# On the existence of absolutely continuous $\sigma$-finite invariant measures for random dynamical systems

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## 1 Introduction

Finding an absolutely continuous finite or $\sigma$-finite infinite invariant measure ($\sigma$-finite acim, for short) for a given system (described by a transformation or a Markov process) is one of the classical problems in ergodic theory. Thus, there are lots of previous researches for this problem (see for example [A97, DS66, Fog69, HK64, In12, In20, Sch95, Th80] and references therein). However, necessary and sufficient conditions for the existence of a $\sigma$-finite acim have still not been well-known. In this paper, for a given Markov operator over a probability space, we give some equivalent conditions for the existence of a $\sigma$-finite acim with certain support property. One of the equivalent conditions is weak almost periodicity of the jump operator with respect to some sweep-out set (which implies the Jacobs-de Leeuw-Glicksberg splitting theorem [E06]). Here the method of jump operators is generalization of the method of jump transformations established in [Sch95, Th80]. Because we consider general Markov operators, we can apply our setting not only to deterministic systems but also to random dynamical systems represented by null-preserving transition probabilities. Our result is applicable to certain one-dimensional random dynamical system arising from intermittent Markov maps with uniformly contractive part.

To be more precise, we consider a probability space $(X, \mathscr{F}, m)$ and a *Markov operator* $P$ defined on $L^1 = L^1(X, \mathscr{F}, m)$ into itself, i.e, $P$ satisfies $Pf \geq 0$ and $\|Pf\|_1 = \|f\|_1$ whenever $f \in L^1_+ = \{g \in L^1 : g \geq 0\}$. The adjoint operator of $P$ is denoted by $P^*$ which is defined on $L^\infty$. Then a finite (resp. $\sigma$-finite) measure $\mu$ on $(X, \mathscr{F})$ is said to be a *finite* (resp. *$\sigma$-finite*) *acim* if $\mu$ is absolutely continuous w.r.t. $m$ ($\mu \ll m$) and the Radon-Nikodym derivative $d\mu/dm$ is a (not identically zero) fixed point of $P$. Notice that the domain of $P$ can be extended to the set of all non-negative measurable functions and the definition of a $\sigma$-finite acim makes sense even if $\mu$ is a $\sigma$-finite infinite measure. When we have a null-preserving transition probability $\mathbb{P}(x, A)$ for $x \in X$ and $A \in \mathscr{F}$ (i.e., $\mathbb{P}(x, N) = 0$ for a.e. $x$ if $m(N) = 0$), which describes our system, the corresponding

Markov operator $P$ is given by,

$$\int_A Pf dm = \int_X f(x)\mathbb{P}(x, A)dm(x)$$

for each $f \in L^1$ and $A \in \mathscr{F}$. If we consider a deterministic system, given by a non-singular transformation $T : X \to X$ (i.e., $m \circ T^{-1} \ll m$), the corresponding transition probability is $\mathbb{P}(x, A) = 1_{T^{-1}A}(x)$. Then the Markov operator associated to a given non-singular transformation is called the Perron-Frobenius operator $P$ given by

$$\int_X Pf \cdot g dm = \int_X f \cdot g \circ T dm$$

for $f \in L^1$ and $g \in L^\infty$. We will show the existence of a $\sigma$-finite acim for a Markov operator in the next section. That is, we can apply our results to both non-singular transformations and null-preserving transition probabilities.

## 2 Main Result

In this section, we present our main results. Our results Theorem 2.2 and Theorem 2.6 give equivalent conditions for the existence of a finite or $\sigma$-finite acim with the maximal support condition for a given Markov operator. Here, "the maximal support condition" means the support of the invariant measure contains a proper sweep-out set (see Definition 2.1). That is, almost all trajectories under the process will eventually concentrate on the support of the invariant measure. Throughout this section supp$\mu$ denotes the support of $\mu$, i.e., supp$\mu = \{\frac{d\mu}{dm} > 0\}$.

In order to state Theorem 2.2, we need the following definition of a sweep-out set.

**Definition 2.1** (A sweep-out set)**.** *For a Markov operator over $L^1(X, \mathscr{F}, m)$, a set $E \in \mathscr{F}$ is called a (P-)* sweep-out set *(w.r.t. m) if* $\lim_{n\to\infty} (P^* I_{E^c})^n 1_X(x) = 0$ *m-a.e. $x \in X$ where $I_{E^c}$ denotes the restriction operator on $E^c$.*

Recall that a Markov operator $P$ is called *weakly almost periodic* if for any $f \in L^1$ the sequence of functions $\{P^n f\}$ is weakly precompact. In the following Theorem 2.2, weak almost periodicity of a Markov operator plays a key role as an equivalent condition for the existence of a finite acim with the maximal support condition.

