A Note on Solutions of Real Options Model with a Quadratic Flow Function*

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1 Introduction

This paper investigates solutions of real options model in which the flow function has a quadratic form. Real options model with a quadratic flow function is often used in a price maker's investment problem, for example, Cournot competition in the duopoly setting. In such models, we use the inverse demand function and derive the optimal production quantities. Of course, we have a quadratic form of the system of value-matching and smooth-pasting conditions when we derive the optimal investment threshold since the flow function has a quadratic form.

Then, we should have simple questions. Are there any possibility of existence of two different thresholds? Is the solution obtained strictly optimal? What is characteristics of thresholds? In this paper, we demand the answer to these questions by investigating Verification Theorem. By so doing, we can find conditions for unique threshold and ensure optimality of the solution.

The simplest example for a quadratic flow function is given in the following example.

Example 1 (Monopoly) We consider that a firm produce an item in a monopoly. Inverse demand function is given by

$$p(x,q) = x - \eta q, \tag{1}$$

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t W_t, \qquad X_0 = x, \tag{2}$$

where q is the production quantity, X_t is the demand shock modeled by a geometric Brownian motion, μ is the instantaneous expected growth rate of X_t, σ (> 0) is the instantaneous volatility of X_t, and W_t is a standard Brownian motion. Profit flow function is given by

$$\pi(x,q) = (p(x,q) - C)q = -\eta q^2 + (x - C)q, \tag{3}$$

where C is the production cost. Then, we find the optimal production quantity, by the first order condition $\partial\pi/\partial q = 0$,

$$q^* = \frac{x - C}{2\eta} \tag{4}$$

and the optimal profit flow

$$\pi^*(x) = \frac{x^2 - 2Cx + C^2}{4\eta}. \tag{5}$$

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2 Problem and Solution

2.1 Settings

We consider the following investment problem:

$$F(x) = \sup_{\tau \in \mathcal{T}} J(x, \tau) = J(x, \tau^*), \tag{6}$$

$$J(x, \tau) = \mathbb{E}\left[\int_{\tau}^{\infty} \mathrm{e}^{-\rho t} f(X_t) \mathrm{d}t - \mathrm{e}^{-\rho\tau} I, \right], \tag{7}$$

where ρ (> 0) is the discount rate, I is the investment cost, $f(x)$ is the quadratic profit flow function

$$f(x) = b_2 x^2 + b_1 x + b_0, \qquad b_2 > 0, \tag{8}$$

and τ is the stopping time

$$\tau = \inf\{t > 0 : X_t \in D\} \tag{9}$$

associated with the stopping region D. We find the optimal stopping time τ^* over the set of admissible stopping times $\mathcal{T}$.

Then, we denote some useful algebra:

$$\begin{aligned} J(x, \tau) &= \mathbb{E}\left[\mathrm{e}^{-\rho\tau}\left(\int_{\tau}^{\infty} \mathrm{e}^{-\rho(t-\tau)} f(X_t) \mathrm{d}t - I\right)\right] \\ &= \mathbb{E}\left[\mathrm{e}^{-\rho\tau}\left(\frac{b_2 X_\tau^2}{\rho - 2\mu - \sigma^2} + \frac{b_1 X_\tau}{\mu - \rho} + \frac{b_0}{\rho} - I\right)\right] \\ &=: \mathbb{E}\left[\mathrm{e}^{-\rho\tau}\left(a_2 X_\tau^2 + a_1 X_\tau + a_0 - I\right)\right] \\ &=: \mathbb{E}\left[\mathrm{e}^{-\rho\tau} g(X_\tau)\right], \end{aligned} \tag{10}$$

that is, we define

$$a_n = \frac{b_n}{\rho - (n\mu + n(n-1)\sigma^2/2)} =: \frac{b_n}{\rho - \delta_n}, \qquad n = 0, 1, 2. \tag{11}$$

We assume

$$\rho > \delta_n, \qquad n \leq 2 \tag{12}$$

for conversion of Eq. (10), which results in $a_2 > 0$.

