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Mixing properties for Markov operator cocycles

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1 Introduction

This paper concerns a cocycle generated by Markov operators, called a *Markov operator cocycle*. Let $(X, \mathcal{A}, m)$ be a probability space, and $L^1(X, m)$ the space of all m-integrable functions on X, endowed with the usual L^1-norm $\|\cdot\|_{L^1(X)}$. An operator $P : L^1(X, m) \to L^1(X, m)$ is called a *Markov operator* if P is linear, positive (i.e. $Pf \geq 0$ m-almost everywhere if $f \geq 0$ m-almost everywhere) and

$$\int_X Pf dm = \int_X f dm \quad \text{for all } f \in L^1(X, m). \tag{1}$$

Markov operators naturally appear in the study of dynamical systems as Perron-Frobenius operators; see (3), Markov processes as integral operators with the stochastic kernels of the processes, and random dynamical systems in the annealed regime as integrations of Perron-Frobenius operators over environmental parameters. For these deterministic/stochastic dynamics, $\{P^n f\}_{n\geq 0}$ is the evolution of density functions of random variables driven by the system. We refer to [8, 10].

A Markov operator cocycle is given by compositions of different Markov operators which are provided with according to the environment $\{\sigma^n(\omega)\}_{n\geq 0}$ driven by a measure-preserving transformation $\sigma : \Omega \to \Omega$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\mathbb{N} \times \Omega \times L^1(X, m) \to L^1(X, m) : (n, \omega, f) \mapsto P_{\sigma^{n-1}(\omega)} \circ P_{\sigma^{n-2}(\omega)} \circ \cdots \circ P_\omega f$$

(see Definition 1.1 more precisely). So, in nature it possess two kinds of randomness:

(i) The evolution of densities at each time are dominated by Markov operators P_ω,

(ii) The selection of each Markov operators is driven by the base dynamics σ.

The aim of this paper is to introduce our results how the *observation* of the randomness of the state space and the environment influences statistical properties of the system, and to give a step to understanding more complicated phenomenon in multi-stochastic systems.

Our focus lies on the mixing property. Recall that a Markov operator $P : L^1(X,m) \to L^1(X,m)$ is said to be *mixing* if

$$\int_X P^n f g dm \to \int_X f dm \int_X g dm \quad \text{as } n \to \infty \tag{2}$$

for any $f \in L^1(X,m)$ and $g \in L^\infty(X,m)$ when $P1_X = 1_X$ (see Remark 1.3 for more general form). Due to (1), this means that two random variables $P^n f$ and g are asymptotically independent so that the system is considered to "mix" the state space well. In other words, the randomness of P in the sense of mixing can be seen through the observables f and g. Hence, for Markov operator cocycles, the strength of the dependence of the observables on ω expresses how one observes the randomness of the state space and the environment. Furthermore, more directly, we can consider different kinds of mixing properties according to whether the environment ω is observed as *a prior event* to the observation of f, g. According to these viewpoints, we will introduce six definitions of mixing for Markov operator cocycles (Definition 1.2), and show that four of them are equivalent when Ω is a compact topological space, while at least two of them are different. In the case when the Markov operator cocycle is generated by a random dynamical system over a mixing driving system, we also show that all of them imply the (conventional) mixing property of the skew-product transformation induced by the random dynamical system.

We further investigate exactness for Markov operator cocycles. Since the observable g in (2) does not appear in the definition of exactness for a Markov operator P (recall that, when $P1_X = 1_X$, P is said to be exact if $\lim_{n\to\infty} \left\| P^n f - \int_X f dm \right\|_{L^1(X)} = 0$ for all $f \in L^1(X,m)$; see also the remark following Definition 1.5), in contrast to the mixing property, we only have one definition of exactness for Markov operator cocycles (Definition 1.5). We will show that Lin's criterion for exactness can be naturally extended to the case of Markov operator cocycles (Section 3), and finally, in the class of asymptotically periodic Markov operator cocycles, we prove Lasota-Mackey type equivalence between mixing, exactness and asymptotic stability (Section 4). See [13] for more precise descriptions including the proofs. Moreover, a random invariant density for Markov operator cocycles is discussed in [14].

