On Possible Limit Functions on a Fatou Component in non-Autonomous Iteration (Research on the Theory of Random Dynamical Systems and Fractal Geometry)

Comerford, Mark; Staniszewski, Christopher

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Mark Comerford
University of Rhode Island

Christopher Staniszewski
Framingham State University

Abstract

The possibilities for limit functions on a Fatou component for the iteration of a single polynomial or rational function are well understood and quite restricted. In non-autonomous iteration, where one considers compositions of arbitrary polynomials with suitably bounded degrees and coefficients, one ought to observe a far greater range of behaviour. We show this is indeed the case and we exhibit a sequence of quadratic polynomials which has a bounded Fatou component on which one obtains as limit functions every member of the classical Schlicht family of suitably normalized univalent functions on the unit disc. The main idea behind this is to make use of dynamics on Siegel discs where high iterates of a single polynomial with a Siegel disc approximate the identity arbitrarily closely on compact subsets of the Siegel disc.

1 Introduction

1.1 Non-Autonomous Iteration

We are concerned with non-autonomous iteration of bounded sequences of polynomials, a relatively new field in complex dynamics. In classical complex dynamics, one studies the iteration of a (fixed) rational function on the Riemann sphere. Often in applications of dynamical systems, noise is introduced, and thus it is natural to consider iteration where the function at each stage is allowed to vary. Here, we study the situation where the functions being applied are polynomials with appropriate bounds on the coefficients and degrees.

Let $d \geq 2$, $M \geq 0$, $K \geq 1$, and let $\{P_m\}_{m=1}^{\infty}$ be a sequence of polynomials where each

$$P_m(z) = a_{d_m,m}z^{d_m} + a_{d_m-1,m}z^{d_m-1} + \cdots\cdots + a_{1,m}z + a_{0,m}$$

is a polynomial of degree $2 \leq d_m \leq d$ whose coefficients satisfy

$$\frac{1}{K} \leq |a_{d_m,m}| \leq K, \quad m \geq 1, \qquad |a_{k,m}| \leq M, \quad m \geq 1,\ 0 \leq k \leq d_m - 1.$$

Such sequences are called *bounded sequences of polynomials* or *simply bounded sequences.* For each $0 \leq m$, we let Q_m be the composition $P_m \circ \cdots\cdots \circ P_2 \circ P_1$ and, for each $0 \leq m < n$, we let $Q_{m,n}$ be the composition $P_n \circ \cdots\cdots \circ P_{m+2} \circ P_{m+1}$. For each $m \geq 0$ define the *mth iterated Fatou set* or simply the *Fatou set* at time m, $\mathcal{F}_m$, by

$$\mathcal{F}_m = \{z \in \hat{\mathbb{C}} : \{Q_{m,n}\}_{n=m}^{\infty} \text{ is a normal family on some neighborhood of } z\}$$

where we take our neighborhoods with respect to the spherical topology on $\hat{\mathbb{C}}$ and let the *mth iterated Julia set* or simply the *Julia set* at time m, $\mathcal{J}_m$, to be the complement $\hat{\mathbb{C}} \setminus \mathcal{F}_m$.

1.2 The Schlicht Class

The *Schlicht* class of functions, commonly denoted by $\mathcal{S}$, is the set of univalent functions defined on the unit disk such that, for all $f \in \mathcal{S}$, we have $f(0) = 0$ and $f'(0) = 1$. This is a well-studied class of functions for which many useful results are known (see [2, 7]). By rescaling, one can often apply these results to an arbitrary univalent function, making the knowledge of this class quite useful in practice.

1.3 Statement of the Main Theorem

Our main goal is to prove the following result:

Theorem 1.1 *There exists a bounded sequence of quadratic polynomials $\{P_m\}_{m=1}^{\infty}$ and a bounded Fatou component V for this sequence such that, for all $f \in \mathcal{S}$, there exists a subsequence $\{P_{m_k}\}_{k=1}^{\infty}$ of $\{P_m\}_{m=1}^{\infty}$ such that $\{Q_{m_k}\}_{k=1}^{\infty}$ converges locally uniformly to f on V.*

The strength of this statement is that *every* member of $\mathcal{S}$ is a limit function on the *same* Fatou component for the *same* polynomial sequence.