2.2 Solution

In order to solve Eq. (6), we can utilize well-known HJB equation:

$$\max\{\mathcal{L}F(x),\ g(x) - F(x)\} = 0, \tag{13}$$

where $\mathcal{L}$ is the differential operator

$$\mathcal{L} = \frac{1}{2}\sigma^2 x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu x \frac{\mathrm{d}}{\mathrm{d}x} - \rho. \tag{14}$$

We consider a twice differential function satisfying Eq. (13) ϕ and the stopping time maximizing $\mathcal{L}\phi$

$$\tau^* = \inf\{t > 0 : X_t \geq X^*\}, \tag{15}$$

where X^* is the optimal investment threshold to be solved.

Then, we have transformed HJB equation

$$\begin{cases} \mathcal{L}\phi(x) = 0, & \text{for } x < X^*, \\ \phi(x) = g(x), & \text{for } x \geq X^*. \end{cases} \tag{16}$$

By value-matching and smooth-pasting conditions, we find

$$\phi(x) = \begin{cases} A_1 x^{\beta_1}, & \text{for } x < X^*, \\ a_2 x^2 + a_1 x + a_0 - I, & \text{for } x \geq X^*, \end{cases} \tag{17}$$

$$A_1 = \frac{a_2 (X^*)^2 + a_1 X^* + a_0 - I}{(X^*)^{\beta_1}}, \tag{18}$$

$$X^* = \frac{-(\beta_1 - 1)a_1 + \sqrt{(\beta_1 - 1)^2 a_1^2 - 4\beta_1(\beta_1 - 2)a_2(a_0 - I)}}{2(\beta_1 - 2)a_2}, \tag{19}$$

where $\beta_1 > 1$ is a positive root of the characteristic equation.

3 Questions

Question 1 Are there any possibility of existence of two different thresholds?

Answer: First, we show Eq. (19) is positive. We can easily prove

$$\rho > \delta_n \Rightarrow \beta_1 > n, \tag{20}$$

so the denominator of Eq. (19) is positive. Signs of square root in Eq. (19) is divided into

$$\underbrace{(\beta_1 - 1)^2 a_1^2}_{+} - \underbrace{4\beta_1(\beta_1 - 2)a_2}_{+}(a_0 - I), \tag{21}$$

therefore, $I > a_0$ is required so that this is positive. If $a_1 \leq 0$ then the numerator of Eq. (19) is positive. If $a_1 > 0$ then

$$\begin{aligned} &((\beta_1 - 1)^2 a_1^2 - 4\beta_1(\beta_1 - 2)a_2(a_0 - I)) - ((\beta_1 - 1)a_1)^2 \\ &= 4\beta_1(\beta_1 - 2)a_2(I - a_0) > 0 \end{aligned} \tag{22}$$

and the numerator of Eq. (19) is positive. Therefore, Eq. (19) can be an investment threshold.

Second, we show another solution

$$X_* = \frac{-(\beta_1 - 1)a_1 - \sqrt{(\beta_1 - 1)^2 a_1^2 - 4\beta_1(\beta_1 - 2)a_2(a_0 - I)}}{2(\beta_1 - 2)a_2} \tag{23}$$

is negative. If $a_1 \geq 0$ then the numerator of Eq. (23) is negative. If $a_1 < 0$ then

$$\begin{aligned}&(-(\beta_1 - 1)a_1)^2 - ((\beta_1 - 1)^2 a_1^2 - 4\beta_1(\beta_1 - 2)a_2(a_0 - I)) \\ &= 4\beta_1(\beta_1 - 2)a_2(a_0 - I) < 0 \end{aligned} \tag{24}$$

and the numerator of Eq. (23) is negative. Therefore, Eq. (23) cannot be an investment threshold and Eq. (19) is the unique solution and the investment threshold. □

Question 2 Is the solution (19) strictly optimal?

Answer: We can utilize Verification Theorem in stochastic control theory.