1.1 Definitions of mixing and exactness

Let $D(X,m)$ and $L^1_0(X,m)$ be subsets of $L^1(X,m)$ given by

$$\begin{aligned} D(X,m) &= \left\{ f \in L^1(X,m) : f \geq 0 \ m\text{-almost everywhere}, \ \|f\|_{L^1(X)} = 1 \right\}, \\ L^1_0(X,m) &= \left\{ f \in L^1(X,m) : \int_X f dm = 0 \right\}. \end{aligned}$$

Note that $P : L^1(X,m) \to L^1(X,m)$ is a Markov operator if and only if $P(D(X,m)) \subset D(X,m)$.

One of the most important examples of Markov operators is the *Perron-Frobenius operator* induced by a measurable and non-singular transformation $T : X \to X$ (that is, the probability measure $m \circ T^{-1}$ is absolutely continuous with respect to m). The Perron-Frobenius operator $\mathcal{L}_T : L^1(X,m) \to L^1(X,m)$ of T is defined by

$$\int_X \mathcal{L}_T f g dm = \int_X f g \circ T dm \qquad \text{for } f \in L^1(X,m) \text{ and } g \in L^\infty(X,m). \tag{3}$$

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $\sigma : \Omega \to \Omega$ is a $\mathbb{P}$-preserving transformation. For a measurable space Σ, we say that a measurable map $\Phi : \mathbb{N}_0 \times \Omega \times \Sigma \to \Sigma$ is a *random dynamical system* on Σ over the driving system σ if

$$\varphi^{(0)}_\omega = \mathrm{id}_\Sigma \quad \text{and} \quad \varphi^{(n+m)}_\omega = \varphi^{(n)}_{\sigma^m \omega} \circ \varphi^{(m)}_\omega$$

for each $n, m \in \mathbb{N}_0$ and $\omega \in \Omega$, with the notation $\varphi_\omega^{(n)} = \Phi(n, \omega, \cdot)$ and $\sigma\omega = \sigma(\omega)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A standard reference for random dynamical systems is the monographs by Arnold [2]. It is easy to check that

$$\varphi_\omega^{(n)} = \varphi_{\sigma^{n-1}\omega} \circ \varphi_{\sigma^{n-2}\omega} \circ \cdots \circ \varphi_\omega \tag{4}$$

with the notation $\varphi_\omega = \Phi(1, \omega, \cdot)$. Conversely, for each measurable map $\varphi : \Omega \times \Sigma \to \Sigma : (\omega, x) \mapsto \varphi_\omega(x)$, the measurable map $(n, \omega, x) \mapsto \varphi_\omega^{(n)}(x)$ given by (4) is a random dynamical system. We call it a random dynamical system induced by φ over σ, and simply denote it by (φ, σ). When Σ is a Banach space and $\varphi_\omega : \Sigma \to \Sigma$ is $\mathbb{P}$-almost surely linear, (φ, σ) is called a *linear operator cocycle*. We give a formulation of Markov operators in random environments in terms of linear operator cocycles.

Definition 1.1. We say that a linear operator cocycle (P, σ) induced by a measurable map $P : \Omega \times L^1(X, m) \to L^1(X, m)$ over σ is a *Markov operator cocycle* (or a *Markov operator in random environments*) if $P_\omega = P(\omega, \cdot) : L^1(X, m) \to L^1(X, m)$ is a Markov operator for $\mathbb{P}$-almost every $\omega \in \Omega$.

Let $(n, \omega, f) \mapsto P_\omega^{(n)} f$ be a Markov operator cocycle induced by $P : \Omega \times L^1(X, m) \to L^1(X, m)$ such that $P_\omega = P(\omega, \cdot)$ is the Perron-Frobenius operator L_{T_ω} associated with a non-singular map $T_\omega : X \to X$ for $\mathbb{P}$-almost every ω. Then, it follows from (3) that $\mathbb{P}$-almost surely

$$\int_X P_\omega^{(n)} f g dm = \int_X f g \circ T_\omega^{(n)} dm, \quad \text{for } f \in L^1(X, m) \text{ and } g \in L^\infty(X, m), \tag{5}$$

where $T_\omega^{(n)} = T_{\sigma^{n-1}\omega} \circ T_{\sigma^{n-2}\omega} \circ \cdots \circ T_\omega$.

We are now in place to give definitions of mixing for Markov operator cocycles. Let K be a space consisting of measurable maps from Ω to $L^\infty(X, m)$.

