ERGODIC THEORY METHODS IN BERNOULLI CONVOLUTIONS FOR ALGEBRAIC PARAMETERS AND SELF-AFFINE MEASURES

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science and Engineering

2021

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Word count 55075

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The University of Manchester

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December 8, 2021

In this thesis we look at several problems in the fractal geometry of iterated function systems. In particular Bernoulli convolutions for algebraic parameters and self-affine measures. In chapter 3, for a given hyperbolic number, we define (and prove existence) a natural measure supported on its spectrum and prove the presence of a local structure in this measure. We also discuss links to the fractal geometry of Bernoulli convolutions, which is also the main motivation for the project. In chapter 4, using ideas from the previous chapter we investigate the absolute continuity of Bernoulli convolutions for hyperbolic parameters. We reduce the absolute continuity to an ergodic theory problem involving cocycles over domain exchange transformations. In the next chapter we study the multifractal spectrum of planar self-affine measures under assumptions on their orthogonal projections. We also assume that the respective set of matrices is dominated. In the last chapter we investigate a toy problem motivated by our attempt to study Bernoulli convolutions for Pisot numbers of high algebraic degree. The problem is related to sparse matrices associated to Pisot numbers. The results provide some unexpected intuitions related to the initial question.

Declaration

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Acknowledgements

Firstly I would like express my sincere gratitude to my supervisor Tom Kempton not only for the beautiful mathematics he shared with me but also for generally being supportive and helpful during my PhD. I want to also thank Charles Walkden for helping making this happen. I need to extent my thanks to everyone in the Analysis&Dynamics group in Manchester for always being friendly and open to discussions. The same is true also for so many people in the general area of dynamics and fractal geometry who also gave me the valuable opportunity to participate in important and exciting research events. All those wonderful people made my PhD experience beautiful. Special thanks to Antti Käenmäki for an exciting collaboration.

Next I would like to thank the analysis students in Manchester, for all the good times we shared, my office-mates and many other friends in ATB for making up an awesome friendly environment. It made my daily life in ATB so much brighter. I should also mention that those already there when I started, made me really feel welcome. I am also thankful to the University of Manchester for awarding me a scholarship.

Finally I would like to deeply thank my parents for their unconditional love and support during my PhD.

Chapter 1

Introduction

Iterated function systems (IFS) are one of the main aspects of fractal geometry. Bernoulli convolutions and self-affine measures are examples of measures generated by IFSes which, despite their elegant simplicity, are not yet fully understood. In this thesis we present new methods in some problems on these systems. These methods are of ergodic theory nature including thermodynamic formalism, skew-products, transition matrices, random matrix products, symbolic dynamics, domain exchange transformations. The Bernoulli convolution ν_β, for $\beta \in (1, 2)$, is the unique probability measure satisfying

$$\nu_\beta = \frac{1}{2}F_0(\nu_\beta) + \frac{1}{2}F_1(\nu_\beta)$$

where $F_i(x) = \beta^{-1}x - i$. The attractor $\mathcal{R}$ of $\{F_0, F_{-1}\}$ is just the closed interval between the fixed points of F_0 and F_{-1}. The main difficulty in understating Bernoulli convolutions comes from that fact that

$$F_0(\mathcal{R}) \cap F_1(\mathcal{R}) \neq \emptyset,$$

as implied by $\beta < 2$. Often in fractal geometry separation conditions are assumed that exclude this type of behaviour. Overlaps in IFS make the fractal geometry especially hard to understand. This is what makes Bernoulli convolutions a useful family of examples as they provide the simplest setting in which overlaps are present. Bernoulli convolutions have been studied since the 1930's but recent exceptional results have renewed interest. Some of these results appeared in [45],[17],[79],[73],[2]. Algebraic numbers are of special importance since if β is a Pisot number then $\dim_H(\nu_\beta) < 1$ (see [38]). Pisot numbers are the only examples known that drop the Hausdorff dimension below 1. Recently it was also proved that $\dim_H(\nu_\beta) = 1$ for transcendental β (see [79]). In the self-affine case we focus on IFSes $\{F_1, ..., F_N\}$ $(N > 1)$ where F_i are affine maps $F_i : \mathbb{R}^d \to \mathbb{R}^d$. We call attractors of such systems self-affine sets. The term self-affine measures is used for natural pushforwads of Bernoulli measures of (Σ, σ) where $\Sigma = \{1, ..., N\}^{\mathbb{N}}$ and σ is the left shift map. In particular they are pushforwards along the function $\pi : \Sigma \to \mathbb{R}^d$ defined by

$$\pi(x_1, x_2, ...) = \lim_{n\to\infty} F_{x_1} \circ ... \circ F_{x_n}(p),$$

where p can be chosen to be any point in $\mathbb{R}^d$ without affecting π.

The main difficulty in self-affine systems is that in most cases the maps F_i contract at different rates for different directions, making the local behaviour of self-affine sets and measures particularly hard to control. Formally, the maps F_i are not conformal, a condition required for a lot of the classical tools which can not be used in this setting. Below we give a brief description of the projects making up this thesis.

1.1 Measures on the Spectra of Algebraic Integers

This project is joint work with Tom Kempton. The two authors had roughly equal contribution to the project. A lot is already understood about Bernoulli convolutions for Pisot parameters. That is because in the Pisot case there is rigid structure present that makes the mathematics simpler. In particular the respective IFS is of finite type. Hyperbolic numbers can be seen as the natural next step to study Bernoulli convolutions of algebraic parameters, beyond the Pisot case. Motivated by problems in Bernoulli convolutions in this chapter we study the following sequence of measures. Assume $\beta \in (1,2)$ is a hyperbolic number. Let μ_n be the countably supported measure on $\mathbb{R}$ defined by

$$\mu_n(\{x\}) = \# \left\{ (a_1, ..., a_n, b_1, ..., b_n) \in \{0,1\}^{2n} : \sum_{i=1}^{n} a_i \beta^{n-i} - \sum_{i=1}^{n} b_i \beta^{n-i} = x \right\}.$$

Let $T_i(x) = \beta x + i$. Notice that the sums in the definition above can be written as $T_{a_n} \circ ... \circ T_{a_1}(0)$ and $T_{b_n} \circ ... \circ T_{b_1}(0)$. So $\mu_n(\{x\})$ counts in how many ways x can be written as $T_{a_n} \circ ... \circ T_{a_1}(0) - T_{b_n} \circ ... \circ T_{b_1}(0)$ where $a_1, ..., a_n, b_1, ..., b_n \in \{0,1\}$. Another way to see μ_n, that hints to Bernoulli convolutions, is as sums of rescaled local measures appearing in

$$\sum_{a_1,...,a_n \in \{0,1\}} \delta_{T_{a_1}^{-1} \circ ... \circ T_{a_n}^{-1}(0)}.$$

The motivation to study these measures are potential applications to Bernoulli convolutions. The measures μ_n are hidden for example in [2] and [53] as well as in chapter 4 of this thesis. Usually in such applications the aim is to prove equidistribution properties for μ_n. We will expand more on applications in chapter

4. We recall that $\beta \in (1,2)$ is called hyperbolic if it is an algebraic integer with no Galois conjugate on the circle. So we can assume that β is an algebraic integer with Galois conjugates $\beta = \beta_1, ..., \beta_d, \beta_{d+1}, ..., \beta_{d+s}$ such that $|\beta_2|, ..., |\beta_d| > 1$ and $|\beta_{d+1}|, ..., |\beta_{d+s}| \in (0,1)$. By considering the Galois conjugates of β we create a multidimensional lift of μ_n that lives on a lattice as we describe below. We set $\bar{T}_i(x_1, ..., x_{d+s}) = (\beta_1 x_1 + i, ..., \beta_{d+s} x_{d+s} + i)$ and define the finitely supported measures $\bar{\mu}_n$ on $\mathbb{C}^n$ by

$$\mu_n(\{x\}) = \# \left\{ (a_1, ..., a_n, b_1, ..., b_n) \in \{0,1\}^{2n} : \bar{T}_{a_1 - b_1} \circ ... \circ \bar{T}_{a_n - b_n}(0) = x \right\},$$

where we should note that $\bar{T}_{a_1-b_1} \circ ... \circ \bar{T}_{a_n-b_n}(0) = \bar{T}_{a_n} \circ ... \circ \bar{T}_{a_1}(0) - \bar{T}_{b_n} \circ ... \circ \bar{T}_{b_1}(0)$. Now the set

$$\{(\beta_1^\kappa, ..., \beta_{d+s}^\kappa) : \kappa \in \{0, ..., d+s-1\}\}$$

can be proven to be independent over the reals so it generates a lattice set

$$\bar{Z} = \left\{ \sum_{\kappa=0}^{d+s-1} a_\kappa (\beta_1^\kappa, ..., \beta_{d+s}^\kappa) : a_0, ..., a_{d+s-1} \in \mathbb{Z} \right\}.$$

It is easy to check that $\bar{T}_i(Z) \subseteq \bar{Z}$ which implies that $\bar{\mu}_n$ lives on $\bar{Z}$. Observe that the maps $\bar{T}_i$ are expanding in coordinates $1, ..., d$ and contracting in coordinates $d+1, ..., d+s$. Motivated by this split in expanding/contracting components we define the following projection maps,

$$\pi_e(x_1, \cdots, x_{d+s}) = (x_1, \cdots, x_d)$$
$$\pi_c(x_1, \cdots, x_{d+s}) = (x_{d+1}, \cdots, x_{d+s}).$$

We prove in theorem 3.1.1 that there is $\lambda > 0$ and a measure $\bar{\mu}$ on $\bar{Z}$ such that

$$\bar{\mu}(\{x\}) = \lim_{n\to\infty} \frac{\bar{\mu}_n(\{x\})}{\lambda^n},$$

for all $x \in \bar{Z}$.

We construct matrices A_{-1}, A_0, A_1 and a vector W, depending only on the number β, so that when $x = T_{x_n} \circ ... \circ T_{x_1}(0)$ then

$$\bar{\mu}_n(\{x\}) = \frac{1}{\lambda^n}(WA_{x_1} \cdot ... \cdot A_{x_n})_1. \tag{1.1}$$

The other entries of $WA_{x_1} \cdot ... \cdot A_{x_n}$ above are equal to the measures of points nearby x, so this vector describes the measure $\bar{\mu}$ locally around x. The main result of this chapter is theorem 3.1.3. Roughly it says that for $v \in \bar{Z}$ and under conditions, when $\pi_c(x)$ and $\pi_c(y)$ are close then $\bar{\mu}(\{x+v\})/\bar{\mu}(\{x\})$ and $\bar{\mu}(\{y+v\})/\bar{\mu}(\{y\})$ tend to be close. In the paper v belongs to a particular set notated as Δ, but the theorem holds more generally by combining translations in Δ. So this tells us in a sense that we understand the way $\bar{\mu}$ evolves as we move to nearby points by looking at the counteractive directions. We believe that this structure is a kind of symmetry that could be exploited to prove equidistribution properties for μ_n. The main idea of the proof is that the approximate position of $\pi_c(x)$ can determine the last few matrices $A_{x_{n-\kappa}}, ..., A_{x_n}$ in equation 1.1, for at least one coding of x. So then we can approximate ratios of the form $\bar{\mu}(\{x+v\})/\bar{\mu}(\{x\})$ by working on the projective space on which the matrices A_i act.

1.2 Absolutely Continuous Bernoulli Convolutions

This project is joint work with Tom Kempton. The author has written most of it while there were challenging points where Kempton contributed. In this chapter we focus on the absolute continuity of Bernoulli convolutions ν_β for hyperbolic β. We link absolute continuity of ν_β to a problem involving a domain exchange transformation. We make the extra assumption that β has another real Galois conjugate of absolute value larger than one. In our context we define domain exchange transformation as follows.

Definition 1.2.1. *Let E be a compact subset of a euclidean space and $T : E \to E$. The map T is call a domain exchange transformation if there are $E_1, ..., E_n$ measurable subsets of E such that following hold.*

- *$\{E_1, ..., E_n\}$ is a partition of E.*
- *The map T is an injection.*
- *If $i \in \{1, ...n\}$ then $T|_{E_i}$ is a translation.*

For a given hyperbolic number β, as above, we construct compact subsets (with non-empty interior) of euclidean spaces $\mathcal{R}$, I which contain zero, a domain exchange transformation $T : D \to D$ where $D = I \times \mathcal{R}$ and a function $f : D \to R^+$ satisfying certain variation conditions. Let π_e and π_c be the projections of $D = I \times \mathcal{R}$ to I and $\mathcal{R}$ respectively. Also for $n \in \mathbb{N}$ we define

$$\omega_n = \sum_{\kappa=0}^{n} \left(\prod_{i=0}^{\kappa-1} \exp\left(f(T^i(0))\right) \right) \delta_{T^\kappa(0)}.$$

The purpose of this construction is theorem 4.1.1 where we essentially claim that under conditions if $\pi_e \omega_n$, once normalised to probability measures, converge to Lebesgue fast enough then ν_β is absolutely continuous.

The exchange of domains T and the measures ω_n come from methods developed in chapter 3. There, β generates a measure $\bar{\mu}$ on a lattice $L \subseteq \mathbb{R}^\kappa$. The construction of sets I and $\mathcal{R}$ implies $D = I \times \mathcal{R} \subseteq \mathbb{R}^{\kappa-1}$ so that $S := \text{int}(D) \times \mathbb{R} \subseteq \mathbb{R}^\kappa$. Here we focus on the set S. We want to study the part of $\bar{\mu}$ that lives on S be moving along the strip-shaped set S. When this process is projected down to $\text{int}(D)$ the exchange of domains T expresses the move from a lattice point to the next as we move along S. The measure ω_n is $\bar{\mu}$ restricted to the first n lattice points and projected down to D. To be more precise we define $\pi_{\text{free}} : \mathbb{R}^\kappa \to \mathbb{R}$ by

$$\pi_{\text{free}}(x_1, ..., x_\kappa) = x_\kappa$$

and $\text{succ}_l : L \cap S \to L \cap S$ by

$$\pi_{\text{free}}(\text{succ}_l(x)) = \min\{\pi_{\text{free}}(y) : y \in L \cap S, \pi_{\text{free}}(y) > \pi_{\text{free}}(x)\}.$$

Now T and succ_l are related by

$$\pi_D \circ \text{succ}_l = T \circ \pi_D$$

where $\pi_D(x_1, ..., x_\kappa) = (x_1, ..., x_{\kappa-1})$. Also it holds that

$$\bar{\mu}(0)\omega_n = \sum_{\kappa=0}^{n} \bar{\mu}(\text{succ}_l^\kappa(0))\delta_{T^\kappa(0)}.$$

We should note that assuming β is a hyperbolic number with Galois conjugates $\beta = \beta_1, ..., \beta_d, \beta_{d+1}, \cdots, \beta_{d+s}, \beta_{d+s+1}$ where $|\beta_1|, ..., |\beta_d| > 1$, $|\beta_{d+1}|, ..., |\beta_{d+s}| < 1$ and $\beta_{d+s+1} \in \mathbb{R} \setminus [-1, 1]$ then

$$L = \left\{ \sum_{i=0}^{d+s} a_i(\beta_1^i, ..., \beta_{d+s+1}^i) : a_0, ..., a_{d+s} \in \mathbb{Z} \right\},$$

which is essentially a multidimensional lift of the spectrum of β. Here we identify $\mathbb{C}$ with $\mathbb{R}^2$ making L a subset of a euclidean space.

1.3 On the Local Dimension Spectrum for Self-Affine Measures

This project is joint work with Tom Kempton and Antti Käenmäki. The three authors have been writing and rewriting each other's texts and they had lively discussions on problem solving. Here we focus on the multifractal formalism of self-affine measures. As it often the case with self-affine measures we use methods from sub-additive thermodynamic formalism.

Definition 1.3.1. *Let ν be a measure on $\mathbb{R}^d$. If the limit*

$$\lim_{r\to 0}\frac{\log \nu(B(x,r))}{\log r}$$

exists we call it the local dimension of ν at x and denote it by $\dim_{loc}(\nu, x)$.

In general the multifractal formalism is concerned with the multifractal spectrum function

$$f(a) = \dim_{\mathrm{H}}\{x \in \mathbb{R}^d : \dim_{\mathrm{loc}}(\nu, x) = a\}.$$

Assume we have an self-affine IFS $\{T_1, ..., T_N\}$ of the form

$$T_i(x) = A_i x + t_i, \quad x \in \mathbb{R}^d,$$

where A_i are $d \times d$ real invertable contractive matrices and $t_i \in \mathbb{R}^d$. Also let π be the associated function that maps $\{1, ..., N\}^{\mathbb{N}}$ to $\mathbb{R}^d$ and μ be a measure on $\{1, ..., N\}^{\mathbb{N}}$. Also for $a \in \{1, ..., N\}^n$ we define $[a]$ to be the set $\{x \in \{1, ..., N\}^{\mathbb{N}} : x(i) = a(i) \quad \text{for} \quad 1 \leqslant i \leqslant n\}$. For $a \in \{1, ..., N\}^n$ we set $A_a = A_{a(1)} \cdot ... \cdot A_{a(n)}$. Below we give a brief description of what is the expected way to express the function f, for $\nu = \pi\mu$, in well behaved situations.

Definition 1.3.2. *Let A be a $d \times d$ matrix and $a_1 \geqslant, ..., \geqslant a_d$ be the singular values of A, that is the eigenvalues of $A^T A$. The singular value function ϕ is defined as*

$$\phi^s(A) = \begin{cases} a_1 ... a_{\kappa-1} a_\kappa^{(s-\kappa+1)}, & \kappa - 1 < s \leqslant \kappa \leqslant d \\ (a_1 ... a_d)^s, & s \geqslant d \end{cases} .$$

We should note that the singular values are the half-lengths of the axes of the ellipsoid $A(D)$ where D is the unit ball. Ideally we expect that the following relation defines a convex function τ of q

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{a \in \{1,...N\}^n} (\phi^{(\tau(q)/(q-1))}(A_a))^{1-q} \mu([a])^q = 0,$$

where we set $\tau(1) = 0$, and that the multifractal spectrum of $\pi\mu$ is given by

$$f(a) = \inf_q (aq - \tau(q)).$$

This is not always true and sometimes refined versions of it are true. Partial results towards this direction where given by Julien Barral and De-Jun Feng in [10] for Lebesgue almost all vectors $t_1, ..., t_N$. This kind of results for randomly chosen parameters are common in the study of self-affine fractals. For example one of the early results was by Kenneth Falconer [25] proving that for almost all choices of translation vectors the dimension of the attractor is given by the value s satisfying

$$1 = \lim_{n\to\infty} \left(\sum_{a \in \{1,...N\}^n} \phi^s(A_a) \right)^{1/n} .$$

We should also mention [8] where similar results proved for fixed translation vectors and almost all $A_1, ...A_N$. A different approach is to study families of self-affine fractals satisfying certain conditions that makes them well behaved. See for example Theorem 1.1 in [27] which is focused on families of self-affine measures satisfying projection properties. Projections give us information about how $\pi\mu$ is distributed on sets of the form $\pi([a])$ for $a \in \{1, ..., N\}^n$ which allows to overcome the obstacle of the almost "degenerate" geometry of $\pi([a])$. Our project in chapter 5 is on this type of approach. We study the multifractal formalism of planar self-affine measures under conditions on their projections.

1.4 Matrices associated to Pisot numbers

Pisot numbers are of special interest in the study of Bernoulli convolutions. This is because Garsia proved in [38] that $\dim_{\mathrm{H}} \nu_\beta < 1$ when β is Pisot and it is conjectured that the inverse is also true. Let $\deg(\beta)$, of an algebraic number β, be the degree of its respective minimal polynomial. This chapter was motivated by an attempt to study $\dim_{\mathrm{H}} \nu_\beta$ when β is a Pisot and $\deg(\beta)$ is high. In particular we wanted to argue that in such cases $\dim_{\mathrm{H}} \nu_\beta$ is close to 1. Ideally we would like to prove that

$$\lim_{n\to\infty} \min\{\dim_{\mathrm{H}}(\nu_\beta) : \deg(\beta) > n\} = 1.$$

As an intermediate step we also considered the question of whether a sequence of Pisot numbers β_n such that $\beta_n \to \phi$ and $\deg(\beta) \to \infty$ satisfies

$$\lim_{n\to\infty} \dim_{\mathrm{H}} \nu_{\beta_n} = 1.$$

There is an advantage in having β_n converging and the number ϕ is chosen because its algebraic properties make the related mathematics especially simple.

The main tool would come from [2]. Let $T_i(x) = \beta x - i$ and

$$S_{\beta,x} = \{T_{\epsilon_n} \circ ... \circ T_{\epsilon_1}(x) : n \in \mathbb{N}, \epsilon_1, ..., \epsilon_n \in \{-1, 0, 1\}\} \cap [-1(\beta - 1), 1(\beta - 1)] \,.$$

We also set $S_{\beta,0} = S_\beta$. Notice that $-1(\beta-1), 1(\beta-1)$ are the fixed points of T_{-1}, T_1 respectively. When β is Pisot and the greedy β-expansion of x is periodic, the set $S_{\beta,x}$ is finite. We denote the elements of $S_{\beta,x}$ by $S^1_{\beta,x} <, ..., < S^{|S_{\beta,x}|}_{\beta,x}$. We define $M_{\beta,x}$ to be the following $|S_{\beta,x}| \times |S_{\beta,x}|$ matrix.

$$M_{\beta,x}(i,j) = \begin{cases} 1/2, \quad T_{-1}(S^i_{\beta,x}) = S^j_{\beta,x} \text{ or } T_1(S^i_{\beta,x}) = S^j_{\beta,x} \\ 1, \quad T_0(S^i_{\beta,x}) = S^j_{\beta,x} \\ 0, \quad \text{otherwise} \end{cases} .$$

Again we set $M_{\beta,0} = M_\beta$. The matrices M_β appear in [2] where it is proven that

$$\dim_{\mathrm{H}}(\nu_\beta) \geqslant \min\left\{1, \frac{\log 2 - \log(\rho(M_\beta))}{\log(\beta)}\right\},$$

providing a lower bound for $\dim_{\mathrm{H}}(\nu_\beta)$. It is observed numerically that when $\deg(\beta)$ is large then a pattern appears in M_β. The plots of such matrices suggest that, as $\deg(\beta)$ increases (and stays bounded away from 2), the set

$$\left\{\left(\frac{i}{|S_\beta|}, \frac{j}{|S_\beta|}\right) \in \mathbb{R}^2 : M_\beta(i,j) \neq 0, \quad 1 \leqslant i, j \leqslant |S_\beta|\right\}$$

looks like a finite approximation of

$$\bigcup_{i=-1}^{1} \{(x, T_i(x)) : |x| < 1/|\beta - 1|\},$$

properly rescaled (this is formalised in definition 6.1.5 and conjecture 4). This is not a total surprise since the matrices are defined as transitions matrices for finite sets closed under the maps T_i but it does suggest equidistribution properties for S_β. Now let for simplicity β_n be a sequence of Pisot numbers converging to the golden ratio and $\deg(\beta_n) \to \infty$. The strategy was to formalise and prove the appearance of the pattern described above and exploit this pattern to understand the limit of the spectral radius proving that $\dim_{\mathrm{H}}(\nu_{\beta_n}) \to 1$. Following this strategy ended up being more difficult than we expected. This is partly because the matrices blow up from the very first steps, making it hard to spot any patterns, and because the spectral properties of sparse matrices are hard to control. For this reason we introduced the matrices $M_{\phi,x}$ as a simplified toy problem where the complexity doesn't come in through β but by changing the starting point x. We always assume that the greedy β-expansion of x is periodic. The result was a proof showing that for x 'relatively typical' two things are true. Firstly, the matrix $M_{\phi,x}$ has very large size and follows the pattern described above. In a formal level it just means that $S_{\phi,x}$ is uniformly equidistributed. Secondly, the spectral radius of $M_{\phi,x}$ is different from what was expected. To be more precise there is $L > 0$ such that for each $\varepsilon > 0$ there is $\delta > 0$ for which

$$d\left(\frac{1}{|S_{\beta,x}|}\sum_{x\in S_{\beta,x}}\delta_x, \frac{1}{\beta-1}\,\mathrm{Leb}\right) < \varepsilon$$

and

$$|\rho(M_{\phi,x}) - L| < \varepsilon$$

when

$$d\left(\frac{1}{n}\sum_{i=0}^{n-1}\delta_{T^i(x)}, \mu\right) < \delta.$$

where T is the beta-expansion map associated to β, μ is its unique absolutely continuous invariant probability measure and d is a natural metric we define between measures. The last inequality above expresses what we mentioned as 'relatively typical'. This ended up being a kind of counterexample for the second part of the strategy. It shows that the pattern described above, on its own, doesn't determine (up to approximation) the spectral radius of a matrix.

Chapter 2

Preliminaries

2.1 Ergodic Theory

Definition 2.1.1. *A pair (X, T) will be called a dynamical system if X is a metric space and T is a measurable mapping from X to itself. A Borel probability measure m on X is called invariant under T iff*

$$m(A) = m(T^{-1}(A))$$

for any Borel set $A \subseteq X$. The probability measure m is called ergodic iff for any Borel set $A \subseteq X$,

$$T^{-1}(A) = A \Rightarrow m(A) \in \{0, 1\}$$

or equivalently

$$T^{-1}(A) \Delta A = 0 \Rightarrow m(A) \in \{0, 1\}.$$

Theorem 2.1.1 (Ergodic theorem)**.** *Let (X, T) be a dynamical system and m an ergodic invariant probability measure of T. If $f \in L^1(X)$ then for m-almost all $x \in X$ we have*

$$\int f dm = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$$

For a proof of the Ergodic theorem see theorem 1.5 in [81]. By applying the ergodic theorem on indicator functions we get the following corollary which shows that invariant measures describe the long term distribution of orbits of T.

Corollary 2.1.1. *Let (X, T) be a dynamical system and m an ergodic invariant probability measure of T. Then for m-almost all $x \in X$ and Borel set $A \subseteq X$ we have*

$$m(A) = \lim_{n\to\infty} \frac{1}{n} \#\{i \in \mathbb{N} : T^i(x) \in A,\ 0 \leqslant i \leqslant n-1\}.$$

Definition 2.1.2. *For a measurable space (X, Σ) and measures μ, ν on Σ, we say that ν is absolutely continuous with respect to μ and write $\nu \ll \mu$ iff*

$$\mu(A) = 0 \Rightarrow \nu(A) = 0$$

for all $A \in \Sigma$. We say that the measures μ and ν are equivalent if $\nu \ll \mu$ and $\mu \ll \nu$.

If μ, ν are σ-finite measures on the measurable space (X, Σ), the Radon-Nikodym theorem ([56], Cor. 7.34) states that $\nu \ll \mu$ iff there exists Σ-measurable $f : X \to [0, \infty)$ such that

$$\nu(A) = \int_A f dm$$

for all $A \in \Sigma$. In this case we write $\nu = f d\mu$.

The following lemma is well known but we include a proof for completeness.

Lemma 2.1.1. *Let μ, ν be invariant probability measures of a dynamical system (X,T) such that μ is ergodic and $\nu \ll \mu$. Then $\mu = \nu$.*

Proof. By the Radon-Nikodym theorem there is a measurable $f : X \to \mathbb{R}$ such that

$$\nu(A) = \int_A f d\mu$$

for all Borel sets $A \subseteq X$. It is enough to prove that $f(x) = 1$ for μ-almost all $x \in X$. Let

$$M = \{x \in X : f(x) < 1\}.$$

Aiming to prove that $\mu(M \backslash T^{-1}(M)) = 0$ we assume, towards a contradiction, that $\mu(M \backslash T^{-1}(M)) \neq 0$. Notice that $\mu(M) = \mu(T^{-1}(M))$ implies $\mu(M \backslash T^{-1}(M)) = \mu(T^{-1}(M) \backslash M)$ so

$$\begin{aligned}
\nu(M) = \int_M f d\mu &= \int_{M \cap T^{-1}(M)} f d\mu + \int_{M \backslash T^{-1}(M)} f d\mu \\
&< \int_{M \cap T^{-1}(M)} f d\mu + \mu(M \backslash T^{-1}(M)) \\
&= \int_{M \cap T^{-1}(M)} f d\mu + \mu(T^{-1}(M) \backslash M)
\end{aligned}$$

and

$$\begin{aligned}
\nu(T^{-1}(M)) = \int_{T^{-1}(M)} f d\mu &= \int_{M \cap T^{-1}(M)} f d\mu + \int_{T^{-1}(M) \backslash M} f d\mu \\
&\geqslant \int_{M \cap T^{-1}(M)} f d\mu + \mu(T^{-1}(M) \backslash M)
\end{aligned}$$

which contradicts $\nu(M) = \nu(T^{-1}(M))$. Hence we have $\mu(M \backslash T^{-1}(M)) = \mu(T^{-1}(M) \backslash M) = 0$ which implies that $\mu(M \Delta T^{-1}(M)) = 0$ which by the ergodicty of μ gives that $\mu(M) \in \{0, 1\}$. Concluding we have that

$$\int_X f d\mu = \nu(X) = 1$$

and either $f(x) < 1$ for μ-almost all $x \in X$ or $f(x) \geqslant 1$ for μ-almost all $x \in X$ which combined imply that $f(x) = 1$ for μ-almost $x \in X$.

□

Definition 2.1.3. *Let X be a metric space, $(\mu_n)_{n \in \mathbb{N}}$ a sequence of Borel measures on X and μ a Borel measure on X. We say that μ is the weak* limit of the sequence $(\mu_n)_{n \in \mathbb{N}}$ iff*

$$\lim_{n \to \infty} \int f d\mu_n = \int f d\mu$$

for all continuous bounded $f : X \to X$.

Remark. *If the metric space X, in the definition above, is compact then the weak* convergence determines a metrizable topology called the weak* topology. In this topology the set of Borel probability measures is compact (see [56], p252, remark 13.14, and p260, Th. 13.29).*

Definition 2.1.4. *Let X be a metric space and μ a Borel measure on X. We say that a Borel set $A \subseteq X$ is a continuity set of μ iff $\mu(\partial A) = 0$.*

The following lemma can be found in [56] as theorem 13.16 in page 253.

Lemma 2.1.2. *Let X be a metric space, $(\mu_n)_{n \in \mathbb{N}}$ a sequence of Borel measures on X and μ a Borel measure on X. Then μ is the weak* limit of the sequence $(\mu_n)_{n \in \mathbb{N}}$ iff*

$$\lim_{n\to\infty} \mu_n(A) = \mu(A)$$

for all continuity sets A of μ.

Definition 2.1.5. *Let (X, Σ_1, m) be a measure space, (Y, Σ_2) a measurable space and $T : X \to Y$ a measurable map. Then the pushforward measure $T(m)$ on Σ_2 is defined to be the one satisfying*

$$T(m)(A) = m(T^{-1}(A))$$

for all $A \in \Sigma_2$.

In ergodic theory sub-additive sequences often arise, so we need the following lemma by Fekete (see [29]and [36]).

Lemma 2.1.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a sub-additive sequence (i.e. $a_{n+m} \leqslant a_n + a_m$ holds). Then*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n},$$

including the possibility that $\lim_{n\to\infty} a_n/n = -\infty$.

We will also refer later to the Wasserstein distance so we include the definition for completeness.

Definition 2.1.6. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$ and $p_1, p_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be such that*

$$p_1(x, y) = x, \quad x, y \in \mathbb{R}^d$$
$$p_2(x, y) = y, \quad x, y \in \mathbb{R}^d.$$

Also we set M to be the set of all Borel probability measures g on $\mathbb{R}^d \times \mathbb{R}^d$ such that $p_1(g) = \mu$ and $p_2(g) = \nu$. The Wasserstein distance $W_1(\mu, \nu)$ between μ and ν is defined as

$$W_1(\mu, \nu) = \inf_{g \in M} \int_{\mathbb{R}^d \times \mathbb{R}^d} ||x - y|| dg(x, y).$$

The intuition behind the definition, in very loose terms, is that we look for the minimum cost for turning μ into ν by moving mass around. The cost for moving a portion of mass is proportional to the size of the mass and the distance covered to move it.

2.2 Thermodynamic Formalism

When we want to understand geometrical dynamical systems we often encode them in symbolic shift spaces which is the focus of this subsection. Roughly, information of a geometric problem is encoded in the symbolic space through a potential, and its pressure and equilibrium states. We will start mentioning some key points of classical thermodynamic formalism and then move to sub-additive thermodynamic formalism.

Fix a natural number N and an $N \times N$ matrix A with entries in $\{0, 1\}$. We set

$$\Sigma = \left\{ a \in \{1, ..., N\}^{\mathbb{N}} : A(a(i), a(i+1)) = 1 \text{ for all } i \in \mathbb{N} \right\}.$$

We will assume that A is irreducible. For $\theta \in (0, 1)$ we define the metric d_θ on Σ satisfying $d_\theta(a, b) = \theta^n$ where n is the first natural number such that $a(n) \neq b(n)$. Finally we define the shift map $\sigma : \Sigma \to \Sigma$ by $\sigma(a)(i) = a(i + 1)$. A main aspect of thermodynamic formalism is describing invariant probability measures of the dynamical system (Σ, σ). We will denote the set of all invariant probability measures of σ by $\mathcal{M}_\sigma(\Sigma)$.

Definition 2.2.1. *The cylinder set* $[x_0, ..., x_n]$*, for* $(x_0, ..., x_n) \in \{1, ..., N\}^{n+1}$*, is defined to be the set*

$$\{a \in \Sigma : a(i) = x(i) \text{ for all } i \in \{0, ..., n\}\}.$$

Definition 2.2.2. *A Borel probability measure* m *on* Σ *is called a Gibbs measure iff there are a continuous* $f : \Sigma \to \mathbb{R}$*,* $P > 0$ *and* $C > 1$ *such that for all* $x \in \Sigma$ *and* $n \in \mathbb{N}$*,*

$$C^{-1} e^{\sum_{i=0}^{n-1} f(\sigma^i(x)) - nP} \leqslant m([x_0, ..., x_{n-1}]) \leqslant C e^{\sum_{i=0}^{n-1} f(\sigma^i(x)) - nP}.$$

In the context of thermodynamic formalism it used to say that a function $f : \Sigma \to \mathbb{R}$ is Holder continuous iff there is $C > 0$ such that for all $x, y \in \Sigma$,

$$|f(x) - f(y)| \leqslant C d_\theta(x, y).$$

It is common to call such a function, a potential. Below we define the pressure of a potential, a quantity which is useful in the construction of invariant measures as well as for expressing exponential growth/decay phenomena in geometric problems.

Definition 2.2.3. *Let* $f : \Sigma \to \mathbb{R}$ *be a Holder continuous function. The pressure* $P(f)$ *of* f *is defined to be*

$$\lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{\bar{x} \in \{1, ..., N\}^n} \sup_{x \in [\bar{x}]} \exp \left(\sum_{i=0}^{n-1} f(\sigma^i(x)) \right) \right).$$

By the Holder continuity it is easy to observe that the pressure can equivalently by defined by

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\sigma^n(x)=x} \exp\left(\sum_{i=0}^{n-1} f(\sigma^i(x)) \right) \right).$$

The existence of the pressure and the variational principle stated bellow are implied by the more general theorem 3 in [71].

Definition 2.2.4. *Let $m \in \mathcal{M}_\sigma(\Sigma)$. The entropy $h_\sigma(m)$ of m is defined by*

$$h_\sigma(m) = \lim_{n\to\infty} \frac{1}{n} \sum_{\bar{x}\in\{1,\dots,N\}^n} \mu([\bar{x}]) \log(m([\bar{x}])).$$

The existence of limit follows from sub-additivety.

Now we are ready to state the variational principle.

Proposition 2.2.1. *Let $f : \Sigma \to \mathbb{R}$ be a Holder continuous function. Then*

$$P(f) = \sup \left\{ h_\sigma(m) + \int f dm : m \in \mathcal{M}_\sigma(\Sigma) \right\}.$$

There is a general theory of pressure and equilibrium states for the case where f is only assumed to be continuous by Walters [80] and Ruelle [68], but we will not need it in this thesis.

Definition 2.2.5. *Let $f : \Sigma \to \mathbb{R}$ be a Holder continuous function. A measure $m \in \mathcal{M}_\sigma(\Sigma)$ is called an equilibrium state of the potential f iff*

$$P(f) = h_\sigma(m) + \int f dm.$$

For each Holder continuous function $f : \Sigma \to \mathbb{R}$ there is a unique equilibrium state m. In addition it satisfies

$$C^{-1}e^{\sum_{i=0}^{n-1} f(\sigma^i(x))-nP(f)} \leqslant m([x_0, ..., x_{n-1}]) \leqslant Ce^{\sum_{i=0}^{n-1} f(\sigma^i(x))-nP(f)},$$

implying that m is Gibbs (see proposition 3.2, comments in page 39, proposition 3.4 and theorem 3.5 in [64]). Now we move on to sub-additive thermodynamic formalism. A sequence of continuous functions $\Phi = (\phi_n)_{n\in\mathbb{N}}$ from Σ to $\mathbb{R}$ is said to be a sub-additive if for all $x \in \Sigma$ and $n, m \in \mathbb{N}$,

$$\phi_{n+m}(x) \leqslant \phi_n(x) + \phi_m(\sigma^n(x)).$$

Definition 2.2.6. *The pressure $P(\Phi)$ of a sub-additive potential $\Phi = (\phi_n)_{n\in\mathbb{N}}$ is defined by*

$$P(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\bar{x}\in\{1,...,N\}^n} \sup_{x\in[\bar{x}]} \exp\left(\phi_n(x)\right) \right).$$

For $m \in \mathcal{M}_\sigma(\Sigma)$ we also set

$$\Lambda(\Phi, \mu) = \lim_{n\to\infty} \frac{1}{n} \int \phi_n dm$$

which exists by sub-additivety.

The variational principle below follows from theorem 1.1 and section 4 in [19].

Proposition 2.2.2. *Let Φ be a sub-additive potential. Then $P(\Phi)$ exists and*

$$P(\Phi) = \sup\left\{h_\sigma(m) + \Lambda(\Phi, m) : m \in \mathcal{M}_\sigma(\Sigma)\right\}.$$

Definition 2.2.7. *Let Φ be a sub-additive potential. A measure $m \in \mathcal{M}_\sigma(\Sigma)$ is called an equilibrium state of Φ iff*

$$P(\Phi) = h_\sigma(m) + \Lambda(\Phi, m).$$

In general the set of equilibrium states can be empty but in most cases, for subadditive potentials that arise from applications to fractal geometry, equilibrium states exist (see [58] and in particular theorem 2.6). We should note though that in some cases the equilibrium state is not unique (see [52] example 6.2).

2.3 Iterated Function Systems

A map $F : \mathbb{R}^d \to \mathbb{R}^d$ is called a contraction iff there is $c < 1$ such that for all $x, y \in \mathbb{R}^d$,

$$|F(x) - F(y)| \leqslant c|x - y|.$$

Definition 2.3.1. *A finite set* $\mathsf{F} = \{F_1, ..., F_N\}$ *of contractions on* $\mathbb{R}^d$ *is called an iterated function system (IFS). The unique non-empty compact set* $\mathcal{A}$ *satisfying*

$$\mathcal{A} = \bigcup_{i=1}^{N} F_i(\mathcal{A}),$$

is called the attractor of F.

For the existence of the unique attractor see [28], theorem 9.1. We can also see an IFS as being driven by probability measures generating a fractal measures supported on the attractor or on subsets of the attractor. To explain this formally we let $\Sigma = \{1, ..., N\}^{\mathbb{N}}$ and for $\bar{a} \in \{1, ..., N\}^n$ we use the cylinder notation $[\bar{a}]$ introduced in the previous subsection. Since the members of F are contractions, for $a \in \Sigma$ we have that

$$\lim_{n\to\infty} \operatorname{diam}\left(F_{a(0)} \circ ... \circ F_{a(n)}(\mathcal{A})\right) = 0$$

also, $F_{a(0)} \circ ... \circ F_{a(n)}(\mathcal{A})$ is a nested sequence of compact subsets of $\mathcal{A}$ so there is a unique $\pi(x) \in \mathcal{A}$ such that

$$\{\pi(x)\} = \bigcap_{n\in\mathbb{N}} F_{a(0)} \circ ... \circ F_{a(n)}(\mathcal{A}).$$

The above defines a function $\pi : \Sigma \to \mathcal{A}$ which is called the projection of the IFS on $\mathcal{A}$. Now we can see that for each Borel probability measure m on Σ we can form the push-forward measure $\pi(m)$. It is also useful to note that for $\bar{a} \in \{1, ..., N\}^n$

$$\pi([\bar{a}]) = F_{\bar{a}(0)} \circ ... \circ F_{\bar{a}(n)}(\mathcal{A}).$$

Often it is assumed that an IFS satisfies the open set condition below. That condition makes the mathematical analysis of the IFS much more tractable.

Definition 2.3.2. *Let* $\mathsf{F} = \{F_1, ..., F_N\}$ *be an IFS on* $\mathbb{R}$*. We say that* F *satisfies the open set condition iff there exists a non-empty bounded open set* $V \subseteq \mathbb{R}^d$ *such that*

$$V \supseteq \bigcup_{i=1}^{N} F_i(V)$$

and

$$F_i(V) \cap F_j(V) = \emptyset$$

for $i, j \in \{1, ..., N\}$ *with* $i \neq j$*.*

Probably the most important way to analyze the fractal behaviour of attractors and projected measures is the Hausdroff dimension. In order to define it we first need to define Hausdorff measures.

Definition 2.3.3. *Given a set $A \subseteq \mathbb{R}^d$, we call a sequence $(D_i)_{i\in\mathbb{N}}$ a δ-cover of A if the following statements hold.*

- $D_i \subseteq \mathbb{R}^d$.
- $\operatorname{diam}(D_i) \leqslant \delta$ *for all* $i \in \mathbb{N}$.
- $A \subseteq \bigcup_{i\in\mathbb{N}} D_i$.

For $A \subseteq \mathbb{R}^d$ and $(s, \delta) \in [0, \infty) \times (0, \infty)$ we define

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=0}^{\infty} \operatorname{diam}(D_i)^s : (D_i)_{i\in\mathbb{N}} \text{ is a } \delta\text{-cover of } A\right\}.$$

which leads us to the definitions of the Hausdorff measure.

Definition 2.3.4. *For s non-negative, the s-dimensional Hausdorff measure $\mathcal{H}^s(A)$ of $A \subseteq \mathbb{R}^d$ is defined by*

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A).$$

For a proof that $\mathcal{H}^s$ is well-defined and that its restriction to the Borel σ-algebra is a measure, see theorem 1 of chapter 2 in [22]. See page 31 of [28] for the proposition below.

Proposition 2.3.1. *Let $A \subseteq \mathbb{R}^d$ then there exists $s_0 \geqslant 0$ such that $\mathcal{H}^s(A) = \infty$ for $s \in [0, s_0)$ and $\mathcal{H}^s(A) = 0$ for $s \in (s_0, \infty)$.*

It is natural see the number s_0 above, where the jump happens, as the dimension of the set A. This leads to the definition of Hausdorff dimension.

Definition 2.3.5. *Let $A \subseteq \mathbb{R}^d$. The Hausdorff dimension $\dim_{\mathrm{H}}(A)$ of A is defined by*

$$\dim_{\mathrm{H}}(A) = \inf \{s \geqslant 0 : \mathcal{H}^s(A) = 0\}.$$

Remark. *It is easy to observe that if $A \subseteq B \subseteq \mathbb{R}^d$ then $\dim_{\mathrm{H}}(A) \leqslant \dim_{\mathrm{H}}(B)$. Also if $(A_i)_{i\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{R}^d$ then*

$$\dim_{\mathrm{H}}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sup_{i\in\mathbb{N}} \dim_{\mathrm{H}}(A_i).$$

Definition 2.3.6. *Let m be a Borel probability measure on $\mathbb{R}^d$. The Hausdorff dimension $\dim_{\mathrm{H}}(m)$ of the measure μ is defined by*

$$\dim_{\mathrm{H}}(m) = \inf \{\dim_{\mathrm{H}}(A) : A \textit{ is a Borel set with } m(A) > 0\}.$$

Sometimes the above is called the lower Hausdorff dimension of m and it is denoted by $\underline{\dim_{\mathrm{H}}}(m)$. In that context the upper Hausdorff dimension $\overline{\dim_{\mathrm{H}}}(m)$ of m is defined by

$$\overline{\dim_{\mathrm{H}}}(m) = \inf \{\dim_{\mathrm{H}}(A) : A \text{ is a Borel set with } m(A) = 1\}.$$

Definition 2.3.7. *Let m be a Borel probability measure on $\mathbb{R}^d$. Then the local dimension $\dim_{\mathrm{loc}}(m,x)$ of m at $x \in \mathbb{R}^d$ is defined by*

$$\dim_{\mathrm{loc}}(m,x) = \lim_{n\to\infty} \frac{\log(m(B(x,r)))}{\log(r)},$$

if it exists, where $B(x,r) = \{z \in \mathbb{R}^d : ||x-z||_2 < r\}$.

Definition 2.3.8. *A Borel probability measure m on $\mathbb{R}^d$ is called exact-dimensional iff for m-almost all $x \in \mathbb{R}^d$*

$$\dim_{\text{loc}}(m, x) = \dim_{\text{H}}(m).$$

As we will see, there are many interesting examples of measures that are exact dimensional. The simplest family of IFSes one can consider are the self-similar IFSes.