**Theorem 2.2** ([T])**.** *Let $P$ be a Markov operator over a probability space $(X, \mathscr{F}, m)$. Then the followings are equivalent.*

1. *There exists a finite acim $\mu$ for $P$ s.t.* supp$\mu$ *is a sweep-out set;*
2. *$\{P^n 1_X\}_n$ is weakly precompact;*
3. *$P$ is weakly almost periodic.*

**Remark 2.3.** *(1) The condition 1 in Theorem 2.2 can be paraphrased:*

*1'. There exists a finite acim $\mu$ for $P$ s.t.* $\lim_{n\to\infty} P^{*n} 1_{\mathrm{supp}\mu}(x) = 1$ *$m$-a.e.* $x \in X$.

*(2) The condition in 1 "* $\mathrm{supp}\mu$ *is a sweep-out set" is a necessary condition of ergodicity of* $(P, m)$*, where* $(P, m)$ *is called ergodic if* $E \in \mathscr{F}$ *with* $P^* 1_E = 1_E$ *implies* $E = \emptyset$ *or* $X$ $(\mathrm{mod}\ m)$*.*

We prepare the methods of inducing and jump to state Theorem 2.6, equivalent conditions for the existence of a $\sigma$-finite acim. The following definition of the induced operator or the jump operator is the generalization of the induced transformation or the jump transformation (see [A97, Fog69, In20, Sch95, Th80, T] for details).

**Definition 2.4** (The induced operator/The jump operator)**.** *For a Markov operator $P$ with a sweep-out set $E$, the* induced operator $P_E$ *is defined by*

$$P_E = I_E P \sum_{n\geq 0} (I_{E^c} P)^n ,$$

*and the* jump operator $\hat{P}_E$ *is defined by*

$$\hat{P}_E = P I_E \sum_{n\geq 0} (P I_{E^c})^n .$$

**Remark 2.5.** *(1) The induced operator $P_E$ and the jump operator $\hat{P}_E$ are also Markov operators over $L^1(X, \mathscr{F}, m)$ as long as $E$ is sweep-out.*

*(2) When $P$ is the Perron-Frobenius operator for some non-singular transformation, the restricted induced operator $P_E I_E$ (defined on $L^1(E, \mathscr{F} \cap E, m\,|_E)$) and the jump operator $\hat{P}_E$ are the Perron-Frobenius operators corresponding to the induced transformation and the jump transformation, respectively.*

The following theorem give equivalent conditions for the existence of a $\sigma$-finite (it might be infinite) acim with the maximal support condition. Equivalent conditions are characterized by the methods of induced operator and jump operator respectively.

**Theorem 2.6** ([T])**.** *Let $P$ be a Markov operator over a probability space $(X, \mathscr{F}, m)$. Then the followings are equivalent.*

1. *There exists a $\sigma$-finite acim $\mu$ for $P$ s.t.* $\mathrm{supp}\mu$ *contains a $P$-sweep-out set $A$ w.r.t. $m$ with $\mu(A) < \infty$;*
2. *There exists a sweep-out set $E$ s.t. the induced operator $P_E$ admits a finite acim $\mu_E$ with* $\mathrm{supp}\mu_E = E$ $(\mathrm{mod}\ m)$*;*
3. *There exists a sweep-out set $E$ s.t. the jump operator $\hat{P}_E$ is weakly almost periodic.*

**Remark 2.7.** *We can apply Theorem 2.2 to the condition 3 in Theorem 2.6. That is, we only have to check weak precompactness of $\{\hat{P}_E^n 1_X\}_n$.*

## 3 Example of Random Dynamical System

In this section, we apply Theorem 2.6 to certain one-dimensional random dynamical system. Our random dynamical system is random iteration of non-uniformly expanding maps which have uniformly contractive part on average. Throughout this section, our phase space $(X, \mathscr{B}(X), \lambda)$ is the unit interval with the Lebesgue measure.

Let $I$ be an at most countable non-empty subset of $\mathbb{N}$ and for each $i \in I$, $J_i$ be also an at most countable non-empty subset of $\mathbb{N}$. We consider $\{T_i : i \in I\}$ a family of piecewise linear Markov maps on the unit interval $X = [0,1]$ with the Lebesgue measure $\lambda$ satisfying:

(a-1) $T_i \mid_{X_n} : X_n \to X_{n-1}$ for $n \geq 2$ and $i \in I$, given by

$$T_i \mid_{X_n} (x) = \frac{n+1}{n-1} x - \frac{1}{n(n-1)};$$

(a-2) $T_i \mid_{X_1} : X_1 \to \bigcup_{k \in J_i} X_k$, a surjective and monotonically increasing map which is piecewise linear in the sense

$$T_i \mid'_{X_1} = \frac{\sum_{k \in J_i} \lambda(X_k)}{\lambda(X_1)}$$

whenever the derivative can be defined, for each $i \in I$

where $X_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$ is the cylinder of rank one for $n \geq 1$. Then the point 0 is the common fixed point for all $T_i$ where the derivative of all $T_i$ is one. We call a family of transformations $\{T_i : i \in I\}$ with the above conditions (a-1) and (a-2) *piecewise linear intermittent Markov maps* (with index $\{J_i\}_{i \in I}$).