The proof relies on a scaled version of the polynomial $P_\lambda(z) = \lambda z(1-z)$ where $\lambda = e^{\frac{2\pi i(\sqrt{5}-1)}{2}}$. As P_λ is conjugate to an irrational rotation on its Siegel disk about 0, which we denote by U_λ, we may find a subsequence of iterates which converges uniformly to the identity on compact subsets of U_λ. We will rescale P_λ so that $\mathcal{K}$, the filled Julia set for the scaled version P of P_λ, is contained in a small disc about 0. This is done so that, for any $f \in \mathcal{S}$, we can use the distortion theorems to control $|f'|$ on a relatively large hyperbolic disk inside U, the scaled version of U_λ.

The initial inspiration for this proof came from Löwner chains (see e.g. [3, 7]), particularly the idea that a univalent function can be expressed as a composition of many univalent functions which are close to the identity. Given our remarks above about iterates of P_λ which converge to the identity locally uniformly on U_λ, this encouraged us to think we might be able to approximate these univalent functions which are close to the identity in some way with polynomials and then compose these polynomials to get an approximation of the desired univalent function on some suitable subset of U_λ, a principle which we like to summarize as 'Do almost nothing and you can do almost anything'.

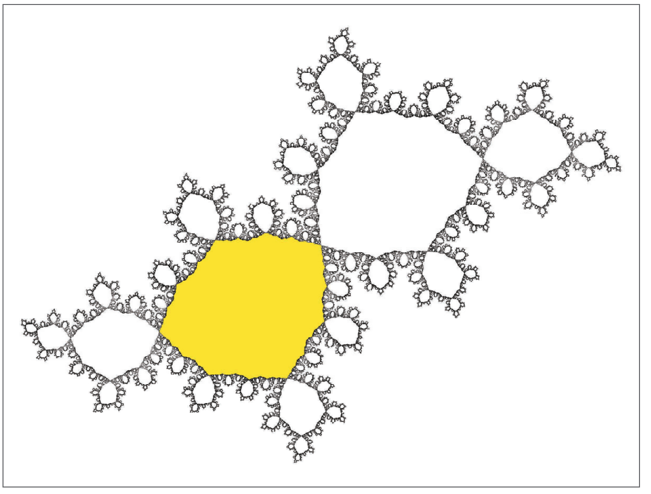


Figure 1: The filled Julia Set for P_λ with Siegel Disc highlighted.

The proof of Theorem 1.1 will follow from an inductive argument, and each step in the induction will be broken up into two phases:

- Phase I: Construct a bounded polynomial composition which approximates given functions from $\mathcal{S}$ on a subset of the unit disk, with arbitrarily small error.
- Phase II: Construct a bounded polynomial composition which corrects the error of the previous sequence to arbitrary accuracy on a slightly smaller subset.

Great care is needed to control the error in the approximations and to ensure that the domain loss that necessarily occurs in each Phase II eventually stabilizes, and that we are left with a non-empty region upon which the desired approximations hold.

To create our polynomial approximations, we use what we call the Polynomial Implementation Lemma. Suppose we want to approximate a given univalent function f with a polynomial composition. Let γ and Γ be two analytic Jordan curves outside $\mathcal{K}$ such that γ is inside Γ while $f(\gamma)$ is still inside Γ. We construct a homeomorphism of the sphere as follows: define it to be f inside γ, the identity outside Γ and extend by interpolation to the region between γ and Γ. The homeomophism can be made quasiconformal, with non-zero dilation (possibly) only on the region between γ and Γ. If we then pull back with a high iterate of P, the support of the dilation becomes small, which will eventually allow us to conclude, that when we straighten, we get a polynomial composition that approximates f closely on a large compact subset of U.

In Phase I, we use the Polynomial Implementation Lemma to create a polynomial composition which approximates a finite set of functions from $\mathcal{S}$. In Phase II, we wish to correct the error from the Phase I composition. This error is defined on a subset of the Siegel disk, but in order to apply the Polynomial Implementation Lemma to create a composition which corrects the error, we need the error to be defined on a region which contains $\mathcal{K}$.

