Convergence to non-minimal quasi-stationary distributions for one-dimensional diffusions and its application to Kummer diffusions (Probability Symposium)

Yamato, Kosuke

Convergence to non-minimal quasi-stationary distributions for one-dimensional diffusions and its application to Kummer diffusions

Kosuke Yamato (Kyoto University)

Abstract

We summarize the following results of the author's recent work [17] without proof. For one-dimensional diffusions killed at the boundaries, the domain of attraction of non-minimal quasi-stationary distributions is studied. We give a general method of reducing the convergence to the tail behavior of the lifetime via a property which we call the first hitting uniqueness. We apply the result to Kummer diffusions with negative drifts and clarify the domain of attraction of each non-minimal quasi-stationary distribution for the processes.

1 Introduction

For a stochastic process X on some state space S with its lifetime ζ, a probability distribution ν on S is called a quasi-stationary distribution if

$$\mathbb{P}_\nu[X_t \in dx \mid \zeta > t] = \nu(dx) \quad \text{for every } t > 0, \tag{1.1}$$

where $\mathbb{P}_\nu$ denotes the underlying probability measure of X with its initial distribution ν. We say that for a quasi-stationary distribution ν, a probability measure μ on S belongs to the domain of attraction of ν if

$$\mu_t(dx) := \mathbb{P}_\mu[X_t \in dx \mid \zeta > t] \xrightarrow[t\to\infty]{} \nu(dx), \tag{1.2}$$

where the convergence is the weak convergence of probability distributions.

It is known that for one-dimensional diffusions, one of the following holds (see e.g., [4, Chapter 6.2]):

(i) There are no quasi-stationary distributions.

(ii) There exists only one quasi-stationary distributions.

(iii) There exists infinitely many quasi-stationary distributions.

When the case (iii) holds, quasi-stationary distributions are naturally parametrized by an interval $(0, \lambda_0]$, where λ_0 is the bottom of the spectrum of the generator. Quasi-stationary distributions are ordered by the stochastic order $\prec$ (recall that $\mu \prec \nu$ means $\mu(x,\infty) \leq \nu(x,\infty)$ for every $x > 0$):

$$\nu_\lambda \prec \nu_{\lambda'} \quad (0 < \lambda' < \lambda \leq \lambda_0), \tag{1.3}$$

where ν_λ is the quasi-stationary distribution corresponding to λ. We will review existence and characterization of quasi-stationary distributions in Section 4.

For the domain of attraction of the minimal quasi-stationary distribution ν_{λ_0} (we also call the only one quasi-stationary distribution for the case (ii) the minimal quasi-stationary distribution) there are many studies (e.g., Mandl [12], Cattiaux et al. [3] and Hening and Kolb [6]) and it is shown that the quasi-stationary distribution attracts all initial distributions with compact support under some mild assumptions on the process X. In contrary to this, there are few studies considering the domain on attraction of non-minimal quasi-stationary distributions. The author only knows two papers: Lladser and San Martín [11] and Martinez, Picco and San Martin [13].

In the present paper, we study the domain of attraction of non-minimal quasi-stationary distributions. Before going on to state our main results, let us fix a setting. We concentrate on the case when the killing only happens at the first hitting time at the boundaries of the state space although the same problem can be considered under more general killing (there are some studies considering the case (Steinsaltz and Evans [15] and Kolb and Steinsaltz [8])). As we will see in Section 4, for existence of non-minimal quasi-stationary distributions, it is necessary that one of the boundaries is natural in the sense of Feller. Hence we may assume without loss of generality that the state spece $S = (0, \infty)$ and the boundary 0 is regular or exit and the boundary ∞ is natural. Note that in this case the lifetime $\zeta = T_0$, where T_0 denotes the first hitting time at 0.