Definition 1.2. A Markov operator cocycle. (P, σ) is called

1. *prior mixing for homogeneous observables* if for $\mathbb{P}$-almost every $\omega \in \Omega$, any $f \in L_0^1(X, m)$ and $g \in L^\infty(X, m)$, it holds that

$$\lim_{n \to \infty} \int_X P_\omega^{(n)} f g dm = 0; \tag{6}$$

2. *posterior mixing for homogeneous observables* if for any $f \in L_0^1(X, m)$, $g \in L^\infty(X, m)$ and $\mathbb{P}$-almost every $\omega \in \Omega$, (6) holds;

3. *prior mixing for inhomogeneous observables in* K if for $\mathbb{P}$-almost every $\omega \in \Omega$, any $f \in L_0^1(X, m)$ and $g \in K$, it holds that

$$\lim_{n \to \infty} \int_X P_\omega^{(n)} f g_{\sigma^n \omega} dm = 0; \tag{7}$$

4. *posterior mixing for inhomogeneous observables in* K if for any $f \in L_0^1(X, m)$, $g \in K$ and $\mathbb{P}$-almost every $\omega \in \Omega$, (7) holds.

In the prior case (the posterior case), the observation of the environment ω is *a prior event* (*a posterior event*, respectively) to the observation of f and g. As the class of inhomogeneous observables K in Definition 1.2, we will consider the following two fundamental classes.

(i) $B(\Omega, L^\infty(X, m))$: the set of all bounded and measurable maps from Ω to $L^\infty(X, m)$.

(ii) $C(\Omega, L^\infty(X, m))$: the set of all bounded and continuous maps from Ω to $L^\infty(X, m)$ (when Ω is a topological space and $\mathcal{F}$ is its Borel σ-field).

Remark 1.3. The above definitions need not require an invariant density map for the Markov operator cocycle (P, σ). We say that a measurable map $h : \Omega \to D(X, m)$ is an *invariant density map* for (P, σ) if $P_\omega h_\omega = h_{\sigma\omega}$ holds for $\mathbb{P}$-almost every $\omega \in \Omega$ where $h_\omega = h(\omega)$. Now we assume that there exist an invariant density map $h : \Omega \to D(X, m)$ for (P, σ) such that for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\lim_{n\to\infty} m\left(\operatorname{supp} P_\omega^{(n)} 1_X \setminus \operatorname{supp} P_\omega^{(n)} h_\omega\right) = 0. \tag{8}$$

Then by (8) and the fact that $P_\omega^{(n)} f - h_{\sigma^n\omega} = P_\omega^{(n)}(f - h_\omega) \in L_0^1(X, m)$ for $f \in D(X, m)$, one can easily check that (P, σ) is prior mixing for homogeneous observables if and only if for $\mathbb{P}$-almost every $\omega \in \Omega$, any $f \in D(X, m)$ and $g \in L^\infty(X, m)$, it holds that

$$\lim_{n\to\infty} \int_X \left(P_\omega^{(n)} f - h_{\sigma^n\omega}\right) g dm = 0.$$

Furthermore, when P_ω is the Perron-Frobenius operator L_{T_ω} associated with a non-singular map $T_\omega : X \to X$, by (5), it is also equivalent to that for $\mathbb{P}$-almost every $\omega \in \Omega$, any $f \in L^1(X, \mu_\omega)$ and $g \in L^\infty(X, m)$,

$$\int_X fg \circ T_\omega^{(n)} d\mu_\omega - \int_X f d\mu_\omega \int_X g d\mu_{\sigma^n\omega} \to 0 \quad \text{as } n \to \infty, \tag{9}$$

where $\mu_\omega = h_\omega m$. See also Remark 2.6. Moreover, we can replace "for any $f \in L^1(X, \mu_\omega)$" in the previous sentence with "for any measurable function $f : \Omega \times X \to \mathbb{R}$ such that $f_\omega = f(\omega, \cdot) \in L^1(X, \mu_\omega)$ $\mathbb{P}$-almost surely", and "f" in (9) with "f_ω". Similar equivalent conditions can be found for other types of mixing in Definition 1.2.