Definition 2.3.9. *An IFS $\mathsf{F} = \{F_1, ..., F_N\}$ on $\mathbb{R}^d$ is called self-similar iff there exist $r_1, ..., r_N \in (0, 1)$ (contraction rates) and $t_1, ..., t_N \in \mathbb{R}^d$ (translation vectors) such that for every $i \in \{1, ..., N\}$ and $x \in \mathbb{R}^d$ we have $F_i(x) = r_i x + t_i$. The attractors of a self-similar IFSes are called self-similar sets. Finally if there are $p_1, ..., p_N \in [0, 1]$ such that $p_1 + ... + p_N = 1$ then a measure m satisfying*

$$m = \sum_{i=1}^{N} p_i F_i(m)$$

is called a self-similar measure.

It is easy to see that given $p_1, ..., p_N$ as above then m always exists, it is unique and it is equal to the projection through F of the Bernoulli measure on $\{1, ..., N\}^{\mathbb{N}}$ corresponding to $p_1, ..., p_N$ (see section 4 in [47]). The fractal geometry of self-similar IFSes satisfying the open set condition is well understood. On the other hand the more general case where overlaps are occur (Open set condition fails) has been proved to be much more difficult. For a better understanding it is worth noting the following basic results on the case where the open set condition holds. For proofs see theorem 9.3 in [28], [65] and [72].

Theorem 2.3.1. *Let $\mathsf{F} = \{F_1, ..., F_N\}$ be a self-similar IFS on $\mathbb{R}^d$ with contraction rates $r_1, ..., r_N \in (0, 1)$. If F satisfies the open set condition then the Hausdorff* *dimension of its attractor is the unique number $s \geqslant 0$ satisfying*

$$\sum_{i=1}^{N} r_i^s = 1.$$

In addition if m is a self-similar measure m corresponding to $p_1, ..., p_N \in (0,1)$ (with $p_1 + ... + p_N = 1$) then

$$\dim_{\mathrm{H}}(m) = \frac{\sum_{i=1}^{N} p_i \log(p_i)}{\sum_{i=1}^{N} r_i \log(r_i)}$$

and the multifractal spectrum

$$f(a) = \dim_{\mathrm{H}}\{x \in \mathbb{R}^d : \dim_{\mathrm{loc}}(m, x) = a\}$$

is equal to the Legendre transform of the function $\tau : \mathbb{R} \to \mathbb{R}$ satisfying

$$\sum_{i=1}^{N} p_i^q r_i^{\tau(q)} = 1.$$

That is

$$f(a) = \inf_{q \in \mathbb{R}} \{\tau(q) + aq\},$$

provided that is finite.

The next natural generalisation is self-affine IFSes.

Definition 2.3.10. *An IFS $\mathsf{F} = \{F_1, ..., F_N\}$ on $\mathbb{R}^d$ is called self-affine iff there exist invertable contracting $d \times d$ matrices $A_1, ..., A_N$ and $t_1, ..., t_N \in \mathbb{R}^d$ (translation vectors) such that for every $i \in \{1, ..., N\}$ and $x \in \mathbb{R}^d$ we have $F_i(x) = A_i x + t_i$. The attractors of a self-affine IFSes are called self-affine sets. Finally if there are $p_1, ..., p_N \in [0,1]$ such that $p_1 + ... + p_N = 1$ then a measure m satisfying*

$$m = \sum_{i=1}^{N} p_i F_i(m)$$

is called a self-affine measure.

Again it is easy to see that given $p_1, ..., p_N$ as above then m always exists, it is unique and it is equal to the projection through F of the Bernoulli measure on $\{1, ..., N\}^{\mathbb{N}}$ corresponding to $p_1, ..., p_N$.

Definition 2.3.11. *If A is a $d \times d$ invertable contracting matrix, the singular values $a_1(A) \leqslant ... \leqslant a_d(A)$ are defined to be the roots of the eigenvalues of $A^{\mathrm{T}}A$. For $s \geqslant 0$ the singular value function $\phi^s(A)$ is defined by*

$$\phi^s(A) = \begin{cases} a_1(A) \cdots a_{\lfloor s \rfloor}(A) a_{\lfloor s \rfloor + 1}(A)^{s - \lfloor s \rfloor} & s \in [0, d] \\ |\det A|^{s/d} & s > d \end{cases}.$$

With the notations of definition 2.3.10, the sequence $(\phi_n^s)_{n\mathbb{N}}$ of real functions on $\{1, ..., N\}^{\mathbb{N}}$ defined by

$$\phi_n^s(i) = \log \phi^s(A_{i(0)} ... A_{i(n)})$$

is a sub-additive potential (see [25], lemma 2.1). We will denote its pressure just by $P(\phi^s)$. The following characteristic result on self-affine sets appeared in [25] as theorem 5.3.

Theorem 2.3.2. *Let $A_1, .., A_N$ be $d \times d$ invertable contracting matrices satisfying $||A_i|| < 1/3$ (operator norm). Then for Lebesgue almost all $t_1, ..., t_N \in \mathbb{R}^d$ the Hausdorff dimension of the attractor of the IFS given by $F_i(x) = A_i x + t_i$, $i \in \{1, ..., N\}$, is the unique $s \geqslant 0$ satisfying*

$$P(\phi^s) = 0.$$

Later the condition $||A_i|| < 1/3$ above was improved to $||A_i|| < 1/2$ by Solomyak, see proposition 3.1 in [75]. There is an analog for projected measures in [49],

Theorem 2.3.3. *Let $A_1, .., A_N$ be $d \times d$ invertable contracting matrices satisfying $||A_i|| < 1/2$ (operator norm). Let $m \in \mathcal{M}_\sigma(\Sigma)$ be ergodic. Then for Lebesgue almost all $t_1, ..., t_N \in \mathbb{R}^d$ if $\pi : \{1, ..., N\}^{\mathbb{N}} \to \mathbb{R}^d$ is the respective projection of the IFS given by $F_i(a) = A_i x + t_i$, $i \in \{1, ..., N\}$, then $\pi(m)$ is exact-dimensional and*

$$\dim_{\mathrm{H}} \pi(m) = \min\{s, d\}$$

where s is the unique non-negative number satisfying

$$h_\sigma(m) + \Lambda(\phi^s, m) = 0.$$

Arguably the main difficulty of self-affine fractal geometry is that usually the singular values $a_i(A_{i(0)}...A_{i(n)})$ decays with different exponential rates for different $i \in \{1, ..., d\}$. This makes the geometry of $\pi([i(0), ..., i(n)])$ not naturally compatible with the geometry of euclidean balls. Finally we mention the following result from [32] (theorem 1.2) in a slightly simpler form.

Theorem 2.3.4. *Let $A_1, .., A_N$ be $d \times d$ invertable contracting matrices, $t_1, ..., t_N \in \mathbb{R}^d$ and $F_i(x) = A_i x + t_i$ for $i \in \{1, ..., N\}$. Let $m \in \mathcal{M}_\sigma(\{1, ..., N\}^{\mathbb{N}})$ be ergodic and $\pi : \{1, ..., N\}^{\mathbb{N}} \to \mathbb{R}^d$ the projection of the IFS $\{F_1, ..., F_N\}$. Then $\pi(m)$ is excact dimensional.*

2.4 Perron theory

Let (G, E, w) be a finite weighted directed graph. That is

- G is a finite set.
- $E \subseteq G^2$.

- $w : E \to (0, \infty)$.

Let $C(G)$ be the vector space of functions from G to $\mathbb{R}$. Then we define the operator $T : C(G) \to C(G)$ by

$$T(f)(x) = \sum_{(y,x)\in E} f(y) \cdot w(y, x).$$

Given a non-negative $d \times d$ matrix A we set $G_A = \{1, ..., d\}$, $E = \{(x, y) \in G^2 : A(x, y) \neq 0\}$ and $w(x, y) = A(x, y)$. For a vector v in $\mathbb{R}^d$ we set $f_v : \{1, ..., d\} \to \mathbb{R}$ such that $f_v(i) = v(i)$. In this case we have

$$T(f_v) = f_{vA}.$$

The above describes a useful viewpoint where we can see non-negative matrices as dynamical processes on graphs. A non-negative matrix A is called irreducible iff G_A is strongly connected (i.e. for any $x, y \in G_A$ there is a path from x to y). Equivalently a non-negative $d \times d$ matrix A is irreducible iff for any $i, j \in \{1, ..., d\}$ there is $n \in \mathbb{N}$ such that $A^n(i, j) > 0$. A matrix is called reducible if it is not irreducible. If A is a non-negative $d \times d$ irreducible matrix then the number $\gcd\{n \in \mathbb{N} : A^n(i, i) > 0\}$ is the same for all $i \in \{1, ..., d\}$ and is called the period of A. The period of A is also equal to the gcd of lengths of closed directed paths on G_A. If the period is equal to 1 then A is called aperiodic. If the period, call it κ, is bigger that one then there is a non-trivial partition $\{S_1, ..., S_\kappa\}$ of G_A such that if (x, y) is a directed edge of G_A then there are $i, j \in \{1, ..., \kappa\}$ such that $x \in S_i$, $y \in S_j$ and $j = i + 1 \mod \kappa$. The sets $S_1, ..., S_\kappa$ will be refered as periodicity classes. In the following theorem vectors are considered as row-vectors.

Theorem 2.4.1. *Perron-Frobenius theorem for primitive matrices*

Let A be a non-negative irreducible aperiodic matrix, also called primitive, then

- *There is an eigenvalue $\rho > 0$ of A such that $|\lambda| < |\rho|$ for every other eigenvalues λ of A.*
- *The eigenvalue ρ is simple.*
- *The eigenvalue ρ has strictly positive left and right eigenvectors. Also, left and right eigenvectors of A are unique up to scalar multiplication.*
- *If w and v are left and right strictly positive eigenvectors of A respectively, so that $w \cdot v^{\mathrm{T}} = 1$, then*

$$\lim_{n\to\infty} \frac{A^n}{\rho^n} = v^{\mathrm{T}} \cdot w.$$

There is a version for non-aperiodic matrices too.

Theorem 2.4.2. *Perron-Frobenius theorem for non-negative irreducible matrices Let A be a non-negative irreducible matrix of period $\kappa > 1$. Then*

- *There is an eigenvalue $\rho > 0$ of A such that either $|\lambda| < |\rho|$ or $(\lambda/\rho)^\kappa = 1$ for every other eigenvalues λ of A.*
- *The eigenvalue ρ is simple.*
- *The eigenvalue ρ has strictly positive left and right eigenvectors. Also, left and right eigenvectors of A are unique up to scalar multiplication.*
- *Let w and v be left and right strictly positive eigenvectors of A respectively, so that $w \cdot v^{\mathrm{T}} = 1$. If i, j in the same periodicity class then*

$$\lim_{n\to\infty} \frac{A^{n\kappa}}{\rho^{n\kappa}} = v^{\mathrm{T}}(i) \cdot w(j).$$

For details and proofs of the above see paragraph 1.3 in [54]. A useful in concept in Perron theory arguments is the projective space of $\mathbb{R}^d$.

Definition 2.4.1. *Let $d \in \mathbb{N}$. For $x \in \mathbb{R}^{d+1}$ let $[x]$ be its linear span*

$$[x] = \{rx \in \mathbb{R}^{d+1} : r \in \mathbb{R}\backslash\{0\}\}.$$

The projective space $\mathbb{R}\,\mathrm{P}^d$ is defined to be the set

$$\{[x] : x \in \mathbb{R}^{d+1}\backslash\{0\}\}.$$

Often the projective space $\mathbb{R}\,\mathrm{P}^d$, or a subset of it, is identified with subsets of $\mathbb{R}^{d+1}$ by choosing a representative element $x' \in [x]$. For example for

$$\{[x] : x \in \mathbb{R}^{d+1} \text{ with strictly positive entries.}\}.$$

we can identify $[x]$ with $x/||x||_1$. Notice that if A is a $(d+1) \times (d+1)$ matrix and $Ax \neq 0$, for $x \in \mathbb{R}^{d+1}$, then

$$[Ay] = [Ax]$$

for all $y \in [x]$. This means that square matrices induce partial actions on the projective space. Finally we mention the very useful Gelfand's spectral radius formula (see [35], p312, Th. A).

Definition 2.4.2. *The spectral radius $\rho(A)$ of a matrix square matrix A is defined by*

$$\rho(A) = \max\{|\lambda| : \ \lambda \textit{ is an eigenvalue of } A. \ \}$$

Theorem 2.4.3. *Gelfand's Formula*

Let A be a square matrix and $||.||$ any matrix norm. Then

$$\lim_{n\to\infty} ||A^n||^{1/n} = \rho(A).$$

2.5 Linear algebra

For any $a_1, ..., a_\kappa \in \mathbb{C}$ the associated Vandermonde matrix V is defined to be the $\kappa \times n$ matrix

$$V(i, j) = a_i^{j-1}.$$

We will later need the following well known lemma.

Lemma 2.5.1. *If $\kappa = n$ then*

$$\det(V) = \prod_{1 \leqslant i < j \leqslant n} (a_j - a_i).$$

The above implies that if $a_1, ..., a_\kappa$ are pairwise different then $\{u_1, ..., u_\kappa\}$, where

$$u_i = (1, a_i, ..., a_i^{n-1}),$$

is independent.

Chapter 3

Measures on the Spectra of Algebraic Integers

Joint work with Tom Kempton

3.1 Introduction

Given a real number $\beta > 1$ and an alphabet $\mathcal{A}$, the spectrum

$$X_{\mathcal{A}}(\beta) := \left\{ \sum_{i=1}^{n} c_i \beta^{n-i} : n \in \mathbb{N}, c_i \in \mathcal{A} \right\}$$

has been the focus of much attention. In particular, when $\mathcal{A} = \{0, \cdots, \lfloor \beta \rfloor\}$ then it is known that $X_{\mathcal{A}}(\beta)$ is uniformly discrete if and only if β is a Pisot number (i.e. an algebraic number, all of whose Galois conjugates have modulus strictly less than one) [3, 18, 31, 38]. Additionally, $X_{\mathcal{A}}(\beta)$ is relatively dense in this setting, making the sets $X_{\mathcal{A}}(\beta)$ Delone sets (uniformly discrete, relatively dense). Delone sets give useful mathematical models for quasicrystals and so the above construction gives a number-theoretic construction of important physical objects.

Much progress has been made on giving dynamical descriptions of sets $X_{\mathcal{A}}(\beta)$ [20, 34, 41]. If β is a Pisot number then $X_{\mathcal{A}}(\beta)$ can be generated by a substitution system [34]. Moreover, for Pisot β there is a naturally related cut and project set which contains $X_{\mathcal{A}}(\beta)$. In all known examples of Pisot β with $\mathcal{A} \subset \mathbb{Z}$ the set $X_{\mathcal{A}}(\beta)$ coincides with this cut and project set, but the question of whether these sets always coincide remains open, and there are some examples with a complex alphabet for which the cut and project set contains finitely many extra points which are not in $X_{\mathcal{A}}(\beta)$ [41]. A generalisation of this cut and project structure to general hyperbolic algebraic integers is given in section 3.4.

We are interested in measures on the sets $X_{\{-1,0,1\}}(\beta)$. In particular, we are interested in what one can say about the measures μ_n given by

$$\mu_n(x) = \frac{1}{4^n}\mathcal{N}_n(x)$$

where

$$\mathcal{N}_n(x) = \#\{a_1 \cdots a_n, b_1 \cdots b_n \in \{0,1\}^n : \sum_{i=1}^{n}(a_i - b_i)\beta^{n-i} = x\}.$$

The measure μ_n is the distribution of the set of differences

$$\sum_{i=1}^{n} a_i\beta^{n-i} - \sum_{i=1}^{n} b_i\beta^{n-i}$$

where each a_i, b_i is picked from $\{0,1\}$ according to the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure.[1] We focus on the case that β is an algebraic integer and a root of a {-1,0,1} polynomial but does not have any Galois conjugates of absolute value one, we call such β hyperbolic.

[1] There has been a lot of recent research into a different class of measures (Patterson measures) on cut and project sets. These are related to diffraction on quasicrystals, where they play the role of the intensity of the Bragg peak [61, 66]. Loosely speaking, the difference between the class of measures that we study and Patterson measures is that our measures incorporate information on the number of different codings $a_1 \cdots a_n$ for which $\sum_{i=1}^{n} a_i\beta^{n-i} = x$, whereas Patterson measures do not. The analogue of $\mu_n(x)$ for the Patterson measure would be (more or less)

$$\gamma_n(x) = \#\{(y,z) \in (X_{\{0,1\}}(\beta))^2 : y - z = x\}.$$

This difference is crucial for our applications.

Broadly, we are interested in the question of whether the measures μ_n, appropriately rescaled, have a limit μ as n tends to infinity, and whether that limit has any 'local structure' analagous to that of the set $X_{\mathcal{A}}(\beta)$. Assuming some technical (but checkable) conditions, our results hold for general hyperbolic β, but all of the ideas behind our proofs are presented in the golden mean case, which is notationally much simpler, and for this reason we prove our results first for the golden mean. The golden mean also has the advantage that the higher dimensional objects which we construct are only two dimensional, and so can be more easily visualised.

Our main theorems are the following.

Theorem 3.1.1. *Let β be hyperbolic. Then there exists a real number $\lambda > 1$, such that for all $x \in X(\beta)$ the limit measure μ given by*

$$\mu(x) := \lim_{n\to\infty} \frac{1}{\lambda^n} \mathcal{N}_n(x)$$

exists and has $\mu(x) \in (0, \infty)$ for $x \in X(\beta)$. Furthermore, the measure μ has infinite total mass.

In the case that β has other Galois conjugates of absolute value larger than one, we prove this theorem by lifting to a measure $\bar{\mu}$ supported on a higher dimensional Delone set, whose projection onto the first coordinate gives μ.

Our second theorem gives an explicit way to calculate $\mu(x)$ using any code of x.

Theorem 3.1.2. *Let β be hyperbolic. There exists a natural number k, a $1 \times k$ vector W, and three $k \times k$ matrices M_{-1}, M_0 and M_1 such that for any $x \in X(\beta)$ and $c_1 \cdots c_n \in \{-1, 0, 1\}^n$ with $x = \sum_{i=1}^n c_i \beta^{n-i}$,*

$$\mu(x) = \frac{1}{\lambda^n} (W M_{c_1} \cdots M_{c_n})_1.$$

Here $(W M_{c_1} \cdots M_{c_n})_1$ denotes the first entry of the row vector $W M_{c_1} \cdots M_{c_n}$.

In fact the vector $WM_{c_1}\cdots M_{c_n}$ also holds information on the values of $\mu(y)$ for other values of $y \in X(\beta)$. There is a set of translations $d_1, \cdots, d_k \in \mathbb{R}$, with $d_1 = 0$, such that, for $x = \sum_{i=1}^n c_i \beta^{n-i}$,

$$\frac{\mu(x + d_i)}{\mu(x)} = \frac{(WM_{c_1}\cdots M_{c_n})_i}{(WM_{c_1}\cdots M_{c_n})_1}.$$

This suggests that one may be able to use a dynamical system to move through the measure μ to calculate its values at different points. We can do this, but we need first to replace the dependence of $\mu(x)$ on the coding of x with a dependence on the position of a point x_c corresponding to x in the 'contracting space'. To describe this, we must first describe a geometric construction related to β-expansions in algebraic bases.

Let β have Galois conjugates $\beta_2 \cdots \beta_d$ of absolute value larger than one and Galois conjugates $\beta_{d+1} \cdots \beta_{d+s}$ of absolute value smaller than one. Define the contracting space $\mathbb{K}_c$ by $\mathbb{K}_c = \mathbb{F}_{d+1} \times \mathbb{F}_{d+2} \times \cdots \times \mathbb{F}_{d+s}$ where $\mathbb{F}_k = \mathbb{R}$ if $\beta_k \in \mathbb{R}$, $\mathbb{F}_k = \mathbb{C}$ if $\beta_k \in \mathbb{C}\backslash\mathbb{R}$. Then, for $i \in \{-1, 0, 1\}$ define the contraction S_i on $\mathbb{K}_c$ by

$$S_i(x_{d+1}, \cdots, x_{d+s}) = (\beta_{d+1}x_{d+1} + i, \cdots, \beta_{d+s}x_{d+s} + i).$$

The maps $\{S_{-1}, S_0, S_1\}$ form an iterated function system on $\mathbb{K}_c$ with an attractor that we denote $\mathcal{R}$. This is a standard construction in numeration/tiling theory, although it is more usual to consider a sub-IFS using only those codes which correspond to greedy β-expansions [1]. To each point $x = \sum_{i=1}^n c_i \beta^{n-i}$ there exists a corresponding point in the contracting space:

$$x_c = \sum_{i=1}^n c_i(\beta_{d+1}^{n-i}, \beta_{d+2}^{n-i}, \cdots, \beta_{d+s}^{n-i}) = S_{c_n} \circ \cdots S_{c_1}(0) \in \mathcal{R}.$$

It is important to stress that the point x_c corresponding to x is independent of the coding $c_1, \cdots, c_n$ of x, this holds since $\beta_{d+1} \cdots \beta_{d+s}$ are Galois conjugates of β.

Theorem 3.1.3. *Assume that Condition 3.4.1 holds. There exists a set* $\Delta = (v_1, \cdots v_k)$ *of translations such that for any* $j \in \{1 \cdots k\}$ *there is a function* $f_j : \mathcal{R} \to \mathbb{R}$ *such that for any* $x \in X(\beta)$ *with* $x + v_j$ *also in* $X(\beta)$ *we have*

$$\log\left(\frac{\mu(x+v_j)}{\mu(x)}\right) = f_j(x_c).$$

Furthermore any $x \in X(\beta)$ *can be reached from* 0 *by applying a finite number of translations from* Δ. *There exists a word* w *and constants* $C_1 > 0$, $C_2 \in (0,1)$ *such that for any* $a_1 \cdots a_n \in \{-1,0,1\}^n$ *which contains* r *non-overlapping copies of the word* w, f_j *varies by at most* $C_1 C_2^{r-1}$ *on* $S_{a_1} \circ \cdots \circ S_{a_n}(\mathcal{R})$.

The final condition on the variation of f_j gives rise to the following continuity properties of f_j.

1. **Continuity almost everywhere:** For any fully supported ergodic measure ν on $\mathcal{R}$, each f_j is continuous ν-almost everywhere

2. **Continuity at most lattice points:** For any fully supported measure m on $\{-1,0,1\}$ and any $\epsilon > 0$ there exists $n \in \mathbb{N}$ and $D \subseteq \{-1,0,1\}^n$ such that $m^n(D) > 1 - \varepsilon$ and

$$|f_j(x) - f_j(y)| < \varepsilon$$

   for all $x, y \in X(\beta)$ with $x_c, y_c \in S_{a_1} \circ \cdots \circ S_{a_n}(\mathcal{R})$ for any $a_1 \cdots a_n \in D$.

These latter two continuity properties follow since ν almost every sequence contains infinitely many copies of the word w, and that for any r and any $\epsilon > 0$ there exists n such that a proportion at least $1 - \epsilon$ of $\{-1,0,1\}$ words of length n contain r non-overlapping occurences of w.

We use this theorem extensively in our follow up article [12]. For now, we limit our application of this theorem to the golden mean case, where we show that the values of $\mu(x)$ can be obtained via a cocycle over an interval exchange transformation on $\mathcal{R} = (-\phi^2, \phi^2)$, see Theorem 3.3.3.

In Section 3.2 we describe some links with the dimension theory of Bernoulli convolutions, which allows us to state some new conjectures about Bernoulli convolutions. In Section 3.3 we prove Theorems 3.1.1, 3.1.2 and 3.1.3 in the special case that β is the golden mean. Finally in Section 3.4 we prove these theorems for the general case of hyperbolic β.

3.2 Links to the Dimension Theory of Bernoulli Convolutions

Our interest in the measures μ stems from a link with the study of the dimension and possible absolute continuity of Bernoulli convolutions ν_β, defined below. We describe here connections with dimension theory for Pisot numbers, links between our work and the question of absolute continuity of ν_β for non-Pisot hyperbolic β are postponed to a follow up article, in which we generalise [53] to give a condition for the absolute continuity of ν_β in terms of the growth of $\mu_n([\frac{-1}{\beta-1}, \frac{1}{\beta-1}])$, which in turn can be stated in terms of rapid equidistribution to Lebesgue measure of the measures $\mu_n|_{[\frac{-1}{\beta-1}, \frac{1}{\beta-1}]}$. We then use the local structure of the measures μ_n described in Theorem 3.1.3 and an analogue of Theorem 3.3.3 to study this equidistribution.

Given a number $\beta \in (1,2)$, the Bernoulli convolution ν_β is the weak* limit of the measures $\nu_{\beta,n}$ given by

$$\nu_{\beta,n} = \sum_{a_1 \cdots a_n \in \{0,1\}^n} \frac{1}{2^n} \delta_{\sum_{i=1}^n a_i \beta^{-i}}$$

where δ_x denotes the Dirac probability measure on x. The measure ν_β is a probability measure on $[0, \frac{1}{\beta-1}]$ and is perhaps the simplest example of a self-similar measure with overlaps. The question of whether ν_β is absolutely continuous for some given parameter β goes back to Jessen and Wintner [48]. Erdős showed that ν_β is singular when β is a Pisot number [21], and indeed Garsia showed that such Bernoulli convolutions have dimension less than one [38]. There has been very substantial progress on the dimension theory of Bernoulli convolutions in the last decade, stemming from the work of Hochman [45], and in particular it is now known that non-algebraic β give rise to Bernoulli convolutions of dimension one [79], whereas for algebraic β there are algorithms to determine whether or not ν_β has dimension one [17, 2]. For a summary of recent research into the dimension theory of Bernoulli Convolutions see [78].

There have been many numerical studies into the dimensions of Bernoulli Convolutions associated with Pisot numbers. The evidence we have suggests that for Pisot numbers of large degree the dimension of the corresponding Bernoulli convolution is close to one [2, 39, 42, 43]. We formalise this conjecture here.

Conjecture 1. *Let β_n be a sequence of Pisot numbers in the interval $(1,2)$ and suppose that the degree of β_n tends to infinity as $n \to \infty$. Then*

$$\dim_H(\nu_{\beta_n}) \to 1.$$

We have not seen this conjecture formally stated before, but it seems consistent with the (admittedly fairly limited) numerical evidence that we have.

The rest of this section is devoted to giving another conjecture on the measures μ_n and showing that this new conjecture would be sufficient to prove Conjecture 1.

It was proved in Hochman [45] that, for algebraic β the dimension of the Bernoulli convolution ν_β is given by

$$\dim_H(\nu_\beta) = \min\left\{1, \frac{H(\beta)}{\log(\beta)}\right\}.$$

Here the Garsia entropy $H(\beta)$ is given by

$$H(\beta) := \lim_{n\to\infty} \frac{1}{n} H_n(\beta)$$

where

$$H_n(\beta) = -\sum_{a_1\cdots a_n\in\{0,1\}^n} \frac{1}{2^n}\log\left(\frac{1}{2^n}\#\{b_1\cdots b_n \in \{0,1\}^n : \sum_{i=1}^{n}(a_i-b_i)\beta^{n-i}=0\}\right).$$

As noted in [2], one can use Jensen's inequality to reverse the order of the summation and the log, to get

$$\begin{aligned} H_n(\beta) &\geqslant -\log\left(\frac{1}{4^n}\#\{a_1\cdots a_n, b_1\cdots b_n \in \{0,1\}^n : \sum_{i=1}^{n}(a_i-b_i)\beta^{n-i}=0\}\right) \\ &= \log(4^n)-\log(\mathcal{N}_n(0)). \end{aligned}$$

In particular, our main theorem, Theorem 3.1.1, introduces a constant λ equal to the exponential growth rate of $\mathcal{N}_n(0)$, using this constant we get

$$H(\beta) \geqslant \log(4) - \log\lambda. \tag{3.1}$$

Our contribution here in the Pisot case is to link the question of how close to being equidistributed μ is to the value of λ, broadly when $\mu|_{[\frac{-1}{\beta-1},\frac{1}{\beta-1}]}$ is well distributed with respect to Lebesgue measure then Equation 3.1 gives a lower bound for the dimension of ν_β which is close to one. Our approach here is more or less that of trying to understand something about the maximal eigenvalue of a matrix by studying the corresponding eigenvector. We use the following elementary lemma from linear algebra.

Lemma 3.2.1. *Let M be a $k\times k$ matrix with maximal eigenvalue ρ and associated left eigenvector $V=(v_1,\cdots,v_k)$ normalised so that $\sum_{i=1}^{k} v_i = 1$. Let $r_i := \sum_{j=1}^{k} M_{i,j}$ denote the ith row sum of M. Then*

$$\rho = \sum_{i=1}^{k} v_i r_i.$$

Let β be a Pisot number and $I_\beta := [\frac{-1}{\beta-1},\frac{1}{\beta-1}]$. Then, as noted before, λ counts the (weighted) growth of the number of words in $\{-1,0,1\}^n$ for which

$\sum_{i=1}^{n} c_i\beta^{n-i} = 0$, the weighting comes from giving each word weight 2^m where m is the number of occurences of letter 0 in the word. Whenever $\sum_{i=1}^{n} c_i\beta^{n-i} = 0$ we have that $\sum_{i=1}^{m} c_i\beta^{m-i}$ is in the interval I_β, and so is in $X(\beta) \cap I_\beta$ which is a finite set $V = \{v_1, \cdots, v_k\}$ thanks to the Garsia Separation Property [37]. We write down a matrix M_0 indexed by $\{v_1, \cdots, v_k\}$ with

$$(M_0)_{i,j} = \begin{cases} 1 & v_j = \beta v_i \pm 1 \\ 2 & v_j = \beta v_i \\ 0 & \text{otherwise} \end{cases}.$$

Then the measure $\mu_{I_\beta} := \frac{1}{\mu(I_\beta)}\mu|_{I_\beta}$ gives mass to v_j equal to the jth entry of the left probability eigenvector of M_0 associated with maximal eigenvalue λ. Furthermore, we can read off the ith row sum r_i of M_0 (associated to point $v_i \in X(\beta) \cap I_\beta$) immediately, since we need only know which of $\beta v_i - 1, \beta v_i$ and $\beta v_i + 1$ lie in I_β.

Let the function $g_\beta : I_\beta \to \{1, 2, 3, 4\}$ be given by

$$g_\beta(x) = \chi_{I_\beta}(\beta x - 1) + 2\chi_{I_\beta}(\beta x) + \chi_{I_\beta}(\beta x + 1).$$

Then $r_j = g_\beta(v_j)$ and so by Lemma 3.2.1 we have

$$\lambda = \sum_{v_j \in V} g_\beta(v_j)\mu_{I_\beta}(v_j) = \int_{I_\beta} g_\beta(x) d\mu_{I_\beta}(x). \tag{3.2}$$

A short calculation gives that if $\mathcal{L}_{I_\beta}$ denotes normalised Lebesgue measure on I_β then

$$\int_{I_\beta} g_\beta(x) d\mathcal{L}_{I_\beta}(x) = \frac{4}{\beta}.$$

We have the following theorem.

Theorem 3.2.1. *Let β_n be a sequence of Pisot numbers and suppose that*

$$W_1(\mu_{I_{\beta_n}}, \mathcal{L}_{I_{\beta_n}}) \to 0$$

where W_1 denotes the Wasserstein metric on the space of probability measures on the Euclidean line. Then $\dim_H(\nu_{\beta_n}) \to 1$.

Proof. The function g_β is a step function on I_β and it is straightforward to give an upper bound for $|\mu_{I_\beta}(A) - \mathcal{L}_{I_\beta}(A)|$ for any of the intervals A upon which the step function is constant in terms of the distance between μ_{I_β} and $\mathcal{L}_{I_\beta}$. These upper bounds are uniform in β. This in turn yields uniform upper bounds on $\int_{I_\beta} g_\beta d\mu_{I_\beta}$, and so by equation 3.2 we have a uniform upper bound on $\lambda(\beta) - \log(\frac{4}{\beta})$ in terms of $W_1(\mu_{I_{\beta_n}}, \mathcal{L}_{I_{\beta_n}})$.

Finally, for Pisot β_n

$$\dim_H(\nu_{\beta_n}) = \frac{H(\beta_n)}{\log(\beta_n)} \geqslant \frac{\log 4 - \log \lambda(\beta_n)}{\log \beta_n} \to \frac{\log 4 - \log\left(\frac{4}{\beta_n}\right)}{\log(\beta_n)} = 1.$$

as required. □

The matrix $M_0(\beta)$ associated to a Pisot number β is very large for β of large degree, and so the numerical evidence we have is limited, but the evidence that we have does suggest that the measures $\mu_{I_{\beta_n}}$ are increasingly well equidistributed for sequences β_n of Pisot numbers in $(1, 2-\epsilon)$ with degree tending to infinity, see Table 3.1. The ϵ here is to exclude the multinacci family, which has different behaviour[2]. In particular we suspect that what allows the multinacci family β_n to behave differently is that the multinacci numbers β_n converge to 2.

Finally, we give our conjecture on the distribution properties of the measures $\mu_{I_{\beta_n}}$. A proof of this conjecture would imply that Conjecture 1 is true by Theorem 3.2.1.

Conjecture 2. *Let $\epsilon > 0$ and let (β_n) be a sequence of Pisot numbers in the interval $(1, 2-\epsilon)$ such that the degree of β_n tends to infinity as n tends to infinity.*

[2]Many structures related to the multinacci family $\beta_n^n - \beta_n^{n-1} - \cdots - 1 = 0$, including the spectrum of β_n, are well understood.

Polynomial	β	Bound	$W_1(\mu_\beta, \text{Leb})$	Matrix Size
$\mathbf{x^3 - x^2 - x - 1}$	1.8393	0.96422	0.13925	7
$x^3 - x^2 - 1$	1.4656	0.999116	0.0547178	51
$x^3 - x - 1$	1.3247	0.99999	0.0286671	181
$\mathbf{x^4 - x^3 - x^2 - x - 1}$	1.9276	0.973329	0.187067	9
$x^4 - x^3 - 1$	1.3803	0.999989	0.0149032	1257
$\mathbf{x^5 - x^4 - x^3 - x^2 - x - 1}$	1.9659	0.983565	0.222569	11
$x^5 - x^4 - x^3 - x^2 - 1$	1.8885	0.982269	0.0803806	745
$x^5 - x^4 - x^3 - x^2 + 1$	1.7785	0.995758	0.0246573	951
$x^5 - x^4 - x^3 - 1$	1.7049	0.993043	0.0356598	339
$x^5 - x^4 - x^3 - x - 1$	1.8124	0.982434	0.0571201	351
$x^5 - x^4 - x^3 + x^2 - 1$	1.4432	0.999982	0.00782515	5423
$x^5 - x^4 - x^2 - 1$	1.5702	0.999862	0.0195581	847
$x^5 - x^3 - x^2 - x - 1$	1.5342	0.999833	0.00890312	2651

Table 3.1: Pisot numbers $\beta \in (1, 2)$ of degree less than six, together with the Wasserstein distance to normalised Lebesgue measure. Multinacci numbers, which have somewhat different behaviour, are in bold.

Then the distance

$$W_1(\mu_{I_{\beta_n}}, \mathcal{L}_{I_{\beta_n}}) \to 0$$

as $n \to \infty$*, and consequently, by Theorem 3.2.1,* $\dim_H(\nu_{\beta_n}) \to 1$.

Remark. *It worth noting that if* β_n *is the multinacci family then tedious but elementary calculations show that* $W_1(\mu_{I_{\beta_n}}, c\delta_0) \to 0$ *where* c *is a suitable normalising factor. We also see that* $\dim_H(\nu_{\beta_n}) \to 1$ *is still true.*

3.3 A First Example: The Golden Mean

In this section we prove our main theorems for the special case that β is equal to the golden mean ϕ. Throughout we use the maps $T_i : \mathbb{R} \to \mathbb{R}$ given by $T_i(x) = \phi x + i$.

Recall that

$$X(\phi) = X_{\{-1,0,1\}}(\phi) = \left\{ \sum_{i=1}^{n} c_i \phi^{n-i} : n \in \mathbb{N}, c_i \in \{-1, 0, 1\} \right\}$$

and that, for $x \in X(\phi)$,

$$\mathcal{N}_n(x) := \#\{a_1 \cdots a_n, b_1 \cdots b_n \in \{0,1\}^n : \sum_{i=1}^{n} (a_i - b_i)\phi^{n-i} = x\}$$

We give the special case of Theorem 3.1.1 for when $\beta = \phi$.

Theorem 3.3.1. *There exists a number $\lambda > 0$ such that limit*

$$\lim_{n\to\infty} \frac{1}{\lambda^n} \mathcal{N}_n(x) =: \mu(x)$$

exists for each $x \in X(\phi)$.

Here λ is easily computed as the maximal eigenvalue of a finite matrix M_0 defined below. This theorem will be proved as part of the proof of Theorem 3.3.2.

There are several ways to describe the measure μ. One could construct an infinite transition matrix corresponding to dynamics on $X(\phi)$ induced by the maps T_0, T_1, T_{-1} such that the values of $\mu(x)$ correspond to entries of the eigenvector corresponding to the maximal eigenvalue. In particular, for any finite K we can describe $\mu|_{X(\phi)\cap[-K,K]}$ by reading off the values of an eigenvector of a finite matrix. We give instead a harder construction which allows us to see local structure in the measure μ.

Lemma 3.3.1. *There exist matrices M_0, M_1, M_{-1}, each of dimensions 17×17 such that for any $x = \sum_{i=1}^{n} c_i \phi^{n-i} \in X(\phi)$ we have*

$$\mathcal{N}_n(x) = (M_{c_1} \cdots M_{c_n})_{1,1}$$

Proof. This proof is similar to the proof of Lemma 3.1 in [2], we are just using a larger digit set.

If $x = \sum_{i=1}^{n} c_i \phi^{n-i}$ for some word $c_1 \cdots c_n \in \{-1,0,1\}^n$ then we start by tracking words $d_1 \cdots d_n \in \{-1,0,1\}^n$ such that

$$\sum_{i=1}^{n} c_i \phi^{n-i} = \sum_{i=1} d_i \phi^{n-i},$$

i.e.

$$\sum_{i=1}^{n}(c_i - d_i)\phi^{n-i} = 0. \tag{3.3}$$

Here d_i represents a difference $a_i - b_i$ where $a_i, b_i \in \{0, 1\}$, and so when counting words we want to double count the case $d_i = 0$ since it corresponds both to $a_i = b_i = 1$ and $a_i = b_i = 0$. This accounts for the 2 in the definition of the matrices M_0, M_1, M_{-1}.

Now the equality 3.3 is equivalent to

$$T_{c_n - d_n} \circ \cdots \circ T_{c_1 - d_1}(0) = 0, \tag{3.4}$$

where each $c_i - d_i \in \{-2, -1, 0, 1, 2\}$. The maps T_i are expanding, and in particular if $x \geqslant 2\phi$ then $T_i(x) \geqslant 2\phi$, and if $x \leqslant -2\phi$ then $T_i(x) \leqslant -2\phi$, for any $i \in \{-2, -1, 0, 1, 2\}$. Thus if equation 3.3 holds then for each $m \leqslant n$ we have

$$T_{c_m - d_m} \circ \cdots \circ T_{c_1 - d_1}(0) \in (-2\phi, 2\phi).$$

By the Garsia separation lemma, or by direct calculation, one can show that there are a finite number of points of the form $T_{c_m - d_m} \circ \cdots \circ T_{c_1 - d_1}(0)$ which lie in $(-2\phi, 2\phi)$ when $c_i, d_i \in \{-1, 0, 1\}$. In fact there are 17 such points, we call the set of such possible values $V = \{v_1, \cdots, v_{17}\}$ with $v_1 = 0$.

Now in general the difference $c_i - d_i$ can take values in $\{-2, -1, 0, 1, 2\}$, but if we know the value of c_i then $c_i - d_i$ can only take three of these values, if $c_i = 1$ then $c_i - d_i$ can take values 0 1 or 2 for example.

Let M_1 be the 17×17 matrix with rows and columns indexed by elements of V, with

$$(M_1)_{ij} = \begin{cases} 1 & v_j = T_0(v_i) \text{ or } v_j = T_{-2}(v_i) \\ 2 & v_j = T_{-1}(v_i) \\ 0 & \text{otherwise} \end{cases}$$

This is the transition matrix for the maps $T_{c_i-d_i}$ where we know $c_i = 1$ and $d_i \in \{-1, 0, 1\}$, the values 1 and 2 occur because we have one way of letting $d_i = a_i - b_i$ equal 1 or -1 but two ways of letting $d_i = 0$.

Similarly, let M_{-1} be the matrix with rows and columns indexed by elements of V, with

$$(M_{-1})_{ij} = \begin{cases} 1 & v_i = T_0(v_j) \text{ or } v_i = T_2(v_j) \\ 2 & v_i = T_1(v_j) \\ 0 & \text{otherwise} \end{cases}$$

and let M_0 be the matrix with rows and columns indexed by elements of V, with

$$(M_0)_{ij} = \begin{cases} 1 & v_i = T_1(v_j) \text{ or } v_i = T_{-1}(v_j) \\ 2 & v_i = T_0(v_j) \\ 0 & \text{otherwise} \end{cases} .$$

Then given $c_1, \cdots c_n \in \{-1, 0, 1\}^n$, the (i, j)th term of the matrix $M_{c_n} \cdots M_{c_1}$ represents the number of $d_1 \cdots d_n \in \{-1, 0, 1\}$ for which

$$T_{c_n-d_n} \circ \cdots T_{c_1-d_1}(v_i) = v_j. \tag{3.5}$$

Again here when we refer to the 'number' of $d_1 \cdots d_n$ we are double counting when $d_i = 0$ because we have two ways of putting $a_i - b_i = 0$.

Thus in order to count equalities of the form (3.4), we need to use (3.5) with $v_i = v_j = v_1 = 0$. We conclude that the number of $a_1 \cdots a_n, b_1 \cdots b_n$ such that $\sum_{i=1}^{n}(a_i - b_i)\phi^{n-i} = x$ is given by the top left entry of the matrix $M_{c_n} \cdots M_{c_1}$, where $c_1 \cdots c_n$ is any $\{-1, 0, 1\}$ code for which $x = \sum_{i=1}^{n} c_i\phi^{n-i}$.

□

We now state and prove Theorem 3.1.2 for the special case that β is equal to ϕ.

Theorem 3.3.2. *Let W be the left eigenvector of M_0 corresponding to the maximal eigenvalue λ, normalised so that $W_1 = \mu(0)$. Then for any $x = \sum_{i=1}^{n} c_i \phi^{n-i} \in X(\phi)$ we have*

$$\mu(x) = \frac{1}{\lambda^n}(WM_{c_1}M_{c_2}\cdots M_{c_n})_1,$$

that is, $\lambda^n \mu(x)$ is the first entry in the 1×17 vector $WM_{c_1}\cdots M_{c_n}$.

Proof. In the previous lemma we showed how to count the number of words $a_1, \cdots a_n, b_1 \cdots b_n$ with $\sum_{i=1}^{n}(a_i - b_i)\phi^i = x$, given knowledge of one code $c_1 \cdots c_n \in \{-1, 0, 1\}^n$ such that

$$x = \sum_{i=1}^{n} c_i \phi^{n-i}. \tag{3.6}$$

Here it was important that the length of the word $c_1 \cdots c_n$ coding x corresponded with the $\mathcal{N}_n$ which we want to calculate. But if equation 3.6 holds then it is also true that

$$x = \sum_{i=1}^{n} c_i \phi^{n-i} + 0\phi^n + 0\phi^{n+1} + \cdots + 0\phi^{n+(k-1)}.$$

So again using Lemma 3.3.1 we see that

$$\begin{aligned} \mathcal{N}_{n+k}(x) &= (M_0^k M_{c_1} \cdots M_{c_n})_{1,1} \\ &= (1\ 0\ 0 \cdots) M_0^k M_{c_1} \cdots M_{c_n} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \end{aligned}$$

If λ is the maximal eigenvalue of M_0 then, since M_0 is primitive, there exists a corresponding eigenvector W such that

$$\frac{1}{\lambda^k}(1\ 0\ 0 \cdots) M_0^k \to W$$

Putting the previous equations together gives that if $x = \sum_{i=1}^{n} c_i\phi^{n-i}$ then

$$\begin{aligned}
\mu(x) &= \lim_{k\to\infty} \frac{1}{\lambda^{n+k}} \mathcal{N}_{n+k}(x) \\
&= \lim_{k\to\infty} \frac{1}{\lambda^k}\frac{1}{\lambda^n}(1\ 0\ 0\cdots)M_0^k M_{c_1}\cdots M_{c_n}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} \\
&= \frac{1}{\lambda^n} W M_{c_1}\cdots M_{c_n}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}.
\end{aligned}$$

□

It is also important to note that if $x = \sum_{i=1}^{n} c_i\phi^{n-i}$ then the vector $\frac{1}{\lambda^n} W M_{c_1}\cdots M_{c_n}$ doesn't just hold information on $\mu(x)$, which is the first entry, but also holds information on the values of μ at other elements of $X(\phi)$.

Lemma 3.3.2. *For v_k the kth element of V we have*

$$\mu(x+v_k) = \frac{1}{\lambda^n}(W M_{c_1} M_{c_2}\cdots M_{c_n})_k,$$

that is, $\lambda^n\mu(x+v_k)$ is the kth entry in the 1×17 vector $W M_{c_1}\cdots M_{c_n}$.

Proof. This follows directly from the proof of the previous lemma and equation 3.5. □

This allows us to start to discuss local structure for μ. We want to describe how one can use dynamics to move through the measure μ and write down the set of pairs $\{(x,\mu(x)) : x \in X(\phi)\}$. To do this, we must first recall the cut and project structure of the set $X(\phi)$.

