Zhu}(\mathbb{V}_c) \simeq \mathbb{C}[\mathfrak{h}, \mathfrak{q}]$.

Let $\mathrm{lrr}(\mathbb{L}_c)$ denote the set of isomorphism classes of \mathbb{Z}_2 -graded irreducible highest weight \mathbb{L}_c -modules and $\mathrm{lrr}(\mathbb{L}_c)/\mathbb{Z}$ denote the set of \mathbb{Z} -orbits with respect to the \mathbb{Z} -action induced by the spectral flow twist. From the classification result in [Ada99, Theorem 7.2] and direct calculation, we obtain the following.

Proposition 3.2. The disjoint union

$$\{\mathcal{L}(\epsilon) := \mathcal{L}_{\frac{4}{3}\epsilon, \frac{4}{3}\epsilon, -1}, \mathbf{\Pi}\mathcal{L}(\epsilon) \mid \epsilon \in \{0, \pm\frac{1}{2}\}\} \sqcup \{\mathcal{L}_j := \mathcal{L}_{-\frac{1}{8}-\frac{2}{3}j^2, \frac{4}{3}j, -1}, \mathbf{\Pi}\mathcal{L}_j \mid j \in \mathbb{C}\}$$

$$\text{(resp. } \{\mathcal{L}(\epsilon), \mathbf{\Pi}\mathcal{L}(\epsilon) \mid \epsilon \in \{0, \frac{1}{2}\}\} \sqcup \{\mathcal{L}_j, \mathbf{\Pi}\mathcal{L}_j \mid j \in S\})$$

gives a complete representative set of $\mathrm{lrr}(\mathbb{L}_c)$ (resp. $\mathrm{lrr}(\mathbb{L}_c)/\mathbb{Z}$), where

$$S := \{x + y\sqrt{-1} \mid -\frac{1}{2} \leq x < \frac{1}{2}, x \notin \{\pm\frac{1}{4}\}, y \in \mathbb{R}\}$$

and $\mathbf{\Pi}$ stands for the \mathbb{Z}_2 -parity reversing functor.

The corresponding irreducible left $\mathrm{Zhu}(\mathbb{L}_{-1})$ -modules are denoted by

$$\{\mathbb{C}(\epsilon), \mathbf{\Pi}\mathbb{C}(\epsilon) \mid \epsilon \in \{0, \pm\frac{1}{2}\}\} \sqcup \{\mathbb{C}_j, \mathbf{\Pi}\mathbb{C}_j \mid j \in \mathbb{C}\}.$$

3.1 Estimate from Above

We set a \mathbb{Z}_2 -graded structure on

$$\mathbf{M} := \bigoplus_{a,b \in \{0,1\}} \mathbb{C}[x_\ell, x_r, y] \psi_{a,b},$$

by $\deg(f\psi_{a,b}) = \overline{a+b}$ for $f \in \mathbb{C}[x_\ell, x_r, y]$. For $j \in \mathbb{C}$, we define a \mathbb{Z}_2 -graded $\mathbb{C}[\mathfrak{h}, \mathfrak{q}]$ -bimodule structure π_j on \mathbf{M} by

$$\begin{aligned} \mathfrak{h}.f\psi_{a,b} &:= x_\ell f\psi_{a,b}, & \mathfrak{q}.f\psi_{a,b} &:= (y + j + a - b)f\psi_{a,b}, \\ f\psi_{a,b}.\mathfrak{h} &:= x_r f\psi_{a,b}, & f\psi_{a,b}.\mathfrak{q} &:= yf\psi_{a,b}. \end{aligned}$$

Then there exists a unique \mathbb{Z}_2 -graded $\mathbb{C}[\mathfrak{h}, \mathfrak{q}]$ -bimodule isomorphism

$$\mathbf{M}_j := (\mathbf{M}, \pi_j) \xrightarrow{\simeq} \text{Zhu}(\mathcal{M}_{h,j,c})$$

such that $\psi_{a,b} \mapsto [(G_{-\frac{1}{2}}^+)^a (G_{-\frac{1}{2}}^-)^b |h, j, c]$ for $a, b \in \{0, 1\}$.

Lemma 3.3. The kernel of the natural surjective $\mathbb{C}[\mathfrak{h}, \mathfrak{q}]$ -bimodule homomorphism

$$\mathbf{M}_{\frac{2}{3}} \simeq \text{Zhu}(\mathcal{M}_{\frac{1}{3}, \frac{2}{3}, -1}) \rightarrow \text{Zhu}(\mathcal{L}(\frac{1}{2}))$$

has a set of explicit generators.