All kinds of mixing in Definition 1.2 were adopted in literature, especially in the form of (9) to discuss mixing for random dynamical systems. For instance, we refer to Baladi et al. [4, 5] and Buzzi [6] for the definition 1, Dragičević et al. [7] for the definition 2, Bahsoun et al. [3] for the definition 3, and Gundlach [9] for the definition 4. Moreover, in the deterministic case (i.e. Ω is a singleton), all the definitions are equivalent to the usual definition of mixing for a single Markov operator [10].

Remark 1.4. Another natural candidate for the class of inhomogeneous observable is the Bochner-Lebesgue space $L^\infty(\Omega, L^\infty(X, m))$, that is, the Kolmogorov quotient (by equality $\mathbb{P}$-almost surely) of the space of all $\mathbb{P}$-essentially bounded and Bochner measurable maps from Ω to $L^\infty(X, m)$ (and (7) is interpreted as it holds under the usual identification between an equivalent class and a representative of the class). However, in the case $K = L^\infty(\Omega, L^\infty(X, m))$, the prior version 3 does not make sense because one can find an equivalent class $[g] \in L^\infty(\Omega, L^\infty(X, m))$ and maps $g_1, g_2 \in [g]$ such that (7) holds for $g = g_1$ while (7) does not hold for $g = g_2$, see Section 2. On the other hand, the posterior version 4 makes sense for $K = L^\infty(\Omega, L^\infty(X, m))$, and indeed, its relationship with posterior mixing for homogeneous observables will be discussed in Theorem 2.2.

By the definitions, we immediately see that the prior mixing implies the posterior mixing (that is, (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) in Definition 1.2). It is also obvious that the prior (posterior) mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$ or $C(\Omega, L^\infty(X, m))$ implies the prior (posterior, respectively) mixing for homogeneous observables.

We next define exactness for Markov operator cocycles.

Definition 1.5. A Markov operator cocycle (P, σ) is called *exact* if for $\mathbb{P}$-almost every $\omega \in \Omega$ and any $f \in L_0^1(X, m)$, it holds that

$$\lim_{n\to\infty} \left\| P_\omega^{(n)} f \right\|_{L^1(X)} = 0. \tag{10}$$

As in Remark 1.3, we can easily see that the exactness of a Markov operator cocycle (P, σ) is equivalent to that for $\mathbb{P}$-almost every $\omega \in \Omega$ and any $f \in D(X, m)$,

$$\lim_{n \to \infty} \left\| P_\omega^{(n)} f - h_{\sigma^n \omega} \right\|_{L^1(X)} = 0.$$

2 Mixing

The following two theorems tell us relations between our several definitions of mixing when Ω is a compact topological space.

Theorem 2.1. *Assume that Ω is a compact topological space. Then, the followings are equivalent:*

1. *(P, σ) is prior mixing for homogeneous observables.*
2. *(P, σ) is posterior mixing for homogeneous observables.*
3. *(P, σ) is prior mixing for inhomogeneous observables in $C(\Omega, L^\infty(X, m))$.*
4. *(P, σ) is posterior mixing for inhomogeneous observables in $C(\Omega, L^\infty(X, m))$.*

Theorem 2.2. *If (P, σ) is posterior mixing for homogeneous observables, then (P, σ) is posterior mixing for inhomogeneous observables in $L^\infty(\Omega, L^\infty(X, m))$.*

The following example gives the difference between prior mixing for homogeneous observables and inhomogeneous observables in $B(\Omega, L^\infty(X, m))$.

Example 2.3. Let $T : X \to X$ be a measurably bijective map (up to zero m-measure sets) preserving m such that the Perron-Frobenius operator L_T associated with T is mixing (note that $L_T 1_X = 1_X$ due to the invariance of m and recall (2)). Note that the baker map is well-known example as such map T. Assume that there is a $\mathbb{P}$-positive measure set Ω_0 such that the forward orbit of $\omega \in \Omega_0$ is not finite and a measurable set (e.g. $\Omega = [0, 1]$ and $\mathbb{P}$ is the Lebesgue measure on Ω), and that $P_\omega = L_T$ for all $\omega \in \Omega_0$. By construction, this Markov operator cocycle (P, σ) is prior mixing for homogeneous observables, but is not prior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$.

We next introduce that our definitions of mixing for Markov operator cocycles naturally lead to the conventional mixing property for skew-product transformations.

