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# Functional renormalization group approach to the two dimensional Bose gas

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**Abstract.** We investigate the small frequency and momentum structure of the weakly interacting Bose gas in two dimensions using a functional renormalization group approach. The flow equations are derived within a derivative approximation of the effective action up to second order in spatial and temporal variables and investigated numerically. The truncation we employ is based on the perturbative structure of the theory and is well described as a renormalization group enhanced perturbation theory. It allows to calculate corrections to the Bogoliubov spectrum and to investigate the damping of quasiparticles. Our approach allows to circumvent the divergences which plague the usual perturbative approach.

## 1. Introduction

The interacting Bose gas is one of the oldest models of quantum field theory. Its excitation spectrum in three dimensions is qualitatively correctly described by the standard Bogoliubov approximation. At small wave vectors the quasiparticles have a Goldstone-mode character with energy linear in momentum while at large wave vectors, where interactions are not dominant, the dispersion approaches the quadratic dispersion of free bosons. The Bogoliubov approximation is the simplest one to obey the Hugenholtz-Pines relation [1] which reads  $\Sigma_{11}(0,0) - \Sigma_{12}(0,0) = \mu$ , where  $\Sigma_{11}(0,0)$  is the normal and  $\Sigma_{12}(0,0)$  the anomalous part of the self-energy evaluated at zero energy and momentum and  $\mu$  is the chemical potential. However, as was shown later by Nepomnyashchy and Nepomnyashchy [2], the exact self-energy further obeys  $\Sigma_{12}(0,0) = 0$ . This relation is violated in the Bogoliubov approximation and also in the Beliaev theory [3]. One puzzling aspect is that the linear dispersion of the low energy excitations in both approximations derive from the incorrect assumption of a finite value for  $\Sigma_{12}(0,0)$ . As shown in [4, 5] the correct structure of the self-energies cannot be obtained from a finite order perturbative analysis. Quite recently, the renormalization group (RG) technique has been successfully employed to investigate the small frequency and momentum structure of the self-energy [6] and recovered the earlier results in [4, 5]. These results have recently been shown to emerge quite naturally also from the nonperturbative or functional renormalization group (FRG) [7, 8, 9] which is perhaps the simplest approach which can account both for the vanishing of the anomalous self-energy and the emergence of the Goldstone-mode. These FRG studies concentrated mainly on the thermodynamic properties of the Bose gas. The dynamical behavior of the Bose gas, e. g. the properties of the self-energies away from the asymptotic Goldstone regime, have so far not been

addressed in a consistent approximation which obeys the Nepomnyashchy relation. Here we briefly summarize how the FRG approach can be employed for this task and demonstrate how it can be used to investigate the dynamics at finite frequency and momenta. For details, we refer to Ref. [10].

The two dimensional situation is perhaps the most interesting case since interaction effects are stronger than in three dimension (they are even stronger in one dimension but here specialized techniques such as bosonization can be applied [11]). In the two dimensional situation, the  $s$ -wave scattering length  $a$ , which is the usual expansion parameter in the three dimensional Bose gas, vanishes. This introduces a strong renormalization of the quasiparticle structure [12]. We shall therefore concentrate below on the two dimensional situation at zero temperature [10]. While we have not attempted this yet, in principle our approach can be directly applied also at finite temperatures. The numerical evaluation would become however more involved.

## 2. Functional renormalization group equations

The starting point is the usual action of the weakly interacting Bose gas of particles with mass  $m$  at zero temperature

$$S[\psi^\dagger, \psi] = \int_{\vec{x}, \tau} \psi^\dagger_{\vec{x}, \tau} \left[ \partial_\tau - \frac{1}{2m} \partial_{\vec{x}}^2 - \mu \right] \psi_{\vec{x}, \tau} + \frac{u_0}{2} \int_{\vec{x}, \tau} \psi^\dagger_{\vec{x}, \tau} \psi^\dagger_{\vec{x}, \tau} \psi_{\vec{x}, \tau} \psi_{\vec{x}, \tau}, \quad (1)$$

where  $\int_{\vec{x}, \tau} = \int d\tau \int d^D x$ . We assume a local contact interaction of strength  $u_0$ . We shall assume that the  $U(1)$ -invariance of the action (1) is broken in the ground state such that the fields  $\psi^\dagger$  and  $\psi$  acquire nonvanishing vacuum averages,

$$\langle \psi^\dagger_K \rangle \equiv \langle \psi_K \rangle = \delta_{K,0} \sqrt{n_0}.$$