Lemma 3.4. Assume that $j \in S$ and set $s := \frac{4}{3}j$. Then the following vector

$$\chi_1 := (4(s-1)L_{-1} + 3(s-1)(s+1)J_{-1} + 4G_{-\frac{1}{2}}^+ G_{-\frac{1}{2}}^-) |-\frac{1}{8}(1+3s^2), s, -1\rangle$$

gives an even singular vector in $\mathcal{M}_{-\frac{1}{8}(1+3s^2), s, -1}$. In addition, the kernel of the natural surjective $\mathbb{C}[\mathfrak{h}, \mathfrak{q}]$ -bimodule homomorphism

$$\mathbf{M}_s \simeq \text{Zhu}(\mathcal{M}_{-\frac{1}{8}(1+3s^2), s, -1}) \rightarrow \text{Zhu}(\mathcal{M}_{-\frac{1}{8}(1+3s^2), s, -1} / \langle \chi_1 \rangle)$$

has a set of explicit generators.

By Lemma 3.3, Lemma 3.4, and [Li99, Proposition 2.10], we obtain the following.

Proposition 3.5 (Koshida–S.). For $\epsilon, \epsilon' \in \{0, \pm\frac{1}{2}\}$, $j, j' \in S$, and $j'' \notin \{\frac{1}{4}\} + \frac{1}{2}\mathbb{Z}$, we have

$$\dim \text{IO} \begin{pmatrix} \mathcal{L}(\epsilon')^\theta \\ \mathcal{L}(\frac{1}{2}) \mathcal{L}(\epsilon) \end{pmatrix} = \begin{cases} (1|0) & \text{if } \epsilon' = \epsilon + \frac{1}{2} \text{ and } \theta = 0 \\ (0|1) & \text{if } (\epsilon, \epsilon') = (\frac{1}{2}, 0) \text{ and } \theta = -1, \\ (0|0) & \text{otherwise,} \end{cases}$$

$$\dim \text{IO} \begin{pmatrix} \mathcal{L}_{j''} \\ \mathcal{L}(\frac{1}{2}) \mathcal{L}(\epsilon) \end{pmatrix} = (0|0),$$

$$\dim \text{IO} \begin{pmatrix} \mathcal{L}_{j''} \\ \mathcal{L}(\frac{1}{2}) \mathcal{L}_j \end{pmatrix} \leq \begin{cases} (1|0) & \text{if } j'' = j + \frac{1}{2}, \\ (0|0) & \text{otherwise,} \end{cases}$$

$$\dim \text{IO} \begin{pmatrix} \mathcal{L}_{j''} \\ \mathcal{L}_j \mathcal{L}_{j'} \end{pmatrix} \leq \begin{cases} (0|1) & \text{if } j'' = j + j' \pm \frac{3}{4}, \\ (0|0) & \text{otherwise.} \end{cases}$$

Remark 3.6. Since we have isomorphisms

$$\mathcal{L}(\pm\frac{1}{2})^{\pm 1} \simeq \mathbf{\Pi}\mathcal{L}(\mp\frac{1}{2}), \quad \mathcal{L}(0)^\vee \simeq \mathcal{L}(0), \quad \mathcal{L}(\pm\frac{1}{2})^\vee \simeq \mathcal{L}(\mp\frac{1}{2}),$$

where $(?)^\vee$ stands for the contragredient dual, the symmetry of intertwining operators (see e.g. [HL95, §7]) and Lemma 2.6 imply

$$\text{IO}\left(\begin{array}{c} \mathcal{L}(0) \\ \mathcal{L}(\frac{1}{2}) \quad \mathcal{L}(-\frac{1}{2}) \end{array}\right) \simeq \text{IO}\left(\begin{array}{c} \mathcal{L}(\frac{1}{2}) \\ \mathcal{L}(\frac{1}{2}) \quad \mathcal{L}(0) \end{array}\right) \simeq \text{IO}\left(\begin{array}{c} \mathcal{L}(\frac{1}{2}) \\ \mathcal{L}(0) \quad \mathcal{L}(\frac{1}{2}) \end{array}\right) \simeq \mathbb{C}^{1|0}$$

and

$$\text{IO}\left(\begin{array}{c} \mathcal{L}(0)^{-1} \\ \mathcal{L}(\frac{1}{2}) \quad \mathcal{L}(\frac{1}{2}) \end{array}\right) \simeq \text{IO}\left(\begin{array}{c} \mathcal{L}(0) \\ \mathcal{L}(\frac{1}{2}) \quad \mathbf{\Pi}\mathcal{L}(-\frac{1}{2}) \end{array}\right) \simeq \mathbb{C}^{0|1}.$$

