On the Existence of a Strongest Die among Dice with an Equally Expected Value (New Trends in Algorithms and Theory of Computation)

On the Existence of a Strongest Die among Dice with an Equally Expected Value

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1 Introduction

A generalized die consists of m-face on each of which a positive number is printed and each face appears equally likely. For instance, a regular die consists of 6 faces with numbers $1, 2, \dots, 6$. When we throw a regular die, each face appears with a probability of $1/6$.

Consider a dice game by two players, where each player selects one die from a dice set, and throws it, and the player with the highest face score wins. In order to design a fair and meaningful game, it is important to avoid the strongest die, that is, to design a non-transitive dice set.

There are many different kinds of dice sets with the goal of constructing non-transitive dice sets. Schaefer and Schweig[3] distribute a set of dice for normal partition, namely, integers from 1 to $m \times n$ are distributed to n m-side dice, and each integer is distributed and appears only once. When there are 3 or more dice, it indicates that there is a non-transitive dice set. Conrey et al.[1] varied the number of faces in a set of dice with the same sum of faces, that is, with equally expected value, and statistically analyzed the proportion of non-transitive dice in it.

In this study, we consider the non-transitive relations of the equally-expected-value dice sets with different face. Then, under the condition of the equally expected value, we prove that the strongest dice does not exist.

2 Preliminary

2.1 Intransitive dice

In our study, we assume each face of a die equally likely to appear. Now there is a die with m faces, the probability of each face appearing in the rolling process is $\frac{1}{m}$, and the number of each face is represented by a_i, so the die with m faces we can represent as $A = \{a_1, a_2, \dots, a_m\}$. As general, when the sides of a die are allowed to have the same number of points, that is $a_1 \le a_2 \le \cdots \le a_m$.

Next, we take any two dice for comparison, $A_1 = \{a_{1,1}, a_{1,2}, \dots a_{1,m}\}$ and $A_2 = \{a_{2,1}, a_{2,2}, \dots a_{2,m}\}$. There are $m \times m$ cases when we roll, and the number of the case that the faces in A_1 are greater than the total number of faces in A_2, namely $|\{(a_{1,i}, a_{2,j}) | a_{1,i} >$

$a_{2,j},\ i = 1, \ldots, m,\ j = 1, \ldots, m\}|$, we denote it as $S_{A_1 \succ A_2}$. Identically, we denote $S_{A_1 \sim A_2}$ as $|\{(a_{1,i}, a_{2,j})|a_{1,i} = a_{2,j},\ i = 1, \ldots, m,\ j = 1, \ldots, m\}|$. So as a game of probability, we are going to have $P(A_1 \succ A_2) = \frac{S_{A_1 \succ A_2}}{m^2 - S_{A_1 \sim A_2}}$, and when this value is more than 1/2, we can say that die A_1 is stronger than A_2, represent as $A_1 \succ A_2$. And when it is equal to 1/2, we define it as $A_1 \sim A_2$.

Of course, there could be a set more than two dice. When we analyze a set of n dice, we want to find the strongest one in this set, but the strongest die does not necessarily exist.

Definition 1. *A set of n dice is non-transitive if $\forall i, A_{i+1} \succ A_i$ and $A_1 \succ A_n$.*

2.2 Regular partition

When we divide the natural numbers from 1 to $m \times n$ into n groups of m, we find that there are no duplicate numbers in each set. With this way to get the dice set, there will be no repeated face, that is, after rolling the dice a result between win and loss will be produced. We define it as a **regular partition** dice set.[3][4]

An n partition $A = \{A_1, A_2, \ldots, A_n\}$ of $[MN] = 1, 2, \ldots, m \times n$ consists of n disjoint subsets with union $[MN]$. We call it a regular n partition when the cardinality $|A_i| = m$ for $i = 1, \ldots, n$. By calculation, we can find that, there are $\frac{(mn)!}{(m!)^n n!}$ regular n partitions of $[MN]$.

Definition 2. *Taking a n m-side dice set $A = \{A_1, A_2, \ldots, A_n\}$, for $|A_i| = m$. If $P(A_1 \succ A_2) = \cdots = P(A_n \succ A_1)$, then we donate it as a balanced dice set.*

Alex Schaefer and Jay Schweig[3][4] show that, as the number of face varies, there is always an intransitive set of 3 dice in regular partition.

Theorem 1 ([3])**.** *For any $m \geq 3$, there exists an intransitive m-side dice set with 3 dice.*

Theorem 2 ([3])**.** *A set of dice A, B, C is balanced if and only if the face-sums of its dice are all equal.*

And with the change of the number of die sets, in regular partition, intransitive dice sets always exist.

Theorem 3 ([4])**.** *For any $n, m \geq 3$, there exists an intransitive m-side dice set of n dice.*

In addition, the relationship between dice in terms of the tournament graph,

Theorem 4 ([4])**.** *Let G be a strong tournament. There is a set of intransitive dice realizing G.*

2.3 Equally expected values

When a regular m-side die is rolled, the probability of each side is $1/m$, that is, as a probability event, the die is playing the game where has an expectation of a roll point. We found that in the regular partition, the expected values of the dice in the intransitive set are very close. At this point, we first put down the setting that the faces of the dice are different to construct such a set of dice, define an m-side die to be an n-tuple $(a_1, \ldots, a_n)$ of non-decreasing positive integers, with $a_1 \leq a_2 \leq \cdots \leq a_n$. The **standard** m-side is $(1, 2, 3, \ldots, m)$. A set of dice is *equally expected* if $\sum A_1 = \sum A_2 = \cdots = \sum A_n = w$, and define **proper** m-side dice to be those with $1 \leq a_i \leq m$ and $\sum a_i = m(m+1)/2$. [1]

2.4 Set of dice in this study

From here, let's make a definition of the dice in next studying.