Theorem 1 (Verification Theorem) *Suppose ϕ is a twice differential function satisfying HJB equation of the stopping problem* (6)–(9). *Then,*

1. *For any stopping time τ,*

$$\phi(x) \geq J(x, \tau). \tag{25}$$

2. *Given the following stopping time exists,*

$$\tau^* = \arg\max\{\mathcal{L}\phi(x)\}, \tag{26}$$

the solution of HJB equation coincides the value function, that is,

$$\phi(x) = F(x), \tag{27}$$

and τ^* is the optimal solution of the problem.

In other words, as long as Verification Theorem holds, solving HJB equation gives the optimal solution of the stopping problem. Now, we have to check the inequalities of HJB equation:

$$\begin{cases} \phi(x) > g(x), & \text{for } x < X^*, \\ \mathcal{L}\phi(x) < 0, & \text{for } x \geq X^*. \end{cases} \tag{28}$$

For $x < X^*$, we have

$$\psi_1(x) := g(x) - \phi(x) = a_2x^2 + a_1x + a_0 - I - A_1x^{\beta_1}, \tag{29}$$

$$\psi_1(0) = a_0 - I < 0, \tag{30}$$

$$\psi_1(X^*) = 0, \tag{31}$$

$$\psi_1'(x) = 2a_2x + a_1 - \beta_1A_1x^{\beta_1 - 1}, \tag{32}$$

$$\psi_1'(0) = a_1, \tag{33}$$

$$\psi_1'(X^*) = g'(X^*) - \phi'(X^*) = 0, \tag{34}$$

$$\psi_1''(x) = 2a_2 - \beta_1(\beta_1 - 1)A_1x^{\beta_1 - 2}. \tag{35}$$

Eq. (35) ensures $\psi_1(x)$ has unique inflection point. Therefore, $\psi_1(x) < 0$.

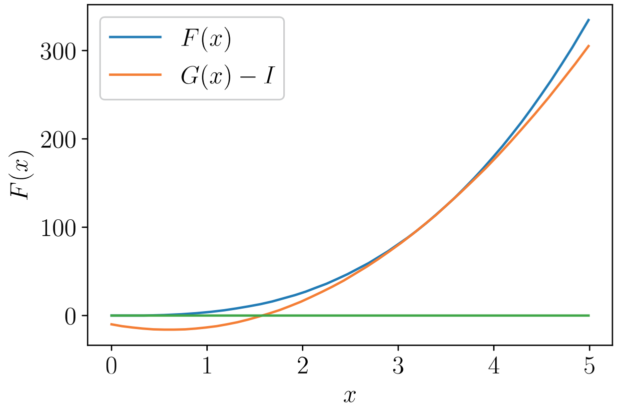

Figure 1: Value function for the base case

For $x \geq X^*$, we have

$$\psi_2(x) := \mathcal{L}\phi(x) = \mathcal{L}g(x) = -b_2x^2 - b_1x - b_0 + \rho I, \tag{36}$$

$$\psi_2(X^*) = \mathcal{L}\phi(X^*) < 0, \tag{37}$$

$$\psi_2(\infty) = -\infty < 0, \tag{38}$$

$$\psi_2'(x) = -2b_2x - b_1. \tag{39}$$

If $b_1 \geq 0$ then we find $\psi_2'(x) < 0$. If $b_1 < 0$ then

$$\psi_2'(\infty) = -\infty < 0, \tag{40}$$

$$\psi_2''(x) = -2b_2 < 0. \tag{41}$$

Therefore, if $\psi_2'(X^*) < 0$ holds then $\psi_2(x) < 0$. □

Question 3 What is characteristics of thresholds?

Answer: To this end, we implement some numerical analyses. We choose the basic parameter set:

$$\mu = 0,\ \sigma = 0.2,\ \rho = 0.1,\ b_2 = 1,\ b_1 = -2,\ b_0 = 1,\ I = 20,$$

then we have

$$a_2 = 16.67,\ a_1 = -20,\ a_0 = 10,\ a_0 - I = -10,\ \beta_1 = 2.79,\ X^* = 3.35.$$

Figure 1 and 2 show value function and sensitivity analysis of X^* w.r.t. σ for the base case, respectively. Figure 3 and 4 show behavior of $\psi_1(x)$ and $\psi_2(x)$ for the base case, respectively.