3.3.1 The Structure of $X(\phi)$

The work of this subsection is well known to experts. We first show that set $X(\phi)$ can be dynamically generated. One can move from a level-n sum to a level-$(n+1)$ sum in the construction of $X(\phi)$ by observing that

$$\sum_{i=1}^{n+1} c_i \phi^{n+1-i} = \phi \left(\sum_{i=1}^{n} c_i \phi^{n-i} \right) + c_{n+1}.$$

Thus with $T_i(x) := \phi x + i$ as before we see that

$$X(\phi) = \{T_{c_n} \circ \cdots \circ T_{c_1}(0) : n \in \mathbb{N}, c_i \in \{-1, 0, 1\}\}. \tag{3.7}$$

As $\phi^2 = \phi + 1$ we can consider multiplication by ϕ in terms of its action on numbers of the form $z_1\phi + z_0$. We let $\pi_e : \mathbb{Z}^2 \to \mathbb{R}$ be given by

$$\pi_e \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} := z_1\phi + z_0$$

and $\pi_c : \mathbb{Z}^2 \to \mathbb{R}$ be given by

$$\pi_c \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} := \frac{-1}{\phi} z_1 + z_0.$$

We will later refer to π_e as projection in the expanding direction and π_c as projection in the contracting direction. Note that $\pi_e : \mathbb{Z}^2 \to \mathbb{R}$ and $\pi_c : \mathbb{Z}^2 \to \mathbb{R}$ are injective (if they were not then $x^2 - x - 1$ would not be the minimal polynomial of ϕ).

Then

$$\phi \left(\pi_e \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \right) = z_1\phi^2 + z_0\phi = (z_1 + z_0)\phi + z_1 = \pi_e \left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \right)$$

and so $T_i : X(\phi) \to X(\phi)$ lifts to a map $\tilde{T}_i : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by

$$\tilde{T}_i \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} + \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

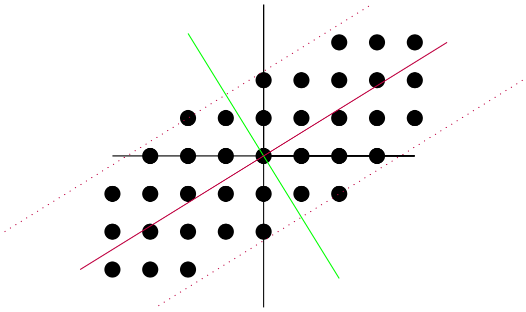

Figure 3.1: The set $\tilde{X}(\phi)$ around the origin, with expanding and contracting eigenvectors shown.

We let

$$\tilde{X}(\phi) := \left\{ \tilde{T}_{c_n} \circ \cdots \circ \tilde{T}_{c_1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} : n \in \mathbb{N}, c_i \in \{-1, 0, 1\} \right\}$$

and have the relation $X(\phi) = \pi_e(\tilde{X}(\phi))$.

One can study the structure of $X(\phi)$ directly on the real line, this was done for example in [34] where the substitution structure of $X(\phi)$ was described. However, some properties of $X(\phi)$ are easier to see if we first study the structure of $\tilde{X}(\phi)$. For example, from equation (3.7) we see that the uniformly discrete set $X(\phi)$ is a subset of the dense set $\{z_1\phi + z_0 : z_1, z_0 \in \mathbb{Z}\}$, but it is not immediately apparent which values of (z_1, z_0) correspond to points in $X(\phi)$.

Lifting to $\tilde{X}(\phi)$ the structure becomes clear. The matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has one expanding eigenvector and one contracting eigenvector, and the maps $\tilde{T}_i$ can be described in terms of their action on points written in terms of these eigenvectors.

Note that if $\pi_c \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} = x$ then

$$\pi_c(\tilde{T}_i \begin{pmatrix} z_1 \\ z_0 \end{pmatrix}) = \frac{-x}{\phi} + i =: S_i(x).$$

Then the system $\{S_0, S_1, S_{-1}\}$ is a contracting iterated function system with attractor $[-\phi^2, \phi^2]$, and so for any point $\begin{pmatrix} z_1 \\ z_0 \end{pmatrix} = \tilde{T}_{a_n} \circ \cdots \tilde{T}_{a_1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in X(\phi)$ we have $\pi_c \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} = S_{a_n} \circ \cdots S_{a_1}(0) \in (-\phi^2, \phi^2)$. The converse is also true and is contained in the following lemma.

Lemma 3.3.3. *The set $\tilde{X}(\phi)$ consists of all pairs $\begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \in \mathbb{Z}^2$ for which $\pi_c \begin{pmatrix} z_1 \\ z_0 \end{pmatrix}$ lies in the interval $(-\phi^2, \phi^2)$.*

Furthermore, if $\pi_c \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \in S_{d_1} \circ \cdots S_{d_k}(-\phi^2, \phi^2)$ for some $d_1, \cdots, d_k \in \{-1, 0, 1\}^k$ then for all sufficiently large n there exists a word $c_1 \cdots c_{n+k} \in \{-1, 0, 1\}^{n+k}$ with $c_{n+k} \cdots c_1 = d_1 \cdots d_k$ and such that

$$\begin{pmatrix} z_1 \\ z_0 \end{pmatrix} = \tilde{T}_{c_{n+k}} \circ \cdots \circ T_{c_1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Proof. One inclusion was proved in the paragraph before the statement of this lemma.

Now let $(z_1, z_0) \in \mathbb{Z}^2$ have $\pi_c(z_1, z_0) \in (-\phi^2, \phi^2)$. We wish to find a word $c_1 \cdots c_n$ such that

$$\begin{pmatrix} z_1 \\ z_0 \end{pmatrix} = \tilde{T}_{c_n} \circ \cdots \tilde{T}_{c_1} \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or equivalently

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \tilde{T}_{c_1}^{-1} \circ \cdots \tilde{T}_{c_n}^{-1} \begin{pmatrix} z_1 \\ z_0 \end{pmatrix}. \tag{3.8}$$

We first observe that for any $\begin{pmatrix} z_1 \\ z_0 \end{pmatrix}$ with $\pi_c \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \in (-\phi^2, \phi^2)$ and $\pi_e \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \in [-\phi, \phi]$ one can find words $c_1 \cdots c_n$ such that Equation 3.8 holds. Since there are only finitely many pairs $\begin{pmatrix} z_1 \\ z_0 \end{pmatrix}$ in this bounded region one can check this observation with a finite calculation.

Now let $\begin{pmatrix} z_1 \\ z_0 \end{pmatrix}$ have $\pi_c \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \in (-\phi^2, \phi^2)$, but place no restriction on $\pi_e \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \in (-\phi, \phi)$. By the IFS construction of the contracting interval, we can choose arbitrarily long words $i_1 \cdots i_n \in \{-1, 0, 1\}$ such that $\tilde{T}_{i_n}^{-1} \circ \cdots \tilde{T}_{i_1}^{-1}(\begin{pmatrix} z_1 \\ z_0 \end{pmatrix})$ still has contracting coordinate in the interval $(-\phi^2, \phi^2)$. But since inverse maps $\tilde{T}_i^{-1}$ contract the expanding direction, the expanding coordinate will eventually lie in $[-\phi, \phi]$, and by the previous paragraph we know that we can return to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Finally we not that if we had $\pi_c \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \in S_{d_1} \circ \cdots S_{d_k}(-\phi^2, \phi^2)$ then we can choose the word $i_1 \cdots i_n$ to start with $d_1 \cdots d_k$.

$\square$

It is worth stressing that the first three quarters of the preceeding proof generalises easily to any algebraic integer β, but the finite check that any integer pair suitably close[3] to the origin can return to the origin under the maps $\tilde{T}_i^{-1}$ needs verifying for each β and we don't know that it is always true.

One interesting consequence of Lemma 3.3.3 is that in order to understand the distance from some point $\tilde{T}_{c_n} \circ \cdots \tilde{T}_{c_1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to its close neighbours in $\tilde{X}(\phi)$, we need only to know about $\pi_c(\tilde{T}_{c_n} \circ \cdots \tilde{T}_{c_1} \begin{pmatrix} 0 \\ 0 \end{pmatrix})$.

[3]For hyperbolic non-Pisot β we will also require that expansions of Galois conjugates are close to the origin, see section 3.4.

Given $x \in X(\phi)$ let $\tilde{x}$ denote the corresponding point in $\tilde{X}(\phi)$ and let $x_c = \pi_c(\tilde{x})$. For $K \in \mathbb{R}$ let $x \in X(\phi)$. Call the set

$$(X(\phi) - x) \cap [-K, K] = \{y - x : y \in X(\phi), y - x \in [-K, K]\}$$

the K-neighbourhood of x.

Lemma 3.3.4. *[Local Structure for $X(\phi)$] For any $K > 0$ there exists a finite partition of $(-\phi^2, \phi^2)$ such that the K-neighbourhood of any $x \in X(\phi)$ depends only upon which partition element of $(-\phi^2, \phi^2)$ x_c lies in.*

Proof. This follows from the analagous statement for $\tilde{X}(\phi)$, which has a fairly direct proof following Lemma 3.3.3, since one needs only to consider which translations in $\mathbb{Z}^2$ can be performed without leaving the contracting window or moving by a distance of more than K in the expanding direction. □

Finally, we outline how to use dynamics to describe the odometer map which maps $x \in X(\phi)$ to $\min\{y \in X(\phi) : y > x\}$.

Let $d : X(\phi) \to \mathbb{R}^+$ denote the distance from $x \in X(\phi)$ to $\min\{y \in X(\phi) : y > x\}$. That is, let d be defined by

$$d(x) = \min\{y \in X(\phi) : y > x\} - x.$$

Proposition 3.3.1. *The odometer map $x \to x + d(x)$ on $X(\phi)$ lifts to the skew-product map $O : X(\phi) \times X_c(\phi) \to X(\phi) \times X_c(\phi)$ by*

$$\tilde{d}(x, x_c) = \begin{cases} (x + 2\phi - 3, x_c - \frac{2}{\phi} - 3) & x_c \in [\phi, \phi^2] \\ (x + \phi - 1, x_c - 1 - \frac{1}{\phi}) & x_c \in (0, \phi) \\ (x + 2 - \phi, x_c + 2 + \frac{1}{\phi}) & x_c \in [-\phi^2, 0] \end{cases}$$

We stress here that the action of O on the contracting direction is of a uniquely ergodic interval exchange transformation.

Proof. The fact that there is some partition of $(-\phi^2, \phi^2)$ telling us how to evolve a skew-product map which is a lift of d follows immediately from Lemma 3.3.4 with $K = \phi - 1$. It is a finite calculation to write down the map exactly. $\square$

3.3.2 An Odometer map for μ

Proposition 3.3.1 dealt with how one can move locally through the set $X(\phi)$ using only knowledge on the position in the contracting direction, we want to build a similar theorem which also incorporates knowlede of the values $\mu(x)$, we do this by building a cocycle over the odometer map O.

Given $x \in X(\phi)$ let x_c denote the corresponding point in the contracting window $(-\phi^2, \phi^2)$. We recall from Lemma 3.3.3 that for $x \in X(\phi)$ and for any word $d_1 \cdots d_k$, x can be written $x = \sum_{i=1}^{n} c_i \phi^{n-i}$ with $c_{n-k+1} \cdots c_n = d_k \cdots d_1$ if and only if $x_c \in S_{d_1} \circ \cdots \circ S_{d_n}(-\phi^2, \phi^2)$.

Now let us map real 1×17 vectors U with strictly positive first entry onto the corresponding projective space by letting

$$(U')_i = \frac{(U)_{i+1}}{(U)_1}$$

for $(1 \leqslant i \leqslant 16)$. in particular, we associate to each $x = \sum_{i=1}^{n} c_i \phi^{n-i} \in X(\phi)$ the corresponding vector $V(x) = (WM_{c_1}M_{c_2} \cdots M_{c_n})'$ considered as an element of real projective space. To be concrete, we define the 1×16 vector $V(x)$ by

$$(V(x))_i = \frac{(WM_{c_1}M_{c_2} \cdots M_{c_n})_{i+1}}{(WM_{c_1}M_{c_2} \cdots M_{c_n})_1} = \frac{\mu(x+v_i)}{\mu(x)}.$$

It follows from the proofs of the previous two statements that these vectors do not depend on the choice of code $c_1 \cdots c_n$ of x. We can also write $V(x)$ as a function $V(x_c)$ of the position in the contracting window.

Consider the metric d on the space of 1×16 non-negative vectors by letting

$$d(U, V) = \max_{i \in \{1, \cdots 16\}} |\log(V_i) - \log(U_i)|.$$

Two vectors U, V are at infinite distance from one another if there exist $i, j \in \{1 \cdots 16\}$ such that $U_i = 0$ $V_i \neq 0$ or $V_i = 0$ $U_i \neq 0$.

Lemma 3.3.5. *Suppose that A is a 17×17 matrix with $A_{1,1} > 0$ such that for any pair of parameters $(i, j) \in \{1, \cdots, 17\}^2$ one of the following holds*

1. *(i, j) is in a zero row, i.e. $(A)_{i',j} = 0$ for all $i' \in \{1, \cdots, 17\}$*

2. *(i, j) is in a zero column, i.e. $(A)_{i,j'} = 0$ for all $j' \in \{1, \cdots, 17\}$*

3. *$(A)_{i,j} > 0$.*

Then there exists a constant $C < 1$ such that, for any 1×17 vectors U, V with positive first entries and with $d(U', V') < \infty$ we have

$$d((UA)', (VA)') < Cd(U', V').$$

Furthermore, there exists $K > 0$ such that, for any any 1×17 vectors U, V with positive first entry (and possibly with $d(U', V') = \infty$),

$$d((UA)', (VA)') < K.$$

This lemma is proved carefully in section 3.4.

Lemma 3.3.6. *The matrix M_0^7 satisfies the condition of Lemma 3.3.5.*

This can be verified by a short calculation.

One can also see that given a 17×17 non-negative matrix B withstrictly positive top left entry and two 1×17 vectors U and V with strictly positive first entries,

$$d((UA)', (VA)') \leqslant d(U', V').$$

This shows that matrices M_0, M_1 and M_{-1} do not expand distances between vectors in our metric.

Finally we are able to state Theorem 3.1.3 in the special case that $\beta = \phi$ and dealing only with nearest neighbours. Recall that, for $x \in X(\phi)$, $d(x) := \min\{y - x : y \in X(\phi), y > x\}$.

Proposition 3.3.2. *For $x \in X(\phi)$ with corresponding point $x_c \in (-\phi^2, \phi^2)$ define $f(x_c)$ by*

$$\log(\mu(x + d(x))) - \log(\mu(x)) = f(x_c).$$

Then f is bounded and is continuous at each $x_c \in X_c(\phi)$ except for 0 and ϕ.

If we defined d' on (ϕ^2, ϕ^2) by $d'(x_c) := d(x)$ then 0 and ϕ are the points in (ϕ^2, ϕ^2) where $d'(x_c)$ is not continuous.

Proof. We have already shown that

$$d(x) = \begin{cases} 2\phi - 3 & x_c \in [\phi, \phi^2) \\ \phi - 1 & x_c \in (0, \phi) \\ 2 - \phi & x_c \in (-\phi^2, 0] \end{cases}$$

One can check that each of $2\phi - 3, \phi - 1$ and $2 - \phi$ correspond to entries v_k of V. Then by Lemma 3.3.2 we see that

$$f(x_c) := \log(\mu(x + d(x))) - \log(\mu(x))$$

appears as the log of a ratio of two entries in the vector $(WM_{c_1} \cdots M_{c_n})$ for any $c_1 \cdots c_n$ coding x. Since both x and $x + d(x)$ have strictly positive mass, the difference of the logs is finite so $f(x_c) \in \mathbb{R}$.

We now discuss the continuity properties of f. Let $x \in X(\phi)$ and $\epsilon > 0$ be given. Let K and C be the quantities introduced in Lemma 3.3.5 associated to M_0^7, and let $r \in \mathbb{N}$ be such that $KC^{r-1} < \epsilon$. Let $c_1 \cdots c_n$ be a code of x containing at least r copies of the word 0000000, this can be done for example by taking any expansion of x and adding lots of zeros to the start.

Now x_c is contained in the interval $S_{c_n} \circ S_{c_{n-1}} \circ \cdots \circ S_{c_1}(-\phi^2, \phi^2)$. Let $y \in X(\phi)$ be another point with $y_c \in S_{c_n} \circ S_{c_{n-1}} \circ \cdots \circ S_{c_1}(-\phi^2, \phi^2)$. Then y can be written $y = \sum_{d=1}^{m} d_i \phi^{m-i}$ for some code $d_1 \cdots d_m$ with $d_{m-n} \cdots d_n = c_1 \cdots c_n$, as in Lemma 3.3.3.

Assume that x_c and y_c lie in the same one of the intervals $(-\phi^2, 0]$,$(0, \phi)$, $[\phi, \phi^2)$ so that $d(x) = d'(x_c) = v_j$. Then

$$\begin{aligned}
|f(x_c) - f(y_c)| &= |\log(WM_{c_1} \cdots M_{c_n})_j - \log(WM_{d_1} \cdots M_{d_m})_j| \\
&= |\log(WM_{c_1} \cdots M_{c_n})_j - \log(WM_{d_1} \cdots M_{d_{m-n-1}} M_{c_1} \cdots M_{c_n})_j| \\
&\leqslant d((WM_{c_1} \cdots M_{c_n})', (\underbrace{WM_{d_1} \cdots M_{d_{m-n-1}}}_{=:U} M_{c_1} \cdots M_{c_n})') \\
&= d((WM_{c_1} \cdots M_{c_n})', (UM_{c_1} \cdots M_{c_n})') \leqslant KC^{r-1} < \epsilon.
\end{aligned}$$

Here the final line follows since $c_1 \cdots c_n$ contains r non-overlapping occurences of the word M_0^7, the first of which guarantees that

$$d((WM_{c_1} \cdots M_{c_n})', (UM_{c_1} \cdots M_{c_n})') < K$$

and the subsequent $r-1$ of which multiply this upper bound by C, thanks to Lemmas 3.3.5 and 3.3.6. □

We have now completed the proofs of analogues of Theorems 3.1.1, 3.1.2, and 3.1.3 in the special case of the golden mean, although the analogue of 3.1.3 we did only for moving to nearest neighbours.

Putting everything together, we get the following theorem which demonstrates how one can move through the measure μ on $X(\phi)$, and how one could start to study it using ergodic theory.

Theorem 3.3.3. *Let the map* $\psi : X(\phi) \times (-\phi^2, \phi^2) \times \mathbb{R}$ *be given by*

$$\phi(x, y, z) = \begin{cases} (x + 2\phi - 3, y - \frac{2}{\phi} - 3, z + f(y)) & y \in [\phi, \phi^2) \\ (x + \phi - 1, y - \frac{1}{\phi} - 1, z + f(y)) & y \in (0, \phi) \\ (x + 2 - \phi, y + 2 + \frac{1}{\phi}, z + f(y)) & y \in (-\phi^2, 0] \end{cases}$$

Then if x is the nth element to the right of 0 *in* $X(\phi)$ *we have that*

$$(x, x_c, \mu(x)) = \psi^n(0, 0, 0).$$

Thus we have that many of the properties of μ can be studied by studying ψ, which is really a skew-product over an interval exchange transformation on the contracting window (ϕ^2, ϕ^2).

3.4 Measures on the spectra of general hyperbolic algebraic integers

In this section we show how to extend the previous work to general hyperbolic algebraic integers and prove Theorems 3.1.1, 3.1.2 and 3.1.3. As stated in the introduction, the motivation is to study measures of the form

$$\mu_n(x) = \frac{1}{4^n}\#\{a_1 \cdots a_n, b_1 \cdots b_n \in \{0,1\}^n : \sum_{i=1}^{n}(a_i - b_i)\beta^{n-i} = x\}.$$

Given β, we lift μ_n to a measure $\bar{\mu}_n$ living on a lattice subset of a multidimensional euclidean space $\mathbb{K}$. We prove that there is $\lambda > 0$ such that $4^n\bar{\mu}_n/\lambda^n$ converges to a measure $\bar{\mu}$. We also prove that there are local patterns in the measure $\bar{\mu}$ that repeat in a way that we understand. This means that we understand how the measure of a lattice point changes when we move to nearby points on the lattice[4]. In particular there is a non-trivial linear subspace $\mathbb{K}_c$ of $\mathbb{K}$ such that the following holds. Under conditions and given a suitable vector d then for typical x the ratio $\frac{\bar{\mu}(x+d)}{\bar{\mu}(x)}$ is determined, up to certain accuracy, by the approximate position

[4]We don't state an analogue of Theorem 3.3.3 for the higher dimensional case since there is no natural choice of 'next point' to move to when we are working in higher dimensional Euclidean space. One could state such results, perhaps by identifying a strip which is infinite in only one direction and describing the dynamics to move through such a strip.

of the orthogonal projection of x on $\mathbb{K}_c$. That is the numbers of the form $\frac{\bar{\mu}(x+d)}{\bar{\mu}(x)}$ are approximately equal for all x projecting on to the same small region of $\mathbb{K}_c$.

Let $\beta = \beta_1 \in (1,2)$ be an algebraic integer with Galois conjugates $\beta_2, ..., \beta_d, \beta_{d+1}, ..., \beta_{d+s}$ such that $|\beta_2|, ..., |\beta_d| > 1$ and $|\beta_{d+1}|, ..., |\beta_{d+s}| \in (0,1)$. Further define $\bar{\beta}^n = (\beta_1^n, ..., \beta_{d+s}^n)$. For this section we let

$$T_i(x_1, ..., x_{d+s}) = (\beta_1 x_1 + i, ..., \beta_{d+s} x_{d+s} + i),$$

these maps are higher dimensional lifts of their analogues in the previous section. For Galois conjugates $\beta_i \in \mathbb{C}$ let $\mathbb{F}_{\beta_i} = \mathbb{R}$ if $\beta_i \in \mathbb{R}$ and $\mathbb{F}_{\beta_i} = \mathbb{C}$ if $\beta_i \in \mathbb{C} \setminus \mathbb{R}$. We define the sets

$$\begin{aligned} \mathbb{K} &:= \prod_{i=1}^{d+s} \mathbb{F}_{\beta_i}, \\ \mathbb{K}_c &:= \{0\}^d \times \mathbb{F}_{\beta_{d+1}} \times ... \times \mathbb{F}_{\beta_{d+s}}, \\ \bar{Z} &:= \{a_{d+s-1}\bar{\beta}^{d+s-1} + ... + a_0\bar{\beta}^0 : a_{d+s-1}, ..., a_0 \in \mathbb{Z}\}, \end{aligned}$$

and

$$\begin{aligned} \bar{X}(\beta) &:= \left\{ \sum_{i=1}^{n} a_i \bar{\beta}^{n-i} : n \in \mathbb{N}, a_1 ..., a_n \in \{-1,0,1\} \right\} \\ &= \{T_{a_n} \circ ... \circ T_{a_1}(0) : n \in \mathbb{N}, a_1 ..., a_n \in \{-1,0,1\}\} \end{aligned}$$

where 0 denotes the origin in $\mathbb{K}$.

The set $\bar{Z}$ is a lattice in $\mathbb{K} \cong \mathbb{R}^{\sum_{i=1}^{d+s} \dim(\mathbb{F}_{\beta_i})}$. That is because $\{\bar{\beta}^0, ..., \bar{\beta}^{d+s-1}\}$ is an independent subset of the real vector space $\mathbb{K}$. That can be checked using the formula for the determinant of the Vandermonde matrix. It is useful to keep in mind that for each $i \in \mathbb{Z}$ we have $T_i(\bar{Z}) \subseteq \bar{Z}$, in particular $\bar{X}(\beta) \subseteq \bar{Z}$.

Notice that all coordinate projections, restricted on $\bar{Z}$, are injective so there is in a sense a natural identification of $\bar{Z}$ to any image of it under a coordinate projection. Here by a coordinate projection we mean any map from $\mathbb{K}$ to itself, of the form $(a_1, ..., a_{d+s}) \mapsto (a_1\kappa_1, ..., a_{d+s}\kappa_{d+s})$ where $\kappa_1, ..., \kappa_{d+s} \in \{0, 1\}$. As in the one dimensional case, we define the measure $\bar{\mu}_n$ on $\bar{Z}$ by

$$\bar{\mu}_n(x) = \frac{1}{4^n}\bar{\mathcal{N}}_n(x)$$

where

$$\bar{\mathcal{N}}_n(x) = \# \left\{ (a_1, ..., a_n, b_1, ..., b_n) \in \{0, 1\}^{2n} : \sum_{i=1}^{n} a_i \bar{\beta}^{n-i} - \sum_{i=1}^{n} b_i \bar{\beta}^{n-i} = x \right\},$$

for $x \in \bar{Z}$. It is immediate that $\bar{\mu}_n(\bar{Z} \backslash \bar{X}(\beta)) = 0$, that $\bar{\mu}_n(x) = \mu_n(x_1)$ and $\bar{\mathcal{N}}_n(x) = \mathcal{N}_n(x_1)$. We set

$$\pi_c(x_1, \cdots, x_{d+s}) = (x_{d+1}, ..., x_{d+s})$$

to be the projection onto the contracting directions, and $S_i := (\pi_c \circ T_i)|_{\mathbb{K}_c}$. The maps S_i are contractions. Let $\mathcal{R}$ be the attractor of the overlapping iterated function scheme $\{S_{-1}, S_0, S_1\}$. We have immediately that

$$\begin{aligned} \pi_c(\bar{X}(\beta)) &= \pi_c \{T_{a_n} \circ ... \circ T_{a_1}(\underline{0}) : n \in \mathbb{N}, a_1 ..., a_n \in \{-1, 0, 1\}\} \\ &= \{S_{a_n} \circ ... \circ S_{a_1}(\underline{0}) : n \in \mathbb{N}, a_1 ..., a_n \in \{-1, 0, 1\}\} \subset \mathcal{R} \end{aligned}$$

since $0 \in \mathcal{R}$.

Definition 3.4.1. *Let $a = (a_1, ..., a_n) \in \{-1, 0, 1\}^n$. We define $[a] := S_{a_1} \circ ... \circ S_{a_n}(\mathcal{R})$.*

Finally we define a set of small differences between points in $\overline{X}(\beta)$.

Definition 3.4.2. *Let*

$$\begin{aligned} \Delta = \{x - y : &x, y \in \overline{X}(\beta) \text{ and} \\ &\exists c_1 \cdots c_n, d_1 \cdots d_n \in \{-1, 0, 1\}^n : T_{c_n} \circ \cdots T_{c_1}(x) = T_{d_n} \cdots T_{d_1}(y)\}. \end{aligned}$$

That is, Δ is the set of differences between points $x, y \in \overline{X}(\beta)$ which can be mapped to the same point in the future by the application of maps T_i. Δ is finite, we write $\Delta = \{v_1, \cdots, v_k\}$ with $v_1 = 0$.

In this section we prove Theorems 3.1.1, 3.1.2 and 3.1.3 by proving higher dimensional analogues. In particular, in subsection 3.4.1 we prove that, for some $\lambda > 0$, the measure $\frac{\bar{\mu}_n}{\lambda^n}$ converges to an infinite stationary measure $\bar{\mu}$ (Proposition 3.4.1, which has Theorem 3.1.1 as a direct corollary.

In subsection 3.4.2 we define matrices A_{-1}, A_0, A_1 playing the role of M_{-1}, M_0, M_1 of the Golden mean example. Given a point $x = T_{a_n} \circ ... \circ T_{a_1}(0)$, where $a_i \in \{-1, 0, 1\}$, we use the matrix $A_{a_1} \cdot ... \cdot A_{a_n}$ to compute the measure $\bar{\mu}$ locally around x (Proposition 3.4.1), which has Theorem 3.1.2 as a direct corollary.

Finally in subsection 3.4.3 we show that information about the position of $\pi_c(x)$ determines the last few elements $a_\kappa, ..., a_n$ of a code of x. This allow us to use arguments involving a modified Birkhoff metric, presented in 3.5.2, on the product $A_{a_1} \cdot ... \cdot A_{a_n}$ to estimate the local measure around x based on information about $\pi_c(x)$. This gives rise to Proposition 3.4.5, which has Theorem 3.1.3 as a corollary, as explained directly after the proof of Proposition 3.4.5.

3.4.1 The limit measure $\bar{\mu}$

We will denote the vector space of signed measures on $\bar{Z}$ by $\mathcal{M}(\bar{Z})$. For $\nu \in \mathcal{M}(\bar{Z})$ we set

$$||\nu|| = \sum_{x \in \bar{Z}} |v(x)|.$$

There is a recursive way to go from $\bar{\mu}_n$ to $\bar{\mu}_{n+1}$ which gives a dynamical description of $\bar{\mu}_n$.

$$\bar{\mu}_{n+1}(x) = \#\left\{(a_1, ..., a_{n+1}, b_1, ..., b_{n+1}) \in \{0,1\}^{2n} : \sum_{i=1}^{n+1} a_i \bar{\beta}^{n+1-i} - \sum_{i=1}^{n+1} b_i \bar{\beta}^{n+1-i} = x\right\}$$
$$= \#\left\{(a_1, ..., a_{n+1}, b_1, ..., b_{n+1}) \in \{0,1\}^{2(n+1)} : T_{a_{n+1}-b_{n+1}}\left(\sum_{i=1}^{n} a_i \beta^{n-i} - \sum_{i=1}^{n} b_i \beta^{n-i}\right) = x\right\}$$
$$= \sum_{(a,b)\in\{0,1\}^2} \#\left\{(a_1, ..., a_n, b_1, ..., b_n) \in \{0,1\}^{2n} : \sum_{i=1}^{n-1} a_i \beta^{n-i} - \sum_{i=1}^{n-1} b_i \beta^{n-i} = T_{a-b}^{-1}(x)\right\}$$
$$= \sum_{(a,b)\in\{0,1\}^2} \bar{\mu}_n(T_{a-b}^{-1}(x)).$$

Definition 3.4.3. *We define the operator L on $\mathcal{M}(Z)$ by letting*

$$(L(\nu))(A) := \sum_{(a,b)\in\{0,1\}^2} \nu(T_{a-b}^{-1}(A)).$$

for $A \subset Z$.

Then $\bar{\mu}_n$ satisfies

$$\bar{\mu}_n = L^n \bar{\mu}_0.$$

Lemma 3.4.1. *For all $n \in \mathbb{N}$ and $y \in \bar{X}(\beta)$ we have $\bar{\mu}_n(y) \leqslant \bar{\mu}_n(0)$.*

Proof. This follows from the Cauchy-Schwarz inequality. Define

$$\mu_n'(x) = \#\left\{a_1, ..., a_n \in \{0,1\}^n : \sum_{i=1}^{n} a_i \bar{\beta}^{n-i} = x\right\}$$

By the construction of $\bar{\mu}_n$ we have that

$$\begin{aligned}\bar{\mu}_n(y) &= \sum_{x\in\bar{Z}} \mu'_n(x)\mu'_n(x+y) \\ &\leqslant \left(\sum_{x\in\bar{Z}} \mu'_n(x)^2\right)^{1/2} \left(\sum_{x\in\bar{Z}} \mu'_n(x+y)^2\right)^{1/2} \\ &\leqslant \left(\sum_{x\in\bar{Z}} \mu'_n(x)^2\right)^{1/2} \left(\sum_{x\in\bar{Z}} \mu'_n(x)^2\right)^{1/2} \\ &= \sum_{x\in\bar{Z}} \mu'_n(x)^2 \\ &= \sum_{x\in\bar{Z}} \mu'_n(x)\mu'_n(x) \\ &= \bar{\mu}_n(0)\end{aligned}$$

□

Now we prove that the measure $\bar{\mu}$ exists. To do this, we show that it exists on arbitrarily large neighbourhoods of the origin. Let

$$I_{\beta_i}(R) = \begin{cases} \left(\frac{-R}{||\beta_i|-1|}, \frac{R}{||\beta_i|-1|}\right), & \beta_i \in \mathbb{R}\backslash\{-1,1\} \\ \left\{z \in \mathbb{C} : |z| < \frac{R}{||\beta_i|-1|}\right\}, & \beta_i \in \{z \in \mathbb{C} : |z| \neq 1\} \backslash \mathbb{R} \end{cases},$$

$B_\beta(R) = \Pi_{i=1}^{d+s} I_{\beta_i}(R)$, and

$$\bar{X}_R(\beta) := \bar{X}(\beta) \cap B_\beta(R).$$

Observe that

$$T_i(\bar{X}(\beta)\backslash\bar{X}_R(\beta)) \subseteq \bar{X}(\beta)\backslash\bar{X}_R(\beta)$$

for $R \geqslant 1$ and $i \in \{-1,0,1\}$. This means that, for $R > 1$ and $x \in \bar{X}_R(\beta)$, any word $a_1 \cdots a_n$ for which $T_{a_n} \circ \cdots T_{a_1}(0) = x$ has that all the intermediate orbit points $T_{a_m} \circ \cdots T_{a_1}(0)$ for $m < n$ also lie in $\bar{X}_R(\beta)$. Thus, for $x \in \bar{X}_R(\beta)$ we can compute $\bar{\mathcal{N}}_n(x)$ just by studying the dynamics of the maps T_i restricted to $\bar{X}_R(\beta)$.

Since $\bar{X}_R(\beta)$ is a bounded subset of a lattice, it is finite, we enumerate its elements $\{x_1, \cdots x_{k_R}\}$ with $x_1 = 0$. Then we write down the matrix

$$\Lambda_R(i,j) = \begin{cases} 1 & \text{if} \quad T_1(x_i) = x_j \text{ or} T_{-1}(x_i) = x_j \\ 2 & \text{if} \quad T_0(x_i) = x_j \\ 0 & \text{otherwise} \end{cases}.$$

which encodes the dynamics on $\bar{X}_R(\beta)$ given by the maps T_i. Then since $\bar{\mathcal{N}}_n(x_j)$ counts the number of length n orbit pieces from 0 to x_j under the maps T_0, T_1, T_{-1}, double counting for each use of T_0, we see that

$$\bar{\mathcal{N}}_n(x_j) = (\Lambda_R^n)_{1,j}.$$

From $T_i(\bar{X}(\beta)\backslash\bar{X}_1(\beta)) \subset \bar{X}(\beta)\backslash\bar{X}_1(\beta)$ we get that the irreducible component of Λ_R that contains the zero point is contained in $\bar{X}_1(\beta)$ so by lemma 3.4.1 we have that the spectral radius of Λ_R is equal to the spectral radius of Λ_1 for all $R > 1$.

Definition 3.4.4. *We set* $\lambda := \rho(\Lambda_1)$.

Now if we knew that the matrices Λ_R were irreducible, the existence of μ would be immediate. As it is we require the following lemma, the proof of which is postponed to the appendix.

Lemma 3.4.2. *Let A be a non-negative $N \times N$ matrix and $e_1 = (1,0,0,...,0) \in \mathbb{R}^N$. Assume that*

i) $A(1,1) > 0$,

ii) there exists $n \in \mathbb{N}$ such that e_1A^n is stricly positive,

iii) $e_1A^n(i) \leqslant e_1A^n(1)$ *for all* $n \in \mathbb{N}$ *and* $i \in \{1,...,N\}$, *then* $\lim_{n\to\infty} e_1 A^n/\rho(A)^n$ *exists.*

Now by the construction of Λ_R and by Lemma 3.4.1 and Lemma 3.4.2 we have the following proposition.

Proposition 3.4.1. *For each* $x \in \bar{X}(\beta)$

$$\bar{\mu}(x) := \lim_{n\to\infty} \frac{\bar{\mathcal{N}}_n(x)}{\lambda^n}$$

exists, defining a measure $\bar{\mu} \in \mathcal{M}(\bar{Z})$.

We conclude this section with three lemmas showing that the measure μ is invariant under L, that $\lambda < 4$, and that the total mass of the measure μ is infinite.

Lemma 3.4.3. $L\bar{\mu} = \lambda\bar{\mu}$

Proof. For all $x \in \bar{X}(\beta)$ we have

$$\begin{aligned} L\bar{\mu}(x) &= \bar{\mu}(T_{-1}^{-1}(x)) + 2\bar{\mu}(T_0^{-1}(x)) + \bar{\mu}(T_1^{-1}(x)) \\ &= \lim_{n\to\infty} \frac{1}{\lambda^n} \left(\bar{\mu}_n(T_{-1}^{-1}(x)) + 2\bar{\mu}_n(T_0^{-1}(x)) + \bar{\mu}_n(T_1^{-1}(x))\right) \\ &= \lim_{n\to\infty} \frac{1}{\lambda^n} L\bar{\mu}_n(x) \\ &= \lambda \lim_{n\to\infty} \frac{1}{\lambda^{n+1}} \bar{\mu}_{n+1}(x) \\ &= \lambda\bar{\mu}(x). \end{aligned}$$

□

For sets X, measures $\nu \in \mathcal{M}(X)$ and measurable sets $A \subset X$ we let $\nu|_A$ be such that $\nu|_A(B) = \nu(A \cap B)$ for all measurable $B \subset X$.

Lemma 3.4.4. $\lambda < 4$

Proof. It is clear that if $\nu \in \mathcal{M}(\bar{Z})$ is such that

$$||\nu|| < \infty$$

then

$$||L\nu|| = 4||\nu||.$$

Note that $L(\bar{\mu}) = \lambda\bar{\mu}$ and

$$L\left(\bar{\mu}|_{\bar{X}_1(\beta)}\right)|_{\bar{X}_1(\beta)} = \lambda\bar{\mu}|_{X_1(\beta)},$$

but

$$\left|\left|\left(L\left(\bar{\mu}|_{\bar{X}_1(\beta)}\right)\right)|_{\bar{Z}\setminus\bar{X}_1(\beta)}\right|\right| > 0$$

since $\bar{X}_1(\beta)$ is not invariant under the maps T_0, T_1, T_{-1}. Then

$$\begin{aligned} 4||\bar{\mu}|_{\bar{X}_1(\beta)}|| &= \left|\left|L\left(\bar{\mu}|_{\bar{X}_1(\beta)}\right)\right|\right| = \left|\left|\left(L\left(\bar{\mu}|_{\bar{X}_1(\beta)}\right)\right)|_{\bar{X}_1(\beta)}\right|\right| + \left|\left|\left(L\left(\bar{\mu}|_{\bar{X}_1(\beta)}\right)\right)|_{\bar{Z}\setminus\bar{X}_1(\beta)}\right|\right| \\ &= \lambda||\bar{\mu}|_{\bar{X}_1(\beta)}|| + \left|\left|\left(L\left(\bar{\mu}|_{\bar{X}_1(\beta)}\right)\right)|_{\bar{Z}\setminus\bar{X}_1(\beta)}\right|\right| > \lambda||\bar{\mu}|_{\bar{X}_1(\beta)}||_1 \end{aligned}$$

giving us $\lambda < 4$. □

Proposition 3.4.2. $||\bar{\mu}|| = \infty$*, i.e., the measure* $\bar{\mu}$ *is infinite.*

Proof. For $n \in \mathbb{N}$ we get

$$||\bar{\mu}|| = \left|\left|\frac{1}{\lambda^n}L^n\bar{\mu}\right|\right| > \left|\left|\frac{1}{\lambda^n}L^n\left(\bar{\mu}|_{\{0\}}\right)\right|\right| = \frac{4^n}{\lambda^n}\bar{\mu}(0).$$

The result follows since $\lambda < 4$, $\bar{\mu}(0) > 0$ and n was arbitrary.

□

3.4.2 Transition Matrices

Let $\Delta = \{v_1, \cdots, v_k\}$ with $v_1 = 0$. We introduce a $k \times k$ matrix with rows/columns corresponding to the points in Δ.

Definition 3.4.5. *For $i \in \{-1, 0, 1\}$ let A_i be the $k \times k$ matrix such that*

$$(A_i)_{m,n} = \begin{cases} 1 & \textit{if} \quad \exists j \in \{-1,1\} : T_{j-i}(v_m) = v_n \\ 2 & \textit{if} \quad T_{-i}(v_m) = v_n \\ 0 & \textit{otherwise} \end{cases}.$$

The matrices A_i describe the evolution of local measure as we move from x to $T_i(x)$, as described in Lemma 3.4.5. Recall that $v_1 = 0, v_2, \cdots v_k$ are the elements of Δ (Definition 3.4.2. We define a vector which describes the local measure around x.

Definition 3.4.6. *We let $v(x) = (\mu(x), \mu(x+v_2), \cdots, \mu(x+v_k))$.*

Lemma 3.4.5. *Let $x \in \bar{X}(\beta)$. Then*

$$\frac{1}{\lambda} v(x) A_i = v(T_i(x)).$$

Proof. We show that

$$(\bar{\mathcal{N}}_n(x), \bar{\mathcal{N}}_n(x+v_2), \cdots, \bar{\mathcal{N}}_n(x+v_k))A_i = (\bar{\mathcal{N}}_{n+1}(T_i(x)), \bar{\mathcal{N}}_{n+1}(T_i(x)+v_2), \cdots, \bar{\mathcal{N}}_{n+1}(T_i(x)+v_k)),$$

the result will follow from this statement.

Note that

$$\bar{\mathcal{N}}_{n+1}(T_i(x)+v_l) = \bar{\mathcal{N}}_n(T_1^{-1}(T_i(x)+v_l)) + \bar{\mathcal{N}}_n(T_{-1}^{-1}(T_i(x)+v_l)) + 2\bar{\mathcal{N}}_n(T_0^{-1}(T_i(x)+v_l)) \tag{3.9}$$

where of course $\bar{\mathcal{N}}_n(y) = 0$ for $y \notin \bar{X}(\beta)$.

Secondly we note that

$$\begin{aligned} T_j(x+v_m) &= T_j(x) + T_0(v_m) \\ &= T_i(x) + T_0(v_m) + j - i \\ &= T_i(x) + T_{j-i}(v_m), \end{aligned}$$

which is equal to $T_i(x)+v_l$ if and only if $T_{j-i}(v_m)=v_l$.

So we can rewrite equation 3.9 to get

$$\begin{aligned}\bar{\mathcal{N}}_{n+1}(T_i(x)+v_l) = &\sum_{m\in\{1,\cdots,k\}} \bar{\mathcal{N}}_n(x+v_m)\chi_{T_{1-i}(v_m)=v_l} \\ &+ \sum_{m\in\{1,\cdots,k\}} \bar{\mathcal{N}}_n(x+v_m)\chi_{T_{-1-i}(v_m)=v_l} \\ &+ 2\sum_{m\in\{1,\cdots,k\}} \bar{\mathcal{N}}_n(x+v_m)\chi_{T_{-i}(v_m)=v_l}.\end{aligned}$$

which is precisely the lth entry of $(\bar{\mathcal{N}}_n(x), \bar{\mathcal{N}}_n(x+v_2), \cdots, \bar{\mathcal{N}}_n(x+v_k))A_i$.

□

Proposition 3.4.1. *Set* $W = v(0) = (\mu(0), \mu(v_2), \cdots, \mu(v_k))$. *Let* $x = \sum_{i=1}^{n} c_i\beta^{n-i}$. *Then*

$$v(x) = \frac{1}{\lambda^n}(WA_{c_1}\cdots A_{c_n}).$$

In particular,

$$\bar{\mu}(x) = \frac{1}{\lambda^n}(WA_{c_1}\cdots A_{c_n})_1,$$

i.e. the first entry of the $1\times k$ *vector* $\frac{1}{\lambda^n}WA_{c_1}\cdots A_{c_n}$.

Proof. This follows immediately from the previous lemma by writing

$$x = T_{a_n}\circ T_{a_{n-1}}\circ\cdots\circ T_{a_1}(0).$$

□

Since the one dimensional measure μ is the projection of $\bar{\mu}$ onto the first coordinate, Theorem 3.1.2 follows as a direct corollary to Proposition 3.4.1.

3.4.3 Approximating local measures via the contractive subspace

Recall that $\mathcal{R}$ is the attractor of the IFS $\{S_{-1}, S_0, S_1\}$ and that $\pi_c(\bar{X}(\beta))\subseteq\mathcal{R}$. We will assume the following condition.

Condition 3.4.1. $\bar{X}(\beta) \cap cl(B_\beta(1)) = \bar{Z} \cap \pi_c^{-1}(\mathcal{R}^o) \cap cl(B_\beta(1))$

This is similar to a condition appearing in Corollary 4.5 of [41]. Here $\mathcal{R}^o$ denotes the interior of the set. Condition 3.4.1 is a condition about two finite sets being equal, and so can be easily checked. In words, the condition says that a finite patch around zero of the set $\bar{X}(\beta)$, which is a higher dimensional analogue of the spectrum of β, can be written as a patch of a cut and project set with window $\mathcal{R}^o$. Condition 3.4.1 implies that the whole set $\bar{X}(\beta)$ can be written as a cut and project set, this is the content of Corollary 3.4.1. In every example we have checked with $\beta \in (1, 2)$ a hyperbolic algebraic unit and alphabet $\mathcal{A} = \{-1, 0, 1\}$, Condition 3.4.1 does indeed hold, but there are examples of Hare, Masáková and Vávra [41] using complex alphabets in which the cut and project set contains extra points.

Lemma 3.4.6. *For each $i \in \{-1, 0, 1\}$ we have $T_i^{-1}(\bar{Z}) \subseteq \bar{Z}$.*

Proof. We need only show that for $x = \sum_{i=0}^{d-1} z_i\beta^i$ where $z_0, \cdots, z_{d-1} \in \mathbb{Z}$ we have that there exist $z'_0, \cdots z'_{d-1}$ such that $\frac{x}{\beta} = \sum_{i=0}^{d-1} z'_i\beta^i$. Once we have shown this for x, the corresponding results for the Galois conjugates follow directly.