Definition 3. *A n m-side dice set* $A_1, \ldots, A_n \subset [mn]$ *is regular partition that satisfies* $|A_1| = \cdots = |A_n| = m$*, and* $A_1 \cup \cdots \cup A_n = [mn]$*. Each die is fair and each face in die with the probability of* $1/m$*. Then setting the die* $A = \{a_1, a_2, \ldots, a_m\}$ *with* $a_1 < a_2 < \cdots < a_m$*. Furthermore, if the expected values of each die is equal in a regular partition, as* $w = \sum A_1 = \cdots = \sum A_n$*, we donate it as a set of dice with regular partition and equally expected values.*

Proposition 1. $S_{A \succ B} + S_{B \succ A} = m^2$

Proposition 2. *A die owns* $w = \frac{m(nm+1)}{2}$ *weight when it is in n partition m-side dice set with equally expected values.*

3 Intransitive in a set of dice with regular partition and equal expected values

In this section, we focus on the existence of intransitive, regular partition and equally expected values dice set. We analyze the specific number of dice set, and then generalize them.

Firstly consider the case of $n = 2$. Taking two m-side dice $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ with the same weight, $w = \sum a_i = \sum b_i = \frac{m(2m+1)}{2}$ can be easily calculated by Proposition 2. If m is odd, there are no two m-sided dice set with the same weight as w always be the odd. When m is even, we can see the following

Proposition 3. *If* $n = 2$ *and* m *is even, two* m*-side dice set with the equal expected values draw each other.*

Next, we are concerned with the case of $n = 3$. When $n = 3$, three m-side dice set with the equally expected values, and let $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_m\}$, $C = \{c_1, c_2, \ldots, c_m\}$, so $w = \sum a_i = \sum b_i = \sum c_i = \frac{m(3m+1)}{2}$ is calculated by Proposition 2.

Theorem 5. *Let* A*,* B*,* C *be a set of three* m*-side dice with equally expected values. If* $A \sim C$*, then* $A \sim B \sim C$*.*

When m is odd, the case A is equal to C does not exist. Then, we are concerned with the case that the magnitude relationship exists, and for the symmetry, setting $A \succ C$.

Theorem 6. *Let* A*,* B*,* C *be a set of three* m*-side dice with equally expected values. If* $A \succ C$*, then* $B \succ A \succ C \succ B$*.*

At this point we arrive at the same conclusion as Theorem 2 by [3], which tells us that the dice set is balanced if and only if the expected values are equal. Here, the balance is the two conditions of Theorems 6 and 7, where all dice are equally divided between each other, or have a magnitude relationship, with equal probabilities.

We can say intransitivity exists in 3 m-side dice set when magnitude relationship exists. But wether magnitude relationship always exists?

It can be seen that when m is odd, $S_{A\succ C}+S_{C\succ A}=m^2$ will be odd as well. Since $S_{A\succ C}$ and $S_{C\succ A}$ are integers, there is no case where $S_{A\succ C}=S_{C\succ A}$, which means magnitude relationship always exists when m is odd.

When a is not all connected and only part of a is connected to each other in the string. Let $a_1=1, a_2=2, \ldots, a_{\frac{m}{2}}=\frac{m}{2}, a_{\frac{m}{2}+1}=3m-\frac{m}{2}+1, \ldots, a_m=3m$, we can find $\forall C, S_{A\succ C}=S_{C\succ A}=\frac{m^2}{2}$.

Proposition 4. *The die A defined above always draws to any other die in regular partition.*

However, Theorem 2 is only limited to the set of three dice. As an example in [4] that

A = 12 6 1
B = 11 5 4
C = 10 8 2
D = 9 7 3

for $P(A\succ B)=P(B\succ C)=P(C\succ D)=P(D\succ A)=\frac{5}{9}$, and $\sum A=19, \sum B=20, \sum C=20, \sum D=19$.

Therefore, when the number of dice increases, the sum of faces and the transitivity no longer have a necessary and sufficient relationship. Next, we are concerned with a situation where the set include four dice.

Theorem 7. *When the number of face m is even and more than 6, the case $A\succ B\succ C$ and $A\succ C$ exists in the 4 m-side equally-expected-values dice set A, B, C, and D.*

From this, we can see that in the set of dice with equally expected values, the relationship exists conditionally. For example, in a set of 3 dice, if two of them are equal to each other, the third die must also be equal to them. If there is a magnitude relationship between two dice, and add to the third die, it will be a non-transitive dice set. Then, considering the four sets of dice, as non-adjacent dice appear, such as B and D in theorem 7, we need to give three relationships to determine the relationship of the fourth die.

In the next, we generalize the number of dice in a die set to consider the existence of an intransitive and equally-expected-value dice set.

Theorem 8. *Let $A_1, \ldots, A_n$ be a set of dice given by a regular partition with an equally expected value. If $A_i\succ A_j$, then there exists A_k such that $A_k\succ A_i$.*

Then using Theorem 8 to analyze the existence of intransitive dice sets.

Corollary 1. *$\forall n\geq 2$, n transitive dice sets with regular partition and equally expected values do not exist.*

4 Conclusion

In this study, we consider the non-transitive relations of the equally-expected-value dice sets with different face. Then, under the condition of the equally expected value, we prove that the strongest dice does not exist.

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