2 Main results

We state our main results without the proof. For the proof, see [17]. One of our main results is a method of reducing the convergence (1.2) to the tail behavior of T_0. For a class $\mathcal{P}$ of initial distributions, we say that the *first hitting uniqueness* holds on $\mathcal{P}$ if

$$\text{the map} \quad \mathcal{P} \ni \mu \longmapsto \mathbb{P}_\mu[T_0 \in dt] \quad \text{is injective.} \tag{2.1}$$

As the class $\mathcal{P}$, we shall take

$$\mathcal{P}_{\mathrm{exp}} = \{\mu \in \mathcal{P}(0,\infty) \mid \mathbb{P}_\mu[T_0 \in dt] = \lambda \mathrm{e}^{-\lambda t} dt \quad (\lambda > 0)\}, \tag{2.2}$$

the set of initial distributions with exponential hitting probabilities, where $\mathcal{P}(0,\infty)$ denotes the set of probability distributions on $(0,\infty)$. The reason we consider the class $\mathcal{P}_{\mathrm{exp}}$ is that if ν is a quasi-stationary distribution, the distribution $\mathbb{P}_\nu[T_0 \in dt]$ is exponentially distributed. Indeed, $\mathbb{P}_\nu[T_0 > t + s \mid T_0 > t] = \mathbb{P}_\nu[X_{t+s} > 0 \mid T_0 > t] = \mathbb{P}_\nu[X_s > 0] = \mathbb{P}_\nu[T_0 > s]$.

The next theorem gives a general method to reduce the convergence (1.2) to the tail behavior of T_0, provided that the first hitting uniqueness holds on $\mathcal{P}_{\mathrm{exp}}$:

Theorem 2.1 ([17, Theorem 1.1]). *Let X be a $\frac{d}{dm}\frac{d}{ds}$-diffusion on $(0,\infty)$ and set*

$$\mu_t(dx) = \mathbb{P}_\mu[X_t \in dx \mid T_0 > t]. \tag{2.3}$$

Assume the first hitting uniqueness holds on $\mathcal{P}_{\rm exp}$ and

$$\mathbb{P}_{\nu_\lambda}[T_0 \in dt] = \lambda e^{-\lambda t} dt \quad \text{for some } \lambda > 0 \text{ and some } \nu_\lambda \in \mathcal{P}(0,\infty). \tag{2.4}$$

Then for $\mu \in \mathcal{P}(0,\infty)$ and $\lambda > 0$, the following are equivalent:

(i) $\lim_{t\to\infty} \frac{\mathbb{P}_\mu[T_0 > t+s]}{\mathbb{P}_\mu[T_0>t]} = e^{-\lambda s}$ $(s > 0)$.

(ii) $\mathbb{P}_{\mu_t}[T_0 \in ds] \xrightarrow[t\to\infty]{} \lambda e^{-\lambda s} ds$.

(iii) $\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda$.

To study concrete sufficient conditions for the convergence (1.2), we introduce the class of processes we call *Kummer diffusions* with negative drifts. A Kummer diffusion $Y^{(0)} = Y^{(\alpha,\beta)}$ $(\alpha > 0,\ \beta \in \mathbb{R})$ is a diffusion on $[0,\infty)$ stopped upon hitting 0 whose local generator $\mathcal{L}^{(0)} = \mathcal{L}^{(\alpha,\beta)}$ on $(0,\infty)$ is

$$\mathcal{L}^{(0)} = \mathcal{L}^{(\alpha,\beta)} = x\frac{d^2}{dx^2} + (-\alpha + 1 - \beta x)\frac{d}{dx}. \tag{2.5}$$

Note that the process $Y^{(0)} = Y^{(\alpha,\beta)}$ is also called a *radial Ornstein-Uhlenbeck process* in some literature (see e.g., [2] and [5]). Write

$$g_\gamma(x) := \mathbb{P}_x[e^{-\gamma T_0^{(0)}}] \quad (\gamma \geq 0), \tag{2.6}$$

which is the Laplace transform of the first hitting time of 0 for $Y^{(0)} = Y^{(\alpha,\beta)}$. Then g_γ is a γ-eigenfunction for $\mathcal{L}^{(0)}$, i.e., $\mathcal{L}^{(0)} g_\gamma = \gamma g_\gamma$ (see e.g., [14, p.292]). We define a Kummer diffusion with a negative drift $Y^{(\gamma)} = Y^{(\alpha,\beta,\gamma)}$ $(\gamma \geq 0)$ as the h-transform of $Y^{(\alpha,\beta)}$ by the function g_γ, that is, the process $Y^{(\alpha,\beta,\gamma)}$ is a diffusion on $[0,\infty)$ stopped at 0 whose local generator on $(0,\infty)$ is

$$\mathcal{L}^{(\gamma)} = \mathcal{L}^{(\alpha,\beta,\gamma)} = \frac{1}{g_\gamma}(\mathcal{L}^{(0)} - \gamma) g_\gamma. \tag{2.7}$$