Here,  $n_0$  is the density of the Bose condensate and  $K = (\vec{k}, i\omega)$  denotes the  $D + 1$ -dimensional energy-momentum. To apply the FRG formalism, we switch to the generating functional of irreducible vertices  $\Gamma(\bar{\phi}, \phi)$  which is related to the Legendre transform of the generating functional of connected Green functions  $\mathcal{G}_c[j, \bar{j}]$  with respect to the averaged fields  $(\bar{\phi}, \phi)^T = (\delta_j, \delta_{\bar{j}})^T \mathcal{G}_c[j, \bar{j}]$  and satisfies the exact flow equation [13]

$$\partial_\Lambda \Gamma_\Lambda(\Phi) = -\frac{1}{2} Tr \left\{ \partial_\Lambda \mathbf{R}_\Lambda \left( \frac{\delta^2 \Gamma_\Lambda}{\delta \Phi \delta \Phi} - \mathbf{G}_{0,\Lambda}^{-1} \right)^{-1} \right\}, \quad (2)$$

where we introduced the vector field  $\Phi = (\bar{\phi}, \phi)^T$ . The block-diagonal inverse free propagator matrix has elements  $[\mathbf{G}_{0,\Lambda}^{-1}]_{K,K'} = \delta_{K,K'} (\pm i\omega - k^2/(2m) + \mu - R_\Lambda(k))$ . The dependence of  $\Gamma_\Lambda$  on the infrared cutoff  $\Lambda$  is generated through the additive regulator  $R_\Lambda(k)$  which is the matrix element of block-diagonal matrix  $\mathbf{R}_\Lambda$ . We choose the Litim regulator which has the form [14]

$$R_\Lambda(k) = (1 - \delta_{k,0}) (2mZ_\Lambda)^{-1} (\Lambda^2 - k^2) \Theta(\Lambda^2 - k^2), \quad (3)$$

where  $Z_\Lambda$  is the wave function renormalization. It is of advantage to introduce the longitudinal field  $\phi_K^l$  and the transversal field  $\phi_K^t$ , which are related to the original fields  $\psi_K$  and  $\psi_K^\dagger$  through  $\phi_K^l = 2^{-1/2}(\psi_K^\dagger + \psi_{-K})$  and  $\phi_K^t = 2^{-1/2}i(\psi_K - \psi_{-K}^\dagger)$ . The new fields are introduced in such a way that only the longitudinal component has a nonvanishing vacuum average,  $\langle \phi_K^l \rangle = \delta_{K,0} \sqrt{2n_0}$  while  $\langle \phi_K^t \rangle = 0$ . In this basis, and using a lowest order local-potential-approximation [15], one can now derive in a straightforward manner (we used the formalism developed in Refs.[16, 17]) the flow equations for the order parameter [7, 8]

$$\partial_\Lambda n_\Lambda = -\frac{1}{2} \int_K \left( \dot{G}_\Lambda^{tt}(K) + 3\dot{G}_\Lambda^{ll}(K) \right), \quad (4)$$

as well as the flow of both diagonal vertex functions and of the mixed vertex function

$$\begin{aligned} \partial_\Lambda \Gamma_{tt}(Q) &= -u_\Lambda \int_K \left( \dot{G}_{ll}(K) - \dot{G}_{tt}(K) \right) \\ &\quad - 2n_\Lambda u_\Lambda^2 \int_K \left( \dot{G}_{ll}(K) G_{tt}(Q-K) + \dot{G}_{tt}(K) G_{ll}(Q-K) + 2\dot{G}_{lt}(K) G_{tl}(Q-K) \right), \end{aligned} \quad (5)$$

$$\begin{aligned} \partial_\Lambda \Gamma_{ll}(Q) &= -u_\Lambda \int_K \left( \dot{G}_{tt}(K) + 3\dot{G}_{ll}(K) \right) \\ &\quad - 2n_\Lambda u_\Lambda^2 \int_K \left( 9\dot{G}_{ll}(K) G_{ll}(Q-K) + \dot{G}_{tt}(K) G_{tt}(Q-K) + 6\dot{G}_{lt}(K) G_{tl}(Q-K) \right), \end{aligned} \quad (6)$$