The result holds because, for β to be a root of a $\{-1, 0, 1\}$-polynomial, it is necessary that the final term a_0 of the minimal polynomial[5] of β is ± 1. Then we use

$$\begin{aligned} 0 &= a_d\beta^d + a_{d_1}\beta^{d-1} + \cdots + a_1\beta + a_0 \\ \implies \frac{1}{\beta} &= \frac{a_d}{-a_0}\beta^{d-1} + \cdots + \frac{a_1}{-a_0}. \end{aligned}$$

and since each of the terms $\frac{a_i}{-a_0}$ are integers, since $a_0 = \pm 1$, we have that dividing by β keeps numbers within the integer lattice as required. □

[5]The fact that β is a root of a $\{-1, 0, 1\}$-polynomial isn't enough to imply that the *minimal* polynomial of β has digits only $\{-1, 0, 1\}$, but it does follow that the largest and smallest terms in the minimal polynomial are ± 1.

Proposition 3.4.3. *Suppose that $x \in \bar{X}(\beta)$ has $\pi_c(x) \in [\varepsilon_1, ..., \varepsilon_n]^o$ for some $\varepsilon_1, ..., \varepsilon_n \in \{-1, 0, 1\}^n$. Then, under condition 3.4.1, there are $a_1, ..., a_\kappa \in \{-1, 0, 1\}$ such that*

$$T_{\varepsilon_1} \circ ... \circ T_{\varepsilon_n} \circ T_{a_\kappa} \circ ... \circ T_{a_1}(0) = x.$$

Recall that $[\varepsilon_1, \cdots, \varepsilon_n]$ is a subset of $\mathcal{R}$ defined in Definition 3.4.1, and that $[\varepsilon_1, \cdots, \varepsilon_n]^o$ is its interior.

Proof. By the iterated function system construction of $\mathcal{R}$, the fact that $\pi_c(x) \in [\varepsilon_1, \cdots, \varepsilon_n]$ gives the existence of arbitrarily long words $a_1, \cdots a_m \in \{-1, 0, 1\}^m$ such that

$$\pi_c(x) \in S_{\varepsilon_1} \circ ... \circ S_{\varepsilon_n} \circ S_{a_1} \circ ... \circ S_{a_m}(\mathcal{R}).$$

This implies that there is $y \in \bar{Z}$ with $\pi_c(y) \in \mathcal{R}$ such that

$$x = T_{\varepsilon_1} \circ ... \circ T_{\varepsilon_n} \circ T_{a_1} \circ ... \circ T_{a_m}(y),$$

the fact that $y \in \bar{Z}$ follows using Lemma 3.4.6 using that $x \in \bar{Z}$. Now $x = (x_1 \cdots, x_d, x_{d+1}, \cdots x_{d+s})$ where the maps T_i are expanding on the first d coordinates and contracting on the final s coordinates. Hence the maps T_i^{-1} contract the first d coordinates and for any $\epsilon > 0$, for large enough m, the point

$$y = (T_{\varepsilon_1} \circ ... \circ T_{\varepsilon_n} \circ T_{a_1} \circ ... \circ T_{a_m})^{-1}(x)$$

must have its first d coordinates within distance ϵ of the box $\Pi_{i=1}^{d} I_{\beta_i}(1)$. But since these points lie in a uniformly discrete set, the first d coordinates must actually lie in the closure of this box.

The final s coordinates must be in $\mathcal{R}^o$, since $\pi_c(x) \in S_{\varepsilon_1} \circ ... \circ S_{\varepsilon_n} \circ S_{a_1} \circ ... \circ S_{a_m}(\mathcal{R}^o)$. Thus

$$(T_{\varepsilon_1} \circ ... \circ T_{\varepsilon_n} \circ T_{a_1} \circ ... \circ T_{a_m})^{-1}(x) \in \overline{Z} \cap \pi_c^{-1}(\mathcal{R}) \cap B_\beta(1),$$

and so by Condition 3.4.1 there exists $b_1 \cdots b_k \in \{-1, 0, 1\}^k$ such that

$$(T_{\varepsilon_1} \circ ... \circ T_{\varepsilon_n} \circ T_{a_1} \circ ... \circ T_{a_m})^{-1}(x) = T_{b_1} \circ \cdots \circ T_{b_k}(0) \in \bar{X}_1(\beta).$$

Then

$$x = T_{\varepsilon_1} \circ ... \circ T_{\varepsilon_n} \circ T_{a_1} \circ ... \circ T_{a_m} \circ T_{b_1} \circ \cdots T_{b_k}(0)$$

as required.

□

Corollary 3.4.1. *Under condition 3.4.1,* $\bar{X}(\beta) = \bar{Z} \cap \pi_c^{-1}(\mathcal{R}^o)$.

This is just the statement of the previous proposition with $\varepsilon_1, \cdots \varepsilon_n$ being the empty word. A similar statement appears as Corollary 4.5 in [41].

Lemma 3.4.7. *Let* $i, j \in \{1, \cdots, k\}$*. Then there exists* $c_1, ..., c_n \in \{-1, 0, 1\}$ *such that*

$$(A_{c_1} \cdot ... \cdot A_{c_n})_{ij} > 0.$$

Proof. The definition of Δ means there exist $a_1 \cdots a_m \in \{-2, -1, 0, 1, 2\}^m$ and $a_{m+1} \cdots a_n \in \{-2, -1, 0, 1, 2\}$ such that $T_{a_m} \circ \cdots \circ T_{a_1}(v_i) = 0$ and $T_{a_{m+1}} \circ \cdots \circ T_{a_n}(0) = v_j$. Then choosing $c_1 \cdots c_m$ such that $a_i - c_i \in \{-1, 0, 1\}$ for each i the result follows directly from the definition of A_i. □

The following lemma is important in defining for us a 'mixing word' $a_n \cdots a_1 \in \{-1, 0, 1\}^n$.

Proposition 3.4.4. *There is a word* $w = w_1, ..., w_n \in \{-1, 0, 1\}^n$ *and* $I, J \subseteq \Delta$ *such that* $0 \notin I, 0 \notin J$ *and* $(A_{w_1} \cdot ... \cdot A_{w_n})_{i,j} = 0 \Leftrightarrow i \in I$ *or* $j \in J$.

Proof. We start by building a set I and a word $w_1, \cdots, w_m$ such that the ith row of $A_{w_1}. \cdots .A_{w_m}$ is a zero row for $i \in I$ and $(A_{w_1}. \cdots .A_{w_m})_{i,1} > 0$ otherwise.

Step 1: Note that for $i \in \{-1, 0, 1\}$, $(A_i)_{1,1} > 0$.

Step 2: The point v_2 is in Δ, and from the definition of Δ and lemma 3.4.7 there exist $w_1 \cdots w_{m_1} \in \{-1, 0, 1\}$ such that

$$(A_{w_1} \cdots A_{w_{m_1}})_{2,1} > 0$$

Step 3: Either the 3rd row of the product $A_{w_1} \cdots A_{w_{m_1}}$ is a zero row, in which case we declare $v_3 \in I$, or there exists $v_p \in \Delta$ with $(A_{w_1} \cdots A_{w_{m_1}})_{3,p} > 0$. As in step 2, since $v_p \in \Delta$ choose a word $w_{m_1+1} \cdots w_{m_2}$ such that

$$(A_{w_{m_1+1}} \cdots A_{w_{m_2}})_{p,1} > 0.$$

Then the product of matrices $A_{w_1} \cdots A_{w_{m_2}}$ has that entry $(3, 1)$ is positive. Furthermore, entry $(2, 1)$ is still positive, since $A_{w_1} \cdots A_{w_{m_1}}$ had entry $(2, 1)$ positive, and then we are post multiplying by matrices with positive top left entry.

Iterating this procedure, we create a word $w_1 \cdots w_{m_k}$ and a set $I \subset \Delta$ such that the ith row of $A_{w_1}. \cdots .A_{w_{m_k}}$ is a zero row for $i \in I$ and $(A_{w_1}. \cdots .A_{w_{m_k}})_{i,1} > 0$ otherwise.

Note that the matrices A_1^T, A_0^T, A_{-1}^T also have top left entry strictly positive and that for any $i \in \{1, \cdots k\}$ there exists a word $c_1 \cdots c_n$ such that $(A_{c_1} \cdots A_{c_n})_{(i,1)} > 0$. So we repeat the above procedure for the matrices A_1^T, A_0^T, A_{-1}^T to create a word $w'_1 \cdots w'_{n_k}$ and a set J such that the jth row of $A^T_{w'_1} \cdots A^T_{w'_{n_k}}$ is a zero row for $j \in J$, and $(A^T_{w'_1} \cdots A^T_{w'_{n_k}})_{(j,1)} > 0$ otherwise.

Taking the transpose once more gives us that the product $A_{w'_{n_k}} \cdots A_{w'_1}$ has a set J of zero columns, and for all other columns the first entry is strictly positive.

Now setting $w_1 \cdots w_n = w_1 \cdots w_{m_k} w'_{n_k} \cdots w'_1$ we see that the product $A_{w_1} \cdots A_{w_n}$ has a set I of zero rows, a set J of zero columns, with all other entries strictly positive as required.

□

Definition 3.4.7. *Let the mixing word $w = w_1, ..., w_n$ and $A_w = A_{w_1} \cdot ... \cdot A_{w_n}$ where $w_1, ..., w_n$ are as in proposition 3.4.4*

Recall that we defined the $1 \times k$ vectors

$$v(x) = (\mu(x), \mu(x + v_2), \cdots, \mu(x + v_k))$$

where $\Delta = (v_1, \cdots, v_k)$ with $v_1 = 0$. Map the space of $1 \times k$ vectors with positive first entry onto projective space by letting $(V')_i = \frac{(V)_{i+1}}{(V)_1}$ for $1 \leqslant i \leqslant 16$, giving

$$v'(x) = \left(\frac{\mu(x + v_2)}{\mu(x)}, \frac{\mu(x + v_3)}{\mu(x)}, \cdots \frac{\mu(x + v_k)}{\mu(x)} \right)$$

As before, define the projective distance by

$$d(U, V) = \max_{i \in \{1, \cdots, k-1\}} |\log((V)_i) - \log((U)_i)| \in [0, \infty].$$

Proposition 3.4.5. *Assume that condition 3.4.1 holds. Then there exist positive constants C_1, C_2 such that for any word $a_1 \cdots a_r \in \{-1, 0, 1\}^n$ and for any $x, y \in \bar{X}(\beta)$ with $\pi_c(x), \pi_c(y) \in [a]^o$,*

$$d(v'(x), v'(y)) < C_1 C_2^{d(a)-1}$$

where $d(a)$ is the number of disjoint occurrences of w in $a = a_1 \cdots a_n$.

Proof. By Lemma 3.4.3 we have that x and y both have expansions ending with the word a, i.e. we can write $x = \sum_{i=1}^{n} c_i \beta^{n-i}$, $y = \sum_{i=1}^{m} d_i \beta^{m-i}$ where both $c_1 \cdots c_n$ and $d_1 \cdots d_m$ end in word $a_r \cdots a_1$.

Then by Lemma 3.4.5 we can write

$$v(x) = \frac{1}{\lambda^n} v(0) A_{c_1} \cdots A_{c_n} = \underbrace{\frac{1}{\lambda^n} v_0 A_{c_1} \cdots A_{c_{n-r}}}_{:=U} A_{a_r} \cdots A_{a_1}$$

and

$$v(y) = \frac{1}{\lambda^n} v(0) A_{d_1} \cdots A_{d_m} = \underbrace{\frac{1}{\lambda^n} v_0 A_{d_1} \cdots A_{d_{m-r}}}_{:=V} A_{a_r} \cdots A_{a_1}$$

But now $a_r \cdots a_1$ contains d occurrences of the mixing word w, the first of which contracts the distance between vectors U and V to at most C_1, and the final $d(a) - 1$ of which each contract the distance by a factor of C_2, as is proved in Appendix 2. Then we have the required result.

$\square$

We note that Theorem 3.1.3 follows as a direct corollary to Propsition 3.4.5, as the vector $v'(x)$ can be written

$$v'(x) = (\exp(f_2(x_c)), \exp(f_3(x_c)), \cdots \exp(f_k(x_c)))$$

and that $d(v'(x), v'(y)) < C_1 C_2^{d(a)-1}$ implies that for each $i \in \{2, \cdots, k\}$ the differences $|\log(f_i(x_c)) - \log(f_i(y_c)| < C_1 C_2^{d(a)-1}$. Projecting $\bar{\mu}$ and the elements of Δ onto their first coordinates we are done.

Finally we show that all elements of $\bar{X}$ can be reached from 0 by applying finitely many translations from the set Δ.

Lemma 3.4.8. *Let* $a_1, ...a_m \in \{-1, 0, 1\}$ *be such that* $a_1\bar{\beta}^{m-1} + ... + a^{m-1}\bar{\beta} + a_m\bar{\beta}^0 = 0$ *and* $a_1 \neq 0$*. Then*

$$\left\{ \sum_{i=0}^{\kappa} x_i : \kappa \in \mathbb{N}, x_1, ..., x_\kappa \in \Delta \right\} = \bar{X}.$$

Proof. Notice that $m \geqslant \deg(\beta) + 1$. We have

$$T_{a_m} \circ ... \circ T_{a_1}(0) = 0$$

hence the set

$$\begin{aligned} B :&= \{T_{a_k} \circ ... \circ T_{a_1}(0) : 1 \leqslant k \leqslant m-1\} \\ &= \left\{ a_1\bar{\beta}^{k-1} + ... + a^{k-1}\bar{\beta} + a_k\bar{\beta}^0 : 1 \leqslant k \leqslant m-1 \right\} \end{aligned}$$

is a subset of Δ. Set

$$\Delta(0) = \left\{ \sum_{i=0}^{\kappa} x_i : \kappa \in \mathbb{N}, x_1, ..., x_\kappa \in \Delta \right\}.$$

The proof is completed by showing inductively that $\bar{\beta}^0, ..., \bar{\beta}^{m-1} \in \Delta(0)$. Indeed $\bar{\beta}^0 \in B \subseteq \Delta$ and if $\bar{\beta}^0, ..., \bar{\beta}^k \in \Delta(0)$, for $\kappa < m-1$, then

$$\bar{\beta}^{k+1} = a_1((a_1\bar{\beta}^{k+1} + ... + a^{k+1}\bar{\beta} + a_{k+2}\bar{\beta}^0) - a_2\bar{\beta}^k - ... - a^k\bar{\beta} - a_{k+2}\bar{\beta}^0) \in \Delta(0).$$

□

3.5 Appendix

3.5.1 Appendix 1: A Perron theory lemma

In this subsection we will prove Lemma 3.4.2

Proof. By bringing the matrix to it's normal form of a reducible matrix, see ([76], p. 51), we can assume that

$$A = \begin{bmatrix} B_1 & * & * & \cdots & * \\ 0 & B_2 & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \cdots & B_h \end{bmatrix}$$

where B_i is a non-negative irreducible square matrix for $i \in \{1, ..., h\}$. By rescaling we can assume that $\rho(A) = 1$. Clearly $1 = \rho(A) = \max\{\rho(B_1), ..., \rho(B_h)\}$ so from assumption iii) we get $\rho(B_1) = 1$. We set

$$S_i := \{j \in \{1, ..., N\} : \text{ The entry } (j,j) \text{ is contained in the } B_i\text{-block }\}.$$

For $i \in \{1, ...h\}$ let

$$V_i := \left\{ u \in \mathbb{R}^N : u(j) = 0 \text{ if } j \notin S_i \right\}$$

and

$$V_{i-} := \left\{ u \in \mathbb{R}^N : u(j) = 0 \text{ if } j \notin \cup_{\kappa=1}^{i-1} S_\kappa \right\}.$$

Define p_i and p_{i-} to be the orthogonal projections of $\mathbb{R}^N$ to the subspaces V_i and V_{i-} respectively. Finally let B_i' to be A where all entries outside the B_i-block are replaced by 0 and B_{i-}' to be A where all the entries of the form (i, j) are replaced by zero if and only if $j \notin \cup_{\kappa=1}^{i-1} S_\kappa$.

We will prove the lemma by proving inductively that $p_i(e_1 A^n)$ converges for $i \in \{1, ..., h\}$. For $i = 1$ we have that $p_i(e_1 A^n) = p_i(e_1 B_1'^n)$ so the statement is true since B_1 is an irreducible aperiodic matrix of spectral radius one. The aperiodicity comes from assumption i). Now we assume that $i \in \{2, ..., h\}$ and $p_{i-}(e_1 A^n)$ converges to some $v' \in \mathbb{R}^N$ aiming to prove that $p_i(e_1 A^n)$ converges.

Case 1 $\rho(B_i) < 1$: We define $T_i : \mathbb{R}^N \to \mathbb{R}^N$ by

$$T_i(x) = x B_i' + p_i(v' A)$$

Since $\rho(B_i) < 1$ there is $u' \in \mathbb{R}^N$ such that $u'(I - B_i') = p_i(v' A)$ so that

$$T_i(x) = (x - u') B_i' + u'.$$

Now, from $\rho(B_i) < 1$ again, we can conclude that $T_i^n(x) \to u'$ for any $x \in \mathbb{R}^N$. Writing

$$p_i(e_1 A^n) = T_i^n(0) + p_i(e_1 A^n) - T_i^n(0)$$

we only need to prove that $p_i(e_1 A^n) - T_i^n(0) \to 0$ to prove the convergence of $p_i(e_1 A^n)$ to u'. Let $\varepsilon > 0$. By the spectral radius formula there exists $C > 0$ such that

$$||B_i'^n|| \leqslant C \left(\rho(B_i) + \delta \right)^n$$

where $\delta > 0$ is chosen such that $\rho(B_i') + \delta < 1$. Also by $p_{i-}(e_1 A^n) \to v'$ we get that there is κ_0 such that $|p_i(v'A) - p_i(p_{i-}(e_1 A^{n-1})A)| < \varepsilon$. Notice that

$$p_i(e_1 A^{\kappa+1}) = p_i(e_1 A^{\kappa}) B_i' + p_i\left(p_{i-}(e_1 A^{\kappa})\right), \quad \kappa \in \{0, ...\}.$$

By iterating the relation above and choosing n large enough we get

$$\begin{aligned}
|p_i(e_1 A^n) - T_i^n(0)| &= \left| \sum_{\kappa=1}^{n} \left(p_i(p_{i-}(e_1 A^{\kappa-1})A) - p_i(v'A)\right) B_i'^{n-\kappa} \right| \\
&\leqslant \left| \sum_{\kappa=1}^{\kappa_0 - 1} \left(p_i(p_{i-}(e_1 A^{\kappa-1})A) - p_i(v'A)\right) B_i'^{n-\kappa} \right| + \sum_{\kappa=\kappa_0}^{n} ||B_i^{n-\kappa}|| \cdot \varepsilon \\
&\leqslant \left| \left(\sum_{\kappa=1}^{\kappa_0 - 1} \left(p_i(v'A) - p_i(p_{i-1}(e_1 A^{\kappa-1})A\right) B_i'^{\kappa_0 - 1 - \kappa} \right) B_i'^{n-\kappa_0+1} \right| \\
&+ \frac{\varepsilon \cdot C}{1 - \rho(B_i) - \delta}
\end{aligned}$$

Since $x B_i'^n \to 0$ for all $x \in \mathbb{R}^N$ the above gives

$$\limsup_{n\to\infty} |p_i^n(e_1 A^n) - T_i^n(0)| \leqslant \frac{\varepsilon \cdot C}{1 - \rho(B_i) - \delta}$$

but since ε was arbitrary we get

$$\lim_{n\to\infty} |p_i^n(e_1 A^n) - T_i^n(0)| = 0$$

completing the inductive step in the case $\rho(B_i) < 1$.

Case 2 $\rho(B_i) = 1$: Now let u' be a left eigenvector of 1 of B_i' with all entries in S_i being positive. There exists such a u' from Perron–Frobenius theorem since B_i is a non-negative irreducible matrix. There are κ_0,$m \in \mathbb{N}$ and $c > 0$ such that all entries in S_i of

$$p_i\left(p_{i-}(e_1 A^n) A^m\right) - cu'$$

are positive for all $n > \kappa_0$. This is true, by choosing c small enough, because of assumption ii) and $p_{i-}(e_1 A^n) \to v'$. Let $\kappa_1 \in \mathbb{N}$ be such that $m(\kappa_1 - 1) > \kappa_0$. The inequalities in the following are to be understood entrywise. For n large enough we have,

$$\begin{aligned} p_i(e_1A^{nm}) &= \sum_{\kappa=1}^{n} \left(p_i\left(p_{i-}\left(e_1A^{m(\kappa-1)}\right)A^m\right)\right)B_i'^{m(n-\kappa)} \\ &\geqslant \sum_{\kappa=\kappa_1}^{n} \left(p_i\left(p_{i-}\left(e_1A^{m(\kappa-1)}\right)A^m\right)\right)B_i'^{m(n-\kappa)} \\ &= \sum_{\kappa=\kappa_1}^{n} \left(p_i\left(p_{i-}\left(e_1A^{m(\kappa-1)}\right)A^m\right)-cu'\right)B_i'^{m(n-\kappa)} + \sum_{\kappa=\kappa_1}^{n} cu'B_i'^{m(n-\kappa)} \\ &\geqslant \sum_{\kappa=\kappa_1}^{n} cu'B_i'^{m(n-\kappa)} = (n-\kappa_1+1)cu'. \end{aligned}$$

The above implies that $||p_i(e_1A^{nm})||_1 \to \infty$ which contradicts assumption iii). Thus case 2 never occurs.

□

3.5.2 Appendix 2: Birkhoff metric arguments

This section is based on methods from [13]. We use a metric of Birkhoff which is equivalent to the metric used in the text above, and in particular the contraction results of this section carry over to the metric used in the main text.

For a vector $x \in \mathbb{R}^n$ we define C_x to be the closure of the set

$$\{y \in \mathbb{R}^n | \forall i \in \{1,..,n\} : y(i) \geqslant 0 \text{ and } (x(i)=0 \Leftrightarrow y(i)=0)\}$$

and $\langle C_x \rangle$ the linear subspace it spans. Also we set ∂C_x and C_x° to be the boundary and the interior of C_x respectively, with respect to the topology of $\langle C_x \rangle$. Let pr be the canonical mapping of $R^n \backslash \{0\}$ to it's projective space. We identify $pr(C_{(1,\dots,1)} \backslash \{0\})$ with $H := C_{(1,\dots,1)} \cap \{x \in \mathbb{R}^n : ||x||_1 = 1\}$ so that $pr(x)$ is identified with $x/||x||_1$. Let a, b be two distinct elements of H such that there is $x \in \mathbb{R}$ with $a, b \in C_x^{\circ} \cap H$. Note that, given a and b, all choices of x give rise to at most one set

C_x. Denote by a', b' the points of $\partial C_x \cap H$ such that a is a convex combination of a' and b and b is a convex combination of a and b'. Define $K_{a,b} : \langle\{a, b\}\rangle \to R^2$ to be the unique linear transformation such that $K_{a,b}(a') = (1, 0)$ and $K_{a,b}(b') = (0, 1)$.

Now for $x \in \mathbb{R}^n \backslash \{0\}$ we define a metric d_x on $C_x^{\circ} \cap H$ by

$$d_x(a, b) = d_2(K_{a,b}(a), K_{a,b}(b)) \qquad a \neq b \in C_x^{\circ} \cap H$$

where

$$d_2\left((x_1, y_1), (x_2, y_2)\right) = \left|\log\left(\frac{y_1 x_2}{x_1 y_2}\right)\right|.$$

The fact that the above defines a metric is Lemma 1 of [13].

Lemma 3.5.1. *Let $x, y \in \mathbb{R}^n \backslash \{0\}$ and T be a linear transformation from $\mathbb{R}^n$ to itself such that $T(C_x \backslash \{0\}) \subseteq C_y^{\circ}$. Then there are $C \in (0, 1)$ and $M > 0$ such that for all $a, b \in C_x^{\circ}$*

$$d_y(pr(T(a)), pr(T(b))) \leqslant C d_x(pr(a), pr(b))$$

and

$$d_y(pr(T(a)), pr(T(b))) \leqslant M$$

Proof. By a trivial compactness argument we can see that $pr(T(C_x \cap H))$ is bounded away from $\partial C_y \cap H$. From that we get that the image of the segment joining $(1, 0)$ and $(0, 1)$ under $T'_{a,b} := K_{pr(T(a)),pr(T(b))} T K^{-1}_{pr(a),pr(b)}$ is bounded away from $\{(1, 0), (0, 1)\}$ uniformly for all distinct $a, b \in C_x^{\circ}$. So there exist $C \in (0, 1)$ and $M > 0$ such that for all distinct $a, b \in C_x^{\circ}$:

$$d_2(T'_{a,b} K_{pr(a),pr(b)}(a), T'_{a,b} K_{pr(a),pr(b)}(b)) \leqslant C d_2(K_{pr(a),pr(b)}(a), K_{pr(a),pr(b)}(b))$$

and

$$d_2(T'_{a,b} K_{pr(a),pr(b)}(a), T'_{a,b} K_{pr(a),pr(b)}(b)) \leqslant M,$$

see ([13], p. 220), which is equivalent to

$$d_y(pr(T(a)), pr(T(b))) \leqslant C d_x(pr(a), pr(b))$$

and

$$d_y(pr(T(a)), pr(T(b))) \leqslant M$$

for all $a, b \in C_x^{\mathrm{o}}$. □

By a similar argument one can also prove the following.

Lemma 3.5.2. *Let $x, y \in \mathbb{R}^n \backslash \{0\}$ and T be a linear transformation from $\mathbb{R}^n$ to itself such that $T(C_x^{\mathrm{o}}) \subseteq C_y^{\mathrm{o}}$. Then for all $a, b \in C_x^{\mathrm{o}}$*

$$d_y(pr(T(a)), pr(T(b))) \leqslant d_x(pr(a), pr(b))$$

Also one can directly check the following two lemmas.

Lemma 3.5.3. *Let $T(x) = x^{\intercal} A$ be a linear transformation from $\mathbb{R}^n$ to itself where A is a $n \times n$ matrix which has only non-negative entries. Then for each $x \in \mathbb{R}^n$ there is a unique set C_y^{o}, for some $y \in \mathbb{R}^n$, such that $T(C_x^{\mathrm{o}}) \subseteq C_y^{\mathrm{o}}$.*

Lemma 3.5.4. *Let $T(x) = x^{\intercal} A$ be a linear transformation from $\mathbb{R}^n$ to itself where A is a $n \times n$ matrix which has only non-negative entries. Also assume that there exist $I, J \subseteq \{1, ..., n\}$ such that $A(i, j) = 0 \Leftrightarrow i \in I \vee j \in J$. Then there is a unique set C_x^{o}, for some $x \in \mathbb{R}^n$, such that for each $z \in \mathbb{R}^n$ if $\prod_{i \notin I} z(i) \neq 0$ then $T(C_z \backslash \{0\}) \subseteq C_x^{\mathrm{o}}$.*

Now the following lemma connects the metric d defined earleier with the metrics d_x defined in this appendix. We omit the full proof because it is a lengthy elementary inspection.

Lemma 3.5.5. *There is a constant $C > 1$, depending only on n, such that for all $x \in \mathbb{R}^n \backslash \{0\}$ and $a, b \in C_x^\circ$ we have*

$$C^{-1} d(a,b) < d_x(a,b) < C d(a,b).$$

Sketch of Proof. We choose arbitrary $x \in \mathbb{R}^n \backslash \{0\}$ and $p, q \in \partial C_x \cap H$. Then we work on the set

$$S := \{tp + (1-t)q : t \in (0,1)\}.$$

We fix a point $a_0 := t_0 p + (1-t_0) q$ in S and the rest of the proof is elementary asymptotic analysis on the formulas we get for $d_x(a_0, tp + (1-t)q)$ and $d(a_0, tp + (1-t)q)$.

□

Finally we conclude that products of matrices indexed by our contracting word contract projective space.

Proposition 3.5.1. *Let $a \in \{-1, 0, 1\}^{\mathbb{N}}$ contain $d(a)$ distinct incidences of the mixing word w. Then for any two non-negative $k \times 1$ vectors U, V, such that $U(1), V(1) > 0$, we have*

$$d(VA_{a_1} \cdots A_{a_n}, WA_{a_1} \cdots A_{a_n}) < C_1 C_2^{d(a)-1}$$

where C_1 and $C_2 \in (0,1)$ are explicit constants.

Proof. Let v be a non-negative $k \times 1$ vector such that $v(1) > 0$. The word $(a_1, ..., a_n)$ can be written as

$$(a_1, ..., a_n) = w_1 * ... * w_m$$

where $*$ is concatenation of words, $w_i \in \cup_{n\in\mathbb{N}}\{-1,0,1\}^n$ and

$$\#\{i \in \{1,...,m\} : w_i = a_c\} = d(a).$$

Set

$$i_{\min} = \min\{i \in \{1,...,m\} : w_i = a_c\}.$$

For each $i \in \{1,...,m\}$ we define A_{w_i} to be $A_{a_{\kappa(i)}} \cdot ... \cdot A_{a_{n(i)}}$ where $w_i = (a_{\kappa(i)},...,a_{n(i)})$. Also for each $i \in \{1,...,m\}$ we set

$$v_i = vA_{w_1} \cdot ... \cdot A_{w_i}$$

and $v_0 = v$. Notice that, by lemma 3.5.3,

$$(C_v^{\mathrm{o}})A_{w_1} \cdot ... \cdot A_{w_i} \subseteq C_{v_i}^{\mathrm{o}}$$

so

$$(C_v^{\mathrm{o}})A_{w_1} \cdot ... \cdot A_{w_{i_{\min}}} \subseteq (C_{v_{i_{\min}-1}}^{\mathrm{o}})A_c.$$

From lemmata 3.5.4 and 3.5.1 there exists $C_1 > 0$ such that

$$\mathrm{diam}_{d_{v_{i_{min}}}}(pr((C_v^{\mathrm{o}})A_{w_1} \cdot ... \cdot A_{w_{i_{\min}}})) \leqslant C_1$$

Now let $i_{\min} < i \leqslant m$, then

$$(C_v^{\mathrm{o}})A_{w_1} \cdot ... \cdot A_{w_{i-1}} \subseteq C_{v_{i-1}}^{\mathrm{o}}.$$

If $w_i \neq a_c$ then by 3.5.3 we see that $C_{v_{i-1}}^{\mathrm{o}} A_{w_i} \subseteq C_{v_i}^{\mathrm{o}}$ so by lemma 3.5.2

$$\mathrm{diam}_{d_{v_{i-1}}} pr((C_v^{\mathrm{o}})A_{w_1} \cdot ... \cdot A_{w_{i-1}}) \leqslant \mathrm{diam}_{d_{v_i}} pr((C_v^{\mathrm{o}})A_{w_1} \cdot ... \cdot A_{w_i}).$$

If $w_i = a_c$ then by lemmata 3.5.4 and 3.5.1

$$\mathrm{diam}_{d_{v_{i-1}}} pr((C_v^{\mathrm{o}})A_{w_1} \cdot ... \cdot A_{w_{i-1}}) \leqslant C\,\mathrm{diam}_{d_{v_i}} pr((C_v^{\mathrm{o}})A_{w_1} \cdot ... \cdot A_{w_i}).$$

So inductively we get

$$\mathrm{diam}_{d_{v_m}}\, pr((C_v^{\circ})A_{w_1} \cdot \ldots \cdot A_{w_m}) < C_1 C_2^{d(a)-1}.$$

The result follows from lemma 3.5.5 since the set

$$\{C_v^{\circ} : v \text{ is a non-negative non-zero } k \times 1 \text{ vector}\}$$

is finite.

□

3.6 Further Questions:

We have a number of further questions on the structure of the sets $X(\beta)$, the measure μ, and on how one can start to study μ using ergodic theory.

Question 1: Is it the case for any integer alphabet $\mathcal{A}$ and for any hyperbolic β one can express $X(\beta)$ (or the higher dimensional analogue $\tilde{X}(\beta)$ in the non-Pisot case) as a cut and project set with window $\mathcal{R}$ (or maybe $\mathcal{R}^o$) defined as the attractor of an iterated function system $\{S_i : i \in \mathcal{A}\}$ where S_i is defined in terms of the Galois conjugates of β of absolute value less than one. We have shown an inclusion in Corollary 3.4.1. This question is also considered in [41].

Question 2: Is it true that, for a sequence of Pisot numbers β_n of increasing degree in any interval $(1, 2-\epsilon)$, the sequence of sets $\frac{1}{\beta_n - 1}\left(X_{\{-1,0,1\}}(\beta_n) \cap \left[\frac{-1}{\beta-1}, \frac{1}{\beta-1}\right]\right)$ equidistribute in $[-1, 1]$. These sets are just pieces of the spectra of $X_{\{-1,0,1\}}(\beta_n)$ renormalised to live on $[-1, 1]$.

In Conjecture 2 we predict that, for such a sequence of Pisot numbers β_n, the distance between measures $\mu_{I_{\beta_n}}$ and normalised Lebesgue measure on I_{β_n} tends to zero as n tends to infinity. Our question here is the corresponding question for the sets $\mathrm{supp}(\mu_{I_{\beta_n}}) = X_{\{-1,0,1\}}(\beta_n) \cap \left[\frac{-1}{\beta-1}, \frac{1}{\beta-1}\right]$. If the answer to Question 1 is positive, then this is a question about the structure of a sequence of cut and project sets.

Question 3: Does further numerical evidence support our Conjectures 1 and 2 on the dimension of Bernoulli convolutions and the distribution of measures $\mu_{I_{\beta_n}}$? The case that β_n is a sequence of Pisot numbers converging to a limit in $(1, 2)$ is of particular interest. In that case the limit must also be a Pisot number.

Question 4: In the special case of the Golden mean, Theorem 3.3.3 describes how the measure μ evolves as one moves through the spectrum. Can one use this theorem, for example, to prove that

$$\lim_{n\to\infty} \sum_{x \in X(\phi)\cap[0,n]} \mu(x)\delta_{x(mod 1)}$$

converges weak$*$ to Lebesgue measure on $[0, 1]$. Inducing on the region $\{(x, y, z) : y \in [0, \phi^2]\}$ we have an irrational rotation in the x direction, and an irrational rotation in the y direction which also gives the weights which tell us how to evolve the measure μ. Then one might believe our question has a positive answer, since the weights $\mu(x)$ are driven by the evolution in the y direction which is somehow independent of our position in the x direction.

Chapter 4

Absolutely Continuous Bernoulli Convolutions

Joint work with Tom Kempton

4.1 Introduction

Bernoulli convolutions are a simple family of overlapping self-similar measures. For $\beta \in (1,2)$ the Bernoulli convolution ν_β is defined be the weak* limit of the sequence $\nu_{\beta,n}$ of probability measures given by

$$\nu_{\beta,n} = \sum_{a_1 \cdots a_n \in \{0,1\}^n} \frac{1}{2^n} \delta_{\sum_{i=1}^n a_i \beta^{-i}}.$$

The question of the absolute continuity of Bernoulli convolutions goes back to work of Erdős in 1939 [21], in which it was shown that the ν_β is singular when β is a Pisot number. These remain the only known examples of singular Bernoulli convolutions. In the other direction, Garsia, Varjú and Kittle have each given examples of classes of absolutely continuous Bernoulli convolutions associated with algebraic parameters [37, 77, 55]. Solomyak showed that the set of $\beta \in (1,2)$ giving rise to singular Bernoulli convolutions has Lebesgue measure zero [74], this result was improved by Shmerkin who showed that the set has Hausdorff dimension zero [73].

If instead of asking for absolute continuity of ν_β we ask whether $\dim_H(\nu_\beta) = 1$ then a lot more is known, mainly stemming from work of Hochman [45]. Several recent articles give conditions under which the Bernoulli convolution associated to an algebraic β has dimension one [17, 16, 39] or show that the Hausdorff dimension can be computed [2]. Most significantly, Varjú has shown that $\dim_H(\nu_\beta) = 1$ whenever β is transcendental [79]. Finally we mention recent papers of Feng and Feng and of Kleptsyn, Pollicott and Vytnova which give remarkable lower bounds for the $\dim_H(\nu_\beta)$ which hold for all $\beta \in (1, 2)$ [33, 57].

In this article we give new ergodic-theoretic conditions for the absolute continuity of Bernoulli convolutions. In particular, we turn the question of the absolute continuity of certain Bernoulli convolutions into a question relating to the ergodic theory of cocycles over uniquely ergodic domain exchange transformations. Our hope is that, with further work, our techniques will give rise to a proof that the Bernoulli convolution ν_β is absolutely continuous whenever $\beta \in (1, 2)$ is algebraic and has at least one Galois conjugate larger than one in absolute value, with no Galois conjugates having absolute value one. Our main theorem is the following.

Theorem 4.1.1. *[Stated Precisely as Theorem 4.5.1.] Assume that $\beta \in (1, 2)$ is an algebraic integer that has no Galois conjugates of absolute value one, and at least one real Galois conjugate of absolute value larger than one. Under assumptions, there exist a fractal $\mathcal{R}$, a set I, a domain exchange transformation $T : I \times \mathcal{R} \to I \times \mathcal{R}$ and a function (which satisfies regularity conditions) $f : \mathcal{R} \to \mathbb{R}^+$ such that, if the projection onto I of the sequence of measures*

$$\sum_{i=1}^{n} f(0)f(T(0)) \cdots f(T^{n-1}(0))\delta_{T^{n-1}(0)}$$

converges to Lebesgue measure sufficiently quickly then the Bernoulli convolution

ν_β *is absolutely continuous.*

If the function f took values in a compact group K then the Santos-Walkden version of the Wiener-Wintner ergodic theorem [70] would give us the convergence that we need. As it is, further work on the ergodic theory of cocycles over domain exchange transformations is needed to use our techniques to prove that certain Bernoulli convolutions are absolutely continuous.

We illustrate our results by first looking at a particular example.

4.1.1 A First Example:

Let $\beta \approx 1.513$ satisfy $\beta^4 = \beta^3 + \beta^2 - \beta + 1$. Then β has one real Galois conjugate $\beta_2 \approx -1.179$ and a pair of complex Galois conjugates which are less than one in modulus. We chose this example because it has no Galois conjugates of absolute value one (essential for our techniques) and because it is of small degree with only one Galois conjugate larger than one in modulus (which makes things easier to compute and to visualise).

Our first result, a special case of Theorem 4.2.1, gives conditions for the absolute continuity of ν_β in terms of the growth of the total number of overlaps at the nth level of the construction of the Bernoulli convolution.

Let $\mathcal{N}_n$ be the number of overlaps at the nth level of the construction of the Bernoulli convolution. This is equal to the number of pairs of words $a_1 \cdots a_n, b_1, \cdots b_n \in \{0,1\}^n$ for which $|\sum_{i=1}^n a_i \beta^{n-i} - \sum_{i=1}^n b_i \beta^{n-i}| < \frac{1}{\beta - 1}$.

Proposition 4.1.1 (Special Case of Theorem 4.2.1)**.** *If there exists $C > 0$ such that $\mathcal{N}_n \leqslant C \left(\frac{4}{\beta}\right)^n$ for all $n \in \mathbb{N}$ then the Bernoulli convolution ν_β is absolutely continuous.*

Unfortunately, estimating $\mathcal{N}_n$ is difficult. The bulk of this paper is dedicated to giving upper bounds via a geometric construction.

We define the measure μ_n on $I := \left[\frac{-1}{\beta-1}, \frac{1}{\beta-1}\right]$ by

$$\mu_n(A) = \#\{a_1 \cdots a_n, b_1 \cdots b_n \in \{0,1\}^n : \sum_{i=1}^{n}(a_i - b_i)\beta^{n-i} \in A\}.$$

Then $\mathcal{N}_n = \mu_n(I)$.

We want to understand the ratio $\frac{\mathcal{N}_{n+1}}{\mathcal{N}_n}$. Given $a_1 \cdots a_n, b_1 \cdots b_n$ contributing to the count for $\mathcal{N}_n$, we ask how many of the four choices of $a_{n+1}, b_{n+1} \in \{0,1\}^2$ give rise to a pair $a_1, \cdots a_{n+1}, b_1 \cdots b_{n+1}$ contributing to the count for $\mathcal{N}_{n+1}$. This boils down to the number of a_{n+1}, b_{n+1} for which

$$\beta\left(\sum_{i=1}^{n}(a_i - b_i)\beta^{n-i}\right) + (a_{n+1} - b_{n+1}) \in I,$$

which in turn depends only on the value of $\sum_{i=1}^{n}(a_i - b_i)\beta^{n-i}$. Using this, we show in Section 4.3 that the ratio $\frac{\mathcal{N}_{n+1}}{\mathcal{N}_n}$ can be expressed as the integral of a step function g with respect to the measure μ_n. This yields the following corollary.

Proposition 4.1.2. *Suppose that the measures μ_n equidistribute with respect to Lebesgue measure on I with certain rate (made precise in Theorem 4.3.1 and the comments afterwards). Then the Bernoulli convolution ν_β is absolutely continuous.*

A corollary of this is that if the measures μ_n equidistribute with respect to Lebesgue measure on I with certain rate then the Bernoulli convolution ν_β is absolutely continuous, see Theorem 4.3.1 and the comments afterwards.

If one draws the points contributing to the count for $\mathcal{N}_n$, that is if one draws the set

$$\left\{\sum_{i=1}^{n}(a_i - b_i)\beta^{n-i} : \text{ each } a_i, b_i \in \{0,1\}\right\} \cap I$$

then no structure is apparent, although the set of points becomes increasingly dense as n increases. Similarly, the measures μ_n do not seem to have any discernable structure when viewed in one dimension.

If however, one includes a second coordinate using the other Galois conjugate larger than one in modulus, then one uncovers the highly structured set

$$X_n = \left\{ \sum_{i=1}^{n} \left((a_i - b_i)\beta^{n-i}, (a_i - b_i)\beta_2^{n-i} \right) : \text{ each } a_i, b_i \in \{0,1\} \right\} \cap (I \times \mathbb{R})$$

We have plotted this set below for $n = 6$.

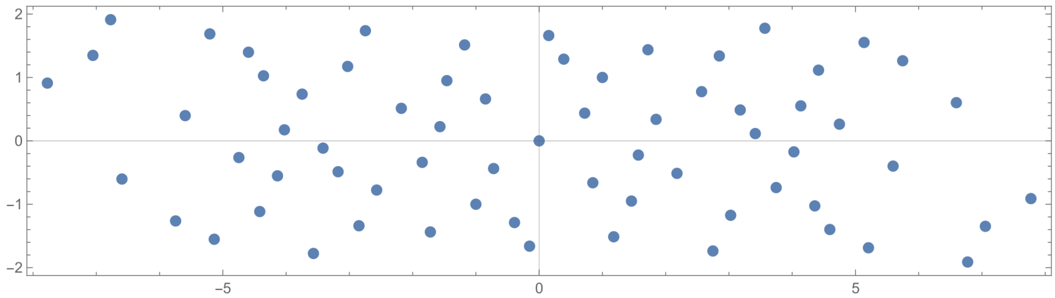

Figure 4.1: The set X_6 reflected across the diagonal.

The measure μ_n lifts naturally to a measure on X_n. As n grows, X_n expands to fill the set

$$X = \left\{ \sum_{i=1}^{n} \left((a_i - b_i)\beta^{n-i}, (a_i - b_i)\beta_2^{n-i} \right) : n \in \mathbb{N}, \text{ each } a_i, b_i \in \{0,1\} \right\} \cap (I \times \mathbb{R})$$

which is uniformly discrete and relatively dense in the strip $(I \times \mathbb{R})$. In fact X is a cut and project set where the cut and project scheme uses a window involving the Galois conjugates less than one in modulus, it can be constructed by a method similar to that of the Rauzy fractal [4].

In order to estimate $\mathcal{N}_n$ we are left with two problems, firstly to work out which elements of X are in X_n, and secondly to work out $\mu_n(x)$ for points $(x, y) \in X_n$. The first problem is easy, we use the y-coordinate $\sum_{i=1}^{n} (a_i - b_i)\beta_2^{n-i}$ as a proxy for the smallest n for which $(x, y) \in X_n$, it is certainly true that

$$X_n \subset \{(x, y) \in X : |y| \leqslant \sum_{i=1}^{n} \beta_2^{n-i}\}$$

and this estimate is good enough for us.

The second problem is much harder, and we rely heavily on our work [11]. We use that there exists $\alpha > 1$ such that, for each $(x, y) \in X$, $\mu(x) := \lim_{n\to\infty} \frac{1}{\alpha^n}\mu_n(x)$ exists. The key result of section 4.4 gives the following corollary, stated more precisely in Theorem 4.4.2.

Proposition 4.1.3. *[Special Case of Theorem 4.4.2] Suppose that the sequence of measures*

$$\sum_{(x,y)\in X: y\in[-\sum_{i=1}^{n}\beta_2^{n-i}, \sum_{i=1}^{n}\beta_2^{n-i}]} \mu(x)\delta_x,$$

once renormalised to have mass one, converges with certain rate to Lebesgue measure. Then the Bernoulli convolution ν_β is absolutely continuous.

The convergence to the Lebesgue measure of sequence of measures above is consistent with numerical evidence. The table below shows the Wasserstein distance of

$$\sum_{(x,y)\in X: y\in[-n/(\beta-1), n/(\beta-1)]} \mu(x)\delta_x,$$

once normalised to have mass one, to the Lebesgue measure for $n = 1, ..., 20$.