If we write

$$\widetilde{Y}^{(\alpha,\beta,\gamma)} := \sqrt{2Y^{(\alpha,\beta,\gamma)}}, \tag{2.8}$$

then the local generator $\widetilde{\mathcal{L}}^{(\alpha,\beta,\gamma)}$ of $\widetilde{Y}^{(\alpha,\beta,\gamma)}$ on $(0,\infty)$ is given as

$$\widetilde{\mathcal{L}}^{(\alpha,\beta,\gamma)} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{1-2\alpha}{2x} - \frac{\beta x}{2} + \frac{\widetilde{g}'_\gamma}{\widetilde{g}_\gamma}\right)\frac{d}{dx}, \tag{2.9}$$

where $\widetilde{g}_\gamma(x) = \widetilde{\mathbb{P}}_x[e^{-\gamma \widetilde{T}_0}]$ denotes the Laplace transform of the first hitting time of 0 for $\widetilde{Y}^{(0)}$ starting from x. When $\alpha = 1/2$ and $\gamma = 0$, the process $\widetilde{Y}^{(1/2,\beta,0)}$ is the Ornstein-Uhlenbeck process and, when $\beta = 0$, the process $\widetilde{Y}^{(\alpha,0,\gamma)}$ is the Bessel process with a negative drift (see e.g., [5]).

A necessary and sufficient condition for existence of non-minimal quasi-stationary distributions for general one-dimensional diffusions will be given by Theorem 4.3. Applying the theorem to Kummer diffusions with negative drifts, we obtain the following:

Proposition 2.2 ([17, Proposition 5.1]). *We classify* $Y^{(\gamma)} = Y^{(\alpha,\beta,\gamma)}$ $(\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0)$ *into the following five cases by* β *and* γ*:*

$$\begin{array}{lll} \textit{Case 1:} & \beta = 0, & \gamma > 0. \\ \textit{Case 2:} & \beta > 0, & \gamma \geq 0. \\ \textit{Case 3:} & \beta < 0, & \gamma > 0. \\ \textit{Case 1':} & \beta = 0, & \gamma = 0. \\ \textit{Case 3':} & \beta < 0, & \gamma = 0. \end{array} \tag{2.10}$$

Then non-minimal quasi-stationary distributions exist if and only if one of the Case 1-3 in (2.10) *holds.*

The following theorem gives a class of initial distributions converging to each non-minimal quasi-stationary distributions, where $L^1(I, \nu)$ denotes the set of integrable functions on I w.r.t. the measure ν. For the definition of quasi-stationary distribution ν_λ and the spectral bottom $\lambda_0^{(\gamma)}$, see Section 4.

Theorem 2.3 ([17, Theorem 1.2]). *Let* $X = Y^{(\gamma)} = Y^{(\alpha,\beta,\gamma)}$ $(\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0)$ *satisfying one of the Case 1-3 in* (2.10) *and let* $\mu \in \mathcal{P}(0,\infty)$. *Then the following holds:*

(i) If the Case 1 holds and $\mu(dx) = \rho(x)dx$ *for some* $\rho \in L^1((0,\infty), dx)$ *and*

$$\log \rho(x) \sim (\delta - 2\sqrt{\gamma})\sqrt{x} \quad (x \to \infty) \tag{2.11}$$

for some $0 < \delta < 2\sqrt{\gamma}$, *then it holds*

$$\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda \tag{2.12}$$

with $\lambda = \gamma - \delta^2/4 \in (0, \lambda_0^{(\gamma)})$, *where* $\lambda_0^{(\gamma)} = \gamma > 0$ *is the spectral bottom.*

(ii) If the Case 2 holds and

$$\mu(x,\infty) \sim x^{-\alpha-\gamma/\beta+\delta}\ell(x) \quad (x \to \infty) \tag{2.13}$$

for some $0 < \delta < \alpha + \gamma/\beta$ *and some slowly varying function* ℓ *at* ∞, *then it holds*

$$\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda \tag{2.14}$$

with $\lambda = \beta(\alpha - \delta) + \gamma \in (0, \lambda_0^{(\gamma)})$, *where* $\lambda_0^{(\gamma)} = \alpha\beta + \gamma > 0$ *is the spectral bottom.*

