Received: 25 June 2021 | Accepted: 20 September 2021

DOI: 10.1002/pamm.202100226

Darcy's law for evolving microstructure

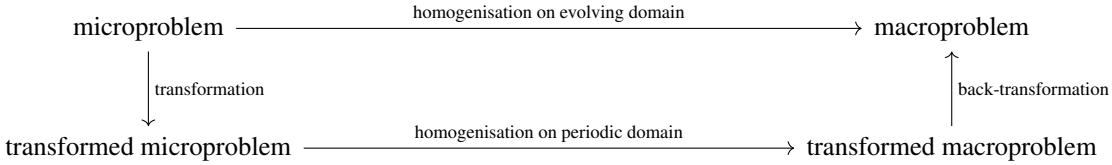
David Wiedemann[1,*] and **Malte A. Peter**[1]

[1] Institute of Mathematics, University of Augsburg, 86135 Augsburg

A Darcy law for evolving microstructure is derived by homogenisation of Stokes flow. We transform the Stokes equations from the locally evolving domain onto a periodic reference domain. There, we pass to the homogenisation limit by employing the method of two-scale convergence. After transforming the limit back to its original two-scale domain, we obtain a Darcy Law for evolving microstructure. This Darcy Law does not only capture the space- and time-dependent permeability, but also includes how the change of the porosity induces pressure. In order to give uniform a priori estimates, we derive a Korn-type inequality for this two-scale transformation method as well as a family of ε-scaled operators $\mathrm{div}_\varepsilon^{-1}$.

1 Introduction

Saturated fluid flow through porous media plays an important role in many chemical, biological and geological applications. There, the Darcy law is widely used as the physical model. However, if the pore structure is evolving in time, the standard Darcy law does not constitute a sufficient model anymore. Therefore, we derive an appropriate model for such processes by homogenising the Stokes flow in porous media evolving in time. In order to pass to the homogenisation limit rigorously, we use the two-scale transformation method (cf. (1)), which was proposed in [2] and substantially developed further in [3]. First, we transform the problem to a periodic domain. There, we derive a new Korn-type inequality for the two-scale transformation method as well as a sequence of ε-scaled operators $\mathrm{div}_\varepsilon^{-1}$, which are right inverses to the divergence operator. Using these results, we obtain uniform a priori estimates which allow to pass to the homogenisation limit in the periodic substitute problem and to derive a two-pressure Stokes system. After a back-transformation, we can derive a new Darcy law for evolving microstructure with a time- and space-dependent permeability tensor. Moreover, the new Darcy law captures compression and suction effects arising from the change of the porosity by a new source term for the pressure.



(1)

2 Microscopic model

On the microscopic scale, we consider the steady state Stokes equations in a time-dependent domain $\Omega_\varepsilon^p(t)$ on a time interval S, where $\Omega_\varepsilon^p(t)$ represents the pore space and $\Omega_\varepsilon^s(t) = \Omega \setminus \overline{\Omega_\varepsilon^p(t)}$ the solid space of a porous medium $\Omega \subset \mathbb{R}^N$ at $t \in S$:

$$-\varepsilon^2 \nu \,\mathrm{div}(2e(v_\varepsilon)) + \nabla p_\varepsilon = f, \qquad \mathrm{div}(v_\varepsilon) = 0 \qquad \text{in } \Omega_\varepsilon^p(t). \tag{2}$$

Thereby, v_ε denotes the fluid velocity, p_ε the pressure field, $e(v_\varepsilon)$ the symmetric gradient of v_ε and f a source term. For the boundary condition, we distinguish between the interface of the pores with the solid part and the outer boundary. At the interface $\Gamma_\varepsilon(t) := \partial\Omega_\varepsilon^p(t) \cap \partial\Omega_\varepsilon^s(t)$, we use a no-slip boundary condition. In our case of an evolving domain, this means that the fluid velocity is equal to the velocity of the boundary's evolution, i.e. $v_\varepsilon = v_{\Gamma_\varepsilon}$. Since the evolution of the domain can cause a change of the total pore volume, we have to enable fluid in- and outflow at the outer boundary. Therefore, we use the pressure boundary condition $p_\varepsilon n - \varepsilon^2 \nu 2e(v_\varepsilon)n = gn$ for a given pressure g at the outer boundary $\partial\Omega_\varepsilon^p(t) \setminus \overline{\partial\Omega_\varepsilon^s(t)}$, where n is the outward unit normal.