One can study the support of the sequence of measures defined in Proposition 4.1.3 using uniquely ergodic domain exchange transformations, in much the same way that one studies greedy β expansions using the Rauzy fractal. We also proved in [11] that one can study the measures (rather than just the support) using a cocycle over this domain exchange transformation. This yields a final corollary (Theorem 4.5.1) which gives a condition for the absolute continuity of the Bernoulli convolution in terms of the ergodic theory of cocycles over domain exchange transformations.

n	$W_1(\cdot, \text{Leb})$
1	0.0257383
2	0.0154008
3	0.0079060
4	0.0068856
5	0.0065858
6	0.0048812
7	0.0038639
8	0.0053756
9	0.0047376
10	0.0049352
11	0.0040242
12	0.0054624
13	0.0030473
14	0.0033527
15	0.0021562
16	0.0028536
17	0.0021284
18	0.0031695
19	0.0018788
20	0.0016524

Table 4.1: Evidence for an equidistribution property of μ.

4.2 A First Condition for Absolute Continuity

There has been a lot of progress in recent years in showing that certain Bernoulli convolutions have dimension one. For algebraic parameters this has based on understanding Garsia entropy, which counts the number of exact overlaps in the level n approximations to the Bernoulli convolution. In this section we explain how good estimates in the total number of overlaps (including partial overlaps) in the level n approximation to the Bernoulli convolution would allow one to understand absolute continuity.

Our starting point is the article [53] of the second author, in which two simple observations were made. The first is that if a self-similar measure ν is absolutely continuous, then the similarity equation which ν satisfies gives rise to a similarity equation for its density h. Furthermore, the measure ν is absolutely continuous if and only if there exists an L^1 function satisfying this density self-similarity equation. In the case of Bernoulli convolutions associated to a parameter $\beta \in (1, 2)$ the statement becomes that the Bernoulli convolution is absolutely continuous if and only if there exists a non-negative L^1 function $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \frac{\beta}{2}(h(\beta x) + h(\beta x - 1)).$$

The second observation of [53] was that one can study the existence of solutions to such equations in terms of functions which count the number of codings of each point x in the level n-construction of the self-similar measure.

In this section we generalise both of these ideas to measures on self-affine carpets with contraction rates in different directions corresponding to Galois conjugates of β, these measures are higher dimensional generalisations of Bernoulli convolutions. We also convert the second observation described above into one involving counting the total number of overlaps in the self-affine construction. When the self-affine measures we study are projected onto their first coordinate they give rise to the Bernoulli convolution, and so absolute continuity of these self-affine measure implies the absolute continuity of the Bernoulli convolution.

4.2.1 The Self-Affine Case

Let $\beta \in (1, 2)$ be a hyperbolic algebraic integer.

We will be interested in diagonal self-affine sets with contraction parameters associated with all but one of the Galois conjugates of β of absolute value larger than one. For this reason we number the Galois conjugates of β in an unusual way, let β have Galois conjugates $\beta = \beta_1, ..., \beta_d, \beta_{d+1}, \cdots, \beta_{d+s}, \beta_{d+s+1}$ where $|\beta_1|, ..., |\beta_d| > 1$, $|\beta_{d+1}|, ..., |\beta_{d+s}| < 1$ and $\beta_{d+s+1} \in \mathbb{R} \setminus [-1, 1]$.

In this section we will focus on $\beta_1, ..., \beta_d$. For $z \in \mathbb{C}$ set $\mathbb{F}_z = \mathbb{R}$ when $z \in \mathbb{R}$ and $\mathbb{F}_z = \mathbb{C}$ when $z \in \mathbb{C} \setminus \mathbb{R}$. Further define

$$\mathbb{K} := \prod_{i=1}^{d} \mathbb{F}_{\beta_i},$$

For $i \in \mathbb{N}$ we define $T_i : \mathbb{K} \to \mathbb{K}$ by

$$T_i(x_1, ..., x_d) = (\beta_1 x_1 + i, ..., \beta_d x_d + i).$$

For $j \in \{1, \cdots d\}$ let

$$I^+_{\beta_j} = \begin{cases} \left[0, \frac{1}{\beta_j - 1}\right], & \beta_j \in (1, \infty) \\ \left\{x \in \mathbb{R} : |x| \in \left[0, \frac{1}{|\beta_j| - 1}\right]\right\}, & x \in (-\infty, -1) \\ \left\{z \in \mathbb{C} : |z| \in \left[0, \frac{1}{|\beta_j| - 1}\right]\right\}, & z \in \mathbb{C} \setminus \mathbb{R} \end{cases}$$

and

$$I^+ = I^+_{\beta_1} \times \cdots \times I^+_{\beta_d}.$$

Define the self-affine measure $\nu_{\underline{\beta}}$ on $\mathbb{K}$ by

$$\nu_{\underline{\beta}} = \frac{1}{2}\left(\nu_{\underline{\beta}} \circ T_0 + \nu_{\underline{\beta}} \circ T_{-1}\right). \tag{4.1}$$

Note that the maps T_i are expanding, and $\nu_{\underline{\beta}}$ is the measure associated to contractions T_0^{-1}, T_{-1}^{-1}. This measure has support contained in I^+. If $\nu_{\underline{\beta}}$ is absolutely continuous then ν_β is absolutely continuous, we aim to prove the absolute continuity of $\nu_{\underline{\beta}}$.

Define an operator P on functions $f : \mathbb{K} \to \mathbb{R}$ by letting

$$Pf = \frac{|\beta_1 \cdot ... \cdot \beta_d|}{2}(f \circ T_0 + f \circ T_{-1}).$$

P preserves the space of non-negative functions that vanish outside I^+ and have integral one. P is a linear operator, and in particular if f is a fixed point of

P then cf is also a fixed point of P for any constant $c > 0$, thus if P has a fixed point of positive finite integral then it has a fixed point of integral one.

Proposition 4.2.1. *Suppose that P has a fixed point which has positive finite integral. Then the self-affine measure $\nu_{\underline{\beta}}$ is absolutely continuous and the fixed point of P of integral one is the density of $\nu_{\underline{\beta}}$.*

Proof. By integrating the fixed point f of P with integral one, we get a probability measure ν' on I^+. In order to check that $\nu' = \nu_{\underline{\beta}}$ we need only check that ν' satisfies the self-affinity equation 4.1, and so it is enough to check that for any $A \subset I^+$ we have

$$\nu'(A) = \frac{1}{2}\left(\nu'(T_0(A)) + \nu'(T_{-1}(A))\right).$$

This then follows immediately from the equation $Pf = f$ using that

$$\begin{aligned}
\nu'(A) &= \int_A f(x_1, \cdots, x_d) d(x_1, \cdots x_d) \\
&= \int_A Pf(x_1, \cdots, x_d) d(x_1, \cdots x_d) \\
&= \frac{|\beta_1 \cdot \ldots \cdot \beta_d|}{2} \int_A f(T_0(x_1, \cdots, x_d)) + f(T_{-1}(x_1, \cdots, x_d)) d(x_1, \cdots, x_d) \\
&= \frac{1}{2}\left(\int_{T_0(A)} f(x_1, \cdots, x_d) d(x_1, \cdots x_d) + \int_{T_{-1}(A)} f(x_1, \cdots, x_d) d(x_1, \cdots x_d)\right) \\
&= \frac{1}{2}\left(\nu'(T_0(A)) + \nu'(T_{-1}(A))\right).
\end{aligned}$$

□

Our goal now is to construct L^1 functions which satisfy $Pf = f$. Let functions f_n be given by

$$f_n := P^n(\chi_{I^+})$$

Here $f_n(x_1, \cdots, x_d)$ gives the number of words $a_1, \cdots, a_n \in \{0, -1\}^n$ for which $T_{a_n} \circ \cdots \circ T_{a_1}(x_1, \cdots, x_d)$ remains in the region I^+, multiplied by $\left(\frac{|\beta_1 \cdot \ldots \cdot \beta_d|}{2}\right)^n$. Equivalently, if we consider the iterated function system on I^+ with contractions

T_0^{-1}, T_1^{-1} then $f_n(x_1, \cdots x_d)$ counts the number of words $a_1 \cdots a_n$ for which $T_{a_1}^{-1} \circ \cdots \circ T_{a_n}^{-1}(I^+)$ covers $(x_1 \cdots, x_d)$, again multiplied by $\left(\frac{|\beta_1 \cdot \ldots \cdot \beta_d|}{2}\right)^n$.

Since the operator P preserves integral, each f_n has integral equal to the integral of f_0, which is the area of I^+.

Lemma 4.2.1. *Suppose that there exists a uniform constant C such that $||f_n||_2 := \int_{I^+}(f_n(x_1, \cdots x_d))^2 d(x_1, \cdots, x_d) < C$ for all $n \in \mathbb{N}$. Then P has a fixed point h of integral one and with bounded L^2 norm.*

Proof. Define

$$g_n(x_1, \cdots, x_d) := \frac{1}{n} \sum_{k=1}^{n} f_k(x_1, \cdots, x_d).$$

then each g_n also has $||g_n||_2 < C$ so, since balls are weakly compact in Hilbert spaces, there is a subsequence of g_n that converges weakly to some $g \in L^2(I^+)$ with $||g||_2 \leqslant C$. Hence by the Banach-Saks theorem there is a subsequence g_{n_κ} of g_n such that

$$\left\| g - \frac{1}{n} \sum_{\kappa=1}^{n} g_{n_\kappa} \right\|_2 \to 0.$$

Furthermore

$$||g_\kappa - P(g_\kappa)||_2 = \frac{1}{\kappa} ||f_1 - f_{\kappa+1}||_2 < \frac{2C}{\kappa}$$

so

$$\begin{aligned} \left\| \frac{1}{n} \sum_{\kappa=1}^{n} g_{n_\kappa} - P\left(\frac{1}{n} \sum_{\kappa=1}^{n} g_{n_\kappa} \right) \right\|_2 &= \left\| \frac{1}{n} \sum_{\kappa=1}^{n} g_{n_\kappa} - \frac{1}{n} \sum_{\kappa=1}^{n} P(g_{n_\kappa}) \right\|_2 \\ &\leqslant \frac{1}{n} \sum_{\kappa=1}^{n} ||g_{n_\kappa} - P(g_{n_\kappa})||_2 \\ &\leqslant \frac{1}{n} \sum_{\kappa=1}^{n} \frac{2C}{n_\kappa}. \end{aligned}$$

Letting n go to infinity in the inequality above we get $||g - P(g)||_2 = 0$ and so g is a fixed point of P. Finally, since g is the limit of a sequence of functions of fixed positive finite integral and $||g||_2 \leqslant C$ we conclude that g has positive finite integral, and so we can normalise it to give a function h of integral 1.

□

We now explain how to bound $||f_n||_2$ in terms of the total number of overlaps at level n of the iterated function system $\{T_0^{-1}, T_{-1}^{-1}\}$. Let

$$\mathcal{N}_n := \#\left\{a_1 \cdots a_n, b_1 \cdots b_n \in \{0,-1\}^{2n} : T_{a_1}^{-1} \circ \cdots T_{a_n}^{-1}(I^+) \cap T_{b_1}^{-1} \circ \cdots T_{b_n}^{-1}(I^+) \neq \varnothing\right\}.$$

The question of whether these contracted regions overlap for given $a_1, \cdots, a_n, b_1, \cdots, b_n$ can be phrased in terms of the forward image of the origin $\underline{0}$.

This gives

$$\begin{aligned}
\mathcal{N}_n &= \#\{a_1 \cdots a_n, b_1 \cdots b_n \in \{0,-1\}^{2n} : |T_{a_1} \circ \cdots \circ T_{a_n}(\underline{0}) - T_{b_1} \circ \cdots \circ T_{b_n}(\underline{0})| \\
&\in I_{\beta_1} \times ... \times I_{\beta_d}\} \\
&= \#\{a_1 \cdots a_n, b_1 \cdots b_n \in \{0,1\}^{2n} : \left|\sum_{i=1}^{n}(a_i - b_i)\beta_j^{n-i}\right| \in I_{\beta_j} \text{ for each} \\
&\quad j \in \{1, \cdots, d\}\}.
\end{aligned}$$

where

$$I_{\beta_j} = \begin{cases} \left[\frac{-1}{\beta_j - 1}, \frac{1}{\beta_j - 1}\right], & \beta_j \in (1, \infty) \\ \left\{x \in \mathbb{R} : |x| \in \left[0, \frac{2}{|\beta_j| - 1}\right]\right\}, & x \in (-\infty, -1) \\ \left\{z \in \mathbb{C} : |z| \in \left[0, \frac{2}{|\beta_j| - 1}\right]\right\}, & z \in \mathbb{C} \setminus \mathbb{R} \end{cases}$$

for $\{1, \cdots, d\}$.

Proposition 4.2.2. *We have*

$$||f_n||_2 \leqslant \lambda(I^+)\left(\frac{|\beta_1 \cdot ... \cdot \beta_d|}{4}\right)^n \mathcal{N}_n$$

Proof. Notice that

$$P^n f = \left(\frac{|\beta_1 \cdot ... \cdot \beta_d|}{2}\right)^n \sum_{a_1,...,a_n \in \{0,-1\}} f \circ T_{a_1} \circ ... \circ T_{a_n}$$

So we have

$$\begin{aligned}
||f_n||_2 &= \int_{I^+} f_n(x) f_n(x) dx \\
&= \int_{I^+} \left(\left(\frac{|\beta_1 \cdot ... \cdot \beta_d|}{2}\right)^n \sum_{a_1,...,a_n \in \{0,-1\}} \chi_{I^+} \circ T_{a_1} \circ ... \circ T_{a_n} \right) \\
&\left(\left(\frac{|\beta_1 \cdot ... \cdot \beta_d|}{2}\right)^n \sum_{b_1,...,b_n \in \{0,-1\}} \chi_{I^+} \circ T_{a_1} \circ ... \circ T_{a_n} \right) dx \\
&= \int_{I^+} \left(\frac{|\beta_1 \cdot ... \cdot \beta_d|^2}{4}\right)^n \sum_{a_1,...a_n,b_1,...,b_n} \chi_{I^+} \circ T_{a_1} \circ ... \circ T_{a_n} \cdot \chi_{I^+} \circ T_{b_1} \circ ... \circ T_{b_n} dx \\
&= \left(\frac{|\beta_1 \cdot ... \cdot \beta_d|^2}{4}\right)^n \sum_{a_1,...a_n,b_1,...,b_n} \int_{I^+} \chi_{I^+} \circ T_{a_1} \circ ... \circ T_{a_n} \cdot \chi_{I^+} \circ T_{b_1} \circ ... \circ T_{b_n} dx
\end{aligned}$$

Notice that in the bound for $||f_n||_2$ given above we need to keep only the terms for $a_1, ..., a_n, b_1, ..., b_n$ such that $\chi_{I^+} \circ T_{a_1} \circ ... \circ T_{a_n} \cdot \chi_{I^+} \circ T_{b_1} \circ ... \circ T_{b_n} \neq 0$, i.e. those $a_1, \cdots a_n, b_1 \cdots, b_n$ involved in the definition of $\mathcal{N}_n$. Furthermore, by noticing that $\int_{I^+} \chi_{I^+} \circ T_{a_1} \circ ... \circ T_{a_n} \cdot \chi_{I^+} \circ T_{b_1} \circ ... \circ T_{b_n} dx$ is at most $\lambda(I^+)|\beta_1 \cdots \beta_d|^{-n}$, we end up with

$$||f_n||_2 \leqslant \lambda(I^+) \left(\frac{|\beta_1 \cdot ... \cdot \beta_d|}{4}\right)^n \mathcal{N}_n$$

as required. □

Combining Proposition 4.2.1, Lemma 4.2.1 and Proposition 4.2.2 gives the following theorem.

Theorem 4.2.1. *Suppose that the total number $\mathcal{N}_n$ of overlaps in the nth level of the iterated function system T_0, T_1 satisfies that*

$$\mathcal{N}_n \leqslant C\left(\frac{4}{|\beta_1 \cdot \ldots \cdot \beta_d|}\right)^n$$

for some constant $C > 0$ and for each $n \in \mathbb{N}$. Then the corresponding self-affine measure $\nu_{\underline{\beta}}$ is absolutely continuous.

We have stated Theorem 4.2.1 for a measure rectangular self-affine set with contraction rates associated to $\beta_1, \cdots \beta_d$ which were all Galois conjugates, since this is how we will apply the result in later sections, but it is worth noting that assumptions on the contraction rates were not used in this section and the theorem holds for any set of contraction rates $\beta_1, \cdots, \beta_d$.

4.3 Measures on the distance set

Theorem 4.2.1 involves counting all pairs $a_1, \cdots, a_n, b_1, \cdots, b_n \in \{0,1\}^{2n}$ for which

$$\left|\sum_{i=1}^{n}(a_i - b_i)\beta_j^{n-i}\right| \in I_{\beta_j} \text{ for each } j \in \{1, \cdots, d\}$$

If we let $\underline{\beta} := (\beta_1, \cdots, \beta_d)$, $\underline{\beta^n} := (\beta_1^n, \cdots, \beta_d^n)$, and

$$I = I_{\beta_1} \times \ldots \times I_{\beta_d}$$

we are counting the number of pairs $a_1, \cdots, a_n, b_1, \cdots, b_n$ for which

$$\sum_{i=1}^{n}(a_i - b_i)\underline{\beta}^{n-i} \in I.$$

Let $\mathcal{D}_n \subset \{0,1\}^{2n}$ be the set of such pairs $a_1, \cdots, a_n, b_1, \cdots, b_n$. It is useful for us to put a measure on the set of such differences. Let

$$\mu_n := \sum_{\{a_1 \cdots a_n, b_1 \cdots b_n \in \mathcal{D}_n\}} \delta_{\sum_{i=1}^{n}(a_i - b_i)\underline{\beta^{n-i}}},$$

for $n \geqslant 1$. This is a sum of weighted Dirac masses, supported on the set I, with total mass $\mathcal{N}_n$.

In going from $\mathcal{N}_n$ to $\mathcal{N}_{n+1}$ it is useful to note that

$$\sum_{i=1}^{n+1}(a_i - b_i)\beta_j^{(n+1)-i} = \beta_j \left(\sum_{i=1}^{n}(a_i - b_i)\beta_j^{n-i}\right) + (a_{n+1} - b_{n+1}),$$

with the difference $(a_{n+1} - b_{n+1})$ taking value $1, -1$, or 0. There are two different ways of getting value 0 here, we can have $a_{n+1} = b_{n+1} = 0$ or $a_{n+1} = b_{n+1} = 1$.

Define an operator Φ on the space of measures on I by letting

$$(\Phi(\mu))(A) := \mu\left(T_1^{-1}(A)\right) + \mu\left(T_{-1}^{-1}(A)\right) + 2\mu\left(T_0^{-1}(A)\right).$$

for $A \subset I$. Note that we only define Φ on measures supported on I and define $\Phi(\mu)$ to also be supported on I, we do not spread mass outside of I.

If we set $\mu_0 = \delta_{\underline{0}}$ then

$$\mu_n = \Phi(\mu_{n-1})$$

for $n \in \mathbb{N}$. Let $|\mu| := \mu(I)$ denote the total mass of a measure μ supported on I. Phrased in this new language, Theorem 4.2.1 yields the following corollary.

Corollary 4.3.1. *Suppose that there exists a constant $C > 0$ such that*

$$|\Phi^n(\delta_{\underline{0}})| \leqslant C\left(\frac{4}{|\beta_1 \cdot \ldots \cdot \beta_d|}\right)^n$$

for all $n \in \mathbb{N}$. Then the self-affine measure $\nu_{\underline{\beta}}$ is absolutely continuous.

We now turn to understanding how measures grow under the operator Φ.

Lemma 4.3.1.

$$|\Phi(\mu)| = \mu(T_1^{-1}(I)) + \mu(T_{-1}^{-1}(I)) + 2\mu(T_0^{-1}(I)).$$

Proof. This is immediate from the definition of Φ. □

Define a step function $g : I \to \mathbb{R}$ by

$$g(x) = \chi_I(T_1(x)) + \chi_I(T_{-1}(x)) + 2\chi_I(T_0(x))$$

Then the previous lemma just says that

$$|\phi(\mu)| = \int g d\mu.$$

We have the following theorem.

Theorem 4.3.1. *Suppose that there exists a constant $C > 1$ such that*

$$\sum_{n=1}^{\infty} \log\left(\frac{|\beta_1 \cdot \ldots \cdot \beta_d|}{4} \frac{1}{|\mu_n|} \int g d\mu_n\right) \leqslant \log(C).$$

Then the self-affine measure $\nu_{\underline{\beta}}$ is absolutely continuous.

Note that $\frac{1}{|\mu_n|} \int g d\mu_n$ is the integral of g with respect to the probability measure $\frac{1}{|\mu_n|}\mu_n$. Secondly, if $\mathcal{L}$ denotes Lebesgue measure on I, normalised to have mass one, then $\int_I g(x) d\mathcal{L}(x) = \frac{4}{|\beta_1 \cdot \ldots \cdot \beta_d|}$. Thus, if the sequence of probability measures $\frac{\mu_n}{|\mu_n|}$ converge weakly to normalised Lebesgue measure $\mathcal{L}$ then

$$\log\left(\frac{|\beta_1 \cdot \ldots \cdot \beta_d|}{4} \frac{1}{|\mu_n|} \int g d\mu_n\right) \to 0.$$

Thus the condition in Theorem 4.3.1 would follow from the sequence $\frac{\mu_n}{|\mu_n|}$ converging weakly to $\mathcal{L}$ with a given rate.

Proof. From Corollary 4.3.1 it is enough to prove that

$$\frac{1}{n} \log(|\mu_n|) \leqslant \frac{C}{n} + \log\left(\frac{4}{|\beta_1 \cdot \ldots \cdot \beta_d|}\right)$$

for some $C > 0$. From Lemma 4.3.1 and the discussion afterwards, for each positive integer k,

$$\frac{|\mu_{k+1}|}{|\mu_k|} = \frac{|\Phi(\mu_k)|}{|\mu_k|} = \frac{1}{|\mu_k|} \int g d\mu_k.$$

Then since $\log(|\mu_0|) = 0$, we have

$$\begin{aligned}
\log(|\mu_n|) &= \sum_{k=0}^{n-1} \log\left(\frac{|\mu_{k+1}|}{|\mu_k|}\right) \\
&= \sum_{k=0}^{n-1} \log\left(\frac{1}{|\mu_k|}\int g d\mu_k\right) \\
&= \sum_{k=0}^{n-1} \log\left(\frac{4}{|\beta_1 \cdot ... \cdot \beta_d|}\right) + \sum_{k=0}^{n-1} \log\left(\frac{|\beta_1 \cdot ... \cdot \beta_d|}{4}\frac{1}{|\mu_k|}\int g d\mu_k\right) \\
&\leqslant n \log\left(\frac{4}{|\beta_1 \cdot ... \cdot \beta_d|}\right) + \log(C)
\end{aligned}$$

by the assumption in the theorem. Then

$$\frac{1}{n}\log(|\mu_n|) \leqslant \log\left(\frac{4}{|\beta_1 \cdot ... \cdot \beta_d|}\right) + \frac{\log(C)}{n}$$

as required.

□

4.4 The limit measure $\bar{\mu}$

In this section we link the measures μ_n with methods appeared in [11]. The goal is to replace the measures μ_n, which evolve in time, with a fixed limit measure $\bar{\mu}$.

First we need to move in a higher dimensional space by considering the rest of the Galois conjugates $\beta_{d+1}, ..., \beta_{d+s+1}$. We set $\bar{\beta}^n = (\beta_1^n, ..., \beta_{d+s+1}^n)$. Set $\bar{T}_i(x_1, ..., x_{d+s+1}) = (\beta_1 x_1 + i, ..., \beta_{d+s+1} x_{d+s+1} + i)$ which acts on the space $\bar{\mathbb{K}} := \prod_{i=1}^{d+s+1} \mathbb{F}_{\beta_i}$. We also define the set

$$\bar{Z} = \{a_{d+s}\bar{\beta}^{d+s} + ... + a_0\bar{\beta}^0 : a_{d+s}, ..., a_0 \in \mathbb{Z}\}.$$

The set $\bar{Z}$ is a lattice in $\bar{\mathbb{K}} \cong \mathbb{R}^{\sum_{i=1}^{d+s+1} \dim(\mathbb{F}_{\beta_i})}$. That is because $\{\bar{\beta}^0, ..., \bar{\beta}^{d+s}\}$ is an independent subset of the real vector space $\bar{\mathbb{K}}$. That can be checked using the formula for the determinant of the Vandermonde matrix. We partition our coordinates into expanding directions $1, \cdots, d$, contracting directions $d+1, \cdots, d+s$ and the free direction $d+s+1$. The dynamics we will introduce is also expanding on the free direction, but we deal with this coordinate separately since we will eventually project in this direction.

We define projections π_e, π_c and π_{free} from $\bar{\mathbb{K}}$ onto subspaces of $\bar{\mathbb{K}}$ corresponding to expanding directions, contracting directions and the free direction respectively. They are given by

$$\begin{aligned} \pi_e(x_1, \cdots, x_{d+s+1}) &= (x_1, \cdots, x_d) \\ \pi_c(x_1, \cdots, x_{d+s+1}) &= (x_{d+1}, \cdots, x_{d+s}) \\ \pi_{free}(x_1, \cdots, x_{d+s+1}) &= x_{d+s+1}. \end{aligned}$$

It is worth noting that π_e, π_c and π_{free} are injective when restricted to $\bar{Z}$. We define a strip $S \subset \bar{\mathbb{K}}$ by

$$S = \{(x_1, \cdots, x_{d+s+1}) \in \bar{\mathbb{K}} : \pi_e(x_1, \cdots, x_{d+s+1}) \in I\}.$$

The following definitions differ from those in [11] in that we restrict both $\bar{\mu}_n$ and $\bar{X}$ to the set S. Let the measure $\bar{\mu}_n$ on S be given by

$$\bar{\mu}_n(x) = \# \left\{ (a_1, ..., a_n, b_1, ..., b_n) \in \{0,1\}^{2n} : \sum_{i=1}^{n} (a_i - b_i)\bar{\beta}^{n-i} = x \right\}$$

for $x \subset S$. We do not give mass to points outside S. The measure $\bar{\mu}_n$ is a weighted sum of Dirac masses supported on the set

$$\begin{aligned} \bar{X} &:= \left\{ \sum_{i=1}^{n} a_i \bar{\beta}^{n-i} : n \in \mathbb{N}, a_1 ..., a_n \in \{-1, 0, 1\} \right\} \cap S \\ &= \left\{ \bar{T}_{a_n} \circ ... \circ \bar{T}_{a_1}(0) : n \in \mathbb{N}, a_1 ..., a_n \in \{-1, 0, 1\} \right\} \cap S, \end{aligned}$$

Notice that for each $i \in \mathbb{Z}$ we have $\bar{T}_i(\bar{Z}) \subseteq \bar{Z}$. In particular $\bar{X} \subseteq \bar{Z}$ so $\bar{X}$ is uniformly discrete in $\bar{\mathbb{K}}$. Note that for $A \subset \bar{\mathbb{K}}$, $\mu_n \circ \pi_e(A) = \bar{\mu}_n(A)$ so the measures

$\bar{\mu}_n$ are just lifts of the measures μ_n of the previous section to a higher dimensional space in which they are uniformly discrete.

Definition 4.4.1. *Let $\mathcal{R} \subseteq I_{\beta_{d+1}} \times ... \times I_{\beta_{d+s}}$ be the attractor of the iterated function system involving the maps $\overline{T}_i$ restricted to contracting coordinates $d+1, \cdots, d+s$.*

The significance of the set $\mathcal{R}$ becomes clear in the condition 4.4.1 below, although one can already observe that

$$\bar{X} \subseteq \{z \in \bar{Z} : \pi_c(z) \in \mathcal{R}, \pi_e(z) \in I\}.$$

We will need the following condition which can be checked in finite time (see [11]) and which holds for all examples we have checked.

Condition 4.4.1. $\bar{X} = \bar{Z} \cap \pi_c^{-1}(\text{int}(\mathcal{R})) \cap S$*,*

Below we have plotted on approximation of $\mathcal{R}$ for the example of section 4.1.1. The following theorem recalls some results of [11] that we will need.

Theorem 4.4.1.

1. *There exists $\lambda > 1$ and a function $f : \bar{X} \to (0, \infty)$ such that for each $x \in \bar{X}$ the sequence of real numbers $\frac{1}{\lambda^n}\bar{\mu}_n(x)$ converges to $f(x)$.*

2. *We have $0 < f(x) \leqslant f(0)$ for each $x \in \bar{X}$.*

Definition 4.4.2. *Define the measure $\bar{\mu}$ on $\bar{X}$ by $\bar{\mu}(A) = \sum_{x \in \bar{X} \cap A} f(x)$.*

As we did with the measures μ_n we define an operator $\bar{\Phi}$ acting on measures on $\bar{X}$ by

$$\bar{\Phi}(\mu)(A) = \mu(\bar{T}_{-1}^{-1}(A)) + 2\mu(\bar{T}_0^{-1}(A)) + \mu(\bar{T}_{-1}^{1}(A))$$

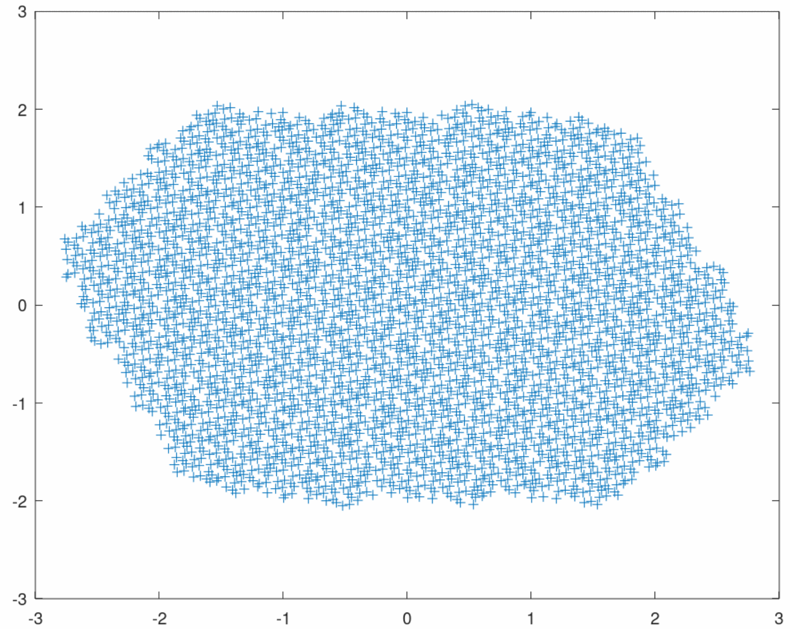

Figure 4.2: An approximation of $\mathcal{R}$ when $\beta^4 = \beta^3 + \beta^2 - \beta + 1$.

for $A \subset S$, and $\bar{\Phi}(\mu)(A) := \bar{\Phi}(\mu)(A \cap S)$ for more general A. Φ does not spread mass outside of the strip S. We have

$$\bar{\mu}_n = \bar{\Phi}^n \delta_0$$

and

$$\bar{\mu} = \frac{1}{\lambda} \bar{\Phi}(\bar{\mu}),$$

see Lemma 4.3 of [11].

We comment that the set $\bar{X}$ is bounded in the coordinates $1, \cdots, d$ since we insist on remaining in the strip S, and it is bounded in the coordinated $d+1, \cdots, d+s$ since the action of the maps $\bar{T}_i$ is contracting on these coordinates and orbits remain in the fractal $\mathcal{R}$. It is only the free direction $d+s+1$ in which $\bar{X}$ is unbounded.

Let

$$R_n = \{x \in \bar{X} : |\pi_{\text{free}}(x)| \leqslant \sum_{i=0}^{n-1} |\beta_{d+s+1}^i|\}$$

The rest of this section is dedicated to proving the following theorem, which replaces the μ_n of Theorem 4.3.1 with $\pi_e(\bar{\mu}|_{R_n})$.

Theorem 4.4.2. *Suppose that $\lambda < 4/|\beta_1 \cdots \beta_d|$ and that there exists a constant C such that*

$$\sum_{n=1}^{\infty} \log \left(\frac{|\beta_1 \cdot ... \cdot \beta_d|}{4} \frac{1}{|\bar{\mu}|_{R_n}|} \int g d\pi_e \bar{\mu}|_{R_n} \right) \leqslant \log(C).$$

Then the self-affine measure $\nu_{\underline{\beta}}$ is absolutely continuous.

Again, we comment that this is really an equidistribution result, requiring that for the probability measure $\frac{1}{|\bar{\mu}|_{R_n}|}\pi_e(\bar{\mu}|_{R_n})$ the mass of certain intervals (involved in the definition of the step function g) is sufficiently close to the Lebesgue measure of those intervals.

4.4.1 Proof of Theorem 4.4.2

In Theorem 4.2.1 we gave a criteria for the absolute continuity of $\nu_{\underline{\beta}}$ in terms of the measure μ_n, which can be easily translated to a criteria involving $\bar{\mu}_n$. In order to relate this to $\bar{\mu}$, we need first to consider the subset of $\bar{X}$ upon which $\bar{\mu}_n$ is supported.

Note that in the free direction our maps $\bar{T}_i$ act by $x \to \beta_{d+s+1}(x) + i$, and so points $\bar{T}_{a_n} \circ \cdots \circ \bar{T}_{a_1}(0)$ must lie in R_n. We have the following lemma.

Lemma 4.4.1.

$$|\bar{\Phi}^n(\delta_0)| \leqslant \frac{\lambda^n}{\bar{\mu}(0)}\bar{\mu}(R_n).$$

Proof. Since $\bar{\Phi}$ is monotone and $\bar{\mu}(0)\delta_0 \leqslant \bar{\mu}$, using $\bar{\Phi}(\bar{\mu})/\lambda = \bar{\mu}$ we have

$$\frac{1}{\lambda^n}\bar{\Phi}^n(\bar{\mu}(0)\delta_0) \leqslant \frac{1}{\lambda^n}\bar{\Phi}^n(\bar{\mu}) = \bar{\mu}.$$

On the other hand from the construction of R_n we have that

$$\frac{1}{\lambda^n}\bar{\Phi}^n(\bar{\mu}(0)\delta_0)(\bar{X} \setminus R_n) = 0.$$

Combining these facts gives

$$|\bar{\Phi}^n(\delta_0)| = \frac{\lambda^n}{\bar{\mu}(0)}\frac{1}{\lambda^n}\bar{\Phi}^n(\bar{\mu}(0)\delta_0)(R_n) \leqslant \frac{\lambda^n}{\bar{\mu}(0)}\bar{\mu}(R_n).$$

□

Lemma 4.4.2. *Assume that $\lambda < \frac{4}{|\beta_1 \cdots \beta_d|}$. Then $\bar{\mu}(R_n)$ grows exponentially in n.*

Proof. We note that the 2^n rectangles $(T_{a_1} \circ \cdots T_{a_n})^{-1}(I^+)$ are each contained in I^+ and each have an area of $\frac{1}{|\beta_1 \cdots \beta_d|^n} \times$ Area(I^+), giving a total area of $\frac{2^n}{|\beta_1 \cdots \beta_d|^n} \times$ Area(I^+). A lower bound for the total number of overlaps comes from assuming these rectangles are evenly spread, in which case one would have that a typical rectangle intersects $\frac{2^n}{|\beta_1 \cdots \beta_d|^n}$ others, giving $\mathcal{N}_n \geqslant \frac{1}{2}\frac{4^n}{|\beta_1 \cdots \beta_d|^n}$.

Then

$$\bar{\mu}(R_n) \geqslant \frac{\mathcal{N}_n}{\lambda^n} = \frac{1}{2}\left(\frac{4}{|\beta_1 \cdots \beta_d|}\frac{1}{\lambda}\right)^n.$$

which grows exponentially by our assumption. □

We stress that λ can be computed by a finite calculation when β has no Galois conjugates of absolute value 1 (as we are assuming throughout this article). Values of λ are computed for many values in [2] and in all examples we have computed satisfy the condition of Lemma 4.4.2.

Lemma 4.4.3. *There exist ϵ_n tending to zero exponentially quickly such that*

$$\begin{aligned}\bar{\mu}(R_{n+1}) &\leqslant \frac{1+\epsilon_n}{\lambda}|\bar{\Phi}(\bar{\mu}|_{R_n})| \\ &= \frac{(1+\epsilon_n)}{\lambda}\int g d\pi_e(\bar{\mu}|_{R_n})\end{aligned}$$

Proof. Let $x \in \bar{X}$ be such that

$$|\pi_{free}(x)| \leqslant -2 + \sum_{i=0}^{n} |\beta^i_{d+s+1}|.$$

Then

$$|\pi_{free}(\bar{T}_i^{-1}(x))| = \left|\frac{\pi_{free}(x) - i}{\beta_{d+s+1}}\right| \leqslant \sum_{i=0}^{n-1} |\beta^i_{d+s+1}|$$

and so $\bar{T}_i^{-1}(x) \in R_n \cup (\bar{\mathbb{K}} \setminus \bar{X})$ for each $i \in \{-1, 0, 1\}$. Hence from $\frac{\bar{\Phi}(\bar{\mu})}{\lambda} = \bar{\mu}$ we get

$$\begin{aligned}\frac{1}{\lambda}\bar{\Phi}(\bar{\mu}|_{R_n})(x) &= \frac{1}{\lambda}\left(\bar{\mu}|_{R_n}(\bar{T}_{-1}^{-1}(x)) + 2\bar{\mu}|_{R_n}(\bar{T}_0^{-1}(x)) + \bar{\mu}|_{R_n}(\bar{T}_1^{-1}(x))\right) \\ &= \frac{1}{\lambda}\left(\bar{\mu}(\bar{T}_{-1}^{-1}(x)) + 2\bar{\mu}(\bar{T}_0^{-1}(x)) + \bar{\mu}(\bar{T}_1^{-1}(x))\right) \\ &= \frac{1}{\lambda}\bar{\Phi}(\bar{\mu})(x) = \bar{\mu}(x).\end{aligned}$$

Thus

$$\begin{aligned}&\bar{\mu}\left(\left\{x \in R_{n+1} : |\pi_{free}(x)| \leqslant -2 + \sum_{i=0}^{n} |\beta^i_{d+s+1}|\right\}\right) \\ =&\frac{1}{\lambda}\bar{\Phi}(\bar{\mu}|_{R_n})\left(\left\{x \in R_{n+1} : |\pi_{free}(x)| \leqslant -2 + \sum_{i=0}^{n} |\beta^i_{d+s+1}|\right\}\right).\end{aligned} \tag{4.2}$$

The diameter of

$$\left\{x \in R_{n+1} : |\pi_{free}(x)| > -2 + \sum_{i=0}^{n} |\beta^i_{d+s+1}|\right\}$$

is uniformly bounded so there is $M > 0$ that depends only on β such that

$$\#\left\{x \in R_{n+1} : |\pi_{free}(x)| > -2 + \sum_{i=0}^{n} |\beta^i_{d+s+1}|\right\} < M$$

for all $n \in \mathbb{N}$. By Theorem 4.4.1 we have $\bar{\mu}(x) \leqslant \bar{\mu}(0)$ for all $x \in \bar{X}$ and so

$$\bar{\mu}\left(\left\{x \in R_{n+1} : |\pi_{free}(x)| > -2 + \sum_{i=0}^{n} |\beta^i_{d+s+1}|\right\}\right) < M\bar{\mu}(0). \tag{4.3}$$

Combining (4.2) and (4.3) we have

$$\begin{aligned}\bar{\mu}(R_{n+1}) &\leqslant \frac{1}{\lambda}\bar{\Phi}(\bar{\mu}|_{R_n})(\bar{X}) + M\bar{\mu}(0)\\ &\leqslant \frac{1}{\lambda}\bar{\Phi}(\bar{\mu}|_{R_n})(\bar{X})(1+\epsilon_n)\end{aligned}$$

Where $\epsilon_n = \frac{M\bar{\mu}(0)}{\frac{1}{\lambda}\bar{\Phi}(\bar{\mu}|_{R_n})(S)}$ tends to zero exponentially fast due to Lemma 4.4.2. Finally we mention that, by the construction of $\bar{\Phi}$

$$|\bar{\Phi}(\bar{\mu}|_{R_n})| = \int g d\pi_e(\bar{\mu}|_{R_n}),$$

this is is just the analogue of Lemma 4.3.1 for the lifted operator $\bar{\Phi}$ rather than Φ. □

Proposition 4.4.1. *If* $\lambda < \frac{4}{|\beta_1 \cdots \beta_d|}$ *there is* $c > 1$ *such that*

$$|\Phi^n(\delta_0)| \leqslant c\frac{\bar{\mu}(R_0)}{\bar{\mu}(0)}\prod_{i=0}^{n-1}\frac{1}{\bar{\mu}(R_i)}\int g d\pi_e(\mu|_{R_i}).$$

Proof. From Lemma 4.4.3 we have

$$\lambda\frac{\bar{\mu}(R_{n+1})}{\bar{\mu}(R_n)} \leqslant (1+\epsilon_n)\frac{\int g d\pi_e(\bar{\mu}|_{R_n})}{\bar{\mu}(R_n)}.$$

The above combined with Lemma 4.4.1 leads to

$$\begin{aligned}
|\Phi^n(\delta_0)| &= |(\bar{\Phi}^n(\delta_0))| \\
&\leqslant \frac{\lambda^n}{\bar{\mu}(0)}\bar{\mu}(R_n) \\
&= \frac{\bar{\mu}(R_0)}{\bar{\mu}(0)}\prod_{i=0}^{n-1}\frac{\lambda\bar{\mu}(R_{i+1})}{\bar{\mu}(R_i)} \\
&\leqslant \frac{\bar{\mu}(R_0)}{\bar{\mu}(0)}\left(\prod_{i=0}^{n-1}(1+\epsilon_i)\frac{1}{|\bar{\mu}(R_i)|}\int g d\pi_e(\bar{\mu}|_{R_i})\right).
\end{aligned}$$

The proof is complete by observing that from Lemma 4.4.2 we have

$$\prod_{i=0}^{\infty}(1+\epsilon_i) < \infty.$$

□

We can now prove Theorem 4.4.2. Assuming, as in the theorem, that

$$\sum_{n=1}^{\infty}\log\left(\frac{|\beta_1\cdot\ldots\cdot\beta_d|}{4}\frac{1}{\bar{\mu}(R_n)}\int g d\pi_e(\bar{\mu}|_{R_n})\right) \leqslant \log(C)$$

gives

$$\prod_{i=0}^{n-1}\frac{1}{|\bar{\mu}(R_n)|}\int g d\pi_e(\bar{\mu}|_{R_n}) \leqslant C\left(\frac{4}{|\beta_1\cdot\ldots\cdot\beta_d|}\right)^n,$$

hence, by Proposition 4.4.1,

$$\mathcal{N}_n = |\phi^n(\delta_0)| \leqslant C'\left(\frac{4}{|\beta_1\cdot\ldots\cdot\beta_d|}\right)^n$$

for some $C' > 0$. Thus the conditions of Corollary 4.3.1 are satisfied and so the measure $\nu_{\underline{\beta}}$ is absolutely continuous. This completes the proof of Theorem 4.4.2.

4.5 Domain Exchange Transformation

Definition 4.5.1. *We define the set the successor function* $\mathrm{succ} : \bar{X} \to \bar{X}$ *by*

$$\pi_{\mathrm{free}}(\mathrm{succ}(x)) = \min\{\pi_{\mathrm{free}}(y) : y \in \bar{X}, \pi_{\mathrm{free}}(y) > \pi_{\mathrm{free}}(x)\}.$$

We will later see that the successor function projects to a domain exchange transformation on $D = I \times \mathcal{R}$. We clarify that in our context a domain exchange transformation is defined as follows.

Definition 4.5.2. *Let* E *be a compact subset of a euclidean space and* $T : E \to E$*. The map* T *is call a domain exchange transformation if there are* $E_1, ..., E_n$ *measurable subsets of* E *such that following hold.*

- $\{E_1, ..., E_n\}$ *is a partition of* E*.*
- *The map* T *is an injection.*
- *If* $i \in \{1, ...n\}$ *then* $T|_{D_i}$ *is a translation.*

Let $\pi_D : \bar{X} \to D$ be given by $\pi_D(x_1, \cdots x_{d+s+1}) = (x_1, \cdots x_{d+s})$. Again we notice that $\pi_D|_{\bar{Z}}$ is injective.

Definition 4.5.3. *Let* w_n *be the measure on* D *defined by*

$$w_n = \sum_{\kappa=0}^{m} \bar{\mu}(\mathrm{succ}^{\kappa}(0))\delta_{\pi_D(\mathrm{succ}^{\kappa}(0))},$$

where m *is the greatest natural number such that*

$$\pi_{\mathrm{free}}(\mathrm{succ}^{m}(0)) \leqslant \sum_{i=0}^{n-1} |\beta^{i}_{d+s+1}|.$$

w_n is the image under projection onto coordinates $1, \cdots, d+s$ of the measure $\bar{\mu}$ restricted in the free direction to the range $[0, \sum_{i=0}^{n-1} |\beta^i_{d+s+1}|]$.

Theorem 4.4.2 gave sufficient conditions for the absolute continuity of $\nu_{\underline{\beta}}$ in terms of convergence to Lebesgue of the measures $\pi_e w_n$, which were projections onto expanding coordinates $1, \cdots, d$ of the measure $\bar{\mu}$ restricted to a bounded region in the free direction.

Here we stress that the successor function projects to a uniquely ergodic domain exchange transformation on $I \times \mathcal{R}$.

Recall that $D = I \times \mathcal{R}$.