By using piecewise linear intermittent Markov maps $\{T_i : i \in I\}$, we define our random dynamical system. For a given probability vector $\{p_i\}_{i \in I}$ (i.e., $p_i \geq 0$ and $\sum_{i \in I} p_i = 1$), *the random iteration of piecewise linear intermittent Markov maps of* $\{T_i, p_i : i \in I\}$ is given by the following transition probability:

$$\mathbb{P}(x, E) = \sum_{i \in I} p_i 1_E(T_i x) \quad (x \in X,\ E \in \mathscr{B}(X)). \tag{3.1}$$

That is, each transformation $T_i$ will be chosen with probability $p_i$ and the selected transformation will be applied to the system. For the transition probability given by (3.1), we can define the Markov operator $P$ on $L^1 = L^1(X, \mathscr{B}, \lambda)$ since each transformation is non-singular w.r.t. $\lambda$, which is determined by

$$\int_A P f d\lambda = \int_X f(x) \mathbb{P}(x, A) dm(x) \quad (A \in \mathscr{B}(X),\ f \in L^1).$$

Equivalently, the Markov operator associated to this random dynamical system is given by $P = \sum_{i \in I} p_i P_i$ where each $P_i$ is the Perron-Frobenius operator corresponding to

$T_i$. We remark that the random iteration of piecewise linear intermittent Markov maps $\{T_i, p_i : i \in I\}$ may not satisfy expanding property on average in the following sense:

$$\sup_{X \setminus U_0} \sum_{i \in I} \frac{p_i}{|T_i'|}(x) < 1 \tag{3.2}$$

where $U_0$ is a small neighborhood of the indifferent fixed point 0. Indeed, if $1 \notin J_i$ for any $i \in I$ then the average of derivatives of $\{T_i, p_i : i \in I\}$ is obviously strictly less than one on $X_1$.

Then by Theorem 2.6 we can show the existence of a $\sigma$-finite acim for random iterations of piecewise linear intermittent Markov maps:

**Proposition 3.1.** *Any random iteration of piecewise linear intermittent Markov maps* $\{T_i, p_i : i \in I\}$ *admits a* $\sigma$*-finite acim.*

**Remark 3.2.** *If* $\{T_i, p_i : i \in I\}$ *satisfies expanding property on average in (3.2) sense, then the existence of a* $\sigma$*-finite acim was already shown in [In20] via the method of inducing. However, for random iterations of non-uniformly expanding maps which do not satisfy expanding property on average, statistical properties including the existence of a* $\sigma$*-finite acim are not well-studied. Therefore, our result could be interpreted as the first step toward the direction of non-uniformly expanding random maps with uniformly contractive part.*

## References

[A97] J. Aaronson, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, 50. American Mathematical Society, (1997).

[DS66] D. W. Dean, and L. Sucheston, On invariant measures for operators, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, **6** (1966), 1–9.

[E06] E. Y. Emel'yanov, *Non-spectral Asymptotic Analysis of One-Parameter Operator Semigroups*, Operator Theory Advances and Applications, Vol.173 (2006).

[Fog69] S. R. Foguel, *The ergodic theory of Markov processes*, Van Nostrand Mathematical Studies, No. 21. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, (1969).

[HK64] A. Hajian, and S. Kakutani, Weakly wandering sets and invariant measures, *Trans. Amer. Math. Soc.*, **110** (1964), 136–151.

[In12] T. Inoue, Invariant measures for position dependent random maps with continuous random parameters, *Studia Math.*, **208** (2012), 11–29.

[In20] T. Inoue, First return maps of random map and invariant measures, *Nonlinearity*, **33** (2020), 249–275.

[Sch95] F. Schweiger, *Ergodic theory of fibred systems and metric number theory*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, (1995).

[Th80] M. Thaler, Estimates of the invariant densities of endomorphisms with indifferent fixed points, *Israel J. Math.*, **37** (1980), 303–314.

[T] H. Toyokawa, $\sigma$-finite invariant densities for eventually conservative Markov operators, *Discrete Continuous Dynam. Systems - A*, to appear.