Projective Hausdorff measure for Cantor sets

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1 Introduction

Throughout this note, F denotes a non-empty compact subset of the Euclidean space $\mathbf{R}^n$.

Let us first recall the definition of Hausdorff measure. Given $s > 0$ and $0 < \delta < 1$, put

$$\mathcal{H}^s_\delta(F) = \inf\{\sum_{U\in\mathcal{U}} |U|^s \mid \mathcal{U} : \delta\text{-cover of } F\},$$

where $|U|$ denotes the diameter of U. A δ-cover $\mathcal{U}$ is a countable collection of subsets $U \subset \mathbf{R}^n$ with $|U| \leq \delta$ covering F. The value $\mathcal{H}^s_\delta(F)$ increases monotonically as δ decreases, and the limit

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F) \quad \in [0,\infty]$$

is called the s-dimensional Hausdorff measure of F.

We introduce a variety of Hausdorff measure, and here is the motivation. The area of a disk of diameter d is $\frac{\pi}{4}d^2$. How about the area of the moon in the sky? The diameter of the moon is about 0.5 degree, thus the area of the moon is $\frac{\pi}{4}0.5^2$ deg^2. This 0.5 degree is not the actual diameter 3500km of the moon but the ratio of the diameter to the distance 380000km between the moon and the earth. Now we introduce the following definition.

Let $p \in \mathbf{R}^n$ be an arbitrary point. Given $s > 0$, $0 < \delta < 1$, and $\rho > 0$, we put

$$\mathcal{H}^s_{\delta,\rho}(F,p) = \inf\{\sum_{U\in\mathcal{U}} \left(\frac{|U|}{d(U,p)}\right)^s \mid \mathcal{U} : \delta\text{-cover of } F - B_\rho(p)\},$$

where $d(U,p)$ is the distance between U and p, and $B_\rho(p)$ the open ρ-ball centered at p. The value $\mathcal{H}^s_{\delta,\rho}(F,p)$ increases monotonically both as δ decreases and as ρ decreases. We call the limit

$$\mathcal{H}^s(F,p) = \lim_{\delta\to 0}\lim_{\rho\to 0} \mathcal{H}^s_{\delta,\rho}(F,p) \quad \in [0,\infty]$$

the s-dimensional projective Hausdorff measure of F at p. We explore its properties.

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2 Basic properties

It is easy to prove the following.

Proposition 1. *Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a similarity transformation with similarity ratio r. Then, for any $s > 0$, $t \geq 0$, and at any point $p \in \boldsymbol{R}^n$, we have*

$$\mathcal{H}^s(f(F), f(p)) = \mathcal{H}^s(F, p).$$

It is also easy to show the following.

Proposition 2. *Suppose that $p \notin F$, and choose positive numbers m, M such that*

$$m < d(x, p) < M$$

for any $x \in F$. Then we have

$$\frac{\mathcal{H}^s(F)}{M^s} \leq \mathcal{H}^s(F, p) \leq \frac{\mathcal{H}^s(F)}{m^s}.$$

The next corollary shows that the projective Hausdorff measure $\mathcal{H}^s(F, p)$ is of interest only when $s = \dim_H(F)$, the Hausdorff dimension.

Corollary 3. *The following hold at any point $p \in \boldsymbol{R}^n$.*

1. *If $s < \dim_H(F)$, then $\mathcal{H}^s(F, p) = \infty$.*
2. *If $\dim_H(F) < s$, then $\mathcal{H}^s(F, p) = 0$.*

This holds even if $p \in F$.

3 As a function of p

Proposition 4. *If $\mathcal{H}^s(F) < \infty$, the projective Hausdorff measure $\mathcal{H}^s(F, p)$ is continuous with respect to p in the complement of F.*

In fact, for any $\epsilon > 0$, $\mathcal{H}^s(F, p)$ is uniformly continuous in the complement of the ϵ-neighborhood of F.