To get around this, we conjugate so that the conjugated error is defined on a region which contains $\mathcal{K}$. This introduces a further problem, namely that we must now cancel the conjugacy with polynomial compositions. A key element of the proof is viewing the expanding map, that is the part of the conjugacy which maps a suitably chosen (and relatively large) subset of U to a set containing $\mathcal{K}$, as a dilation in the correct conformal coordinates. An inevitable loss of domain occurs in using these conformal coordinates, but we are, in the end, able to create a Phase II composition which corrects the error of the Phase I approximation on a (slightly smaller) compact subset of U. What allows us to control the loss of domain, is that, while the loss of domain is unavoidable, the accuracy of the Phase II correction is completely at our disposal. This ultimately allows us to control loss of domain. We then implement a fairly lengthly inductive argument to prove the theorem, getting better approximations to more functions in the Schlicht class with each stage in the induction, and ensuring that the region upon which the approximation holds does not shrink to nothing.

2 Useful Tools

Two of the more well-known tools we use are distortion theorems for univalent mappings [2, 7] and the Carathéodory topology for pointed domains [1, 4, 5, 8]. A lesser known tool is the hyperbolic derivative (see [9]). Ordinary derivatives are useful for estimating how points move apart under iteration when using the Euclidean metric. In our case, we need a notion of a derivative taken with respect to the hyperbolic metric.

Let R, S be hyperbolic Riemann surfaces with metrics

$$\begin{aligned} \mathrm{d}\rho_R &= \sigma_R(z)|\mathrm{d}z|, \\ \mathrm{d}\rho_S &= \sigma_S(z)|\mathrm{d}z|, \end{aligned}$$

respectively, and let $f : W \subset R \to S$ be analytic. Define the *hyperbolic derivative:*

$$f^{\natural}_{R,S}(z) := f'(z)\frac{\sigma_S(f(z))}{\sigma_R(z)}, \quad z \in R$$

Note that the hyperbolic derivative satisfies the chain rule, i.e. if R, S, T, are hyperbolic Riemann surfaces with f defined on a set $W \subset R$ and g defined on a set $X \subset f(W) \subset S$ and mapping X into T, then

$$(g \circ f)^{\natural}_{R,T} = (g^{\natural}_{S,T} \circ f) \cdot f^{\natural}_{R,S}$$

Let $K \subset W$ be a relatively compact subset of R. Define the *hyperbolic Lipschitz bound* as

$$\|f^{\natural}_{R,S}\|_K := \sup_{z \in K} |f^{\natural}_{R,S}(z)|.$$

3 The Polynomial Implementation Lemma

Let $\kappa > 1$ be a scaling factor and set $P(z) = \frac{1}{\kappa} P_\lambda(\kappa z)$. Let U be the Siegel Disc for P. Let $\Omega, \Omega' \subset \mathbb{C}$ be the Jordan domains with the analytic boundary curves γ and Γ (defined earlier), respectively, and such that $\mathcal{K} \subset \Omega \subset \overline{\Omega} \subset \Omega'$. Suppose f is univalent on a neighborhood of $\overline{\Omega}$ and recall that $f(\gamma)$ is still inside Γ.

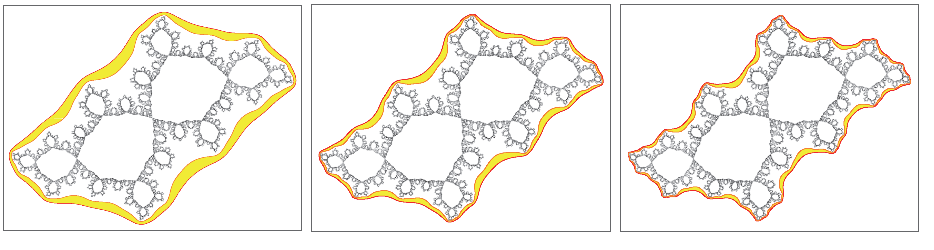
Lemma 3.1 *(The Polynomial Implementation Lemma) Let P_λ, U_λ, κ, P, U, $\{n_k\}_{k=1}^{\infty}$, Ω, Ω', γ, Γ, and f be as above. Suppose $A \subset U$ is open and relatively compact. Then for all M, ϵ, δ positive, if $\hat{A}$ is a δ-neighborhood of A with respect to ρ_U as above and $\|f^{\natural}\|_{\hat{A}} \le M$, there exists $k_0 \ge 1$ (depending on M, ϵ, A, and δ) such that for each $k_1 \ge k_0$ there exists a (17+κ)-bounded finite sequence of quadratic polynomials $\{P_m^{n_{k_1}}\}_{m=1}^{n_{k_1}}$ such that $Q_{n_{k_1}}^{n_{k_1}}$ is univalent on A and*