Recall that $(X, \mathcal{A}, m)$ and $(\Omega, \mathcal{F}, \mathbb{P})$ are probability spaces, and $\sigma : \Omega \to \Omega$ is a $\mathbb{P}$-preserving transformation. We further assume that σ is invertible and *mixing*. Let (P, σ) be a Markov operator cocycle induced by the Perron-Frobenius operator corresponding to a non-singular transformation $T_\omega : X \to X$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Assume that there is an invariant density map $h : \Omega \to D(X, m)$ of (P, σ) and define a measurable family of measures $\{\mu_\omega\}_{\omega \in \Omega}$ by $\mu_\omega(A) = \int_A h_\omega dm$ for $A \in \mathcal{A}$, so that we have $(T_\omega)_* \mu_\omega = \mu_{\sigma\omega}$ due to (5).

Consider the skew-product transformation $\Theta : \Omega \times X \to \Omega \times X$ defined by $\Theta(\omega, x) = (\sigma\omega, T_\omega x)$ with the measure ν on $\Omega \times X$,

$$\nu(A) = \int_\Omega \mu_\omega(A_\omega) d\mathbb{P}(\omega) \quad \text{for } A \in \mathcal{F} \otimes \mathcal{A},$$

where $A_\omega := \{x \in X : (\omega, x) \in A\}$ denotes the ω-section. Then, $(\Omega \times X, \mathcal{F} \otimes \mathcal{A}, \nu)$ becomes a probability space, and ν is an invariant measure for Θ, namely the Perron-Frobenius operator L_Θ corresponding to Θ with respect to ν satisfies $L_\Theta 1_{\Omega \times X} = 1_{\Omega \times X}$ ν-almost everywhere.

Theorem 2.4. *If (P, σ) is prior mixing for inhomogeneous observables in $B(\Omega, L^\infty\ (X, m))$, then Θ is mixing, that is, for any $A, B \in \mathcal{F} \otimes \mathcal{A}$,*

$$\lim_{n\to\infty} \nu(\Theta^{-n} A \cap B) = \nu(A)\nu(B). \tag{11}$$

Remark 2.5. In the case of prior mixing for homogeneous observables, as in the proof of Theorem 2.4, we can derive the convergence

$$\nu(\Theta^{-n}(F_1 \times A_1) \cap (F_2 \times A_2)) \to \nu(F_1 \times A_1)\nu(F_2 \times A_2) \quad (n \to \infty) \tag{12}$$

for any $F_1, F_2 \in \mathcal{F}$ and $A_1, A_2 \in \mathcal{A}$.

Consequently, every mixing considered in this paper imply the conventional mixing for skew-product transformations since any measurable set in $\mathcal{F} \otimes \mathcal{A}$ is approximated by countable rectangle sets in $\mathcal{F} \times \mathcal{A}$. Therefore, we conclude that prior/posterior mixing for (in)homogeneous observables are natural definitions of mixing for random dynamical systems.

Remark 2.6. When σ is an invertible $\mathbb{P}$-preserving mixing transformation, by considering a skew-product transformation, the conventional definition of mixing for a random dynamical system (T, σ) can be derived from our definitions of mixing as follows. From the definition of mixing for homogeneous observables, for any $A_1, A_2 \in \mathcal{A}$ and $\mathbb{P}$-almost every ω,

$$\mu_\omega\left(T_\omega^{(-n)} A_1 \cap A_2\right) - \mu_{\sigma^n\omega}(A_1)\mu_\omega(A_2) \to 0 \quad (n \to \infty).$$

On the other hand, from the definition of mixing for inhomogeneous measurable observables, for any $A, B \in \mathcal{A} \otimes \mathcal{F}$ and $\mathbb{P}$-almost every ω,

$$\mu_\omega\left(T_\omega^{(-n)} A_{\sigma^n\omega} \cap B_\omega\right) - \mu_{\sigma^n\omega}(A_{\sigma^n\omega})\mu_\omega(B_\omega) \to 0 \quad (n \to \infty).$$

where A_ω denotes the ω-section of A. One can see that the above two forms of mixing for a random dynamical systems (T, σ) are equivalent.

3 Exactness

As a characterization of exactness which is well-known for one non-singular transformation (see [1]), we have the generalization of Lin's theorem [11] as follows. For each $\omega \in \Omega$, P_ω^* denotes the adjoint operator of P_ω defined by

$$\int_X P_\omega f g dm = \int_X f P_\omega^* g dm$$

for $f \in L^1(X, m)$ and $g \in L^\infty(X, m)$, and we will use the notation

$$P_\omega^{(n)*} = P_\omega^* \circ P_{\sigma\omega}^* \circ \cdots \circ P_{\sigma^{n-1}\omega}^*$$

for $\omega \in \Omega$ and $n \geq 1$.