Definition 4.5.4. *Let*

$$W = \{x \in \bar{\mathbb{K}} : \pi_c(x) \in \mathrm{int}(\mathcal{R}), \pi_e(x) \in I\}$$

and define $T' : D \to \bar{Z}$ by $T'(x) = u$ where

$$\pi_{\mathrm{free}}(y+u) = \min\left\{\pi_{\mathrm{free}}(z) : z \in \left(y+\bar{Z}\right) \cap W \textit{ and } \pi_{\mathrm{free}}(z) > \pi_{\mathrm{free}}(y)\right\}$$

for any $\pi_D(y) = x$.

It follows from the geometry of W that T' is well defined and that $T'(D)$ is finite. So there are $D_1, ..., D_N \subseteq D$ and $u_1, ..., u_N \in \bar{Z}$ such that $\{D_1, ..., D_N\}$ is a partition of D and

$$x \in D_i \Rightarrow T'(x) = u_i.$$

Notice that when $x \in S \cap \bar{Z}$ then $x + T'(\pi_D(x)) = \mathrm{succ}(x)$.

Lemma 4.5.1. *The map $T : D \to D$ defined by*

$$T(x) = x + \pi_D(T'(x))$$

defines a domain exchange transformation $(T, D_1, ..., D_N)$.

Proof. We only need to prove that T is injective. Let, aiming for a contradiction, $x, y \in D$ such that $T(x) = T(y)$. We can choose $x', y' \in S$ with $\pi_D(x') = x$ and $\pi_D(y') = y$ such that $x' + T'(x) = y' + T'(y)$ since $\pi_D(x' + T'(x)) = T(x) = T(y) = \pi_D(y' + T'(y))$ and we can freely determine $\pi_{free}(x')$ and $\pi_{free}(y')$. Notice that $y' = x' + T'(x) - T'(y) \in x' + \bar{Z}$ so $x' \neq y' \Rightarrow \pi_{\text{free}}(x') \neq \pi_{\text{free}}(y')$. Assume, without loss of generality, that $\pi_{\text{free}}(y') < \pi_{\text{free}}(x')$. We have $\pi_{\text{free}}(y') < \pi_{\text{free}}(x') < \pi_{\text{free}}(x' + T'(x)) = \pi_{\text{free}}(y' + T'(y))$ which contradicts the definition of T' since $x' = y' + T'(y) - T'(x) \in y' + \bar{Z}$. □

Notice that, under condition 4.4.1, $\pi_D(\text{succ}^n(0)) = T^n(0)$ since Theorem 4.4.1 implies $\bar{X} = \bar{Z} \cap W$. For $x \in D$, we define $s(x)$ to be the unique i such that $x \in D_i$. Now we move on to give a characterization of the measures w_n which shows that they have a special structure that could be used to prove equidistribution properties, such as theorem 4.4.2 demands for the absolute continuity of ν_β. The main ingredient of the proof is theorem 1.3 of [11]. For this reason we need to impose the same condition which appears in that theorem and define the set Δ which also appears in it, as we do below.

Definition 4.5.5. *Let*

$$\Delta = \{x - y : x, y \in \bar{X} \text{ and } \exists c_1 \cdots c_n, d_1 \cdots d_n \in \{-1, 0, 1\}^n : \bar{T}_{c_n} \circ \cdots \bar{T}_{c_1}(x) = \bar{T}_{d_n} \cdots \bar{T}_{d_1}(y)\}.$$

That is, Δ is the set of differences between points $x, y \in \bar{X}$ which can be mapped to the same point in the future by the application of maps $\bar{T}_i$. Before we state proposition 4.5.1 we set S_i to be the maps $\bar{T}_i$ restricted to the contracting coordinates $d+1, ..., d+s$.

Proposition 4.5.1. *Under condition 4.4.1, there are functions $\bar{f}_1, ..., \bar{f}_N : \mathcal{R} \to \mathbb{R}^+$ such that*

i) There exists a word w and constants $C_1 > 0$, $C_2 \in (0,1)$ such that for any $a_1 \cdots a_n \in \{-1,0,1\}^n$ which contains r non-overlapping copies of the word w, $\bar{f}_i$ varies by at most $C_1 C_2^{r-1}$ on $S_{a_1} \circ \cdots \circ S_{a_n}(\mathcal{R})$.

ii) If m is the greatest natural number such that

$$\pi_{\text{free}}(\text{succ}^m(0)) \leqslant \sum_{i=0}^{n-1} |\beta^i_{d+s+1}|,$$

then

$$w_n = \bar{\mu}(0) \sum_{\kappa=0}^{m} \left(\prod_{i=0}^{\kappa-1} \exp\left(\bar{f}_{s(T^i(0))}(\pi_c(T^i(0)))\right)\right) \delta_{T^\kappa(0)}$$

Proof. From Theorem 1.3 in [11], for each $i \in \{1, ..., N\}$ there are $\bar{f}_i : \mathcal{R} \to \mathbb{R}^+$ satisfying i) such that $\bar{f}_i(\pi_c(x)) = \log(\bar{\mu}(x+u_i)) - \log(\bar{\mu}(x))$ for all $x \in \bar{X}$. We construct f_i by writing u_i as a sum of members of the set Δ and summing the respective functions given by the theorem. We have

$$
\begin{aligned}
\bar{\mu}(\mathrm{succ}^n(0)) &= \bar{\mu}(0)\prod_{i=0}^{n-1}\frac{\bar{\mu}(\mathrm{succ}^{i+1}(0))}{\bar{\mu}(\mathrm{succ}^{i}(0))}\\
&= \bar{\mu}(0)\prod_{i=0}^{n-1}\frac{\bar{\mu}(\mathrm{succ}^{i}(0)+u_{s(\pi_D(\mathrm{succ}^i(0)))})}{\bar{\mu}(\mathrm{succ}^{i}(0))}\\
&= \bar{\mu}(0)\prod_{i=0}^{n-1}\exp\left(\bar{f}_{s(\pi_D(\mathrm{succ}^i(0)))}(\pi_c(\mathrm{succ}^i(0)))\right)\\
&= \bar{\mu}(0)\prod_{i=0}^{n-1}\exp\left(\bar{f}_{s(T^i(0))}(\pi_c(T^i(0)))\right)
\end{aligned}
$$

so if m is the greatest natural number such that

$$
\pi_{\mathrm{free}}(\mathrm{succ}^m(0)) \leqslant \sum_{i=0}^{n-1}|\beta^i_{d+s+1}|
$$

then

$$
\begin{aligned}
w_n &= \sum_{\kappa=0}^{m}\bar{\mu}(\mathrm{succ}^\kappa(0))\delta_{\pi_D\circ\ \mathrm{succ}^\kappa(0)}\\
&= \bar{\mu}(0)\sum_{\kappa=0}^{m}\left(\prod_{i=0}^{\kappa-1}\exp\left(\bar{f}_{s(T^i(0))}(\pi_c(T^i(0)))\right)\right)\delta_{T^\kappa(0)},
\end{aligned}
$$

concluding ii).

□

Recall that Theorem 4.4.2 gave a condition for the absolute continuity of $\nu_{\underline{\beta}}$ in terms of the measures $\pi_e(\bar{\mu})$. In Definition 4.5.3 we introduced the measures w_n which were projections of weighted Dirac measures along an orbit of the successor function *succ*, and in Proposition 4.5.1 we explain how the weights appear as a cocycle over the dynamical system T. Combining these ideas in one theorem gives the following.

Theorem 4.5.1. *Assume that $\lambda < 4/|\beta_1 ... \beta_d|$ and condition 4.4.1 holds. Then there exists a domain $D = I \times \mathcal{R}$, a domain exchange transformation $T : D \to D$ and a function $f : D \to \mathbb{R}^+$ with $f(x) = \exp(\bar{f}_{s(x)}(\pi_c(x)))$ such that if the projection onto I of the sequence of measures*

$$w_n = \sum_{i=1}^{n} f(0) f(T(0)) \cdots f(T^{n-1}(0)) \delta_{T^{n-1}(0)}$$

converge to Lebesgue measure sufficiently quickly, in the sense that

$$\sum_{n=1}^{\infty} \log \left(\frac{|\beta_1 \cdot ... \cdot \beta_d|}{4} \frac{1}{|w_n|} \int g d\pi_e w_n|_{R_n} \right) \leqslant \log(C),$$

then the measure $\nu_{\underline{\beta}}$ is absolutely continuous.

Proof. The theorem follows from theorem 4.4.2, lemma 4.5.1 and proposition 4.5.1 after observing that

$$\pi_e \bar{\mu}|_{R_n}(x) = \begin{cases} \pi_e w_n(x) + \pi_e w_n(-x), & x \in I \setminus \{0\} \\ \pi_e w_n(0), & x = 0 \end{cases}.$$

□

Chapter 5

On the Local Dimension Spectrum for Self-Affine Measures

JOINT WORK WITH ANTTI KÄENMÄKI AND TOM KEMPTON

5.1 Introduction

In this article we are concerned with the dimension spectrum of self-affine measures on $\mathbb{R}^2$. Given a set of invertable matrices $A_1, \cdots, A_k$ of norm less than one, and a collection of translation vectors $v_1, \cdots, v_k$, we let the maps $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T_i \begin{pmatrix} x \\ y \end{pmatrix} = A_i \begin{pmatrix} x \\ y \end{pmatrix} + v_i.$$

Then given a probability vector $(p_1, \cdots, p_k)$, we let the self-affine measure μ be the unique probability measure satisfying

$$\mu = \sum_{i=1}^{k} p_i \mu \circ T_i^{-1}.$$

The local dimension spectrum $\bar{f}$ of μ is then given by

$$\bar{f}(\alpha) = \dim_H\{x \in \mathbb{R}^2 : \dim_{loc}(\mu, x) = \alpha\}.$$

Here $\dim_{loc}$ is the local dimension, given by

$$\dim_{loc}(\mu, x) := \lim_{r \to 0} \frac{\log(\mu(B(x, r)))}{\log r}$$

where it exists. The dimension spectrum gives an important way of quantifying fractal properties of the measure μ, it is well understood for self-similar measures without overlaps, and some progress has been made in understanding both the overlapping self-similar and the self-affine cases.

In particular, given a set of invertable matrices $A_1, \cdots, A_k$ with norm less than $\frac{1}{2}$ and a probability vector $(p_1, \cdots, p_k)$, Barral and Feng [10] were able to give a formula for part of the corresponding dimension spectrum which holds for almost every set of translation vectors $v_1, \cdots, v_k$. The part of their work giving the almost everywhere result uses the transversality technique and is very much in the spirit of earlier work of Falconer giving an almost everywhere result for the Hausdorff dimension of μ [25].

The Hausdorff dimension result of Falconer has been generalised to replace the almost everywhere condition with specific conditions on orthogonal projections of μ [27, 5], or with exponential separation conditions on the collection of matrices $A_1, \cdots, A_k$ [6, 46]. Our goal in this work is to similarly replace the almost everywhere condition of Barral and Feng with conditions on projections of the measure μ. We also assume that our set of matrices is dominated, we give more details in the next section. Our results also cover the more general case of pushforwards of quasi-Bernoulli measures.

5.2 Preliminaries

Let $(A_1, \ldots, A_N) \in GL_2(\mathbb{R})^N$ be a tuple of contractive invertible 2×2-matrices. If $(v_1, \ldots, v_N) \in (\mathbb{R}^2)^N$ is a tuple of translation vectors, then the tuple $(T_1, \ldots, T_N)$ of invertible contractive affine maps given by

$$T_i(x) = A_i x + v_i$$

is called an *affine iterated function system (affine IFS)*. Given an affine IFS, there exists a unique non-empty compact set $X \subset \mathbb{R}^2$ such that

$$X = \bigcup_{i=1}^{N} T_i(X).$$

We may assume that each map T_i maps the unit disk $D \subset \mathbb{R}^2$ inside itself. Indeed, if this is not the case, then one can rescale each v_i by a constant such that each T_i maps D inside itself. Note that this does not affect any dimension properties since it is a linear rescaling of X.

Let μ be a Borel probability measure supported on X. The *local dimension* of μ at x is

$$\dim_{\mathrm{loc}}(\mu, x) = \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r}$$

provided the limit exists. If the limit does not exist, then the corresponding upper and lower limits are denoted by $\overline{\dim}_{\mathrm{loc}}(\mu, x)$ and $\underline{\dim}_{\mathrm{loc}}(\mu, x)$, respectively. The local dimension of μ is intrinsically connected to the dimension of the subsets of X: it is sufficiently easy to see that

$$\operatorname*{ess\,inf}_{x \sim \mu} \underline{\dim}_{\mathrm{loc}}(\mu, x) = \underline{\dim}_{\mathrm{H}}(\mu),$$

where $\underline{\dim}_{\mathrm{H}}(\mu) = \inf\{\dim_{\mathrm{H}}(A) : A \subset X \text{ is a Borel set such that } \mu(A) > 0\}$ is the *lower Hausdorff dimension* of μ (see [44], theorem 2.3). Let $s \geqslant 0$ and define the *s-level set* of X with respect to μ to be

$$X(\mu, s) = \{x \in X : \dim_{\mathrm{loc}}(\mu, x) = s\}.$$

We are interested in determining the Hausdorff dimension of the level sets. Let us next go through preliminaries needed in the work.

5.2.1 Shift Space

Let $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ be the collection of all infinite words obtained from alphabet $\{1, \ldots, N\}$. If $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$, then we define $\mathtt{i}|_n = i_1 \cdots i_n$ for all $n \in \mathbb{N}$. The empty word $\mathtt{i}|_0$ is denoted by $\varnothing$. Define $\Sigma_n = \{\mathtt{i}|_n : \mathtt{i} \in \Sigma\}$ for all $n \in \mathbb{N}$ and $\Sigma_* = \bigcup_{n\in\mathbb{N}} \Sigma_n \cup \{\varnothing\}$. Thus Σ_* is the collection of all finite words. The length of $\mathtt{i} \in \Sigma_* \cup \Sigma$ is denoted by $|\mathtt{i}|$. The concatenation of two words $\mathtt{i} \in \Sigma_*$ and $\mathtt{j} \in \Sigma_* \cup \Sigma$ is denoted by $\mathtt{ij}$. Let σ be the left shift operator defined by $\sigma\mathtt{i} = i_2 i_3 \cdots$ for all $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$. If $\mathtt{i} \in \Sigma_n$ for some n, then we set $[\mathtt{i}] = \{\mathtt{j} \in \Sigma : \mathtt{j}|_n = \mathtt{i}\}$. The set $[\mathtt{i}]$ is called a *cylinder set*.

Given an affine IFS $(T_1, \ldots, T_N)$, where $T_i(x) = A_i x + v_i$, the *canonical projection* $\pi \colon \Sigma \to X$ is defined by

$$\pi(\mathtt{i}) = \lim_{n\to\infty} T_{\mathtt{i}|_n}(0) = \sum_{n=1}^{\infty} A_{\mathtt{i}|_{n-1}} v_{i_n}$$

for all $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$. Here $T_{\mathtt{i}} = T_{i_1} \circ \cdots \circ T_{i_n}$ and $A_{\mathtt{i}} = A_{i_1} \cdots A_{i_n}$ for all $\mathtt{i} = i_1 \cdots i_n \in \Sigma_n$ and $n \in \mathbb{N}$. It is easy to see that $\pi(\Sigma) = X$. If $\mu \in \mathcal{M}(\Sigma)$, where $\mathcal{M}(\Sigma)$ denote the collection of all Borel probability measures on Σ, then we denote the pushforward measure of μ under π by $\pi\mu = \mu \circ \pi^{-1}$. We say that a measure $\mu \in \mathcal{M}(\Sigma)$ is *fully supported* if each cylinder has positive measure.

5.2.2 Lyapunov Dimension

We shall consider maps $\theta \colon \Sigma_* \to (0, \infty)$ which we refer to as *potentials*. We say that a potential θ is *sub-multiplicative* if $\theta(\mathtt{ij}) \leqslant \theta(\mathtt{i})\theta(\mathtt{j})$ for all $\mathtt{i}, \mathtt{j} \in \Sigma_*$. A potential θ is *super-multiplicative* if the inverse $1/\theta$ is sub-multiplicative. We furthermore say that a potential θ is *almost-multiplicative* if there is a constant $C \geqslant 1$ such that $C\theta$ is sub-multiplicative and $C^{-1}\theta$ is super-multiplicative, and *multiplicative* if the constant C can be chosen to 1. Let $\mathcal{M}_\sigma(\Sigma)$ denote the collection of all σ-invariant Borel probability measures on Σ. For a sub-multiplicative potential θ and $\nu \in \mathcal{M}_\sigma(\Sigma)$, we define

$$\Lambda(\theta,\nu) = \lim_{n\to\infty} \frac{1}{n} \int_\Sigma \log \theta(\mathtt{i}|_n)\,\mathrm{d}\nu(\mathtt{i}).$$

The following lemma guarantees that Λ is well-defined.

Lemma 5.2.1. *If θ is a sub-multiplicative potential and $\nu \in \mathcal{M}_\sigma(\Sigma)$, then $\Lambda(\theta,\nu)$ exists and*

$$\Lambda(\theta,\nu) = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_\Sigma \log \theta(\mathtt{i}|_n)\,\mathrm{d}\nu(\mathtt{i}).$$

Furthermore, $\nu \mapsto \Lambda(\theta,\nu)$ defined on $\mathcal{M}_\sigma(\Sigma)$ is upper semi-continuous in the weak topology.*

Proof. The sequence $(\int_\Sigma \log\theta(\mathtt{i}|_n)\,\mathrm{d}\nu(\mathtt{i}))_{n\in\mathbb{N}}$ is sub-additive and therefore, by Fekete's Lemma, $\Lambda(\theta,\nu)$ exists and is equal to

$$\inf_{n\in\mathbb{N}} \frac{1}{n} \int_\Sigma \log \theta(\mathtt{i}|_n)\,\mathrm{d}\nu(\mathtt{i}).$$

The second claim is a direct consequence of the first claim as each $\nu \mapsto \frac{1}{n}\sum_{\mathtt{i}\in\Sigma_n} \nu([\mathtt{i}])\log\theta(\mathtt{i})$ is continuous. □

Let $(A_1,\ldots,A_N) \in GL_2(\mathbb{R})^N$. For $\mathtt{i} \in \Sigma_*$ we define $\alpha_1(\mathtt{i})$ and $\alpha_2(\mathtt{i})$ to be the lengths of the major and minor semi-axis of the ellipse $A_\mathtt{i}(D)$ respectively, where $D \subset \mathbb{R}^2$ is the unit disc. Note that $\alpha_1(\mathtt{i}) = \|A_\mathtt{i}\|$ and $\alpha_2(\mathtt{i}) = \|A_\mathtt{i}^{-1}\|^{-1}$ for all $\mathtt{i} \in \Sigma_*$. The potential $\mathtt{i} \mapsto \alpha_1(\mathtt{i})$ is thus sub-multiplicative and $\mathtt{i} \mapsto \alpha_2(\mathtt{i})$ is super-multiplicative. We define the *Lyapunov exponents* of $\nu \in \mathcal{M}_\sigma(\Sigma)$ by

$$\lambda_1(\nu) = \Lambda(\alpha_1,\nu) = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_\Sigma \log \alpha_1(\mathtt{i}|_n)\,\mathrm{d}\nu(\mathtt{i}),$$
$$\lambda_2(\nu) = -\Lambda(1/\alpha_2,\nu) = \sup_{n\in\mathbb{N}} \frac{1}{n} \int_\Sigma \log \alpha_2(\mathtt{i}|_n)\,\mathrm{d}\nu(\mathtt{i}).$$

Recall that the *entropy* of $\nu \in \mathcal{M}_\sigma(\Sigma)$ is

$$h(\nu) = -\lim_{n\to\infty} \frac{1}{n} \sum_{\mathtt{i}\in\Sigma_n} \nu([\mathtt{i}]) \log \nu([\mathtt{i}]) = \inf_{n\in\mathbb{N}} -\frac{1}{n} \sum_{\mathtt{i}\in\Sigma_n} \nu([\mathtt{i}]) \log \nu([\mathtt{i}]).$$

We say that a measure $\mu \in \mathcal{M}(\Sigma)$ is *sub-multiplicative* if the potential $\mathtt{i} \mapsto \mu([\mathtt{i}])$ is sub-multiplicative. The other definitions on potentials can be used with measures in a similar manner. Regardless, almost-multiplicative measures are more commonly known as *quasi-Bernoulli* measures and multiplicative measures as *Bernoulli* measures. The *cross-entropy* of a sub-multiplicative measure μ relative to $\nu \in \mathcal{M}_\sigma(\Sigma)$ is defined to be

$$h(\mu, \nu) = -\Lambda(\mu, \nu) = \sup_{n\in\mathbb{N}} -\frac{1}{n} \sum_{\mathtt{i}\in\Sigma_n} \nu([\mathtt{i}]) \log \mu([\mathtt{i}]).$$

The *Lyapunov dimension* of a measure $\nu \in \mathcal{M}_\sigma(\Sigma)$ is given by

$$\dim_{\mathrm{L}}(\nu) = \min\left\{ -\frac{h(\nu)}{\lambda_1(\nu)}, 1 - \frac{h(\nu) + \lambda_1(\nu)}{\lambda_2(\nu)}, -\frac{2h(\nu)}{\lambda_1(\nu) + \lambda_2(\nu)} \right\}.$$

See [49] for a relation between $\dim_{\mathrm{L}}(\nu)$ and $\dim_{\mathrm{H}}(\pi\nu)$ when ν is ergodic and the translation vectors are chosen randomly according to the Lebesgue measure. Finally, the *Lyapunov cross-dimension* $\dim_{\mathrm{L}}(\mu, \nu)$ of a sub-multiplicative measure μ relative to $\nu \in \mathcal{M}_\sigma(\Sigma)$ is

$$\dim_{\mathrm{L}}(\mu, \nu) = \min\left\{ -\frac{h(\mu, \nu)}{\lambda_1(\nu)}, 1 - \frac{h(\mu, \nu) + \lambda_1(\nu)}{\lambda_2(\nu)}, -\frac{2h(\mu, \nu)}{\lambda_1(\nu) + \lambda_2(\nu)} \right\}.$$

In other words, the Lyapunov cross-dimension is obtained by replacing the entropy $h(\nu)$ in the definition of the Lyapunov dimension by the cross-entropy $h(\mu, \nu)$. We should emphasize that despite we see this as a symbolic analog of the local dimension of $\pi\mu$ for $\pi\nu$-almost all points, there are examples where $\dim_{\mathrm{L}}(\mu, \nu)$ gives a different value, even if the translation vectors are chosen randomly according to the Lebesgue measure. In a discussion, Thomas Jordan gave us the following example

$$(A_1, A_2) = \left(\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \right),$$

where μ is the $\left(\frac{1}{3}, \frac{2}{3}\right)$-Bernoulli measure and ν is the $\left(\frac{1}{2}, \frac{1}{2}\right)$-Bernoulli measure.

5.2.3 Domination

We say that $\mathsf{A} = (A_1, \ldots, A_N) \in GL_2(\mathbb{R})^N$ is *dominated* if there exist constants $C > 0$ and $0 < \tau < 1$ such that

$$\alpha_2(\mathtt{i}) \leqslant C\tau^n \alpha_1(\mathtt{i})$$

for all $\mathtt{i} \in \Sigma_n$ and $n \in \mathbb{N}$. Note that if A is dominated, then $\lambda_2(\nu) < \lambda_1(\nu)$ for all $\nu \in \mathcal{M}_\sigma(\Sigma)$. Let $\mathbb{RP}^1$ denote the real projective line, which is the set of all straight unit line segments centred at the origin in $\mathbb{R}^2$ and which we identify with $[0, \pi)$. We call a proper subset $\mathcal{C} \subset \mathbb{RP}^1$ a *multicone* if it is a finite union of closed projective intervals. We say that a multicone $\mathcal{C} \subset \mathbb{RP}^1$ is *strongly invariant* for A if $A_i\mathcal{C} \subset \mathcal{C}^o$ for all $i \in \{1, \ldots, N\}$, where $\mathcal{C}^o$ is the interior of $\mathcal{C}$. For example, the first quadrant is strongly invariant for any tuple of positive matrices. By [14, Theorem B], A has strongly invariant multicone if and only if A is dominated. If A is a dominated tuple of invertible matrices then the collection $\{A_1^{-1} \ldots, A_N^{-1}\}$ is also dominated and thus it has a strongly invariant multicone. Also if A is dominated, then [15, Lemma 2.2] imply that the potential $\mathtt{i} \mapsto \alpha_1(\mathtt{i})$ is almost-multiplicative. Since $|\det(A_\mathtt{i})| = \alpha_1(\mathtt{i})\alpha_2(\mathtt{i})$ for all $\mathtt{i} \in \Sigma_*$ and the determinant is multiplicative, we see that also $\mathtt{i} \mapsto \alpha_2(\mathtt{i}) = \alpha_1(\mathtt{i})^{-1}|\det(A_\mathtt{i})|$ is almost-multiplicative.

Lemma 5.2.2. *Let θ be an almost-multiplicative potential and $\nu \in \mathcal{M}_\sigma(\Sigma)$. If $\nu_k \to \nu$ in the weak* topology, then*

$$\lim_{k \to \infty} \Lambda(\theta, \nu_k) = \Lambda(\theta, \nu).$$

Proof. By Lemma 5.2.1, we have $\limsup_{k\to\infty} \Lambda(\theta, \nu_k) \leqslant \Lambda(\theta, \nu)$. Since C/θ is sub-multiplicative for some $C \geqslant 1$, Lemma 5.2.1 implies that

$$\Lambda(C^{-1}\theta, \nu) = -\Lambda(C/\theta, \nu) = \sup_{n\in\mathbb{N}} \left(\frac{1}{n} \int_\Sigma \log \theta(\mathtt{i}|_n) \, \mathrm{d}\nu(\mathtt{i}) - \frac{1}{n} \log C \right).$$

Therefore, as each $\nu \mapsto \frac{1}{n} \sum_{\mathtt{i}\in\Sigma_n} \nu([\mathtt{i}]) \log \theta(\mathtt{i})$ is continuous, we see that $\nu \mapsto \Lambda(C^{-1}\theta, \nu) = \Lambda(\theta, \nu)$ is lower semi-continuous and thus $\liminf_{k\to\infty} \Lambda(\theta, \nu_k) \geqslant \Lambda(\theta, \nu)$. □

Let $(T_1, \ldots, T_N)$, where $T_i(x) = A_i x + v_i$, be an affine IFS and $\nu \in \mathcal{M}_\sigma(\Sigma)$ be a quasi-Bernoulli measure. If $\mathsf{A} = (A_1, \ldots, A_N)$ is dominated, then, by [7, Theorem 2.6] and [49, proof of Theorem 4.3(a)],

$$\dim_{\mathrm{loc}}(\pi\nu, x) = \underline{\dim}_{\mathrm{H}}(\pi\nu) \leqslant \dim_{\mathrm{L}}(\nu)$$

for ν-almost all $x \in X$. We say that A is *strongly irreducible* if there are no finite set of lines in $\mathbb{R}^2$ which is invariant under all of the matrices in A. Suppose that A is dominated and strongly irreducible and $(v_1, \ldots, v_N)$ is chosen such that the *strong open set condition* holds, i.e. there is a bounded open set $U \subset \mathbb{R}^2$ such that $U \cap X \neq \emptyset$, $\bigcup_{i=1}^N T_i(U) \subset U$, and $T_i(U) \cap T_j(U) = \emptyset$ whenever $i \neq j$. It follows from [6, Theorem 1.2 and the associated footnote] that under these assumptions

$$\underline{\dim}_{\mathrm{H}}(\pi\nu) = \dim_{\mathrm{L}}(\nu)$$

for all quasi-Bernoulli measures $\nu \in \mathcal{M}_\sigma(\Sigma)$.

5.2.4 Equilibrium State

Let θ be a sub-multiplicative potential. We define the *pressure* of θ by setting

$$P(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathtt{i}\in\Sigma_n} \theta(\mathtt{i}) = \inf_{n\in\mathbb{N}} \frac{1}{n} \log \sum_{\mathtt{i}\in\Sigma_n} \theta(\mathtt{i}).$$

As in Lemma 5.2.1, the existence of the limit above and the equality are guaranteed by Fekete's Lemma. By [51, Lemma 2.2], we see that

$$P(\theta) \geqslant h(\nu) + \Lambda(\theta, \nu)$$

for all $\nu \in \mathcal{M}_\sigma(\Sigma)$. A measure $\nu \in \mathcal{M}_\sigma(\Sigma)$ for which

$$P(\theta) = h(\nu) + \Lambda(\theta, \nu)$$

is called the *equilibrium state* for θ. If θ is an almost-multiplicative potential, then, by [51, S3], there exists a unique equilibrium state for θ which furthermore is a quasi-Bernoulli measure.

Lemma 5.2.3. *Let $(\theta_k)_{k\in\mathbb{N}}$ be a sequence of sub-multiplicative potentials and let $\nu_k \in \mathcal{M}_\sigma(\Sigma)$ be an equilibrium state for θ_k for each $k \in \mathbb{N}$. If there exist a measure $\nu \in \mathcal{M}_\sigma(\Sigma)$ and a sub-multiplicative potential θ such that $\nu_k \to \nu$ in the weak* topology and $\theta_k(\mathtt{i})^{1/|\mathtt{i}|} \to \theta(\mathtt{i})^{1/|\mathtt{i}|}$ uniformly in Σ_* as $k \to \infty$, then*

$$\lim_{k\to\infty} \Lambda(\theta_k, \nu_k) - \Lambda(\theta, \nu_k) = 0$$

and ν is an equilibrium state for θ.

Proof. Recall that, by [81, Theorem 8.2] and Lemma 5.2.1, $\limsup_{k\to\infty} h(\nu_k) \leqslant h(\nu)$ and $\limsup_{k\to\infty} \Lambda(\theta, \nu_k) \leqslant \Lambda(\theta, \nu)$. If $\varepsilon > 0$, then the uniform convergence of θ_k implies that there exists $k_0 \in \mathbb{N}$ such that $\log\theta(\mathtt{i}) - \varepsilon|\mathtt{i}| \leqslant \log\theta_k(\mathtt{i}) \leqslant \log\theta(\mathtt{i}) + \varepsilon|\mathtt{i}|$ for all $\mathtt{i} \in \Sigma_*$ and

$$\Lambda(\theta, \nu_k) - \varepsilon \leqslant \Lambda(\theta_k, \nu_k) \leqslant \Lambda(\theta, \nu_k) + \varepsilon$$

for all $k \geqslant k_0$. Therefore, $\lim_{k\to\infty} \Lambda(\theta_k, \nu_k) - \Lambda(\theta, \nu_k) = 0$ and

$$\begin{aligned} P(\theta) = \lim_{k\to\infty} P(\theta_k) &= \lim_{k\to\infty} h(\nu_k) + \Lambda(\theta_k, \nu_k) \\ &\leqslant \limsup_{k\to\infty} h(\nu_k) + \limsup_{k\to\infty} \Lambda(\theta, \nu_k) + \varepsilon \\ &\leqslant h(\nu) + \Lambda(\theta, \nu) + \varepsilon. \end{aligned}$$

By letting $\varepsilon \downarrow 0$, we see that ν is an equilibrium state for θ. □

Let $\mathsf{A} = (A_1, \ldots, A_N) \in GL_2(\mathbb{R})^N$ be a tuple of contractive invertible matrices. For each $s \geqslant 0$, define a potential φ^s by setting

$$\varphi^s(\mathtt{i}) = \begin{cases} \alpha_1(\mathtt{i})^s, & \text{if } 0 \leqslant s < 1, \\ \alpha_1(\mathtt{i})\alpha_2(\mathtt{i})^{s-1}, & \text{if } 1 \leqslant s < 2, \\ |\det(A_{\mathtt{i}})|^{s/2}, & \text{if } 2 \leqslant s < \infty, \end{cases}$$

for all $\mathtt{i} \in \Sigma_*$. Since $\alpha(\mathtt{i})\alpha_2(\mathtt{i})^{s-1} = \alpha_1(\mathtt{i})^{2-s}|\det(A_{\mathtt{i}})|^{s-1}$, the *singular value function* φ^s is sub-multiplicative. Therefore, the pressure $P(\varphi^s)$ is well-defined for all $s \geqslant 0$. By [52, Lemma 2.1], the function $s \mapsto P(\varphi^s)$ defined on $[0, \infty)$ is continuous, convex on intervals $(0, 1)$ and $(1, \infty)$, strictly decreasing, and there exists a unique $s \geqslant 0$ such that $P(\varphi^s) = 0$. This unique $s \geqslant 0$ is called the *affinity dimension* and it is denoted by $\dim_{\text{aff}}(\varphi^s)$.

If $\nu \in \mathcal{M}_\sigma(\Sigma)$, then

$$\Lambda(\varphi^s, \nu) = \begin{cases} s\lambda_1(\nu), & \text{if } 0 \leqslant s < 1, \\ \lambda_1(\nu) + (s-1)\lambda_2(\nu), & \text{if } 1 \leqslant s < 2, \\ \frac{s}{2}(\lambda_1(\nu) + \lambda_2(\nu)), & \text{if } 2 \leqslant s < \infty, \end{cases}$$

where $\lambda_1(\nu)$ and $\lambda_2(\nu)$ are the Lyapunov exponents. It is straightforward to see that the Lyapunov dimension $\dim_{\text{L}}(\nu)$ is the unique $s \geqslant 0$ for which $h(\nu) + \Lambda(\varphi^s, \nu) = 0$. By [50, Theorem 2.6], there exists an equilibrium state ν for φ^s. Note that if A is dominated, then φ^s is almost-multiplicative and there is only one equilibrium state for φ^s which furthermore is a quasi-Bernoulli measure. Note that an equilibrium state has maximal possible Lyapunov dimension,

$$\dim_{\text{L}}(\nu) = \max\{\dim_{\text{L}}(\eta) : \eta \in \mathcal{M}_\sigma(\Sigma)\} = \dim_{\text{aff}}(\varphi^s).$$

Similarly, the Lyapunov cross-dimension $\dim_{\text{L}}(\mu, \nu)$ of a sub-multiplicative measure μ relative to $\nu \in \mathcal{M}_\sigma(\Sigma)$ is the unique $s \geqslant 0$ for which $h(\mu, \nu) + \Lambda(\varphi^s, \nu) = 0$.

5.3 Local Dimension from Projections

In this section we generalise the ideas of [27] to study the following question. Let μ and ν be measures on a self-affine set. What can be said about the ν-almost everywhere value of the local dimension of μ? Before stating our theorems, we need to define projective linear transformations and the Furstenberg measure.

5.3.1 Projective Linear Transformations

Let $\mathsf{A} = (A_1, \ldots, A_N) \in GL_2(\mathbb{R})^N$ be a tuple of contractive invertible matrices. Given $i \in \{1, \ldots, N\}$ there exists a unique map $\phi_i : \mathbb{RP}^1 \to \mathbb{RP}^1$ such that, for $\theta \in [0, \pi)$, straight lines centred at the origin at angle θ to the horizontal are mapped to straight lines centred at the origin at angle $\phi_i(\theta)$ by the action of A_i^{-1}. If A is dominated and $\mathcal{C}_2$ is a strongly invariant multicone of $\{A_1^{-1}, \cdots, A_N^{-1}\}$ then each map ϕ_i is a strict contraction of $\mathcal{C}_2$.

Now let the Furstenberg measure ν_F be the stationary measure on $\mathbb{RP}^1$ associated to the maps ϕ_i chosen according the measure ν. Alternatively, ν_F is the unique probability measure on $\mathbb{RP}^1$ such that for ν-almost every sequence $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$ the sequence of measures

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{\phi_{i_k} \circ \cdots \circ \phi_{i_1}(\theta)}$$

converges weak* to ν_F. The support of ν_F is contained in $\mathcal{C}_2$. See [9] for more details on the Furstenberg measure.

We also define $\pi_\theta : X \to [-1, 1]$ to be the map obtained by projecting the self-affine set X to the diameter of the unit disc which is perpendicular to θ, identified isometrically with $[-1, 1]$. We identify, without confusion, π_θ and $\pi_\theta \circ \pi$.

Theorem 5.3.1. *Let A be dominated and assume that $(T_1, ..., T_N)$ satisfies the strong separation condition. Let $\mu \in \mathcal{M}(\Sigma)$ be a quasi-Bernoulli measure and $\nu \in \mathcal{M}_\sigma(\Sigma)$ be ergodic and quasi-Bernoulli. Then*

1. There exists a number $d \in [0,1]$ such that for $\nu_F \times \nu$-almost all $(\theta, \mathtt{i})$ the local dimension of $\pi_\theta \mu$ at $\pi_\theta(\mathtt{i})$ is d.

2. For ν-almost all $\mathtt{i} \in \Sigma$ it is true that $dim_{loc}(\pi(\mu), \pi(\mathtt{i})) = \alpha$ where,

$$\alpha = d + \frac{h(\mu|\nu) + d\lambda_1(\nu)}{-\lambda_2(\nu)}.$$

5.3.2 Proofs

Throughout this section we assume that A is dominated and $\mathcal{C}_2$ is a strongly invariant multicone of $\{A_1^{-1}, ..., A_N^{-1}\}$. Furthermore, we assume that $\mu \in \mathcal{M}(\Sigma)$ is a quasi-Bernoulli measure and let $\nu \in \mathcal{M}_\sigma(\Sigma)$ be quasi-Bernoulli and ergodic. Finally we assume that $(T_1, ..., T_N)$ satisfies the strong separation condition. The proof of Theorem 5.3.1 proceeds via a number of lemmata, we begin by discussing dynamics on pairs $(\theta, \mathtt{i})$ of angles in $\mathcal{C}_2$ and points in Σ.

Let $(\overline{\Sigma}, \sigma)$ be the extension of (Σ, σ) to a two-sided shift space. Set $P : \overline{\Sigma} \to \mathbb{RP}^1 \times \Sigma$ to be the map defined by

$$P(...i_{-2}i_{-1}i_0i_1i_2...) = \left(\lim_{n\to\infty} \phi_{i_0}\phi_{i_{-1}}...\phi_{i_{-n}}(\theta), i_1i_2...\right)$$

for some $\theta \in \mathcal{C}_2$, the choice of which does not affect P.

Let $\overline{\nu}$ be the extension of ν to the two sided shift $\overline{\Sigma}$, i.e. the unique shift invariant measure on $\overline{\Sigma}$ satisfying $\nu[i_1 \cdots i_n] = \overline{\nu}[i_1 \cdots i_n]$ for any $i_1 \cdots i_n \in \Sigma_*$. Since ν is ergodic it follows that $\overline{\nu}$ is ergodic. The measure $P(\overline{\nu})$ on $\mathbb{RP}^1 \times \Sigma$ is the pushforward of $\overline{\nu}$ under the map P.

The following lemma is essentially Lemma 3.1. of [27].

Lemma 5.3.1. *The map $P \circ \sigma \circ P^{-1} : \mathbb{RP}^1 \times \Sigma \to \mathbb{RP}^1 \times \Sigma$ is well defined and the system $(\mathbb{RP}^1 \times \Sigma, P(\overline{\nu}), P \circ \sigma \circ P^{-1})$ is ergodic. Furthermore $P(\overline{\nu})$ is equivalent to the product measure $\nu_F \times \nu$.*

We now prove the first claim of Theorem 5.3.1. Given a two sided sequence $\underline{\mathtt{i}} \in \overline{\Sigma}$ let $\theta, \mathtt{i}$ be such that $P(\underline{\mathtt{i}}) = (\theta, \mathtt{i})$ and define

$$g(\underline{\mathtt{i}}) = \dim_{loc}(\pi_\theta(\mu), \pi_\theta(\mathtt{i})).$$

Lemma 5.3.2. *We have*

$$g(\underline{\mathtt{i}}) \leqslant g(\sigma(\underline{\mathtt{i}}))$$

An immediate consequence of this lemma is that, since $\overline{\nu}$ is an ergodic σ-invariant measure on $\overline{\Sigma}$, g is equal to some constant d for $\overline{\nu}$ almost every $\underline{\mathtt{i}}$. Then since $P(\overline{\nu})$ is equivalent to $\nu_F \times \nu$ we will have that $\dim_{loc}(\pi_\theta(\mu), \pi_\theta(\mathtt{i})) = d$ for $\nu_F \times \nu$ almost every $(\theta, \mathtt{i})$, completing the proof of statement 1 of Theorem 5.3.1. We now prove Lemma 5.3.2.

Proof. First express μ as the sum of μ restricted to cylinder $[i_1]$ and μ restricted to the complement of this cylinder, giving

$$\begin{aligned} \dim_{loc}(\pi_\theta(\mu), \pi_\theta(\mathtt{i})) &= \dim_{loc}(\pi_\theta(\mu|_{[i_1]}) + \pi_\theta(\mu|_{[i_1]^c}), \pi_\theta(\mathtt{i})) \\ &\leqslant \dim_{loc}(\pi_\theta(\mu|_{[i_1]}), \pi_\theta(\mathtt{i})). \end{aligned}$$

But by applying T_i^{-1} we see

$$\dim_{loc}(\pi_\theta(\mu|_{[i_1]}), \pi_\theta(\mathtt{i})) = \dim_{loc}(\pi_{\phi_i(\theta)}(\mu), \pi_{\phi_i(\theta)}(\sigma(\mathtt{i}))) \tag{5.1}$$

and so the previous inequality becomes

$$\dim_{loc}(\pi_\theta(\mu), \pi_\theta(\mathtt{i})) \leqslant \dim_{loc}(\pi_{\phi_i(\theta)}(\mu), \pi_{\phi_i(\theta)}(\sigma(\mathtt{i})))$$

which is the statement $g(\underline{\mathtt{i}}) \leqslant g(\sigma(\underline{\mathtt{i}}))$ that we wanted to prove.

The equation 5.1 follows directly from a more precise statement (Lemma 3.2) in [27], see also [24] where this was used extensively to give conditions under which the projected measures $\pi_\theta(\mu)$ have the same Hausdorff dimension for all θ. □

The following lemma appears in [27] as Lemma 4.2, it allows us to compare the measure of a ball in our self-affine set with the measure of an ellipse multiplied by the projected measure of a certain interval.

Lemma 5.3.3. *There are numbers $C > 0$ and $0 < \rho_1 < \rho_2$ such that for each $\mathtt{i} \in \Sigma$, $\theta \in \mathcal{C}_2$ and $n \in \mathbb{N}$,*

$$\begin{aligned} & C^{-1}\mu\left(B\left(\pi(\mathtt{i}), \rho_1\alpha_2(\mathtt{i}|_n)\right)\right) \\ & \quad \leqslant \mu([\mathtt{i}|_n])\pi_{\phi_{i_n}\cdots\phi_{i_1}(\theta)}\mu\left(B\left(\pi_{\phi_{i_n}\cdots\phi_{i_1}(\theta)}(\sigma^n(\mathtt{i})), \frac{\alpha_2(\mathtt{i}|_n)}{\alpha_1(\mathtt{i}|_n)}\right)\right) \\ & \quad \leqslant C\mu\left(B\left(\pi(\mathtt{i}), \rho_2\alpha_2(\mathtt{i}|_n)\right)\right). \end{aligned}$$

From now on we set

$$d(\theta, \mathtt{i}, n) := \frac{\log\left(\pi_{\phi_{i_n}\cdots\phi_{i_1}(\theta)}\mu\left(B\left(\pi_{\phi_{i_n}\cdots\phi_{i_1}(\theta)}(\sigma^n(\mathtt{i})), \frac{\alpha_2(\mathtt{i}|_n)}{\alpha_1(\mathtt{i}|_n)}\right)\right)\right)}{\log\left(\frac{\alpha_2(\mathtt{i}|_n)}{\alpha_1(\mathtt{i}|_n)}\right)}.$$

For large n, $\left(\frac{\alpha_2(\mathtt{i}|_n)}{\alpha_1(\mathtt{i}|_n)}\right)$ is small and so for many pairs $(\theta, \mathtt{i})$ we would expect the above quantity to be close to the local dimension of the projected measure $\pi_{\phi_{i_n}\cdots\phi_{i_1}(\theta)}\mu$ at $\pi_{\phi_{i_n}\cdots\phi_{i_1}(\theta)}(\sigma^n(\mathtt{i}))$. With this in mind, let

$$G(\theta, \mathtt{i}, \epsilon) := \left\{n \in \mathbb{N} : |d(\theta, \mathtt{i}, n) - d| < \epsilon\right\}.$$

Also for $k, \epsilon > 0$ we set

$$G_{k,\epsilon} := \left\{(\theta, \mathtt{i}) : \left|\frac{\log \pi_\theta\mu\left(B\left(\pi_\theta(\mathtt{i}), r\right)\right)}{\log r} - d\right| < \epsilon, \forall r < k\right\}$$

Since $P(\overline{\nu})$ is equivalent to $\nu_F \times \nu$, by the definition of the number d, we have that for $P(\overline{\nu})$-a.e. $(\theta, \mathtt{i})$ in $\mathbb{RP}^1 \times \Sigma$

$$\lim_{r\to 0} \frac{\log \pi_\theta\mu\left(B\left(\pi_\theta(\mathtt{i}), r\right)\right)}{\log r} = d.$$

Hence for all $\epsilon > 0$,

$$\lim_{k\to 0} P(\overline{\nu})\left(G_{k,\epsilon}\right) = 1.$$

Lemma 5.3.4. *For $P(\overline{\nu})$-almost every $(\theta, \mathtt{i}) \in \mathbb{RP}^1 \times \Sigma$ and all $\epsilon > 0$ it is true that*

$$\lim_{N\to\infty} \frac{1}{N} |G(\theta, \mathtt{i}, \epsilon) \cap \{1, ..., N\}| = 1.$$

Proof. Let $\delta > 0$ be arbitrary. From the observation above there exists $k > 0$ such that $P(\bar{\nu})(G_{k,\epsilon}) > 1 - \delta$. Also, by domination, for every $\mathtt{i} \in \Sigma$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$

$$\frac{\alpha_2(\mathtt{i}|_n)}{\alpha_1(\mathtt{i}|_n)} < k.$$

Now by observing that

$$\left(P \circ \sigma \circ P^{-1}\right)^n (\theta, \mathtt{i}) = (\phi_{i_n}...\phi_{i_1}(\theta), \sigma^n(\mathtt{i}))$$

and because $(\mathbb{RP}^1 \times \Sigma, P(\overline{\nu}), P \circ \sigma \circ P^{-1})$ is ergodic, for $P(\overline{\nu})$-almost every $(\theta, \mathtt{i})$ we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} |G(\theta, \mathtt{i}, \epsilon) \cap \{1, ..., n\}| &= \lim_{n\to\infty} \frac{1}{n} |\{n \in \{1, ..., n\} : |d(\theta, \mathtt{i}, n) - d| < \epsilon\}| \\
&\geqslant \lim_{n\to\infty} \frac{1}{n} \left|\left\{n \in \{1, ..., n\} : \frac{\alpha_2(\mathtt{i}|_n)}{\alpha_1(\mathtt{i}|_n)} < k \quad \text{and} \quad \left(P \circ \sigma \circ P^{-1}\right)^n (\theta, \mathtt{i}) \in G_{k,\epsilon}\right\}\right| \\
&= P(\bar{\nu})(G_{k,\epsilon}) > 1 - \delta.
\end{aligned}$$

Since δ was arbitrary the proof is complete.