3.2 Estimate from Below

Let $\Lambda = \mathbb{Z}a^+ + \mathbb{Z}a^-$ be a lattice equipped with the non-degenerate symmetric integer form defined by $\langle a^\pm, a^\pm \rangle = \pm 1$ and $\langle a^+, a^- \rangle = 0$. The Friedan–Martinec–Shenker (FMS) bosonization is an embedding of $\langle \beta\gamma \rangle$ to the lattice vertex algebra V_Λ defined by:

$$\langle \beta\gamma \rangle \rightarrow V_\Lambda; \quad \beta \mapsto |a^+ + a^-\rangle, \quad \gamma \mapsto -a_{-1}^+ | -a^+ - a^-\rangle.$$

For $(\ell, \lambda) \in \mathbb{C}^2$, we define

$$\Pi_\ell(\lambda) := \Pi(\ell a^- + \lambda(a^+ + a^-) + \mathbb{Z}(a^+ + a^-)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{\Lambda, \ell a^- + (n+\lambda)(a^+ + a^-)},$$

where $\mathcal{F}_{\Lambda, \alpha}$ stands for the Heisenberg Fock \mathcal{F}_Λ -module of highest weight α . Then this carries a natural twisted $\langle \beta\gamma \rangle$ -module structure via the FMS bosonization. More precisely, we obtain the next lemma.

Lemma 3.7. Let σ be the involution of $\langle \beta\gamma \rangle$ defined in §2.3. For $\ell \in \frac{1}{2}\mathbb{Z}$, the vector space $\Pi_\ell(\lambda)$ carries a natural $\sigma^{2\ell}$ -twisted strongly graded $\langle \beta\gamma \rangle$ -module structure. Moreover, the $\sigma^{2\ell}$ -twisted $\langle \beta\gamma \rangle$ -module $\Pi_\ell(\lambda)$ is irreducible if $\lambda \notin \mathbb{Z}$.

Now, by using Corollary 2.4, we obtain the following.

Proposition 3.8. For $j \in S$ and $b \in \mathbb{Z}$, we have isomorphisms

$$\langle \beta\gamma \rangle \otimes V_{\mathbb{Z}} \simeq \bigoplus_{\theta \in \mathbb{Z}} \mathbf{\Pi}^\theta \left(\mathcal{L}(0)^{-\theta} \otimes \mathcal{F}_{\mathbb{Z}, \sqrt{\frac{4}{3}}\theta} \oplus \mathcal{L}(\frac{1}{2})^{-\theta} \otimes \mathcal{F}_{\mathbb{Z}, \sqrt{\frac{4}{3}}(\theta + \frac{1}{2})} \right)$$

and

$$\mathbf{\Pi}_{\frac{1-b}{2}}(2j + \frac{1}{2}) \otimes V_{\mathbb{Z}} \simeq \bigoplus_{\theta \in \frac{1}{2}\mathbb{Z}} \mathbf{\Pi}^\theta \mathcal{L}_{j+\theta} \otimes \mathcal{F}_{\mathbb{Z}, \sqrt{\frac{4}{3}}(j+\theta + \frac{3b}{4})}$$

of \mathbb{Z}_2 -graded strongly graded $L_{-1} \otimes \mathcal{F}_{\mathbb{Z}}$ -modules.

The next proposition follows from the result by D. Adamović and V. Pedić.

Proposition 3.9 ([AP19]). For any

$$(\ell_i, \lambda_i) \in \mathbb{Z} \times (\mathbb{C} \setminus \mathbb{Z}) \quad (i \in \{1, 2, 3\}, \lambda_1 + \lambda_2 \notin \mathbb{Z}),$$

we obtain the following fusion rule among $\langle \beta\gamma \rangle$ -modules:

$$\dim \text{IO} \left(\begin{array}{c} \Pi_\ell(\lambda_3) \\ \Pi_{\frac{1}{2}}(\lambda_1) \quad \Pi_{\frac{1}{2}}(\lambda_2) \end{array} \right) = \begin{cases} 1 & \text{if } \ell \in \{0, 1\} \text{ and } \lambda_3 = \lambda_1 + \lambda_2, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.10 (Koshida–S.). Assume that

$$j, j_1, j_2 \in S, \quad j_3 := j_1 + j_2 \pm \frac{3}{4} \notin \left\{ \frac{1}{4} \right\} + \frac{1}{2}\mathbb{Z}.$$

Then we have

$$\dim \text{IO} \left(\begin{array}{c} \mathcal{L}_{j+\frac{1}{2}} \\ \mathcal{L}_{(\frac{1}{2})} \quad \mathcal{L}_j \end{array} \right) = (1|0), \quad \dim \text{IO} \left(\begin{array}{c} \mathcal{L}_{j_3} \\ \mathcal{L}_{j_1} \quad \mathcal{L}_{j_2} \end{array} \right) = (0|1).$$

Moreover, the Verlinde formula in the sense of [Sat19, Conjecture 5.1] correctly computes the above fusion rules.

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