Second, we choose $b_1 = 0.5$, then we have $a_1 = 5$, $X^* = 1.15$ and others are same with the base case. Figure 5 and 6 show value function and sensitivity analysis of X^* w.r.t. σ for $b_1 = 0.5$, respectively. Figure 7 and 8 show behavior of $\psi_1(x)$ and $\psi_2(x)$ for $b_1 = 0.5$, respectively.

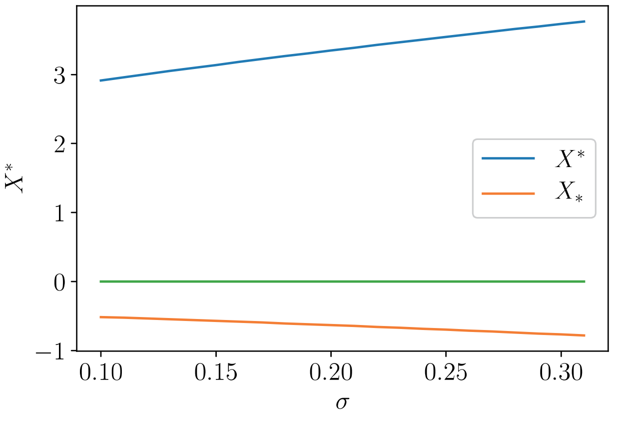

Figure 2: Sensitivity analysis of X^* w.r.t. σ for the base case

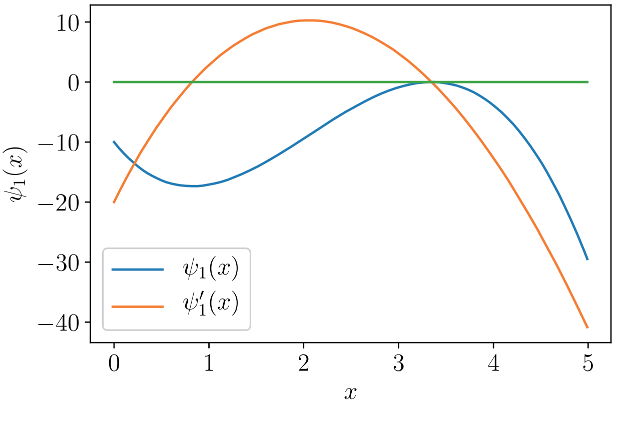

Figure 3: Behavior of $\psi_1(x)$ for the base case

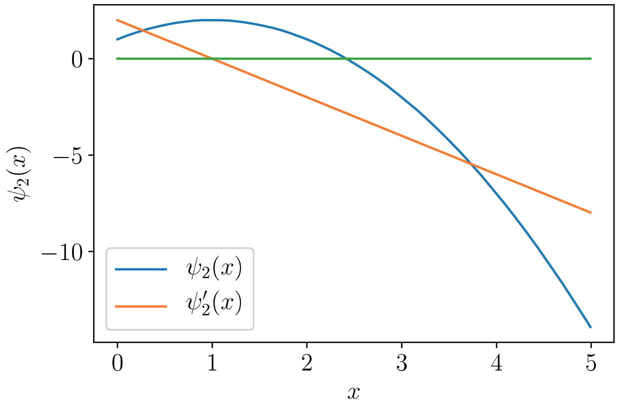

Figure 4: Behavior of $\psi_2(x)$ for the base case

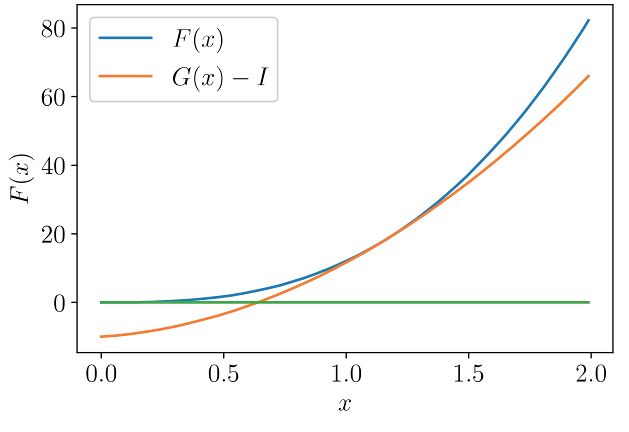

Figure 5: Value function for $b_1 = 0.5$

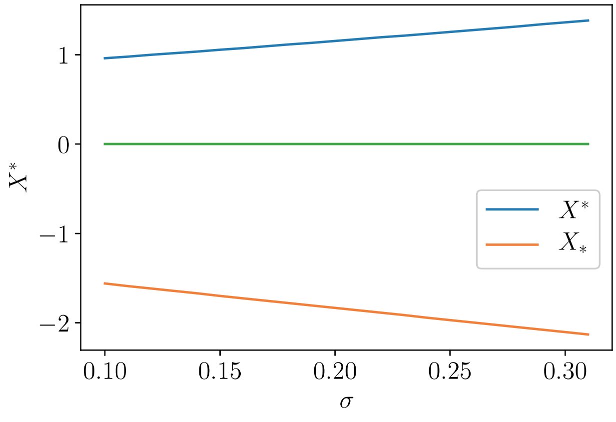

Figure 6: Sensitivity analysis of X^* w.r.t. σ for $b_1 = 0.5$

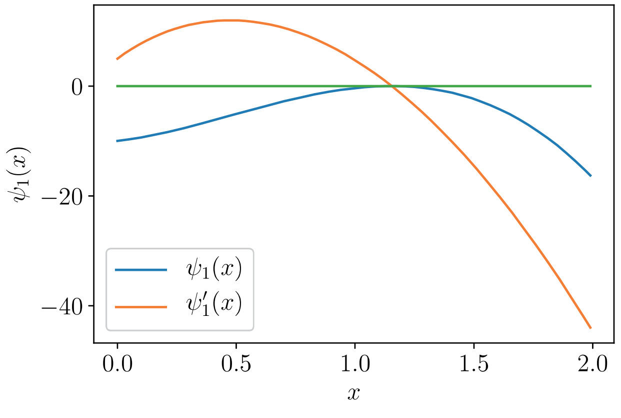

Figure 7: Behavior of $\psi_1(x)$ for $b_1 = 0.5$

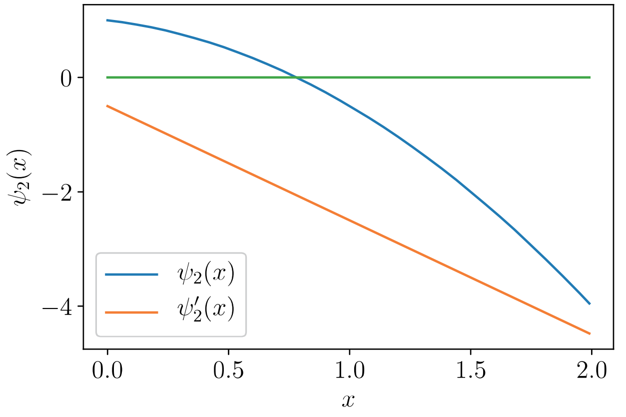

Figure 8: Behavior of $\psi_2(x)$ for $b_1 = 0.5$

Next, we choose $b_0 = -1$, then we have $a_0 = -10$, $X^* = 4.22$ and others are same with the base case. Figure 9 and 10 show value function and sensitivity analysis of X^* w.r.t. σ for $b_0 = -1$, respectively. Figure 11 and 12 show behavior of $\psi_1(x)$ and $\psi_2(x)$ for $b_0 = -1$, respectively.

Finally, we choose $I = 5$, then we have $a_0 - I = 5$, $X^* = 2.25$ and others are same with the base case so that the requirement $I > a_0$ is no longer satisfied. Figure 13 and 14 show value function and sensitivity analysis of X^* w.r.t. σ for $I = 5$, respectively. Figure 15 and 16 show behavior of $\psi_1(x)$ and $\psi_2(x)$ for $I = 5$, respectively. We can see that X_* is positive in Figure 14 and $\psi_1(x)$ is positive in Figure 15 which means the solution is not optimal. Therefore, the requirement $I > a_0$ must be satisfied in order to the optimality.