(iii) If the Case 3 holds and

$$\mu(x,\infty) \sim x^{-1+\gamma/\beta+\delta}\ell(x) \quad (x \to \infty) \tag{2.15}$$

for some $0 < \delta < 1 - \gamma/\beta$ *and some slowly varying function* ℓ *at* ∞. *then it holds*

$$\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda \tag{2.16}$$

with $\lambda = -\beta(1 - \delta) + \gamma \in (0, \lambda_0^{(\gamma)})$, *where* $\lambda_0^{(\gamma)} = -\beta + \gamma > 0$ *is the spectral bottom.*

We will compare Theorem 2.3 with previous studies in Remarks 3.2 and 3.4.

3 Comparison with previous studies

There are many studies on quasi-stationary distributions as we saw in Section 1. As far as the author knows, however, most of them studies the minimal quasi-stationary distributions and there are only two studies considering the domain of attraction of non-minimal quasi-stationary distributions; Martinez, Picco and San Martin [13] and Lladser and San Martin [11].

Firstly, Martinez, Picco and San Martin [13] studied Brownian motions with negative drifts and showed convergence to non-minimal quasi-stationary distributions under the assumptions on tail behavior of the initial distribution:

Theorem 3.1 ([13, Theorem 1.1])**.** *Let B_t be a standard Brownian motion and let $\alpha > 0$ and consider the process*

$$X_t = B_t - \alpha t. \tag{3.1}$$

For an initial distribution μ on $(0, \infty)$ assume $\mu(dx) = \rho(x)dx$ for some $\rho \in L^1((0, \infty), dx)$ satisfying

$$\log \rho(x) \sim -(\alpha - \delta)x \quad (x \to \infty) \tag{3.2}$$

for some $\delta \in (0, \alpha)$. Then it holds

$$\mathbb{P}_\mu[X_t \in dx \mid T_0 > t] \xrightarrow[t\to\infty]{} \nu_\lambda(dx), \tag{3.3}$$

with

$$\lambda = (\alpha^2 - \delta^2)/2 \quad \text{and} \quad \nu_\lambda(dx) = C_\lambda e^{-\alpha x} \sinh(x\sqrt{\alpha^2 - 2\lambda})dx \tag{3.4}$$

for the normalizing constant C_λ.

Remark 3.2. When $\alpha = 1/2, \beta = 0$ and $\gamma > 0$, the process $\sqrt{2Y^{(1/2,0,\gamma)}}$ is a Brownian motion with a negative drift $-\sqrt{2\gamma}t$. Hence this theorem is generalized by (i) of Theorem 2.3.

Secondly, Lladser and San Martin [11] studied Ornstein-Uhlenbeck processes:

Theorem 3.3 ([11, Theorem 1.1])**.** *Let $\alpha > 0$. Let X be the solution of the following SDE:*

$$dX_t = dB_t - \alpha X_t dt, \tag{3.5}$$

where B is a standard Brownian motion. For an initial distribution μ on $(0, \infty)$ assume $\mu(dx) = \rho(x)dx$ for some $\rho \in L^1((0, \infty), dx)$ satisfying

$$\rho(x) \sim x^{-2+\delta}\ell(x) \quad (x \to \infty) \tag{3.6}$$

for some $\delta \in (0,1)$ *and a slowly varying function* ℓ *at* ∞*. Then it holds*

$$\mathbb{P}_\mu[X_t \in dx \mid T_0 > t] \xrightarrow[t\to\infty]{} \nu_\lambda(dx) \tag{3.7}$$

with

$$\lambda = \alpha(1-\delta) \quad \text{and} \quad \nu_\lambda(dx) = C_\lambda \psi_{-\lambda}(x) \mathrm{e}^{-\alpha x^2} dx \tag{3.8}$$

for the normalizing constant C_λ*, where* $u = \psi_{-\lambda}$ *denotes the unique solution for the following differential equation:*

$$\frac{1}{2}\frac{d^2}{dx^2}u - \alpha x \frac{d}{dx}u = -\lambda u, \quad \lim_{x\to+0} u(x) = 0, \quad \lim_{x\to+0}\frac{d}{dx}u(x) = 1 \quad (x \in (0,\infty)). \tag{3.9}$$

Remark 3.4. In Theorem 2.3 (ii), if $\mu(dx) = \rho(x)dx$ for $\rho \in L^1((0,\infty), dx)$ and

$$\rho(x) \sim x^{-\alpha-\gamma/\beta+\delta-1}\ell(x) \quad (x \to \infty), \tag{3.10}$$

for a slowly varying function ℓ, then (2.13) holds from Karamata's theorem [1, Proposition 1.5.8]. Hence (ii) of Theorem 2.3 is an extension of [11, Theorem 1.1].