1. *$\rho_U(Q_{n_{k_1}}^{n_{k_1}}(z), f(z)) < \epsilon$ for all $z \in A$,*
2. *$\|(Q_{n_{k_1}}^{n_{k_1}})^{\natural}\|_A \le M(1+\epsilon)$,*
3. *$Q_{n_{k_1}}^{n_{k_1}}(0) = 0$.*

The idea of the proof is as follows: suppose we want to approximate f with a polynomial composition. Define

$$F(z) = \begin{cases} f(z) & z \in \overline{\Omega} \\ z & z \in \hat{\mathbb{C}} \setminus \Omega' \end{cases}$$

and extend F to a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ using interpolation (e.g., using conformal coordinates). If we precompose this with a high iterate of P and pull back, the area of the region between the preimage of γ and Γ becomes small, while the support of the pullback of the dilatation is contained in the preimage of the conformal annulus $\Omega \setminus \overline{\Omega}$, as the figure below illustrates:



When we straighten, this allows us to conclude that the solution to the Beltrami equation at time 0, ψ_0^N, converges locally uniformly to the identity on $\mathbb{C}$ (as does $(\psi_0^N)^{-1}$) as $N \to \infty$. Further, as P is conjugate to an irrational rotation on the Siegel disc, denoted U, we have that there exists a subsequence of iterates $\{P^{\circ n_k}\}$ which converges locally uniformly to the identity on U. This eventually allows us to approximate the map f with a polynomial composition on a (large) compact subset of the Siegel disc, along the lines of the diagram below:

$$\begin{tikzcd} {} \arrow[r, "P^{\circ n_k} \approx Id"] \arrow[d, "\psi_0^{n_k} \approx Id"'] & {} \arrow[d, "F = f \text{ on } U"] \\ {} \arrow[r, "Q_{n_k}^{n_k} \approx f"'] & {} \end{tikzcd}$$

The Polynomial Implementation Lemma approximates a single univalent function on a compact subset of the Siegel disc U, and can thus be seen as a weak version of our main result (Theorem 1.1).

4 Phase I

For any $R > 0$, define $U_R := \{z \in \mathbb{C} \,:\, \rho_U(0, z) < R\}$. Choose $0 < r_0 < R_0 \leq \frac{\pi}{2}$ and restrict ourselves to $R \in [r_o, R_0]$. The upper bound $\frac{\pi}{2}$ is chosen so that the disc U_R as well as its image under any conformal mapping whose domain of definition contains U is star-shaped (see [6] Lemma 2.10 for details).

Lemma 4.1 *(Phase I) Let P_λ, U_λ, κ, P, and U be as above. Let $R_0 > 0$ be given and let U_{R_0} also be as above. Then, for all $\epsilon > 0$, and $N \in \mathbb{N}$, if $\{f_i\}_{i=0}^{N+1}$ is a collection of mappings with $f_i \in \mathcal{S}$ for $i = 0, 1, 2, \cdots, N+1$ with $f_0 = f_{N+1} = \mathrm{Id}$, there exists $\kappa_0 = \kappa_0(R_0) > 0$, $M_N = M_N(\epsilon, N) \in \mathbb{N}$, such that for all $\kappa \geq \kappa_0$, there exists a $(17+\kappa)$-bounded finite sequence of quadratic polynomials $\{P_m\}_{m=1}^{(N+1)M_N}$ such that for all $1 \leq i \leq N+1$,*

1. $Q_{iM_N}(0) = 0$,
2. Q_{iM_N} *is univalent on* U_{2R_0},
3. $Q_{iM_N}(U_{2R_0}) \subset U_{4R_0}$,
4. $\rho_U(f_i(z), Q_{iM_N}(z)) < \epsilon$ *on* U_{2R_0},
5. $\|Q_{iM_N}^\natural\|_{U_{R_0}} \leq 7$.