$$\begin{aligned} \partial_\Lambda \Gamma_{lt}(Q) &= -n_\Lambda u_\Lambda^2 \int_K \left\{ \left( \dot{G}_{tt}(K) + 3\dot{G}_{ll}(K) \right) G_{lt}(Q-K) \right. \\ &\quad \left. + \dot{G}_{lt}(K) \left( G_{tt}(Q-K) + 3G_{ll}(Q-K) \right) \right\}, \end{aligned} \quad (7)$$

where  $G_{ij}$  with  $i, j = l, t$  are the matrix elements of the full propagator matrix  $\mathbf{G}$  and  $\dot{G}_{ij}$  are the single scale propagators which represent the matrix elements of the matrix

$$\dot{\mathbf{G}}_\Lambda = -\mathbf{G}_\Lambda [\partial_\Lambda \mathbf{R}_\Lambda] \mathbf{G}_\Lambda. \quad (8)$$

Flow equations for the interaction  $u_\Lambda$  and factor  $Z_\Lambda$  can be extracted from Eqs.(5-7).

### 2.1. Calculation of spectral density

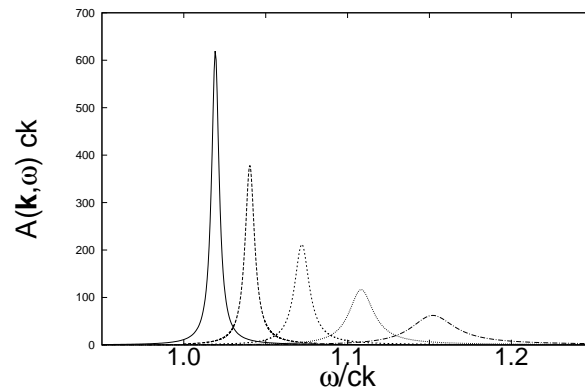
The FRG approach was first applied to calculate the full momentum dependence of non-homogeneous correlation functions in Ref. [18], where a sharp cutoff procedure was employed. Later approaches [15, 19] proposed different schemes of how correlation functions at finite momenta can be combined with local-potential-approximation (LPA) type truncations of the action. We shall follow here an approximation similar to the one used in Ref. [15], which has the most direct route to the self-energy in the symmetry broken state. It provides a feasible way to solve the flow of the vertex functions (5-7) and to obtain information about the bosonic self-energies and the dynamics. Using the analytical continuation technique from Ref. [20], we can finally compute the spectral density from which we can extract e. g. the quasiparticle dispersion and damping [10]. The damping arises from quasiparticles which decay into pairs of quasiparticles (Beliaev damping). In three dimensions the damping is proportional to  $k^5$  [21, 22], whereas in two dimension the damping is proportional to  $k^3$  [23, 24]. While perturbative approaches to the self-energy are plagued by divergences (see e.g. [22]), our calculation yields self-energies which are infrared convergent. Figure (1) shows a representative scan of the spectral density at a finite momentum. A more detailed discussion can be found in [10].

### 2.2. Hard core interaction limit: Schick's results

Let us make a connection of our RG approach to the results obtained by Schick for hard core Bose particles in two dimension [12]. Solving analytically the initial flow equations for large  $mu_\Lambda \gg 1$  one finds that the coupling constant decreases (initially) like  $1/\log(\Lambda_0/\Lambda)$ , in accordance with the vanishing of the two dimensional  $s$ -wave scattering length [25]. Furthermore, we find that there is a crossover energy scale which separates the Goldstone regime from the high momentum regime with quadratic dispersion. This energy scale is given by

$$\mu_* = -4\pi n_0 / [m \log(a^2 n_0)], \quad (9)$$

where  $a$  is the radius of the hard core bosons. Eq. (9) is exactly the result obtained via diagrammatic resummation by Schick [12]. The hard core limit is thus correctly described as a limiting case of our model which describes particles with a finite interaction and thus a soft core.



**Figure 1.** Spectral density for a value  $k/k_c = 0.2, 0.3, 0.4, 0.5, 0.6$  where  $k_c = 2mc$  is the crossover scale of the spectrum and  $c$  is the sound velocity of the Bose gas.

### 3. Conclusions

In summary, we have presented an extension of the FRG approach to the interacting Bose gas which allows calculations of its dynamical properties. In contrast to the conventional resummation techniques, our approach yields self-energies which are consistent with the  $U(1)$ -symmetry which impose the constraints on the self-energies at zero momentum and frequency  $\Sigma_{11}(0, 0) = \mu$  and  $\Sigma_{12}(0, 0) = 0$ .

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