□

4 Conclusion

In this paper, we have investigated solutions of real options model in which the flow function has a quadratic form. We have shown the solution is unique and optimal as long as $a_2 > 0$ and $I > a_0$ are satisfied. Additionally, $a_1 > 0$ and $a_0 < 0$ could be allowed. Also we find investment threshold is larger when σ is larger which means the same characteristics as the case with a linear flow function.

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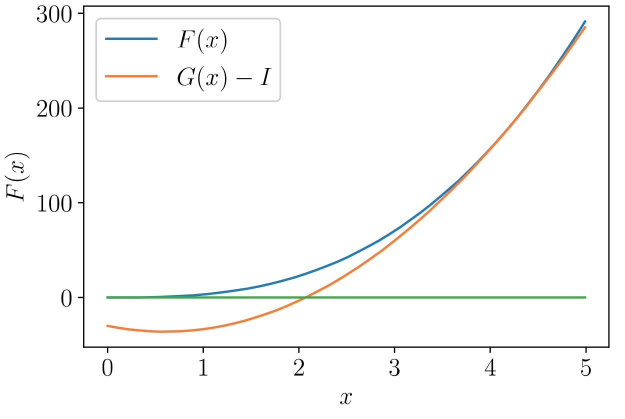

Figure 9: Value function for $b_0 = -1$

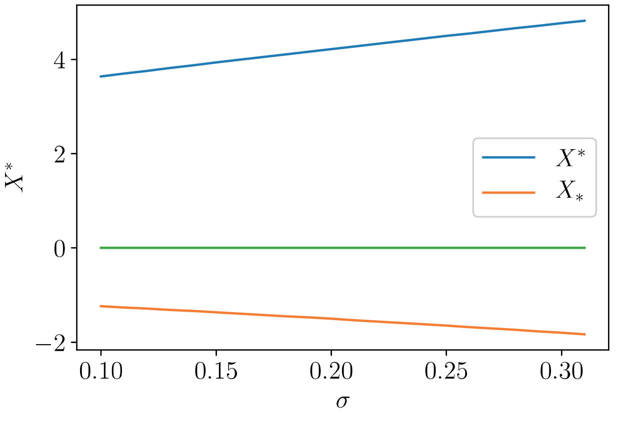