□

For all $\epsilon > 0$ and $\nu_F \times \nu$-almost every $(\theta, \mathtt{i}) \in \mathbb{RP}^1 \times \Sigma$ we can choose, by the lemma above, a strictly increasing sequence n_k of density 1 such that $n_k \in G(\theta, \mathtt{i}, \epsilon)$. By the ergodicity of ν we can additionally assume the properties

$$\lim_{\kappa\to\infty}\frac{1}{n}\log(\mu([\mathtt{i}|_{n_k}])) = -h(\mu,\nu),$$
$$\lim_{\kappa\to\infty}\frac{1}{n}\log(\alpha_1(\mathtt{i}|_{n_k})) = \lambda_1(\nu),$$
$$\lim_{\kappa\to\infty}\frac{1}{n}\log(\alpha_2(\mathtt{i}|_{n_k})) = \lambda_2(\nu).$$

Now Lemma 5.3.3 gives

$$\begin{aligned}
&\limsup_{k\to\infty}\frac{\log\pi\mu\left(B\left(\pi(\mathtt{i}),\rho_1\alpha_2\left(\mathtt{i}|_{n_k}\right)\right)\right)}{\log\left(\rho_1\alpha_2\left(\mathtt{i}|_{n_k}\right)\right)}\\
&\leqslant \limsup_{k\to\infty}\left[\frac{\log\left(C\mu\left([i_1\cdots i_{n_k}]\right)\right)}{\log\left(\rho_1\alpha_2\left(\mathtt{i}|_{n_k}\right)\right)} + \frac{\log\pi_{\phi_{i_{n_k}\cdots i_1(\theta)}}\mu\left(B\left(\pi_{\phi_{i_{n_k}\cdots i_1(\theta)}}\left(\sigma^{n_k}(\mathtt{i})\right),\frac{\alpha_2\left(\mathtt{i}|_{n_k}\right)}{\alpha_1\left(i|_{n_k}\right)}\right)\right)}{\log\left(\rho_1\alpha_2\left(\mathtt{i}|_{n_k}\right)\right)}\right]\\
&= \limsup_{k\to\infty}\left(\frac{\log\left(C\mu([\mathtt{i}|_{n_k}])\right)}{\log\left(\rho_1\alpha_2\left(\mathtt{i}|_{n_k}\right)\right)} + d(\theta,\mathtt{i},n_k)\frac{\log\left(\alpha_2\left(\mathtt{i}|_{n_k}\right)/\alpha_1\left(\mathtt{i}|_{n_k}\right)\right)}{\log\left(\rho_1\alpha_2\left(\mathtt{i}|_{n_k}\right)\right)}\right)\\
&\leqslant \frac{-h(\mu|\nu)}{\lambda_2(\nu)} + (d+\epsilon)\frac{\lambda_2(\nu)-\lambda_1(\nu)}{\lambda_2(\nu)}.
\end{aligned}$$

Since the upper and lower limits of $\mu(B(x,r))/\log(r)$ as $r\to 0$ are determined by any sequence $r_\kappa\to 0$ such that $\log r_{\kappa+1}/\log r_\kappa\to 1$, by taking $r_\kappa=\rho_1\alpha_2(\mathtt{i}|_{n_\kappa}))$ and recalling that ϵ is arbitrary, we conclude that $\overline{\dim}_{\mathrm{loc}}(\mu,\pi(\mathtt{i}))=\alpha$. A similar argument shows that $\underline{\dim}_{\mathrm{loc}}(\mu,\pi(a))=\alpha$. This completes the proof of Theorem 5.3.1.

5.4 Differentiability of the Pressure

Let $(A_1,\ldots,A_N)\in GL_2(\mathbb{R})^N$ be dominated and $\mu\in\mathcal{M}(\Sigma)$ be a quasi-Bernoulli measure. For each $q\in\mathbb{R}$ and $s\geqslant 0$, following [26], we consider the almost-multiplicative potential $\psi^{q,s}$ defined by

$$\psi^{q,s}(\mathtt{i}) = \mu([\mathtt{i}])^q\varphi^s(\mathtt{i})^{1-q}.$$

If $\nu \in \mathcal{M}_\sigma(\Sigma)$, then

$$\Lambda(\psi^{q,s}, \nu) = -qh(\mu, \nu) + (1-q)\Lambda(\varphi^s, \nu).$$

Since $\psi^{q,s}$ is almost-multiplicative, the pressure $P(\psi^{q,s})$ is well-defined and there exists a unique equilibrium state for $\psi^{q,s}$ which furthermore is a quasi-Bernoulli measure. The next lemma collects some elementary properties of the pressure function.

Lemma 5.4.1. *If $(A_1, \ldots, A_N) \in GL_2(\mathbb{R})^N$ is a dominated tuple of contractive matrices and $\mu \in \mathcal{M}(\Sigma)$ is a fully supported quasi-Bernoulli measure, then the following seven properties hold:*

1. *The function $(q, s) \mapsto P(\psi^{q,s})$ is continuous on $\mathbb{R} \times [0, \infty)$.*
2. *For each $q < 1$ the function $s \mapsto P(\psi^{q,s})$ is strictly decreasing with $P(\psi^{q,0}) \geqslant 0$ and $\lim_{s\to\infty} P(\psi^{q,s}) = -\infty$.*
3. *For each $q > 1$ the function $s \mapsto P(\psi^{q,s})$ is strictly increasing with $P(\psi^{q,0}) \leqslant 0$ and $\lim_{s\to\infty} P(\psi^{q,s}) = \infty$.*
4. *For each $q \neq 1$, there exists unique $s(q) \in [0, \infty)$ so that $P(\psi^{q,s(q)}) = 0$.*
5. *The function $q \mapsto s(q)$ is continuous on $\mathbb{R} \setminus \{1\}$.*
6. *For each $q \in \mathbb{R}$ the function $s \mapsto P(\psi^{q,s})$ convex on connected components of $[0, \infty) \setminus \{1, 2\}$.*
7. *For each $s \in [0, \infty) \setminus \{1, 2\}$ the function $q \mapsto P(\psi^{q,s})$ convex on $\mathbb{R}$.*

Proof. Although the proof is a simple modification of [52, Lemma 2.1], we present the full details for the convenience of the reader. We prove the claims only for $s \in [0, 2)$; the case $s \geqslant 2$ is left to the reader. Let $p, q \in \mathbb{R}$ and $s, t \in [0, 2)$. Writing

$$\underline{\alpha} = \min_{i\in\{1,\ldots,N\}} \alpha_2(i), \quad \overline{\alpha} = \max_{i\in\{1,\ldots,N\}} \alpha_1(i), \quad K = \max_{i\in\{1,\ldots,N\}} C\mu([i])^{-1},$$

where $C \geqslant 1$ is the constant given by the quasi-Bernoulli assumption, we see that $0 < \underline{\alpha} \leqslant \overline{\alpha} < 1 < K$. Furthermore, let

$$\overline{\alpha}(q,s,t) = \begin{cases} \overline{\alpha}, & \text{if } (1-q)(s-t) \geqslant 0, \\ \underline{\alpha}, & \text{if } (1-q)(s-t) < 0, \end{cases}$$

and

$$\underline{\alpha}(q,s,t) = \begin{cases} \underline{\alpha}, & \text{if } (1-q)(s-t) \geqslant 0, \\ \overline{\alpha}, & \text{if } (1-q)(s-t) < 0. \end{cases}$$

Then we have $K^{-|\mathtt{i}|} \leqslant \mu([\mathtt{i}]) \leqslant K^{|\mathtt{i}|}$ and

$$\varphi^t(\mathtt{i})^{1-q}\underline{\alpha}(q,s,t)^{(s-t)(1-q)|\mathtt{i}|} \leqslant \varphi^s(\mathtt{i})^{1-q} \leqslant \varphi^t(\mathtt{i})^{1-q}\overline{\alpha}(q,s,t)^{(s-t)(1-q)|\mathtt{i}|}$$

for all $\mathtt{i} \in \Sigma_*$. Since $\underline{\alpha}^{t|\mathtt{i}|} \leqslant \varphi^t(\mathtt{i}) \leqslant \overline{\alpha}^{t|\mathtt{i}|} \leqslant \underline{\alpha}^{-t|\mathtt{i}|}$, we have

$$\varphi^t(\mathtt{i})^{1-p}\underline{\alpha}^{t|p-q||\mathtt{i}|} \leqslant \varphi^t(\mathtt{i})^{1-q} \leqslant \varphi^t(\mathtt{i})^{1-p}\underline{\alpha}^{-t|p-q||\mathtt{i}|}$$

for all $\mathtt{i} \in \Sigma_*$. As $K^{-|p-q||\mathtt{i}|} \leqslant \mu([\mathtt{i}])^{q-p} \leqslant K^{|p-q||\mathtt{i}|}$ for all $\mathtt{i} \in \Sigma_*$, we see that

$$\begin{aligned} \psi^{q,s}(\mathtt{i}) &\leqslant \mu([\mathtt{i}])^p\mu([\mathtt{i}])^{q-p}\varphi^t(\mathtt{i})^{1-q}\overline{\alpha}(q,s,t)^{(s-t)(1-q)|\mathtt{i}|} \\ &\leqslant \psi^{p,t}(\mathtt{i})K^{|p-q||\mathtt{i}|}\underline{\alpha}^{-t|p-q||\mathtt{i}|}\overline{\alpha}(q,s,t)^{(s-t)(1-q)|\mathtt{i}|} \end{aligned} \tag{5.2}$$

and, similarly,

$$\psi^{q,s}(\mathtt{i}) \geqslant \psi^{p,t}(\mathtt{i})K^{-|p-q||\mathtt{i}|}\underline{\alpha}^{t|p-q||\mathtt{i}|}\underline{\alpha}(q,s,t)^{(s-t)(1-q)|\mathtt{i}|} \tag{5.3}$$

for all $\mathtt{i} \in \Sigma_*$. It follows that

$$\begin{aligned} -|p-q|\log K + t|p-q|\log\underline{\alpha} + (s-t)(1-q)\log\underline{\alpha}(q,s,t) \\ \leqslant P(\psi^{q,s}) - P(\psi^{p,t}) \\ \leqslant |p-q|\log K - t|p-q|\log\underline{\alpha} + (s-t)(1-q)\log\overline{\alpha}(q,s,t) \end{aligned}$$

and the function $(q,s) \mapsto P(\psi^{q,s})$ is thus clearly continuous on $\mathbb{R} \times [0,2)$. This shows (1). In particular, if $q < 1$ and $s > t$, then the above estimate shows that

$P(\psi^{q,s}) - P(\psi^{q,t}) \leqslant (s-t)(1-q)\log\overline{\alpha}(q,s,t) = (s-t)(1-q)\log\overline{\alpha} < 0$, and if $q > 1$ and $s > t$, we get $P(\psi^{q,s}) - P(\psi^{q,t}) \geqslant (s-t)(1-q)\log\overline{\alpha} > 0$. These observations show (2)–(4). Notice that (5) follows immediately from (1).

Finally, let us prove (6) and (7). Fix $p, q \in \mathbb{R}$ and $0 < \lambda < 1$. Let $s, t \in [0,2) \setminus \{1\}$ be such that $\lceil s \rceil = \lceil t \rceil$. Since $\varphi^{\lambda t + (1-\lambda)s}(\mathtt{i}) = \varphi^t(\mathtt{i})^\lambda \varphi^s(\mathtt{i})^{1-\lambda}$, we have

$$\begin{aligned}\mu([\mathtt{i}])^q \varphi^{\lambda t+(1-\lambda)s}(\mathtt{i})^{1-q} &= \mu([\mathtt{i}])^q \varphi^t(\mathtt{i})^{\lambda(1-q)} \varphi^s(\mathtt{i})^{(1-\lambda)(1-q)} \\ &= \Big(\mu([\mathtt{i}])^q \varphi^t(\mathtt{i})^{1-q}\Big)^\lambda \Big(\mu([\mathtt{i}])^q \varphi^s(\mathtt{i})^{1-q}\Big)^{1-\lambda}\end{aligned}$$

and

$$\begin{aligned}\psi^{\lambda p+(1-\lambda)q,s}(\mathtt{i}) &= \mu([\mathtt{i}])^{\lambda p+(1-\lambda)q} \varphi^s(\mathtt{i})^{1-\lambda p-(1-\lambda)q} \\ &= \Big(\mu([\mathtt{i}])^p \varphi^s(\mathtt{i})^{1-p}\Big)^\lambda \Big(\mu([\mathtt{i}])^q \varphi^s(\mathtt{i})^{1-q}\Big)^{1-\lambda}\end{aligned}$$

for all $\mathtt{i} \in \Sigma_*$. Therefore, by Hölder's inequality, we see that

$$\sum_{\mathtt{i}\in\Sigma_n} \psi^{q,\lambda t+(1-\lambda)s}(\mathtt{i}) \leqslant \Big(\sum_{\mathtt{i}\in\Sigma_n} \psi^{q,t}(\mathtt{i})\Big)^\lambda \Big(\sum_{\mathtt{i}\in\Sigma_n} \psi^{q,s}(\mathtt{i})\Big)^{1-\lambda}$$

and

$$\sum_{\mathtt{i}\in\Sigma_n} \psi^{\lambda p+(1-\lambda)q,s}(\mathtt{i}) \leqslant \Big(\sum_{\mathtt{i}\in\Sigma_n} \psi^{p,s}(\mathtt{i})\Big)^\lambda \Big(\sum_{\mathtt{i}\in\Sigma_n} \psi^{q,s}(\mathtt{i})\Big)^{1-\lambda}$$

for all $n \in \mathbb{N}$. The claims follow now by taking logarithm, dividing by n, and letting $n \to \infty$. □

Remark. *Let $0 < \alpha < 1$ and $O \in O(2)$ be an orthogonal matrix. If we consider the tuple $(\alpha O, \ldots, \alpha O) \in GL_2(\mathbb{R})^N$ and the uniform distribution $\mu \in \mathcal{M}_\sigma(\Sigma)$, then $P(\psi^{q,s}) = \log N^{1-q}\alpha^{s(1-q)}$. In this case the function $(q,s) \mapsto P(\psi^{q,s})$ is not convex since its Hessian is indefinite as it is an antidiagonal matrix having $-\log\alpha$ in the antidiagonal.*

Let us next study the differentiability of the pressure. We will first recall some basic facts from convex analysis. Let $U \subset \mathbb{R}$ be an open set and let $f\colon U \to \mathbb{R}$

be convex. It is well known that such f is continuous. We say that $G \in \mathbb{R}$ is a *sub-derivative* of f at $x \in U$ if

$$f(y) - f(x) \geqslant G(y - x)$$

for all $y \in U$. It is straightforward to see that any sub-derivative is contained in $[f'_-(x), f'_+(x)]$, where $f'_-(x)$ and $f'_+(x)$ are the left and right derivatives of f at x, respectively; see [67, Theorem 23.2]. Therefore, f is differentiable at x if and only if it has only one sub-derivative at x; see [67, Theorem 25.1].

Proposition 5.4.1. *Let $(A_1, \ldots, A_N) \in GL_2(\mathbb{R})^N$ be dominated tuple of contractive matrices and $\mu \in \mathcal{M}(\Sigma)$ be a quasi-Bernoulli measure. If $(q_0, s_0) \in \mathbb{R} \times (0, \infty) \setminus \{1, 2\}$ and ν is the equilibrium state for ψ^{q_0,s_0}, then the partial derivatives of $(q, s) \mapsto P(\psi^{q,s})$ are*

$$\partial_q Q(q, s_0)|_{q=q_0} = -h(\mu, \nu) - \Lambda(\varphi^{s_0}, \nu)$$

and

$$\partial_s Q(q_0, s)|_{s=s_0} = \begin{cases} (1 - q_0)\lambda_1(\nu), & \text{if } 0 < s_0 < 1, \\ (1 - q_0)\lambda_2(\nu), & \text{if } 1 < s_0 < 2, \\ (1 - q_0)\frac{\lambda_1(\nu)+\lambda_2(\nu)}{2}, & \text{if } 2 < s_0 < \infty \end{cases}$$

provided that they exist.

Proof. We prove the result only for $(q_0, s_0) \in \mathbb{R} \times (0, 2) \setminus \{1\}$; the case $(q_0, s_0) \in \mathbb{R} \times (2, \infty)$ is left to the reader. To simplify notation, we write $Q(q, s) = P(\psi^{q,s})$ for all $(q, s) \in \mathbb{R} \times (0, 2) \setminus \{1\}$. Fix $(q_0, s_0) \in \mathbb{R} \times (0, 2) \setminus \{1\}$ and let ν be the equilibrium state for ψ^{q_0,s_0}.

Let us first assume that the partial derivative $\partial_q Q(q, s_0)|_{q=q_0}$ exists. Recall that, by Lemma 5.4.1(7), the function $q \mapsto Q(q, s_0)$ is convex. Since $\Lambda(\psi^{q,s}, \nu) =$

$-qh(\mu,\nu) + (1-q)\Lambda(\varphi^s,\nu)$, we see that

$$\begin{aligned} Q(q,s_0) - Q(q_0,s_0) &\geqslant h(\nu) + \Lambda(\psi^{q,s_0},\nu) - h(\nu) - \Lambda(\psi^{q_0,s_0},\nu) \\ &= (-h(\mu,\nu) - \Lambda(\varphi^{s_0},\nu))(q-q_0) \end{aligned}$$

for all $q \in \mathbb{R}$. Therefore, $-h(\mu,\nu) - \Lambda(\varphi^{s_0},\nu)$ is a sub-derivative of the convex function $q \mapsto Q(q,s_0)$ at q_0. As the partial derivative $\partial_q Q(q,s_0)|_{q=q_0}$ exists, we have $\partial_q Q(q,s_0)|_{q=q_0} = -h(\mu,\nu) - \Lambda(\varphi^{s_0},\nu)$.

Let us then assume that the partial derivative $\partial_s Q(q_0,s)|_{s=s_0}$ exists. Recall that, by Lemma 5.4.1(6), the function $s \mapsto Q(q_0,s)$ is convex on connected components of $(0,2)\setminus\{1\}$. Since $\Lambda(\psi^{q,s},\nu) = -qh(\mu,\nu) + (1-q)\Lambda(\varphi^s,\nu)$ and $\Lambda(\varphi^s,\nu) = \Lambda(\varphi^{s_0},\nu) + (s-s_0)\lambda_{\lceil s_0\rceil}(\nu)$, we see that

$$\begin{aligned} Q(q_0,s) - Q(q_0,s_0) &\geqslant h(\nu) + \Lambda(\psi^{q_0,s},\nu) - h(\nu) - \Lambda(\psi^{q_0,s_0},\nu) \\ &= (1-q_0)\lambda_{\lceil s_0\rceil}(\nu)(s-s_0) \end{aligned}$$

for all all $s \in (0,2)\setminus\{1\}$ with $\lceil s\rceil = \lceil s_0\rceil$. Therefore, $(1-q_0)\lambda_{\lceil s_0\rceil}(\nu)$ is a sub-derivative of the convex function $s \mapsto Q(q_0,s)$ at s_0. As the partial derivative $\partial_s Q(q_0,s)|_{s=s_0}$ exists, we have $\partial_s Q(q_0,s)|_{s=s_0} = (1-q_0)\lambda_{\lceil s_0\rceil}(\nu)$. □

Let us next show that $(q,s) \mapsto P(\psi^{q,s})$ is differentiable on $\mathbb{R}\times(0,\infty)\setminus\{1,2\}$ which then allows us to apply Proposition 5.4.1. To prove this requires tools from thermodynamic formalism. The following lemma allows us to employ Lemma 5.2.3.

Lemma 5.4.2. *Let $(q_k,s_k)_{k\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}\times(0,\infty)\setminus\{1,2\}$ converging to $(q,s)\in\mathbb{R}\times(0,\infty)\setminus\{1,2\}$. Then $\psi^{q_k,s_k}(\mathtt{i})^{1/|\mathtt{i}|} \to \psi^{q,s}(\mathtt{i})^{1/|\mathtt{i}|}$ uniformly in Σ_* as $k\to\infty$.*

Proof. Following notation of the proof of Lemma 5.4.1, the estimates (5.2) and

(5.3) give

$$K^{-|q_k-q|}\underline{\alpha}^{s_k|q_k-q|}\underline{\alpha}(q,s,s_k)^{(s-s_k)(1-q)} \leqslant \left(\frac{\psi^{q,s}(\mathtt{i})}{\psi^{q_k,s_k}(\mathtt{i})}\right)^{1/|\mathtt{i}|}$$
$$\leqslant K^{|q_k-q|}\underline{\alpha}^{-s_k|q_k-q|}\overline{\alpha}(q,s,s_k)^{(s-s_k)(1-q)}$$

for all $\mathtt{i} \in \Sigma_*$ and $k \in \mathbb{N}$. The claim follows. □

Before proving the differentiability, let us recall some further facts from convex analysis. Let $U \subset \mathbb{R}$ be an open set and let $f\colon U \to \mathbb{R}$ be convex. Let $D \subset U$ be the set of points where f is differentiable. If $z_1, z_2 \in D$ and $x \in U$ such that $z_1 \leqslant x \leqslant z_2$, then, by [67, Theorem 24.1], $f'(z_1) \leqslant G \leqslant f'(z_2)$ for all sub-derivatives G at x. It also follows that the set $U \setminus D$ is at most countable and f' is continous on D; see [67, Theorem 25.3]. In particular, D is dense in U and if f is differentiable on U, then it is continuously differentiable on U.

Proposition 5.4.2. *Let $(A_1, \ldots, A_N) \in GL_2(\mathbb{R})^N$ be dominated tuple of contractive matrices and $\mu \in \mathcal{M}(\Sigma)$ be a quasi-Bernoulli measure. Then the function $(q,s) \mapsto P(\psi^{q,s})$ is differentiable on $\mathbb{R} \times (0,\infty) \setminus \{1,2\}$.*

Proof. We prove the result only on $\mathbb{R} \times (0,2) \setminus \{1\}$; the case $\mathbb{R} \times (2,\infty)$ is left to the reader. To simplify notation, we write $Q(q,s) = P(\psi^{q,s})$ for all $(q,s) \in \mathbb{R} \times (0,2) \setminus \{1\}$. To see that Q is differentiable on $\mathbb{R} \times (0,2) \setminus \{1\}$, it suffices to show that both partial derivatives of Q exist at each point of $\mathbb{R}\times(0,2)\setminus\{1\}$. Indeed, assuming this is the case, then using Proposition 5.4.1 combined with Lemmata 5.2.3, 5.2.2 and 5.4.2 it is straightforward to prove to prove the continuity of the partial derivatives which then implies the differentiability.

Fix $(q_0, s_0) \in \mathbb{R} \times (0,2) \setminus \{1\}$. By Lemma 5.4.1(7), we know that the partial derivative $\partial_q Q(q, s_0)$ exists for all, except possibly at countably many points of $\mathbb{R}$. Relying on this, choose two sequences $(q_k^-)_{k\in\mathbb{N}}$ and $(q_k^+)_{k\in\mathbb{N}}$ of points in $\mathbb{R}$ with

$q_k^- \uparrow q_0$ and $q_k^+ \downarrow q_0$ as $k \to \infty$ so that the partial derivatives $\partial_q Q(q, s_0)|_{q=q_k^-}$ and $\partial_q Q(q, s_0)|_{q=q_k^+}$ exist.

Let ν_k^- and ν_k^+ be the equilibrium states associated to $\psi^{q_k^-, s_0}$ and $\psi^{q_k^+, s_0}$, respectively. Then, by Lemmata 5.4.2 and 5.2.3, any limit point of the sequences $(\nu_k^-)_{k\in\mathbb{N}}$ and $(\nu_k^+)_{k\in\mathbb{N}}$ must be the unique equilibrium state ν of ψ^{q_0,s_0}. Thus, $\nu_k^- \to \nu$ and $\nu_k^+ \to \nu$ by the compactness of $\mathcal{M}_\sigma(\Sigma)$. Let G_1 be a sub-derivative of $q \mapsto Q(q, s_0)$ at q_0. It follows from Proposition 5.4.1 that

$$\begin{aligned} -h(\mu, \nu_k^-) - \Lambda(\varphi^{s_0}, \nu_k^-) = \partial_q Q(q, s_0)|_{q=q_k^-} &\leqslant G_1 \\ &\leqslant \partial_q Q(q, s_0)|_{q=q_k^+} = -h(\mu, \nu_k^+) - \Lambda(\varphi^{s_0}, \nu_k^+), \end{aligned}$$

where both bounds converge to the same value by Lemmata 5.2.2 and 5.2.3. Hence, $G_1 = \partial_q Q(q, s_0)|_{q=q_0}$.

Now we show that other partial derivative exists. By Lemma 5.4.1(6), we know that the partial derivative $\partial_s Q(q_0, s)$ exists for all, except possibly at countably many points on $[0, \infty)$. Relying on this, choose two sequences $(s_k^-)_{k\in\mathbb{N}}$ and $(s_k^+)_{k\in\mathbb{N}}$ of points in $(0,2) \setminus \{1\}$ with $\lceil s_k^- \rceil = \lceil s_k^+ \rceil = \lceil s_0 \rceil$, $s_k^- \uparrow s_0$, and $s_k^+ \downarrow s_0$ as $k \to \infty$ so that the partial derivatives $\partial_s Q(q_0, s_k^-)|_{s=s_k^-}$ and $\partial_s Q(q_0, s_k^+)|_{s=s_k^+}$ exist.

Similarly as before, let η_k^- and η_k^+ be the equilibrium states associated to ψ^{q_0, s_k^-} and ψ^{q_0, s_k^+}, respectively, and notice that $\eta_k^- \to \nu$ and $\eta_k^+ \to \nu$, where ν is the unique equilibrium state of ψ^{q_0,s_0}. Let G_2 be a sub-derivative of $s \mapsto Q(q_0, s)$ at s_0. It follows from Proposition 5.4.1 that

$$\begin{aligned} (1 - q_0)\lambda_{\lceil s_k^- \rceil}(\eta_k^-) = \partial_s Q(q_0, s)|_{s=s_k^-} &\leqslant G_2 \\ &\leqslant \partial_s Q(q_0, s)|_{s=s_k^+} = (1 - q_0)\lambda_{\lceil s_k^+ \rceil}(\eta_k^+), \end{aligned}$$

where both bounds converge to the same value by Lemmata 5.2.2 and 5.2.3. Hence, $G_2 = \partial_s Q(q_0, s)|_{s=s_0}$ finishing the proof. □

5.5 Multifractal Formalism

The quantity $s(q)$ defined in Lemma 5.4.1(4)–(5) will be used throughout this section. The *L^q-spectrum* is the function $\tau\colon \mathbb{R} \to \mathbb{R}$ defined by $\tau(q) = (q-1)s(q)$ and the *multifractal spectrum* is the function $f\colon [0,\infty) \to \mathbb{R}$ defined by

$$f(\alpha) = \sup\{\dim_{\mathrm{L}}(\nu) : \nu \in \mathcal{M}_\sigma(\Sigma) \text{ such that } \dim_{\mathrm{L}}(\mu,\nu) = \alpha\}.$$

We say that τ and f form a *Legendre transform pair* at (q,α) if $f(s) = q\alpha - \tau(q)$.

Proposition 5.5.1. *For each $q > 0$ with $s(q) \in (0,2)\backslash 1$, the symbolic L^q-spectrum τ is continuously differentiable on a neighborhood of q with*

$$\tau'(q) = \begin{cases} \frac{\Lambda(\mu,\nu)-\Lambda(\varphi^{s(q)},\nu)}{\lambda_1(\nu)} + s(q) & s(q) \in (0,1) \\ \frac{\Lambda(\mu,\nu)-\Lambda(\varphi^{s(q)},\nu)}{\lambda_2(\nu)} + s(q) & s(q) \in (1,2) \end{cases},$$

where ν is the equilibrium state for $\psi_{s,s(q)}$.

Proof. Recalling Proposition 5.4.1, implicit differentiation gives

$$s'(q) = \begin{cases} -\frac{\Lambda(\mu,\nu)-\Lambda(\varphi^{s(q)},\nu)}{(1-q)\lambda_1(\nu)} & s(q) \in (0,1) \\ -\frac{\Lambda(\mu,\nu)-\Lambda(\varphi^{s(q)},\nu)}{(1-q)\lambda_2(\nu)} & s(q) \in (1,2) \end{cases}.$$

The claim follows since $\tau'(q) = (q-1)s'(q) + s(q)$. □

Theorem 5.5.1. *Let $q > 0$ be such that $s(q), \tau'(q)$ and $q\tau'(q) - \tau(q)$ are all elements of $(0,1)$ (call this case 1) or all elements of $(1,2)$ (case 2). Then τ and f form a Legendre transform pair at $(q, \tau'(q))$ and $f(\tau'(q)) = \dim_{\mathrm{L}}(\nu)$, where ν is the equilibrium state for $\psi_{q,s(q)}$.*

Proof. It suffices to show that

$$q\tau'(q) - \tau(q) = f(\tau'(q)) = \dim_{\mathrm{L}}(\nu).$$

Case 1: We assume $s(q), \tau'(q)$ and $q\tau'(q) - \tau(q) \in (0,1)$. By the definition of $s(q)$, we have $P(\psi_{q,s(q)}) = 0$. Therefore, if $\eta \in \mathcal{M}_\sigma(\Sigma)$ is such that $\dim_{\mathrm{L}}(\mu, \eta) = \tau'(q)$, which is equivalent to $\Lambda(\mu, \eta) = \Lambda(\varphi^{\tau'(q)}, \eta)$, we have

$$\begin{aligned}
0 = P(\psi_{q,s(q)}) &\geqslant h(\eta) + \Lambda(\psi_{q,s(q)}, \eta) \\
&= h(\eta) + (1-q)\Lambda(\varphi^{s(q)}, \eta) + q\Lambda(\mu, \eta) \\
&= h(\eta) + (1-q)\Lambda(\varphi^{s(q)}, \eta) + q\Lambda(\varphi^{\tau'(q)}, \eta) \\
&= h(\eta) + (1-q)(s\lambda_1(\eta)) + q(\tau'(q)\lambda_1(\eta)) \\
&= h(\eta) + \lambda_1(\eta)((1-q)s(q) + q\tau'(q)) \\
&= h(\eta) + \Lambda(\varphi^{(1-q)s(q)+q\tau'(q)}, \eta) \\
&= h(\eta) + \Lambda(\varphi^{q\tau'(q)-\tau(q)}, \eta).
\end{aligned}$$

Case 2: The argument runs through almost identically, we abbreviate it slightly. If $\eta \in \mathcal{M}_\sigma(\Sigma)$ is such that $\dim_{\mathrm{L}}(\mu, \eta) = \tau'(q)$, again this is $\Lambda(\mu, \eta) = \Lambda(\varphi^{\tau'(q)}, \eta)$, we have

$$\begin{aligned}
0 &\geqslant h(\eta) + \Lambda(\psi_{q,s(q)}, \eta) = h(\eta) + (1-q)\Lambda(\varphi^{s(q)}, \eta) + q\Lambda(\mu, \eta) \\
&= h(\eta) + (1-q)\Lambda(\varphi^{s(q)}, \eta) + q\Lambda(\varphi^{\tau'(q)}, \eta) \\
&= h(\eta) + \lambda_1(\eta) + (q\tau'(q) - (1-q)s(q) - 1)\lambda_2(\eta) \\
&= h(\eta) + \Lambda(\varphi^{q\tau'(q)-\tau(q)}, \eta).
\end{aligned}$$

In both cases $\dim_{\mathrm{L}}(\eta) \leqslant q\tau'(q) - \tau(q)$. We have shown that $f(\tau'(q)) \leqslant q\tau'(q) - \tau(q)$.

Note that if the equilibrium state ν satisfies $\dim_{\mathrm{L}}(\mu, \nu) = \tau'(q)$ and $\dim_{\mathrm{L}}(\nu) = q\tau'(q) - \tau(q)$ then $f(\tau'(q)) \geqslant \dim_{\mathrm{L}}(\nu) = q\tau'(q) - \tau(q)$ and proof is finished. Indeed this is the case. For the first claim by Proposition 5.5.1, we have in case 1 that

$$\begin{aligned}
\Lambda(\mu, \nu) &= (\tau'(q) - s(q))\lambda_1(\nu) + \Lambda(\varphi^{s(q)}, \nu) \\
&= (\tau'(q) - s(q) + s(q))\lambda_1 = \Lambda(\phi^{\tau'(q)}, \nu)
\end{aligned}$$

and in case 2 that

$$\begin{aligned}\Lambda(\mu,\nu) &= \Lambda(\varphi^{s(q)},\nu)+(\tau'(q)-s(q))\lambda_2(\nu)\\ &= \lambda_1(\nu)+(s(q)-1)\lambda_2(\nu)+(\tau'(q)-s(q))\lambda_2(\nu)\\ &= \lambda_1(\nu)+(\tau'(q)-1)\lambda_2(\nu)=\Lambda(\phi^{\tau'(q)},\nu).\end{aligned}$$

In each case $\lambda(\mu,\nu)=\Lambda(\phi^{\tau'(q)},\nu)$ which is equivalent to $\dim_{\mathrm{L}}(\mu,\nu)=\tau'(q)$. For the second claim in case 1 we have

$$0=P(\psi_{q,s(q)})=h(\nu)+\lambda(\psi_{q,s(q)},\nu)=h(\nu)-qh(\mu,\nu)+(1-q)s(q)\lambda_1(\nu)$$

which is equivalent to

$$\frac{-h(\nu)}{\lambda_1(\nu)}=-\tau(q)+q\left(\frac{-h(\mu,\nu)}{\lambda_1(\nu)}\right).$$

From Proposition 5.5.1 we have $\tau'(q)=-h(\mu,\nu)/\lambda_1(\nu)$ while from $q\tau'(q)-\tau(q)\in(0,1)$ we get $\dim_{\mathrm{L}}(\nu)=-h(\nu)/\lambda_1(\nu)$. So the equation above can be written as

$$\dim_{\mathrm{L}}(\nu)=q\tau'(q)-\tau(q).$$

In case 2 we have that

$$0=P(\psi_{q,s(q)})=h(\nu)+\lambda(\psi_{q,s(q)},\nu)=h(\nu)+\lambda_1(\nu)+(s(q)-1)(1-q)\lambda_2(\nu)-qh(\mu,\nu),$$

which is equivalent to

$$1-\frac{h(\nu)+\lambda_1(\nu)}{\lambda_2(\nu)}=-\tau(q)+q\left(1-\frac{h(\mu,\nu)+\lambda_1(\nu)}{\lambda_2(\nu)}\right).$$

From Proposition 5.5.1 we have $\tau'(q) = 1 - (h(\mu,\nu) + \lambda_1(\nu))/\lambda_2(\nu)$ while from $q\tau'(q) - \tau(q) \in (1,2)$ we get $\dim_{\mathrm{L}}(\nu) = 1 - (h(\nu) + \lambda_1(\nu))/\lambda_2(\nu)$. So the equation above can be written as

$$\dim_{\mathrm{L}}(\nu) = q\tau'(q) - \tau(q).$$

So the second claim is also true and the proof is finished.

□

Remark. *Notice that in the last part of the proof above, where we prove that*

$$\dim_{\mathrm{L}}(\nu) = q\tau'(q) - \tau'(q),$$

we didn't use the conditions on $\tau'(q)$ so for this weaker result we can drop $\tau'(q)$ from cases 1 and 2.

5.6 Beyond *symbolic* Multifractal formalism

Let $\mathsf{A} = (A_1, \ldots, A_N) \in GL_2(\mathbb{R})^N$ be a tuple of contractive invertible matrices and $\mu \in \mathcal{M}(\Sigma)$ be a quasi-Bernoulli measure. Throughout this section we assume that A is dominated and strongly irreducible. Furthermore we assume that $(T_1, ..., T_N)$ satisfies the strong separation condition. Also we set

$$E_a = \{x \in \mathbb{R}^2 : \dim_{loc}(\pi\mu, x) = a\}.$$

Proposition 5.6.1. *Let $q \in (0,1)$ and $a \in \mathbb{R}$. Then*

$$\dim_{\mathrm{H}}(E_a) \leqslant qa - \tau(q).$$

Proof. As it is argued in page 17 of [10], it follows from Theorem 6(a) in [26] and Proposition 2.5(iv) in [62]. □

The arguments of the following two propositions are an adjustment of Theorem 4.1 in [59].

Proposition 5.6.2. *Let* $q > 1$*,* $a \in \mathbb{R}$ *and* $\tau(q)/(q-1) \in (1,2)$*. If there is* $C > 1$ *such that for all* $\phi \in \mathbb{R}$*,* $x \in \mathbb{R}$ *and* $r > 0$ *we have* $\pi_\phi\mu(B(\pi_\phi(x), r)) \leqslant C\,\mathrm{Leb}(B(\pi_\phi(x), r))$ *then*

$$\dim_{\mathrm{H}}(E_a) \leqslant qa - \tau(q).$$

Proof. Let $\epsilon, \epsilon_1, \delta > 0$. If ν is an equilibrium state of the potential $\alpha_2^{\epsilon_1}\psi_{q,s}$ then

$$P(\alpha_2^{\epsilon_1}\psi_{q,s}) = h_\sigma(\nu) + \Lambda(\psi_{q,s}, \nu) + \epsilon_1\lambda_2(\nu) \leqslant P(\psi_{q,s}) + \epsilon_1\lambda_2(\nu) < 0,$$

so there is $\gamma < 0$ and $C_2 > 0$ such that

$$\sum_{|i|=n} \alpha_2(i)^{\epsilon_1}\phi^s(i)^{1-q}\mu([i])^q \leqslant C_2 e^{\gamma n}. \tag{5.4}$$

Let n_0 be large enough so that $|i| \geqslant n_0 \Rightarrow \alpha_2(i) < \delta$ and ρ_1 be as in the Lemma 5.3.3. The family

$$\left\{B(\pi(x), \rho_1\alpha_2(x|_n)) : n \geqslant n_0, x \in \Sigma, \frac{\log(\pi\mu(B(\pi(x), \rho_1\alpha_2(x|_n))))}{\log(\alpha_2(x|_n))} \leqslant a + \epsilon\right\}$$

is a Vitali covering of E_a so by the Vitali covering lemma (see [23], thm 1.10) there is a subfamily V of the family which contains pairwise disjoint sets and satisfies

$$\sum_{B\in V} \mathrm{diam}(B)^{(1-q)s+aq+q\epsilon+\epsilon_1} = \infty \quad \text{or} \quad \mathcal{H}^{(1-q)s+aq+q\epsilon+\epsilon_1}(E_a \backslash (\cup V)) = 0 \tag{5.5}$$

For convenience, given $i \in \Sigma^*$, we set

$$V_i = V \cap \left\{ B(\pi(x), \rho_1 \alpha_2(x|_{|i|})) : x \in [i], \frac{\log(\pi\mu(B(\pi(x), \rho_1\alpha_2(x|_{|i|}))))}{\log(\alpha_2(x|_{|i|}))} \leqslant a + \epsilon \right\}$$

and for $n \in \mathbb{N}$ we set $V_n = \cup_{|i|=n} V_i$. Notice that since the elements of V are pairwise disjoint there is $c > 0$ such that $\#V_i \leqslant c\alpha_1(q)/\alpha_2(q)$. We have

$$\begin{aligned}
\sum_{B \in V_n} \operatorname{diam}(B)^{(1-q)s+aq+q\epsilon+\epsilon_1} &= \sum_{|i|=n} \sum_{B \in V_i} \operatorname{diam}(B)^{(1-q)s+aq+q\epsilon+\epsilon_1} \\
&\leqslant \sum_{|i|=n} \sum_{B \in V_i} \operatorname{diam}(B)^{(1-q)s+aq+q\epsilon+\epsilon_1} \left(\frac{\pi\mu(B)}{a_2(i)^{a+\epsilon}} \right)^q \\
&= (\rho_1 2)^{(1-q)s+aq+q\epsilon+\epsilon_1} \sum_{|i|=n} \sum_{B \in V_i} \alpha_2(i)^{(1-q)s+aq+q\epsilon+\epsilon_1} \left(\frac{\pi\mu(B)}{a_2(i)^{a+\epsilon}} \right)^q \\
&= (\rho_1 2)^{(1-q)s+aq+q\epsilon+\epsilon_1} \sum_{|i|=n} \sum_{B \in V_i} \alpha_2(i)^{(1-q)s+\epsilon_1} \pi\mu(B)^q.
\end{aligned}$$

Set $s' = (1-q)s + aq + q\epsilon + \epsilon_1$. From Lemma 5.3.3 there is $c_1 > 0$ such that the last expression in the above calculation is lower or equal to

$$(\rho_1 2)^{s'} c_1^q \sum_{|i|=n} \sum_{B \in V_i} \alpha_2(i)^{(1-q)s+\epsilon_1} \pi_{\phi_{i_n} \circ \ldots \circ \phi_{i_1}(\theta)} \mu \left(B(\pi_{\phi_{i_n} \circ \ldots \circ \phi_{i_1}(\theta)}(\sigma^n(x_B)), \frac{\alpha_2(i)}{\alpha_1(i)}) \right)^q,$$

where $\pi(x_B)$ is the center of B. From our statement hypothesis the expression above is is lower or equal to

$$C^q (\rho_1 2)^{s'} c_1^q \sum_{|i|=n} \sum_{B \in V_i} \alpha_2(i)^{(1-q)s+\epsilon_1} \left(\frac{\alpha_2(i)}{\alpha_1(i)} \mu([i]) \right)^q.$$

Now since $\#V_i \leqslant c\alpha_1(q)/\alpha_2(q)$ the expression above is less than or equal to

$$cC^q (\rho_1 2)^{s'} c_1^q \sum_{|i|=n} \frac{\alpha_1(i)}{\alpha_2(i)} \alpha_2(i)^{(1-q)s+\epsilon_1} \left(\frac{\alpha_2(i)}{\alpha_1(i)} \mu([i]) \right)^q = cC^q (\rho_1 2)^{s'} c_1^q \sum_{|i|=n} \alpha_2(i)^{\epsilon_1} \phi^s(i)^{1-q} \mu([i])^q.$$

From this bound and 5.4 we conclude that there is $M > 0$, which does not depend on δ, such that

$$\sum_{B \in V} \operatorname{diam}(B)^{s'} < M,$$

so from 5.5 we have that

$$\begin{aligned}
\mathcal{H}_\delta^{s'}(E_a) &\leqslant \mathcal{H}_\delta^{s'}(E_a \cap (\cup V)) + \mathcal{H}_\delta^{s'}(E_a \backslash (\cup V)) \\
&\leqslant \mathcal{H}_\delta^{s'}(\cup V) + \mathcal{H}^{s'}(E_a \backslash (\cup V)) \\
&\leqslant \sum_{B \in V} \operatorname{diam}(B)^{s'} < M.
\end{aligned}$$

Since δ was arbitrary we have that $\mathcal{H}^{s'}(E_a) < M$ so $\dim_{\mathrm{H}}(E_a) \leqslant (1-q)s + aq + q\epsilon + \epsilon_1$. Since ϵ and ϵ_1 were arbitrary the result follows. □

Proposition 5.6.3. *Let $q > 1$, $a \in \mathbb{R}$ and $\tau(q)/(q-1) \in (0,1)$. If there is $C > 1$ and $\rho_3 > 0$ such that for all $i = (i_1, ..., i_n) \in \Sigma^*$ and $x \in [i]$ we have $B(\pi(x), \rho_3 \alpha_1(i)) \leqslant C\mu([i])$ then*

$$\dim_{\mathrm{H}}(E_a) \leqslant qa - \tau(q).$$

Proof. Let $\epsilon, \epsilon_1, \delta > 0$. If ν is an equilibrium state of the potential $\alpha_1^{\epsilon_1}\psi_{q,s}$ then

$$P(\alpha_1^{\epsilon_1}\psi_{q,s}) = h_\sigma(\nu) + \Lambda(\psi_{q,s}, \nu) + \epsilon_1\lambda_1(\nu) \leqslant P(\psi_{q,s}) + \epsilon_1\lambda_1(\nu) < 0,$$

so there is $\gamma < 0$ and $C_2 > 0$ such that

$$\sum_{|i|=n} \alpha_1(i)^{\epsilon_1}\phi^s(i)^{1-q}\mu([i])^q \leqslant C_2 e^{\gamma n}. \tag{5.6}$$

Let n_0 be large enough so that $|i| \geqslant n_0 \Rightarrow \alpha_1(i) < \delta$. The family

$$\left\{ B(\pi(x), \rho_3\alpha_1(x|_n)) : n \geqslant n_0, x \in \Sigma, \frac{\log(\pi\mu(B(\pi(x), \rho_3\alpha_1(x|_n))))}{\log(\alpha_1(x|_n))} \leqslant a + \epsilon \right\}$$

is a Vitali covering of E_a so by the Vitali covering lemma (see [23], thm 1.10) there is a subfamily V of the family which contains pairwise disjoint sets and satisfies

$$\sum_{B \in V} \operatorname{diam}(B)^{(1-q)s+aq+q\epsilon+\epsilon_1} = \infty \quad \text{or} \quad \mathcal{H}^{(1-q)s+aq+q\epsilon+\epsilon_1}(E_a \backslash (\cup V)) = 0 \tag{5.7}$$

For convenience, given $i \in \Sigma^*$, we set

$$V_i = V \cap \left\{ B(\pi(x), \rho_3\alpha_1(x|_{|i|})) : x \in [i], \frac{\log(\pi\mu(B(\pi(x), \rho_3\alpha_1(x|_{|i|}))))}{\log(\alpha_1(x|_{|i|}))} \leqslant a + \epsilon \right\}$$

and for $n \in \mathbb{N}$ we set $V_n = \cup_{|i|=n} V_i$. Notice that since the elements of V are pairwise disjoint there is $c > 0$ such that $\#V_i \leqslant c$. We have

$$\begin{aligned}
\sum_{B \in V_n} \operatorname{diam}(B)^{(1-q)s+aq+q\epsilon+\epsilon_1} &= \sum_{|i|=n} \sum_{B \in V_i} \operatorname{diam}(B)^{(1-q)s+aq+q\epsilon+\epsilon_1} \\
&\leqslant \sum_{|i|=n} \sum_{B \in V_i} \operatorname{diam}(B)^{(1-q)s+aq+q\epsilon+\epsilon_1} \left(\frac{\pi\mu(B)}{a_1(i)^{a+\epsilon}} \right)^q \\
&= (\rho_3 2)^{(1-q)s+aq+q\epsilon+\epsilon_1} \sum_{|i|=n} \sum_{B \in V_i} \alpha_1(i)^{(1-q)s+aq+q\epsilon+\epsilon_1} \left(\frac{\pi\mu(B)}{a_1(i)^{a+\epsilon}} \right)^q \\
&= (\rho_1 3)^{(1-q)s+aq+q\epsilon+\epsilon_1} \sum_{|i|=n} \sum_{B \in V_i} \alpha_1(i)^{(1-q)s+\epsilon_1} \pi\mu(B)^q.
\end{aligned}$$

Set $s' = (1-q)s + aq + q\epsilon + \epsilon_1$. From our statement hypothesis the expression above is is less than or equal to

$$C^q(\rho_3 2)^{s'} c_1^q \sum_{|i|=n} \sum_{B\in V_i} \alpha_1(i)^{(1-q)s+\epsilon_1}\mu([i])^q.$$

Now since $\#V_i \leqslant c$ the expression above is lower or equal to

$$cC^q(\rho_3 2)^{s'} c_1^q \sum_{|i|=n} \alpha_1(i)^{(1-q)s+\epsilon_1}\mu([i])^q. = cC^q(\rho_3 2)^{s'} c_1^q \sum_{|i|=n} \alpha_1(i)^{\epsilon_1}\phi^s(i)^{1-q}\mu([i])^q.$$

From this bound and 5.6 we conclude that there is $M > 0$, which does not depend on δ, such that

$$\sum_{B\in V} \operatorname{diam}(B)^{s'} < M,$$

so from 5.7 we have that

$$\begin{aligned}
\mathcal{H}_\delta^{s'}(E_a) &\leqslant \mathcal{H}_\delta^{s'}(E_a \cap (\cup V)) + \mathcal{H}_\delta^{s'}(E_a \backslash (\cup V)) \\
&\leqslant \mathcal{H}_\delta^{s'}(\cup V) + \mathcal{H}^{s'}(E_a \backslash (\cup V)) \\
&\leqslant \sum_{B\in V} \operatorname{diam}(B)^{s'} < M.
\end{aligned}$$

Since δ was arbitrary we have that $\mathcal{H}^{s'}(E_a) < M$ so $\dim_{\mathrm{H}}(E_a) \leqslant (1-q)s + aq + q\epsilon + \epsilon_1$. Since ϵ and ϵ_1 were arbitrary the result follows. □

Below are the two main theorems of this section. We emphasize that these theorems hold under the conditions mentioned in the top of this section.