Theorem 3.1. *Let (P, σ) be a Markov operator cocycle and $S = \{g \in L^\infty(X, m) : \|g\|_{L^\infty} \leq 1\}$ the unit ball in $L^\infty(X, m)$. Then the following are equivalent for each $\omega \in \Omega$.*

1. *$f \in L^1(X, m)$ satisfies $\left\|P_\omega^{(n)} f\right\|_{L^1(X)} \to 0$ as $n \to \infty$;*
2. *$f \in L^1(X, m)$ satisfies $\int_X f g dm = 0$ for any $g \in \bigcap_{n\geq 1} P_\omega^{(n)*} S$.*

Consequently, (P,ω) is exact if and only if $\bigcap_{n\geq 1} P_\omega^{(n)*}S = \{c1_X : c \in \mathbb{R}\}$ for $\mathbb{P}$-almost every $\omega \in \Omega$.

As an immediate corollary of Theorem 3.1, we have:

Corollary 3.2. *If a Markov operator cocycle (P,σ) is derived from non-singular transformations T_ω, that is, each P_ω is the Perron-Frobenius operator associated to T_ω. Then (P,ω) is exact if and only if for $\mathbb{P}$-almost every $\omega \in \Omega$,*

$$\bigcap_{n\geq 1} \left(T_\omega^{(n)}\right)^{-1} \mathcal{A} = \{\emptyset, X\} \pmod{m}.$$

4 Asymptotic periodicity

In the arguments of conventional Markov operators, it is known that mixing and exactness are equivalent properties in the asymptotically periodic class [10]. In this section, we introduce a similar result to the conventional one for our definitions of mixing and exactness for Markov operator cocycles under the following definition of asymptotic periodicity, which is studied in [12].

Definition 4.1 (Asymptotic periodicity)**.** A Markov operator cocycle (P,σ) is said to be *asymptotically periodic* if there exist an integer r, finite collections $\{\lambda_i\}_{i=1}^r \subset B\left(\Omega, (L^1(X,m))'\right)$ and $\{\varphi_i\}_{i=1}^r \subset B\left(\Omega, D(X,m)\right)$ satisfying that $\{\varphi_i^\omega\}_{i=1}^r$ have mutually disjoint supports for $\mathbb{P}$-almost every $\omega \in \Omega$, and there exists a permutation ρ_ω of $\{1,\dots,r\}$ such that

$$P_\omega \varphi_i^\omega = \varphi_{\rho_\omega(i)}^{\sigma\omega} \quad \text{and} \quad \lim_{n\to\infty} \left\| P_\omega^{(n)} \left(f - \sum_{i=1}^r \lambda_i^\omega(f) \varphi_i^\omega \right) \right\|_{L^1(X)} = 0 \tag{13}$$

for every $f \in L^1(X,m)$, $1 \leq i \leq r$ and $\mathbb{P}$-almost every $\omega \in \Omega$, where $\lambda_i^\omega = \lambda_i(\omega)$, $\varphi_i^\omega = \varphi_i(\omega)$ and $\rho_\omega^n := \rho_{\sigma^{n-1}\omega} \circ \cdots \circ \rho_\omega$.

Furthermore, if in addition $r = 1$, then (P,σ) is said to be *asymptotically stable.*

Note that when (P,σ) is asymptotically periodic,

$$h_\omega = \frac{1}{r} \sum_{i=1}^r \varphi_i^\omega$$

becomes an invariant density for (P,σ).

For an asymptotically periodic single Markov operator, exactness and mixing coincide with $r = 1$ for the representation of asymptotic periodicity (see Theorem 5.5.2 and 5.5.3 in [10]). The following theorem leads a Markov operator cocycles version of them.

Theorem 4.2. *Let (P,σ) be an asymptotically periodic Markov operator cocycle. Then the followings are equivalent.*

1. *(P,σ) is exact;*
2. *(P,σ) is prior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X))$;*
3. *(P,σ) is posterior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X))$;*
4. *(P,σ) is asymptotically stable.*

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