Using the Polynomial Implementation Lemma multiple times, we construct a single polynomial composition that approximates each f_i at a prescribed iterative time on U_{R_0}.

We remark that one can view Phase I as a weak form of our main theorem in that it allows to to approximate finitely many elements of $\mathcal{S}$ with arbitrary accuracy using a finite composition of quadratic polynomials. Phase I is thus intermediate in strength between the Polynomial Implementation Lemma and our main result.

5 Phase II

Let $G(z)$ be the Green's function for P, and fix $h_0 \in (0, \infty)$, where in practice h_0 is an upper bound on the values of the Green's function G (again, see [6] Lemma 2.10 for details).

Lemma 5.1 *(Phase II) There exist an upper bound $\tilde{\epsilon}_1 > 0$ and a function $\delta : (0, \tilde{\epsilon}_1] \to (0, \frac{r_0}{4})$, with $\delta(x) \to 0_+$ as $x \to 0_+$, both of which depend on the choice of κ, h_0, and the bounds r_0, R_0 for R, such that, for all $\epsilon_1 \in (0, \tilde{\epsilon}_1]$, there exists an upper bound $\tilde{\epsilon}_2 > 0$, depending on ϵ_1, κ, h_0, and r_0, R_0, such that, for all $\epsilon_2 \in (0, \tilde{\epsilon}_2]$, $R \in [r_0, R_0]$, and all functions $\mathcal{E}$ univalent on U_R with $\mathcal{E}(0) = 0$ and $\rho_U(\mathcal{E}(z), z) < \epsilon_1$ for $z \in U_R$, there exists a $(17 + \kappa)$-bounded composition $\mathbf{Q}$ of quadratic polynomials such that*

1. *$\mathbf{Q}$ is univalent on a neighborhood of $\overline{U_{R-\delta(\epsilon_1)}}$,*
2. $$\rho_U(\mathbf{Q}(z), \mathcal{E}(z)) < \epsilon_2, \qquad \text{for all} \quad z \in U_{R-\delta(\epsilon_1)},$$
3. *$\mathbf{Q}(0) = 0$.*

Because we will be using the Polynomial Implementation Lemma repeatedly to construct our polynomial composition, we need to interpolate functions outside of $\mathcal{K}$. However, $\mathcal{E}$ is only defined on a subset of U and hence we will need to map a suitable subset of U on which $\mathcal{E}$ is defined to a domain which contains $\mathcal{K}$, and correct the conjugated error using the Polynomial Implementation Lemma. The trick to doing this is that we choose our subset of U such that the mapping to blow this subset up to U can be expressed as a high iterate of a map which is defined on the whole of the Green's domain V_h, where $V_h := \{z \in \mathbb{C} \ : \ G(z) < h\}$, not just on this subset. This will allow us to interpolate outside $\mathcal{K}$. Further, we will then use the Polynomial Implementation Lemma once more to 'undo' the conjugating map and its inverse. The two key considerations in the proof are as follows:

- Controlling loss of domain (measured by the function δ in the statement above).
- Showing that the error in our polynomial approximation to the function $\mathcal{E}$ (measured by the quantity ϵ_2 above) is mild.

In controlling loss of domain, one main difficulty will arise in converting between the hyperbolic metrics of different domains, U and V_{2h}, where $V_{2h} := \{z \in \mathbb{C} \;:\; G(z) < 2h\}$. The techniques for controlling loss of domain will be the so-called Target and Fitting Lemmas (see [6] for full details), and the fact that $(V_{2h}, 0) \to (U, 0)$ in the Carathéodory topology as $h \to 0_+$. To approximate $\mathcal{E}$ itself rather than this conjugated version, we then wish to 'cancel' the conjugacy, so 'During' is bookended by 'Up' and 'Down' portions, in which we apply the Polynomial Implementation Lemma to get polynomial compositions which are arbitrarily close to the conjugating map and its inverse.