4 Existence and characterization of quasi-stationary distributions

Here we review some previous studies on quasi-stationary distributions.

Let X be a $\frac{d}{dm}\frac{d}{ds}$-diffusion on $I = [0,b)$ or $[0,b]$ $(0 < b \leq \infty)$ stopped at 0. We assume

$$\mathbb{P}_x[T_y < \infty] > 0 \quad (x \in I \setminus \{0\},\ y \in [0,b)), \tag{4.1}$$

where T_y denotes the first hitting time of y. We also assume that the boundary b is not exit in the sense of Feller and that the boundary b is reflecting when it is regular. Note that from (4.1), the boundary 0 is regular or exit. Define a function $u = \psi_\lambda$ as the unique solution of the following equation:

$$\frac{d}{dm}\frac{d}{ds}u(x) = \lambda u(x), \quad \lim_{x\to+0} u(x) = 0, \quad \lim_{x\to+0}\frac{d}{ds}u(x) = 1 \quad (x \in (0,b), \lambda \in \mathbb{R}). \tag{4.2}$$

Since the boundary 0 is regular or exit, the function ψ_λ always exists. The operator $L = -\frac{d}{dm}\frac{d}{ds}$ defines a non-negative definite self-adjoint operator on $L^2(I, dm) := \{f : I \to \mathbb{R} \mid \int_I |f|^2 dm < \infty\}$. Here we assume the Dirichlet boundary condition at 0 and the Neumann boundary condition at b if the boundary b is regular. We denote the infimum of the spectrum of L by $\lambda_0 \geq 0$.

When the boundary b is not natural, it is known that there is a unique quasi-stationary distribution (noting that Takeda [16] showed the corresponding result for general Markov processes with the *tightness property*):

Proposition 4.1 (see e.g., [10, Lemma 2.2, Theorem 4.1])**.** *Assume the boundary b is not natural. Then it holds $\lambda_0 > 0$ and the function $\psi_{-\lambda_0}$ is strictly positive and integrable w.r.t. dm and, there is a unique quasi-stationary distribution given as*

$$\nu_{\lambda_0}(dx) = \lambda\psi_{-\lambda_0}(x)dm(x), \quad \mathbb{P}_{\nu_{\lambda_0}}[T_0 \in dt] = \lambda_0 \mathrm{e}^{-\lambda_0 t}dt. \tag{4.3}$$

Moreover, for every probability distribution μ on $(0, b)$ with a compact support, it holds

$$\mu_t \xrightarrow[t\to\infty]{} \nu_{\lambda_0}. \tag{4.4}$$

We now assume the boundary b is natural. Then it holds

$$\mathbb{P}_x[T_b < \infty] = 0 \quad (x \in (0, b)), \tag{4.5}$$

and

$$\frac{s(x) - s(0)}{s(M) - s(0)} = \mathbb{P}_x[T_M < T_0] \quad (0 < x < M < b) \tag{4.6}$$

(see e.g., Itô [7]). Taking limit $M \to b$, we have from (4.5)

$$\frac{s(x) - s(0)}{s(b) - s(0)} = \mathbb{P}_x[T_0 = \infty]. \tag{4.7}$$

Hence it follows

$$\mathbb{P}_x[T_0 < \infty] = 1 \quad \text{for some / any } x > 0 \quad \Leftrightarrow \quad s(b) = \infty. \tag{4.8}$$

Since $\mathbb{P}_\nu[T_0 \in dt]$ follows an exponential distribution, by (4.1) it holds $\mathbb{P}_\nu[T_0 = \infty] < 1$ and therefore $\mathbb{P}_\nu[T_0 = \infty] = 0$, which implies $s(b) = \infty$. We recall the following good properties for the function ψ_λ:

Proposition 4.2 ([4, Lemma 6.18])**.** *Suppose the boundary b is natural and $s(b) = \infty$. Then for $\lambda > 0$ the following hold:*

(i) For $0 < \lambda \le \lambda_0$, the function $\psi_{-\lambda}$ is strictly positive on $I \setminus \{0\}$ and

$$1 = \lambda \int_0^b \psi_{-\lambda}(x)dm(x). \tag{4.9}$$

(ii) For $\lambda > \lambda_0$, the function $\psi_{-\lambda}$ change signs on I.