3 Transformation to a periodic reference domain

We assume that there exists a reference cell $Y = [0,1]^N$ with a reference open pore space $Y^p \subset Y$ and a reference solid space $Y^s = Y \setminus \overline{Y^p}$ such that $Y_\#^p := \mathrm{int}(\bigcup_{k\in\mathbb{Z}^N} k + \overline{Y^p})$ is connected and has a Lipschitz boundary. We denote the interface of the reference cell by $\Gamma := \partial Y^p \cap \partial Y^s$. For a macroscopic domain Ω, we define the ε-periodic reference domains by $\Omega_\varepsilon^p := \Omega \cap \varepsilon Y_\#^p$ and the corresponding interfaces by $\Gamma_\varepsilon := \partial\Omega_\varepsilon^p \cap \partial Y_\#^p$. Then, we assume that $\Omega_\varepsilon^p(t) \subset \Omega$ is a locally periodic domain, i.e. there

* Corresponding author: e-mail david.wiedemann@math.uni-augsburg.de, phone +49 821 598 5474, fax +49 821 598 2193

exists a family of locally periodic transformations $\psi_\varepsilon : S \times \Omega_\varepsilon^p \to \Omega$ such that $\Omega_\varepsilon^p(t) = \psi_\varepsilon(t, \Omega_\varepsilon^p)$ and $\Gamma_\varepsilon(t) = \psi_\varepsilon(t, \Gamma_\varepsilon)$ (for the concept of locally periodic transformations, see [3]). We define $\Psi_\varepsilon(t,x) = D_x\psi_\varepsilon(t,x)$, $J_\varepsilon(t,x) = \det(\Psi_\varepsilon(t,x))$ and $A_\varepsilon = J_\varepsilon \Psi_\varepsilon^{-1}$ and denote the transformation with ψ_ε by $\hat{\cdot}_\varepsilon$, i.e. $\hat{v}_\varepsilon(t,x) = v_\varepsilon(t, \psi_\varepsilon(t,x))$, $\hat{f}_\varepsilon(t,x) = f(t, \psi_\varepsilon(t,x)), \dots$. In order to state the following weak form for (2), we substitute $\hat{w}_\varepsilon := \hat{v}_\varepsilon - \partial_t \psi_\varepsilon$ and $\hat{q}_\varepsilon := \hat{p}_\varepsilon - \hat{g}_\varepsilon$ and transform (2) via ψ_ε from the locally evolving domain onto the periodic reference domain $S \times \Omega_\varepsilon^p$. The weak form of the transformed Stokes equations is given by: Find $(w_\varepsilon(t), p_\varepsilon(t)) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ such that

$$\int_{\Omega_\varepsilon} \nu\varepsilon^2 A_\varepsilon(t,x) e_{\Psi_\varepsilon}(\hat{w}_\varepsilon(t,x)) : \nabla\varphi(x) - \hat{q}_\varepsilon(t,x) \operatorname{div}(A_\varepsilon(t,x)\varphi(x)) dx = \int_{\Omega_\varepsilon} J_\varepsilon(t,x) \hat{f}_\varepsilon(t,x) \cdot \varphi(x) dx$$
$$- \int_{\Omega_\varepsilon} A_\varepsilon^\top(t,x) \nabla \hat{g}_\varepsilon(t,x) \cdot \varphi(x) dx - \int_{\Omega_\varepsilon} \nu\varepsilon^2 A_\varepsilon(t,x) e_{\Psi_\varepsilon}(\partial_t \psi_\varepsilon(t,x)) : \nabla\varphi(x) dx, \tag{3}$$

$$\operatorname{div}(A_\varepsilon(t,x) v_\varepsilon(t,x)) = -\operatorname{div}(A_\varepsilon(t,x) \partial_t \psi_\varepsilon(t,x)) \tag{4}$$

for any $\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$, where e_{Ψ_ε} denotes the transformed symmetric gradient, i.e. $e_{\Psi_\varepsilon}(v) := \Psi_\varepsilon^{-\top} \nabla v + (\Psi_\varepsilon^{-\top} \nabla v)^\top$ and $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon^p) := \{v \in H^1(\Omega_\varepsilon^p) \mid v|_{\Gamma_\varepsilon} = 0\}$.