One of the most crucial features of the proof is viewing the expanding portion (in 'Up') of the conjugacy as a dilation in the correct conformal coordinates. Define ψ_{2h} to be the unique conformal map $\psi_{2h} : V_{2h} \to \mathbb{D}$, normalized so that $\psi_{2h}(0) = 0$ and $\psi'_{2h}(0) > 0$. We let $\tilde{V}_{2h}$ be the largest conformal disc (measured using the hyperbolic metric of V_{2h}) about 0 such that $\tilde{V}_{2h} \subset U_R$. The expanding map in the conjugacy is then defined to be the unique conformal map $\varphi_{2h} : \tilde{V}_{2h} \to V_{2h}$, normalized so that $\varphi_{2h}(0) = 0$ and $\varphi'_{2h}(0) > 0$. As $\tilde{V}_{2h}$ is round in the conformal coordinates of V_{2h}, i.e., $\psi_{2h}(\tilde{V}_{2h})$ is a disc (about 0), we may view φ_{2h} as a composition of many smaller dilations. These (conjugated) dilations can be chosen so small so that they are defined on (a neighbourhood of) $\overline{V_h}$, which in particular contains the filled Julia set $\mathcal{K}$, while the dilated $\overline{V}_h$ is still inside V_{2h}. This is what allows us to approximate a small dilation using the Polynomial Implementation Lemma (Lemma 3.1) and eventually approximate φ_{2h}. The 'Down' portion of the conjugacy turns out to be easier. See [6] for full details.

6 Proof of Main Result

We use an inductive argument to prove the following lemma, from which our main result (Theorem 1.1) follows quickly:

Lemma 6.1 *There exists a sequence of quadratic polynomials $\{P_m\}_{m=1}^{\infty}$ such that the following hold:*

1. *$\{P_m\}_{m=1}^{\infty}$ is $(17+\kappa_0)$-bounded,*
2. *$Q_m(\overline{U}_{\frac{1}{20}}) \subset U_{\frac{1}{10}}$ for infinitely many m,*
3. *For all $f \in \mathcal{S}$, there exists a subsequence $\{Q_{m_k}\}_{k=1}^{\infty}$ such that $Q_{m_k} \rightrightarrows f$ on $U_{\frac{1}{20}}$ as $k \to \infty$.*

As $\mathcal{S}$ is a normal family, we can approximate it locally on $\mathbb{D}$ with a finite net of functions $\{f_i\}_{i=0}^{N+1} \in \mathcal{S}$, with $f_0 = f_{N+1} = Id$. The base case of the induction begins with an application of Phase I (Lemma 4.1) (which we can view as preceded by a trivial application of Phase II since there is as yet no error to correct) in which we approximate a finite net of functions from $\mathcal{S}$ on a reasonably large relatively compact subset of the Siegel disc U. In the induction step, we then apply Phase II to correct the error in the previous approximation to arbitrary (finer) accuracy (i.e., the accuracy of this correction does not depend on the error in the Phase I step preceding it) on a slightly smaller subset of U. As the process is repeated ad infinitum, we must ensure that this loss of domain eventually stabilizes. Crucial to controlling the loss of domain is the fact that as the size of the incoming error (ϵ_1 in the statement of Phase II) goes to zero, so too does the loss of domain (measured by the quantity $\delta(\epsilon_1)$ in the statement) that occurs in a Phase II application. The error in our new polynomial approximation (ϵ_2 in the statement of Phase II) must then pass through the subsequent Phase I in the course of which we also pick up a new error. However, due to the estimate on the hyperbolic derivative in Part *5.* of the statement of Phase I (Lemma 4.1) and the fact that the error bound ϵ in Phase I is as small as desired, the total error and thus the loss of domain in the Phase II for the next step can be made as small as we wish. Continuing in this way, we are eventually able to prove our main result.

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