Example 5. *For the interval $[m, M] \subset \boldsymbol{R}$ $(0 \leq m < M)$, the following holds*

$$\mathcal{H}^1([m, M], 0) = \int_m^M \frac{dx}{x} = \log M - \log m.$$

In particular, we have

$$\mathcal{H}^1([0, 1], 0) = \infty.$$

Example 6. *For $0 < r < 1$, consider*

$$F = \bigcup_{n=0}^{\infty} [r^n, r^n M_n] \cup \{0\} \subset \boldsymbol{R},$$

where M_n $(n = 0, 1, \dots)$ are sequence of numbers satisfying $M_n > 1$ and $rM_n < 1$. We have

$$\mathcal{H}^1(F, 0) = \sum_{n=0}^{\infty} \mathcal{H}^1([r^n, r^n M_n], 0)$$
$$= \sum_{n=0}^{\infty} \mathcal{H}^1([1, M_n], 0)$$

We can choose M_n so that

$$\mathcal{H}^1(F, 0) < \infty.$$

This example shows that $\mathcal{H}^s(F, p)$ is not continuous with respect to p in general, for, $r^n \to 0$ as $n \to \infty$ whereas, by Example 5,

$$\mathcal{H}^1(F, r^n) = \infty \qquad \text{and} \qquad \mathcal{H}^1(F, 0) < \infty.$$

4 The Cantor set

Let us investigate the case of Cantor set. The Cantor set C_a is the limit set of the iterated function system generated by

$$f_0(x) = ax \qquad \text{and} \qquad f_1(x) = ax + (1 - a),$$

where $0 < a < 1/2$. It is well-known that the Hausdorff dimension of C_a is

$$s = \dim_H C_a = -\log 2 / \log a,$$

and its s-dimensional Hausdorff measure is

$$\mathcal{H}^s(C_a) = 1.$$

Theorem 7. *For $s = \dim_H C_a$, we have $\mathcal{H}^s(C_a, p) = \infty$ if and only if $p \in C_a$*

Theorem 8. *As a $[0, \infty]$-valued function, the projective Hausdorff measure $\mathcal{H}^s(C_a, p)$ for $s = \dim_H(C_a)$ is continuous with respect to $p \in \boldsymbol{R}$.*

The projective Hausdorff measure $\mathcal{H}^s(C_a, p)$ for $s = \dim_H(C_a)$ is continuous with respect to p if $p \notin C_a$ by Proposition 4.

When $p \in C_a$, we can show that

$$\lim_{q \to p} \mathcal{H}^s(C_a, q) = \infty.$$

Theorem 9. *The projective Hausdorff measure of C_a for $s = \dim_H(C_a)$ satisfies*

$$\mathcal{H}^s(C_a, f_i(p)) \geq \mathcal{H}^s(C_a, p) \qquad (i = 0, 1)$$

for any p. The inequality is strict unless $p \in C_a$, in which case $\mathcal{H}^s(C_a, f_i(p)) = \mathcal{H}^s(C_a, p) = \infty$.

Proof. Since C_a is the disjoint union of $f_0(C_a)$ and $f_1(C_a)$, we have

$$\mathcal{H}^s(C_a, f_i(p)) = \mathcal{H}^s(f_0(C_a), f_i(p)) + \mathcal{H}^s(f_1(C_a), f_i(p)).$$

By Proposition 1, $\mathcal{H}^s(C_a, p)$ is equal to the first term on the right hand side if $i = 0$, and to the second term if $i = 1$. □

In conclusion, the projective Hausdorff measure $\mathcal{H}^s(C_a, p)$ for $s = \dim_H(C_a)$ is a continuous function of p (Theorem 8), increases monotonically when we apply f_0 and f_1 to p (Theorem 9), and $\mathcal{H}^s(C_a, p) = \infty$ if and only if $p \in C_a$ (Theorem 7).

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