4 A priori estimates

In order to derive uniform a priori estimates for w_ε and p_ε, we show the following uniform Korn-type inequality for the two-scale transformation method:

$$\left|\left|\Psi_\varepsilon^{-\top}(t) \nabla\varphi + (\Psi_\varepsilon^{-\top}(t) \nabla\varphi)^\top\right|\right|_{L^2(\Omega_\varepsilon^p)} \geq C \left|\left|\nabla\varphi\right|\right|_{L^2(\Omega_\varepsilon^p)} \text{ for every } w_\varepsilon \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon^p). \tag{5}$$

However, we still cannot use the existing theory for the derivation of the a priori estimates because of the inhomogeneous div-condition (4). Instead, we construct a family of ε-scaled operators $\operatorname{div}_\varepsilon^{-1} : L^2(\Omega_\varepsilon^p) \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon^p)$ such that

$$\operatorname{div} \circ \operatorname{div}_\varepsilon^{-1} = id_{L^2(\Omega_\varepsilon^p)}, \qquad \varepsilon \left|\left|\nabla \operatorname{div}_\varepsilon^{-1}(f)\right|\right|_{L^2(\Omega_\varepsilon^p)} \leq C \left|\left|f\right|\right|_{L^2(\Omega_\varepsilon^p)} \quad \text{for every } f \in L^2(\hat{\Omega}_\varepsilon). \tag{6}$$

These estimates allow us to derive the uniform a priori estimate

$$\left|\left|w_\varepsilon(t)\right|\right|_{L^2(\Omega_\varepsilon^p)} + \varepsilon \left|\left|\nabla w_\varepsilon(t)\right|\right|_{L^2(\Omega_\varepsilon^p)} + \left|\left|q_\varepsilon(t)\right|\right|_{L^2(\Omega_\varepsilon^p)} \leq C \tag{7}$$

for the solutions $(w_\varepsilon(t), q_\varepsilon(t))$ of (3)–(4).

5 Homogenisation and back-transformation

Using the two-scale convergence method and modifying the approach of [1], we pass to the limit $\varepsilon \to 0$ in (3)–(4). Thereby, we use the fact that the transformations ψ_ε two-scale converge to a limit transformation $\psi_0 : S \times \Omega \times Y^p \to Y$, which describes the pore $Y_x^p(t) := \psi_0(t,x,Y^p)$ at a time $t \in S$ and a macroscopic point x in Ω with its porosity $\Theta(t,x) = |Y_x^p(t)|$. In the limit, we obtain a two-pressure Stokes system on the cylindrical domain $S \times \Omega \times Y^p$ with a microscopic incompressibility condition and a macroscopic compressibility condition. We transform this system with ψ_0 back on its actual two-scale coordinates $\{(t,x,y) \in S \times \Omega \times Y \mid y \in Y_x^p(t)\}$. Then, we separate the y-dependence and obtain the following Darcy law:

$$v(t,x) = \frac{1}{\nu} K(t,x)(f(t,x) - \nabla p(t,x)), \quad \operatorname{div}(v(t,x)) = -\frac{d}{dt}\Theta(t,x) \quad \text{in } \Omega, \tag{8}$$

with boundary condition $p = p_b$ on $\partial\Omega$ and permeability tensor $(K(t,x))_{ij} = \int_{Y_x^p} u_i(t,x,y) \cdot e_j dy$, where e_j is the jth unit vector and u_i is the y-periodic solution of

$$-\Delta_y u_i(t,x,y) + \nabla_y \pi(t,x,y) = e_i, \qquad \operatorname{div}_y(u_i(t,x,y)) = 0 \qquad \text{in } Y_x^p(t), \qquad\qquad v = 0 \qquad \text{in } Y \setminus \overline{Y_x^p(t)}. \tag{9}$$

By combining both equations of (8) and the pressure boundary condition, we obtain a well-posed second-order elliptic equation for the pressure p. The new term $\frac{d}{dt}\Theta(t,x)$ constitutes an additional source term for the pressure and captures the compression and suction of the fluid arising from the change of the porosity. After solving the equation for the pressure, the fluid velocity can be computed explicitly.

Acknowledgements Open access funding enabled and organized by Projekt DEAL.

References

[1] G. Allaire, in: Progress in partial differential equations: calculus of variations, applications, edited by C. Bandle et al., Pitman Research Notes in Mathematics Series 267 (Longman Higher Education, New York, 1992), pp. 109–123.
[2] M. A. Peter, C. R. Mecanique **335**, 357–362 (2007).
[3] D. Wiedemann, The two-scale transformation method, arXiv (2021).