Figure 10: Sensitivity analysis of X^* w.r.t. σ for $b_0 = -1$

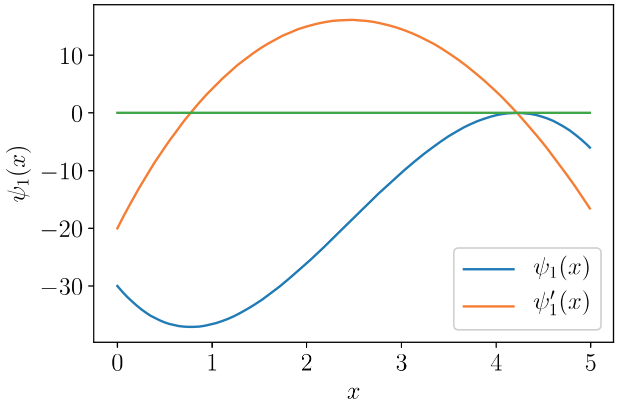

Figure 11: Behavior of $\psi_1(x)$ for $b_0 = -1$

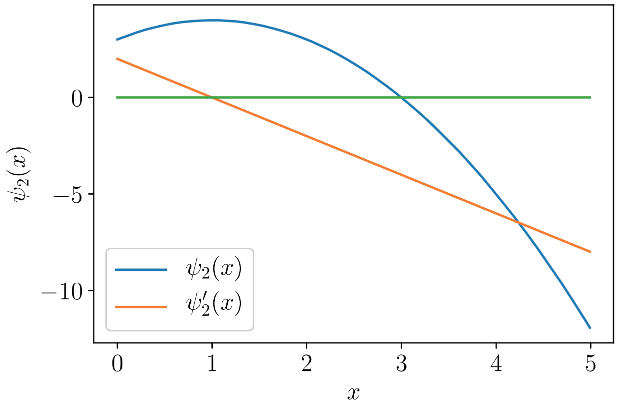

Figure 12: Behavior of $\psi_2(x)$ for $b_0 = -1$

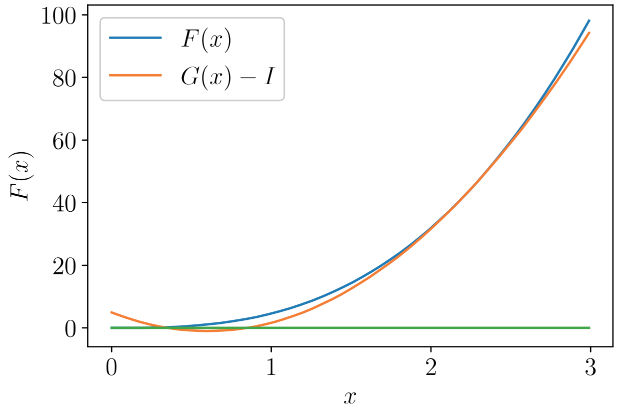

Figure 13: Value function for $I = 5$

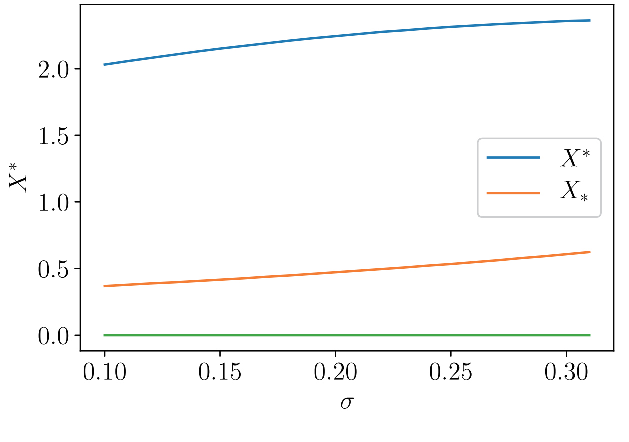

Figure 14: Sensitivity analysis of X^* w.r.t. σ for $I = 5$

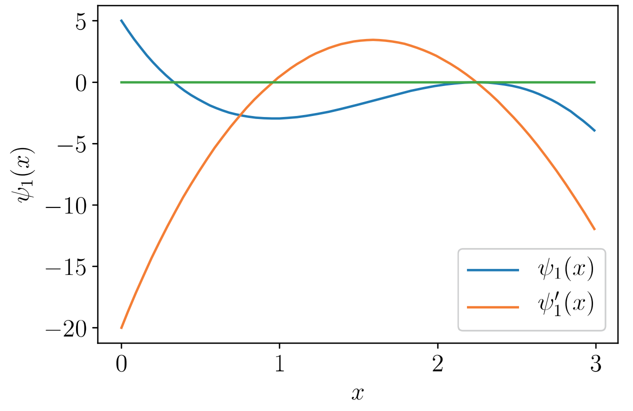

Figure 15: Behavior of $\psi_1(x)$ for $I = 5$

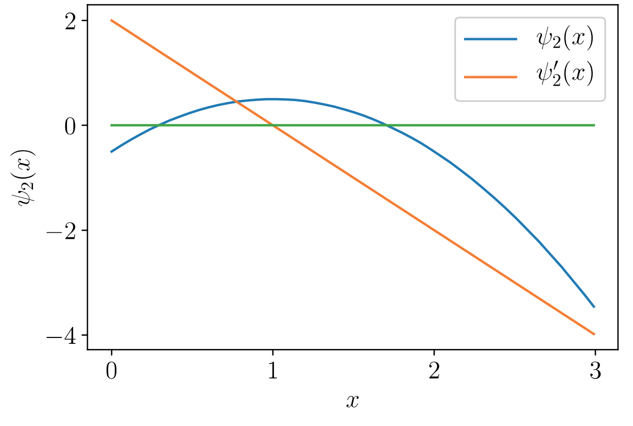

Figure 16: Behavior of $\psi_2(x)$ for $I = 5$