Now we state a necessary and sufficient condition for existence of non-minimal quasi-stationary distributions:

Theorem 4.3 ([4, Theorem 6.34] and [9, Theorem 3, Appendix I])**.** *Suppose the boundary b is natural. Then a non-minimal quasi-stationary distribution exists if and only if*

$$\lambda_0 > 0 \quad \textit{and} \quad s(b) = \infty. \tag{4.10}$$

This condition is equivalent to

$$m(d,b) < \infty \quad \textit{for some } d \in (0,b) \quad \textit{and} \quad \limsup_{x \to b} s(x)m(x,b) < \infty. \tag{4.11}$$

In this case, a probability measure ν is a quasi-stationary distribution if and only if

$$\nu(dx) = \lambda \psi_{-\lambda}(x) dm(x) =: \nu_\lambda(dx), \quad \mathbb{P}_{\nu_\lambda}[T_0 \in dt] = \lambda \mathrm{e}^{-\lambda t} dt \quad \textit{for some } 0 < \lambda \le \lambda_0. \tag{4.12}$$

Here we note that as [4] only dealt with the case the boundary 0 is regular, the proof also works in the case the boundary 0 is exit.

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.

[2] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[3] P. Cattiaux, P. Collet, A. Lambert, S. Martínez, S. Méléard, and J. San Martín. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, 37(5):1926–1969, 2009.

[4] P. Collet, S. Martínez, and J. San Martín. *Quasi-stationary distributions*. Probability and its Applications (New York). Springer, Heidelberg, 2013. Markov chains, diffusions and dynamical systems.

[5] A. Göing-Jaeschke and M. Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349, 2003.

[6] A. Hening and M. Kolb. Quasistationary distributions for one-dimensional diffusions with singular boundary points. *Stochastic Process. Appl.*, 129(5):1659–1696, 2019.

[7] K. Itô. *Essentials of stochastic processes*, volume 231 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2006. Translated from the 1957 Japanese original by Yuji Ito.

[8] M. Kolb and D. Steinsaltz. Quasilimiting behavior for one-dimensional diffusions with killing. *Ann. Probab.*, 40(1):162–212, 2012.

[9] S. Kotani and S. Watanabe. Kreĭn's spectral theory of strings and generalized diffusion processes. In *Functional analysis in Markov processes (Katata/Kyoto, 1981)*, volume 923 of *Lecture Notes in Math.*, pages 235–259. Springer, Berlin-New York, 1982.

[10] J. Littin. Uniqueness of quasistationary distributions and discrete spectra when ∞ is an entrance boundary and 0 is singular. *J. Appl. Probab.*, 49(3):719–730, 2012.

[11] M. Lladser and J. San Martín. Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process. *J. Appl. Probab.*, 37(2):511–520, 2000.

[12] P. Mandl. Spectral theory of semi-groups connected with diffusion processes and its application. *Czechoslovak Math. J.*, 11(86):558–569, 1961.

[13] S. Martinez, P. Picco, and J. San Martin. Domain of attraction of quasi-stationary distributions for the Brownian motion with drift. *Adv. in Appl. Probab.*, 30(2):385–408, 1998.

[14] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 2.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[15] D. Steinsaltz and S. Evans. Quasistationary distributions for one-dimensional diffusions with killing. *Trans. Amer. Math. Soc.*, 359(3):1285–1324, 2007.

[16] M. Takeda. Existence and uniqueness of quasi-stationary distributions for symmetric Markov processes with tightness property. *J. Theoret. Probab.*, 32(4):2006–2019, 2019.

[17] K. Yamato. A unifying approach to non-minimal quasi-stationary distributions for one-dimensional diffusions. *arXiv:2012.12971*, 2020.