Theorem 5.6.1. *Let $q > 0$.*

i) Assume that for all $\phi \in \mathbb{R}$ and $x \in X$ we have $\dim_{\mathrm{loc}}(\pi_\phi\mu, \pi_\phi(x)) = 1$. If $\tau(q)/(q-1), q\tau'(q) - \tau(q) \in (1,2)$ then

$$\dim_{\mathrm{H}}(E_{\tau'(q)}) \geqslant q\tau'(q) - \tau(q).$$

If in addition $q < 1$ then

$$\dim_{\mathrm{H}}(E_{\tau'(q)}) = q\tau'(q) - \tau(q).$$

ii) If $q, \tau(q)/(q-1), q\tau'(q) - \tau(q) \in (0,1)$ then

$$\dim_{\mathrm{H}}(E_{\tau'(q)}) = q\tau'(q) - \tau(q).$$

Proof. We start with i). Let ν to be the equilibrium state of $\psi_{q,s}$. From Theorem 5.5.1 and the related remark we have that

$$\dim_{\mathrm{L}}(\nu) = q\tau'(q) - \tau(q).$$

We also know that $\dim_{\mathrm{L}}(\nu) = \underline{\dim}_{\mathrm{H}}(\pi\nu)$ (see the end of subsection 5.2.3) so the equation above can be written as

$$\underline{\dim}_{\mathrm{H}}(\pi\nu) = q\tau'(q) - \tau(q).$$

From Proposition 5.5.1 we have that

$$\tau'(q) = 1 - \frac{h(\mu,\nu) + \lambda_1(\nu)}{\lambda_2(\nu)},$$

so from Theorem 5.3.1 we have $\pi\nu(E_{\tau'(q)}) = 1$ implying

$$\dim_{\mathrm{H}}(E_{\tau'(q)}) \geqslant \underline{\dim}_{\mathrm{H}}(\pi\nu) = q\tau'(q) - \tau(q).$$

In case $q < 1$ the above becomes an equality from Proposition 5.6.1. The argument for ii) is essentially a simplification of the proof of Proposition 5.5. in [10]. Similarly to i) we have that

$$\underline{\dim}_{\mathrm{H}}(\pi\nu) = q\tau'(q) - \tau(q).$$

From Theorem 5.3.1 we have that there is $t \geqslant 0$ such that $\pi\nu(E_t) = 1$ implying

$$\dim_{\mathrm{H}}(E_t) \geqslant \underline{\dim}_{\mathrm{H}}(\pi\nu) = q\tau'(q) - \tau(q), \tag{5.8}$$

so from Proposition 5.6.1 we have

$$q\tau'(q) - \tau(q) \leqslant \dim_{\mathrm{H}}(E_t) \leqslant qt - \tau(q),$$

giving $\tau'(q) \leqslant t$. But for every $i \in \Sigma$ and $n \in \mathbb{N}$, since $\pi([i|_n])$ is a subset of $T_{i|_n}(D)$, we have $\pi([i]) \subseteq B(\pi(i), 2a_1(i|_n))$ so

$$\overline{\dim}_{\mathrm{loc}}(\mu, \pi(i)) = \limsup_{n\to\infty} \frac{\log(\pi\mu(B(\pi(i), 2a_1(i|_n))))}{\log(a_1(i|_n))} \leqslant \limsup_{n\to\infty} \frac{\log(\mu([i|_n]))}{\log(a_1(i|_n))}$$

which implies that $t \leqslant -h(\mu,\nu)/\lambda_1(\nu) = \tau'(q)$ so $\tau'(q) = t$. Now inequality (5.8) becomes

$$\dim(E_{\tau'(q)}) \geqslant q\tau'(q) - \tau(q).$$

The inverse inequality follows from Proposition 5.6.1.

□

Theorem 5.6.2. *Let $q > 1$.*

i) Assume that there is $C > 1$ *such that for all* $\phi \in \mathbb{R}$, $x \in$ *and* $r > 0$ *we have* $\pi_\phi\mu(B(\pi_\phi(x), r)) \leqslant C\,\mathrm{Leb}(B(\pi_\phi(x), r))$. *If* $\tau(q)/(q-1), q\tau'(q) - \tau(q) \in (1, 2)$ *then*

$$\dim_{\mathrm{H}}(E_{\tau'(q)}) = q\tau'(q) - \tau(q).$$

ii) Assume that here is $C > 1$ *and* $\rho_3 > 0$ *such that for all* $i = (i_1, ..., i_n) \in \Sigma^*$ *and* $x \in [i]$ *we have* $B(\pi(x), \rho_3\alpha_1(i)) \leqslant C\mu([i])$. *If* $\tau(q)/(q-1), q\tau'(q)-\tau(q) \in (0, 1)$ *then*

$$\dim_{\mathrm{H}}(E_{\tau'(q)}) = q\tau'(q) - \tau(q).$$

Proof. The proof is almost identical to the proof of i) in Theorem 5.6.1. The difference here is that for claim ii), $\pi\nu(E_{\tau'(q)}) = 1$ is implied directly from the respective assumption and that for both i) and ii) we get upper bounds from Propositions 5.6.2 and 5.6.3.

□

Chapter 6

Matrices associated to Pisot numbers

6.1 Introduction

Bernoulli convolutions arise from arguably the simplest family of iterated function systems with overlaps.

Definition 6.1.1. *The Bernoulli convolution* ν_β*, for* $\beta \in (1, 2)$ *is the unique probability measure on* $[0, \beta/(\beta-1)]$ *which satisfies*

$$\nu_\beta = \frac{1}{2}F_0(\nu_\beta) + \frac{1}{2}F_1(\nu_\beta)$$

where $F_i(t) = \beta^{-1}t + i$*.*

The questions about Bernoulli convolutions that are of interest are mainly related to their dimension or absolutely continuity. In [21] Erdos proved that ν_β is singular if β is a Pisot number. Later Garsia proved in [38] that $\dim_{\mathrm{H}}(\nu_\beta) < 1$ if β is a Pisot number. Pisot numbers are numbers that are greater than one such their minimal polynomial is monic who's other roots are of absolute value smaller than one. In [74] Solomyak proved the remarkable result that ν_β is absolutely continuous for almost all $\beta \in (1,2)$. In the direction of the Hausdorff dimension there have been recent important results that shed some light to the subject and also raised the interest for further investigation. Many of these results are based on Hochman's influential article [45]. An implication of Hochman's results is that

$$\dim_{\mathrm{H}}(\{\beta \in (1,2) : \dim_{\mathrm{H}}(\nu_\beta) < 1\}) = 0.$$

Varju observed in [17] that Hochman's results also give a formula for the dimension of Bernoulli convolutions for algebraic parameters in terms of the Garsia entropy which will be defined below. This way we can further study the case of algebraic β. Another especially motivating article came from Breuillard and Varju (see [16]) an implication of which is the following

$$\{\beta \in (1,2) : \dim_{\mathrm{H}}(\nu_\beta) < 1\} \subseteq \overline{\{\beta \in (1,2) : \dim_{\mathrm{H}}(\nu_\beta) < 1\} \cap \overline{\mathbb{Q}}},$$

where $\overline{\mathbb{Q}}$ is the set of algebraic numbers. This result suggested that we can find all $\beta \in (1,2)$ that give dimension less than one by focusing on the algebraic numbers values of β. For example, since the set of Pisot numbers is closed (see [69]), if we could show that $\dim_{\mathrm{H}}(\nu_\beta) = 1$ for non-Pisot algebraic β then that would mean that the Pisot numbers are the only numbers that give dimension less than one. Recently another impressive result came from Varju proving that $\dim_{\mathrm{H}}(\nu_\beta) = 1$ for all transcendental $\beta \in (1,2)$ (see [79]). This last result alone, clearly reduces the problem of determining when we have $\dim_{\mathrm{H}}(\nu_\beta) < 1$ to the case where β is algebraic.

This chapter was motivated by our attempt to understand the Hausdorff dimension of Bernoulli convolutions ν_β when β is a Pisot number of high degree. By degree of an algebraic number we mean the degree of its minimal polynomial. In [2] a matrix $M(\beta)$ was introduced that provides lower bounds for $\dim_{\mathrm{H}}(\nu_\beta)$ when β is hyperbolic. We focused on understanding this lower bound for Pisot numbers of high degree. As the degree grows bigger the size of $M(\beta)$ does so as well, giving us sparse matrices. We unfortunately were not able to control the spectral properties required of these matrices. Instead we defined a different family of sparse matrices in a similar way that $M(\beta)$ is defined but with significantly lower complexity. The result about these toy-model matrices gave some partial insight as to why we can not understand the matrices $M(\beta)$ since they serve as a counterexamples in some related conjectures we hoped to use as intermediate steps in the initial plan.

Definition 6.1.2. *For $a_1, ..., a_n \in \{0,1\}$ we define the following notation*

$$N(a_1, ..., a_n) := \left\{ (b_1, ..., b_n) \in \{0,1\}^n : \sum_{i=1}^{n} b_i \beta^i = \sum_{i=1}^{n} a_i \beta^i \right\}.$$

We also set

$$H_n(\beta) := \sum_{a_1,...,a_n \in \{0,1\}} \frac{1}{2^n} \log \left(\frac{\# N(a_1, ..., a_n)}{2^n} \right).$$

The Garsia entropy $H(\beta)$ of ν_β is defined as

$$H(\beta) = \lim_{n \to \infty} \frac{1}{n} H_n(\beta).$$

It is observed in [17] that Hochman's results in [45] imply

$$\dim_{\mathrm{H}}(\nu_\beta) = \min \left\{ \frac{H(\beta)}{\log(\beta)}, 1 \right\}. \tag{6.1}$$

In [2] the authors introduce a matrix $M(\beta)$, for hyperbolic β, such that

$$\log 2 - \log \rho(M(\beta)) \leqslant H(\beta),$$

which combined with 6.1 gives

$$\min\left(\frac{\log 2 - \log \rho(M(\beta))}{\log \beta}, 1\right) \leqslant \dim_{\mathrm{H}}(\nu_\beta). \tag{6.2}$$

We set $T_i(t) = \beta t - i$. The matrix $M(\beta)$ is naturally related to a directed graph $V(\beta)$ which we will identify, without confusion, with it's set of nodes. $V(\beta)$ is defined as $V(\beta) = \bigcup_{n=0}^{\infty} V_{\beta,n}$ where

$$V_{\beta,n} = \left\{ \sum_{i=0}^{n} \varepsilon_i \beta^{n-i} \quad | \quad \epsilon_i \in \{-1,0,1\} \quad \text{and} \quad \left| \sum_{i=0}^{n} \varepsilon_i \beta^{n-i} \right| < \frac{1}{\beta - 1} \right\}.$$

The set of edges of $V(\beta)$ is $\{(x,y) \in (-1/(\beta-1), 1/(\beta-1))^2 \mid \exists i \in \{-1,0,1\} : T_i x = y\}$. They also prove that $V(\beta)$ is finite when β is hyperbolic. Assume that $V(\beta) = \{x_1, ..., x_n\}$ and $x_1 < ... < x_n$. The matrix $M(\beta)$ is defined as follows:

$$(M)_{i,j} = \begin{cases} 1/2 & \text{if} \quad \exists \kappa \in \{-1,1\} : T_\kappa(x_i) = x_j \\ 1 & \text{if} \quad T_0(x_i) = x_j \\ 0 & \text{otherwise} \end{cases}$$

So far we know that the matrix $M(\beta)$ provides an algorithm to get lower bounds for explicit examples of β. Given the minimal polynomial of β we can construct $V(\beta)$, form $M(\beta)$ and calculate the maximal eigenvalue $\rho(M(\beta))$. Then $(\log 2 - \log \rho(M(\beta))/\beta \leqslant \dim_{\mathrm{H}}(\beta)$. Since understating eigenvalues of large matrices is very hard in general, not much is know about the behaviour of $\beta \mapsto \rho(M(\beta))$. Let us describe a possible approach to study $\rho(M(\beta))$. If we identify $u \in \mathbb{R}^n$ with the measure $\sum_{i=1}^{n} u(i)\delta_{x_i}$ then we can write

$$uM(\beta) = \sum_{i=-1}^{1} p_i T_i(u)|_I \tag{6.3}$$

where $(p_{-1}, p_0, p_1) = (1/2, 1, 1/2)$. This connection of $M(\beta)$ to the dynamics allows to observe that if $V(\beta)$ is in some sense equidistributed then we can understand how a plot of $M(\beta)$ would look like. Let L be in operator acting on signed measures on $[-1/(\beta-1), 1/(\beta-1)]$ defined as

$$L_1(u) = \sum_{i=-1}^{1} p_i T_i(u).$$

Observe that if $f : [-1/(\beta-1), 1/(\beta-1)] \to \mathbb{R}$ is continuous then

$$L_1(f d\lambda) = L_2(f) d\lambda$$

where

$$L_2 f(x) = \sum_{i=-1}^{1} (p_i/\beta) \cdot f(T_i^{-1}(x)).$$

Definition 6.1.3. *We will denote both the normalised maximal left eigenvector of $M(\beta)$ and the measure it defines on $[\frac{-\beta}{\beta-1}, \frac{\beta}{\beta-1}]$ as $\mu(\beta)$.*

Definition 6.1.4. *Let a be a real number greater than $log(6/\beta)/log(\beta)$. We define $\mathcal{B}$ to be the space of Holder continuous functions from $[-1/(\beta-1), 1/(\beta-1)]$ to $\mathbb{R}$ with exponent a. Also define $L : \mathcal{B} \setminus \{0\} \longrightarrow \mathcal{B} \setminus \{0\}$ as $Lf = L_2 f / ||L_2 f||_1$*

Lemma 6.1.1. *If $f \in \mathcal{B} \setminus \{0\}$ then $L^n f \to 1$.*

Proof. For $f \in \mathcal{B}$ and $n \in \mathbb{N}$ set

$$\Delta_{f,n} = sup\left\{ |f(x) - f(y)| : (x, y) \in \left[\frac{-1}{\beta-1}, \frac{1}{\beta-1}\right]^2 \text{ and } \quad |x - y| \leqslant \frac{2(1/\beta)^n}{\beta-1} \right\}.$$

Also for a word $(x_1...x_k) \in \{-1,0,1\}^\kappa$ define $I_{x_1,\ldots,x_\kappa}$ to be the function interval $T_{x_\kappa}^{-1} \circ \ldots \circ T_{x_1}^{-1}\left(\left[\frac{-1}{\beta-1}, \frac{1}{\beta-1}\right]\right)$. Next set $\phi_{x_1,\ldots,x_\kappa} = T_{x_1} \circ \ldots \circ T_{x_\kappa}|_{I_{x_1,\ldots,x_\kappa}}$. Finally set $a_n = ||L_2 L^{n-1} f||_1^{-1}$ for $n \geqslant 1$. Then it's easy to see that

$$L^n f = \left(\prod_{i=1}^n a_i\right) \sum_{x_1 \ldots x_n \in \{-1,0,1\}^n} \frac{\prod_{\kappa=1}^n p_{x_\kappa}}{\beta^n} f \circ \phi_{x_1,\ldots,x_n}^{-1}$$

so that

$$\begin{aligned} \Delta_{L^n f,0} \leqslant& \left(\prod_{i=1}^n a_i\right) \sum_{x_1 \ldots x_n \in \{-1,0,1\}^n} \frac{\prod_{\kappa=1}^n p_{x_\kappa}}{\beta^n} \Delta_{f,n} \\ \leqslant& 2^n \cdot 3^n \cdot \left(\frac{1}{\beta}\right)^n \Delta_{f,n} = \left(\frac{6}{\beta}\right)^n \Delta_{f,n} \end{aligned}$$

but, since $f \in \mathcal{B}$, $|x-y| < 2(1/\beta)^n(\beta-1)^{-1}$ implies

$$|f(x) - f(y)| \leqslant C \left(\frac{2}{\beta-1}\right)^a \left(\frac{1}{\beta}\right)^{na}$$

so that $\Delta_{f,n} \leqslant C(2/(\beta-1))^a \, (1/\beta)^{na}$ which leads to

$$\Delta_{L^n f,0} \leqslant C \left(\frac{6}{\beta}\right)^n \left(\frac{2}{\beta-1}\right)^a \left(\frac{1}{\beta}\right)^{na} = C \left(\frac{2}{\beta-1}\right)^a \left[\left(\frac{1}{\beta}\right)^a \left(\frac{6}{\beta}\right)\right]^n \to 0.$$

Finally from $\Delta_{L^n f,0} \to 0$ and $||L^n f||_1 = 1$ we have that $||L^n f - 1||_\infty \to 0$.

□

Corollary 6.1.1. *If $\mu = f d\lambda$ and $f \in \mathcal{B} \setminus \{0\}$ then $L_1^n(\mu) \overset{weak^*}{\longrightarrow} \lambda$.*

The above lemma can been seen as a motivation to ask if the left eigenvector of the matrix $M(\beta)$ is in some sense equidistributed when the $V(\beta)$ is big and ε-dense for small ε. The reason we are interested in the distribution of the eigenvectors is the following lemma.

Lemma 6.1.2. *Let β_n be a sequence of Pisot numbers, bounded away from $\{1, 2\}$, such that $\mu(\beta_n) \overset{weak^*}{\longrightarrow} \lambda$. Then $\left|\rho(M(\beta_n)) - \frac{2}{\beta_n}\right| \to 0$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\mu(\beta_n) \overset{\text{weak}^*}{\longrightarrow} \lambda$ and the lengths of the intervals bellow are bounded away from zero, there is $n_0 \in \mathbb{N}$ such that

$$\left| \mu(\beta_n) \left(\left[\frac{-1}{\beta_n - 1}, \frac{-1}{\beta_n(\beta_n - 1)} \right) \bigcup \left(\frac{1}{\beta_n(\beta_n - 1)}, \frac{1}{\beta_n - 1} \right] \right) - \frac{\beta_n - 1}{\beta_n} \right| < \varepsilon,$$

$$\left| \mu(\beta_n) \left(\left[\frac{-1}{\beta_n(\beta_n - 1)}, \frac{\beta_n - 2}{\beta_n - 1} \right) \bigcup \left(\frac{2 - \beta_n}{\beta_n - 1}, \frac{1}{\beta_n(\beta_n - 1)} \right] \right) - \frac{\beta_n - 1}{\beta_n} \right| < \varepsilon$$

and

$$\left| \mu(\beta_n) \left(\left[\frac{\beta_n - 2}{\beta_n - 1}, \frac{2 - \beta_n}{\beta_n - 1} \right] \right) - \frac{2 - \beta_n}{\beta} \right| < \varepsilon.$$

For $i \in \{1, ..., |V(\beta_n)|\}$ we set $w_i = \mu(\beta_n)(\{x_i\})$ where $V(\beta_n) = \{x_1, ..., x_{|V(\beta_n)|}\}$ and $x_1 < ... < x_{|V(\beta_n)|}$. Also $(e_1, ..., e_{|V(\beta_n)|})$ will be the standard basis of $\mathbb{R}^{|V(\beta_n)|}$.

Then

$$\begin{aligned}
\rho(M(\beta_n)) &= \left\| \left(\sum_{i=1}^{|V(\beta_n)|} w_i e_i \right) M(\beta_n) \right\|_1 \\
&= \left\| \sum_{i=1}^{|V(\beta_n)|} w_i e_i M(\beta_n) \right\|_1 \\
&= \sum_{i=1}^{|V(\beta_n)|} w_i ||e_i M(\beta_n)||_1 \\
&= \sum_{i=1}^{|V(\beta_n)|} w_i \sum_{j=1}^{|V(\beta_n)|} M_{i,j}(\beta_n)
\end{aligned}$$

by the definition of $M(\beta_n)$ the last sum is equal to

$$\begin{aligned}
&\frac{1}{2} \sum_{x \in V_1(\beta_n)} \mu(\beta_n)(\{x\}) + \frac{3}{2} \sum_{x \in V_2(\beta_n)} \mu(\beta_n)(\{x\}) + 2 \sum_{x \in V_3(\beta_n)} \mu(\beta_n)(\{x\}) \\
=&\frac{1}{2}\mu(\beta_n)(V_1(\beta_n)) + \frac{3}{2}\mu(\beta_n)(V_2(\beta_n)) + 2\mu(\beta_n)(V_3(\beta_n))
\end{aligned}$$

where

$$V_i(\beta_n) = \left\{ x_\kappa \in V(\beta_n) : \sum_{j=1}^{|V(\beta_n)|} M_{\kappa,j}(\beta_n) = \sum_{j=-1}^{i-2} p_j \right\},$$

but

$$V_1(\beta_n) = \left(\left[\frac{-1}{\beta_n - 1}, \frac{-1}{\beta_n(\beta_n - 1)} \right) \bigcup \left(\frac{1}{\beta_n(\beta_n - 1)}, \frac{1}{\beta_n - 1} \right] \right) \bigcap V(\beta_n)$$

so $|\mu(\beta_n)(V_1(\beta_n)) - (\beta_n - 1)/\beta_n| < \varepsilon$. Similarly $|\mu(\beta_n)(V_2(\beta_n)) - (\beta_n - 1)/\beta_n| < \varepsilon$ and $|\mu(\beta_n)(V_3(\beta_n)) - (2 - \beta_n)/\beta| < \varepsilon$. Also

$$\frac{1}{2}(\beta_n - 1)/\beta_n + \frac{3}{2}(\beta_n - 1)/\beta_n + 2(2 - \beta_n)/\beta = 2/\beta_n$$

hence

$$|\rho(M(\beta_n)) - 2/\beta_n| < (1/2 + 3/2 + 2)\varepsilon = 4\varepsilon.$$

Since ε was arbitrary that completes the proof. □

The lemma above combined with inequality 6.2 gives the following

Lemma 6.1.3. *Let β_n be a sequence of hyperbolic numbers such that $\mu(\beta_n) \stackrel{weak^*}{\longrightarrow} \lambda$. Then $\dim_{\mathrm{H}}(\nu_{\beta_n}) \to 1$.*

All this leads us to the following conjectures 3 and 4 and suggests, as a strategy, to prove it based on the naive conjecture 5 bellow.

Conjecture 3. *Let β_n be a sequence of Pisot numbers, bounded away from $\{1, 2\}$, such that deg$(\beta_n) \to \infty$ and $\beta_n \to \beta \in (1, 2)$. Then $\dim_{\mathrm{H}}(\nu_{\beta_n}) \to 1$.*

Definition 6.1.5. *Let $S \subseteq [0, 1/(\beta - 1)]$ be finite. We will call an ε-perturbation of S any strictly increasing map $\psi : S \to [0, 1/(\beta - 1)]$ such that $|\psi(x) - x| < \varepsilon$ for all $x \in S$.*

Conjecture 4. *Let* Λ *be a finite subset of* $\mathbb{Z}$ *and* $p : \Lambda \to (0, \infty)$. *Also set* $G_n = \{1/n, ..., n/n\}$ *for each* $n \in \mathbb{N}$. *For each* $\varepsilon > 0$ *there is* $\delta > 0$ *and* $n_0 \in \mathbb{N}$ *such that if* A *is an* $n \times n$ *irreducible matrix for some* $n > n_0$ *and there is a* δ*-perturbation* ψ *of* G_n *for which*

$$A_{i,j} = \begin{cases} p(\kappa), & T_\kappa(\psi(i/n)) = \psi(j/n) \text{ and } \kappa \in \Lambda \\ 0, & \text{otherwise} \end{cases}$$

and $T_\kappa(\psi(G_n)) \subseteq \psi(G_n)$ *for all* $\kappa \in \Lambda$, *then*

$$\left| \rho(A) - \sum_{\kappa \in \Lambda} \frac{p(\kappa)}{\beta} \right| < \varepsilon.$$

Conjecture 5. *Let* β *be a Pisot number. Let* V *be a finite subset of* $[-1/(\beta - 1), 1/(\beta - 1)]$ *such that* $L_1(V) \subseteq V$. *Also let* M *be a matrix indexed by* V *and defined as*

$$(M)_{x,y} = \begin{cases} 1/2 & \text{if } \exists \kappa \in \{-1, 1\} : T_\kappa(x) = y \\ 1 & \text{if } T_0(x) = y \\ 0 & \text{otherwise} \end{cases}.$$

Assume also that M *is irreducible. Then, if* V *is big and* ε*-dense for small* ε, *the spectral radius of* M *is close to* $2/\beta$.

The matrix $M(\beta)$ has been very difficult to study. We initially tried analysing the family of Pisot numbers satisfying $\beta_n^n - \beta_n^{n-1} - \beta_n^{n-2} = 1$. Notice that $\beta_n \to \phi$. The size of $M(\beta_n)$ grows very fast making hard to detect any special structure. Also studying the spectral radius of large matrices seems to be surprisingly hard. What follows is a toy problem that serves as simplification of the questions above.

For $x \in (0, 1/(\beta - 1))$ we define the graph $V(\beta, x)$ as $V(\beta, x) = \bigcup_{n=0}^{\infty} V_{\beta,x,n}$ where

$$V_{\beta,x,n} = \left\{ T_{\varepsilon_n} \circ ... \circ T_{\varepsilon_0}(x) \quad | \quad \varepsilon_i \in \{0,1\} \quad \text{and} \quad 0 < T_{\varepsilon_n} \circ ... \circ T_{\varepsilon_0}(x) < \frac{1}{\beta - 1} \right\}.$$

The set of edges of $V(\beta)$ is $\{(x, y) \in (0, 1/(\beta - 1))^2 \quad | \quad \exists i \in \{0, 1\} : T_i x = y\}$. Assume that x has a periodic greedy β-expansion which, as we will say later, makes $V(\beta, x)$ finite. The matrix $M(\beta, x)$ is defined as the adjacency matrix of (V, E). We focus on the case of $\beta = \phi = (\sqrt{5} + 1)/2$, which is where the simplification happens. This way we keep the dynamics fixed and the size of the matrices is getting big by choosing different starting points x instead. The matrix $M(\beta)$ is completely determined by the minimal polynomial defining β. In the toy problem the matrix $M(\phi, x)$ is completely determined by the greedy expansion of x which can be expressed by the following equation

$$\beta^d x + \sum_{i=0}^{d-1} c_{d-i} \beta^i = x$$

where $c_1, ..., c_d, c_1, ..., c_d, ...$ is the greedy expansion of x. The equation above can been seen as an analogue of the minimal polynomial equation in the case of $M(\beta)$. In section 6.2 we fix $\beta = \phi$ and describe the spectral radius of $M(\phi, x)$ using the beta-expansion map

$$T(x) = \begin{cases} \beta x, & x \in [0, 1/\beta] \\ \beta x - 1, & x \in (1/\beta, 1] \end{cases}.$$

Let $\mathcal{P}_T$ be the set of all $x \in (0, 1)$ that are periodic under T. We prove that for random enough, in a certain sense, $x \in \mathcal{P}_T$ the spectral radius $\rho(M(\phi, x))$ is approximately equal to an explicit number LE. The number LE is expressed as a Lyapunov exponent of random matrix products. The randomness of x, for us roughly means that the orbit of x is relatively equidistributed in respect to the unique absolutely continuous measure of T. It is useful to keep in mind that $M(\phi, x)$ tends to be of large size, as it will become clearer in the next section. We start by proving a formula expressing $\rho(M(\phi, x))$ in terms of 3x3-matrix products.

It is known (see [63]) that $\tau_\beta := T|_{[0,1]} : [0,1] \to [0,1]$ is a dynamical system with an invariant absolutely continuous probability measure μ_β defined by

$$\mu_\beta(E) = \int_E h_\beta d\lambda$$

where λ is the normalised Lebesgue measure on $[0, \phi]$ and

$$h_\beta(t) = \begin{cases} \frac{1+3\beta}{5\beta} & t \in [0, 1/\beta) \\ \frac{2+\beta}{5\beta} & t \in [1/\beta, 1] \end{cases} .$$

Theorem 6.1.1 below, is the main result of this section. The theorem involves a metric d on probability measures on $[0,1]$ which is defined later on in terms of a symbolic space (see definition 6.2.5).

Theorem 6.1.1. *Let $x \in \mathcal{P}_T$ and denote by μ_x the normalised counting measure on the orbit of x. For each $\varepsilon > 0$ there is $\delta > 0$ such that if $d(\mu_\beta, \mu_x) < \delta$ then $|\log(\rho(M(\beta, x))) - LE| < \varepsilon$.*

Roughly in section 6.3 we are going to prove that if the invariant under T probability measure, supported on a periodic orbit of a point $x \in \mathcal{P}_T$, is close to μ_β then $V(\phi, x)$ is evenly spread on $(0, \phi)$. Formally we prove the following theorem.

Theorem 6.1.2. *For $x \in \mathcal{P}_T$ we denote by μ_x the normalised counting measure on the orbit of x. Let I be any subinterval of $[0, \phi]$. For each $\varepsilon > 0$ there is $\delta > 0$ such that if $d(\mu_\beta, \mu_x) < \delta$ then*

$$\left|\frac{\#V(\beta,x)\cap I}{\#V(\beta,x)}-\lambda(I)\right|<\varepsilon.$$

In loose terms, the theorem above says that if the distribution of the orbit of x approximates the measure μ_β then the distribution of $V(\beta,x)$ approximates the Lebesgue measure (properly normalised). Numerical evidence, based on equation 6.9 below, suggest that

$$\frac{\log(2)-\log(LE)}{\log(\beta)}>1.004,$$

which together with theorem 6.1.2 implies that conjecture 4 is wrong. Finally in section 6.4 we prove a connection of the number LE to the Lebesgue almost everywhere value of the local dimension of the Bernoulli convolution ν_ϕ.

6.2 The spectral radius

We start be giving a combinatorial interpretation of $\rho(M(\phi,x))$.

Lemma 6.2.1.

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\#\left\{(\varepsilon_1,...,\varepsilon_n)\in\{0,1\}^n : T_{\varepsilon_n}\circ...\circ T_{\varepsilon_1}(x)\in(0,1)\right\}\right)=\log(\rho(M(\phi,x))).$$

Proof. If $x\in\mathcal{P}_T$ then $M(\phi,x)$ is irreducible. This is an immediate consequence of lemma 6.3.3 which is proved in the next section. Let p be the the period of $M(\phi,x)$. Since the irreducible blocks of $M(\phi,x)^p$ are primitive with spectral radius equal to $\rho(M(\phi,x))^p$ we have that

$$\frac{M(\phi,x)^{pn}}{\rho(M(\phi,x))^{pn}}$$

converges, as $n \to \infty$, to a limit matrix which we will cal $M_{\phi,x}$. Let $v \in \mathbb{R}^{\#V(\phi,x)} \backslash \{0\}$ and $\kappa \in \{0, ..., p-1\}$ then

$$\lim_{n\to\infty} \frac{vM(\phi,x)^{np+\kappa}}{\rho(M(\phi,x))^{np+\kappa}} = \frac{vM(\phi,x)^{\kappa}}{\rho(M(\phi,x))^{\kappa}} M_{\phi,x}$$

so

$$\lim_{n\to\infty} \frac{||vM(\phi,x)^{(n+1)p+\kappa}||}{||vM(\phi,x)^{np+\kappa}||} = \lim_{n\to\infty} \frac{\rho(M(\phi,x))^{p}||vM(\phi,x)^{(n+1)p+\kappa}||/\rho(M(\phi,x))^{(n+1)p+\kappa}}{||vM(\phi,x)^{np+\kappa}||/\rho(M(\phi,x))^{np+\kappa}} \tag{6.4}$$

$$= \rho(M(\phi,x))^{p}. \tag{6.5}$$

By writing

$$||vM(\phi,x)^{np+\kappa}|| = ||vM(\phi,x)^{\kappa}|| \frac{||vM(\phi,x)^{\kappa+p}||}{||vM(\phi,x)^{\kappa}||} ... \frac{||vM(\phi,x)^{np+\kappa}||}{||vM(\phi,x)^{(n-1)p+\kappa}||}$$

and equation 6.4 we have get

$$\lim_{n\to\infty} \frac{1}{n} \log\left(||vM(\phi,x)^{np+\kappa}||\right) = p\log(\rho(M(\phi,x)))$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{np+\kappa} \log\left(||vM(\phi,x)^{np+\kappa}||\right) = \log(\rho(M(\phi,x)))$$

but since κ was arbitrary we have

$$\lim_{n\to\infty} \frac{1}{n} \log\left(||vM(\phi,x)^{n}||\right) = \log(\rho(M(\phi,x))).$$

From the definition of $M(\phi,x)$ if we set v to be the vector corresponding to giving value 1 to x and value 0 to every other element of $V(\phi,x)$ then

$$||vM(\phi,x)^n|| = \#\{(\varepsilon_1,...,\varepsilon_n)\in\{0,1\}^n : T_{\varepsilon_n}\circ...\circ T_{\varepsilon_1}(x)\in(0,1)\}$$

so from the discussion above we conclude

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\#\{(\varepsilon_1,...,\varepsilon_n)\in\{0,1\}^n : T_{\varepsilon_n}\circ...\circ T_{\varepsilon_1}(x)\in(0,1)\}\right) = \log(\rho(M(\phi,x))).$$

□

In the following two lemmata we show that we can compute

$$\sum_{\varepsilon_1,...,\varepsilon_n\in\{0,1\}}\delta_{T_{\varepsilon_n}\circ...\circ T_{\varepsilon_1}(x)}$$

in terms of the the orbit of x under T, using matrix products.

Lemma 6.2.2. *For each $x\in(0,\beta)$ and $n\in\mathbb{N}$*

$$\{T_{\varepsilon_n}\circ...\circ T_{\varepsilon_1}(x) : (\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n\}\cap(0,\beta)\subseteq\{T^n(x),T^n(x)+1/\beta,T^n(x)+1\}.$$

Proof. We are going to prove it by induction on n. For $n=0$ there is nothing to prove so let's assume that the statement is true for some fixed n and prove it for $n+1$. We start by the case where $x' := T^n(x)\leqslant 1/\beta$. Then $T(x')=\beta x$, $T_1(x')\notin(0,1)$ and $T_0(x'+1)\notin(0,1)$ hence by the inductive step

$$\begin{aligned}&\left\{T_{\varepsilon_{n+1}}\circ...\circ T_{\varepsilon_1}(x) : (\varepsilon_{n+1},...,\varepsilon_1)\in\{0,1\}^{n+1}\right\}\cap(0,\beta)\\&\subseteq\{T_0(x'),T_0(x'+1/\beta),T_1(x'+1/\beta),T_1(x'+1)\}\\&=\{\beta x',\beta x'+1/\beta,\beta x'+1\}=\{T^{n+1}(x),T^{n+1}(x)+1/\beta,T^{n+1}(x)+1\}.\end{aligned}$$

For the second case we have $x' := T^n(x) > 1/\beta$. Then $T(x') = \beta x - 1$, $T_0(x'+1) \notin (0,1)$, $T_1(x'+1) \notin (0,1)$ and $T_0(x'+1/\beta) \notin (0,1)$ hence by the inductive step

$$\begin{aligned}
&\left\{T_{\varepsilon_{n+1}} \circ ... \circ T_{\varepsilon_1}(x) : (\varepsilon_{n+1}, ..., \varepsilon_1) \in \{0,1\}^{n+1}\right\} \cap (0,\beta) \\
&\subseteq \{T_0(x'), T_1(x'), T_1(x'+1/\beta)\} \\
&= \{\beta x' - 1, \beta x'\} \subseteq \{T^{n+1}(x), T^{n+1}(x) + 1/\beta, T^{n+1}(x) + 1\}
\end{aligned}$$

which completes the second case and the proof. □

Definition 6.2.1. *For each $x \in (0,\beta)$ and $n \in \mathbb{N}$ we define*

$$\begin{aligned}
v_1(x,n) &:= \#\{(\varepsilon_n, ..., \varepsilon_1) \in \{0,1\}^n : T_{\varepsilon_n} \circ ... \circ T_{\varepsilon_1}(x) = T^n(x)\}, \\
v_2(x,n) &:= \#\{(\varepsilon_n, ..., \varepsilon_1) \in \{0,1\}^n : T_{\varepsilon_n} \circ ... \circ T_{\varepsilon_1}(x) = T^n(x) + 1/\beta\} \cdot \chi_{(0,\beta)}(T^n(x) + 1/\beta), \\
v_3(x,n) &:= \#\{(\varepsilon_n, ..., \varepsilon_1) \in \{0,1\}^n : T_{\varepsilon_n} \circ ... \circ T_{\varepsilon_1}(x) = T^n(x) + 1\} \cdot \chi_{(0,\beta)}(T^n(x) + 1)
\end{aligned}$$

and $v(x,n) = (v_1, v_2, v_3)$.

Lemma 6.2.3. *Let $x \in (0,\beta)$. If $T^n(x) \in (0, 1 - 1/\beta)$ then*

$$v(x, n+1) = v(x,n)A_{0'}.$$

If $T^n(x) \in (1 - 1/\beta, 1/\beta)$ then

$$v(x, n+1) = v(x,n)A_{0''}.$$

If $T^n(x) \in (1/\beta, 1)$ then

$$v(x, n+1) = v(x,n)A_1.$$

where

$$A_{0'} := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A_{0''} := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad A_1 := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proof. Set $x' = T^n(x)$. Then

$$x' \longrightarrow \begin{cases} T_0(x') = \beta x' & \text{for } x' \in (0,1) \\ T_1(x') = \beta x' - 1 & \text{for } x' \in (1/\beta, 1] \end{cases}$$

$$x' + \frac{1}{\beta} \longrightarrow \begin{cases} T_0(x' + \frac{1}{\beta}) = \beta x' + 1 & \text{for } x' \in [0, 1 - 1/\beta) \\ T_1(x' + \frac{1}{\beta}) = \beta x' & \text{for } x' \in (0,1) \end{cases}$$

$$x' + 1 \longrightarrow \begin{cases} T_0(x' + 1) = \beta x' + \beta & \text{for } x' \in \emptyset \\ T_1(x' + 1) = \beta x' + \frac{1}{\beta} & \text{for } x' \in [0, 1/\beta) \end{cases}$$

where the conditions on x' on the right rise by demanding $T_i(x') \in (0, \beta)$. Putting the information above together and remembering that

$$T^{n+1}(x) = T(x') = \begin{cases} \beta x' & x' \in [0, 1/\beta] \\ \beta x' - 1 & x' \in (1/\beta, 1] \end{cases}$$

the proof of the lemma follows.

□

Remark. *If $x \in (1, \beta)$ then there exists $\kappa \in \mathbb{N}$ such that for $m < \kappa$ we have $\{T^m(x)\} = \{T_{\varepsilon_m} \circ ... \circ T_{\varepsilon_1}(x) : (\varepsilon_m, ..., \varepsilon_1) \in \{0,1\}^n\} \cap (0, \beta)$, $T^m(x) \in (1, \beta)$ and $T^n(x) \in [0,1]$ for all $n \geqslant \kappa$.*

By the remark above we can assume that $x \in [0,1]$. Now let $f_0 : [0,1] \to [0,1]$ be defined as $f_0(t) = \beta^{-1}t$ and $f_1 : [0,1/\beta] \to [0,1]$ as $f_1(t) = \beta^{-1}(t+1)$. We also set $f_{0'} = f_0|_{[0,1/\beta]}$ and $f_{0''} = f_0|_{[1/\beta,1]}$. A sequence $\{a_i\}_{i\in\mathbb{N}} \in \{0', 0'', 1\}^{\mathbb{N}}$ is called admissible if or all $i \in \mathbb{N}$

$$A^{\sigma}_{a_i,a_{i+1}} = 1$$

where A^{σ} is the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

indexed by $(0', 0'', 1)$ and similarly $(a_1, ..., a_n) \in \{0', 0'', 1\}^n$, for any $n \in \mathbb{N}$, is called admissible if $A^{\sigma}_{a_i,a_{i+1}} = 1$ for all $i \in \{1, ..., n-1\}$. Let Σ be the set of infinite admissible sequences and $\sigma : \Sigma \to \Sigma$ be the left shift map. Also we set Σ^* to be the set of all finite admissible words with letters in $\{0', 0'', 1\}$ and $\Sigma_{\mathcal{P}}$ to be the set of all non-constant σ-periodic elements of Σ. We define the cylinder set notation as $[a_1, ..., a_n] = \{(x_1, ...) \in \Sigma | \forall i \in \{1, ..., n\} : x_i = a_i\}$ for $(a_1, ..., a_n) \in \Sigma^*$. Finally we define the function $\pi : \Sigma \to [0, 1/(\beta-1)]$ by

$$\{\pi(a_1, ...)\} = \bigcap_{n\in\mathbb{N}} f_{a_1} \circ ... \circ f_{a_n}(\text{Domain}(f_{a_n})).$$

which is well defined since the maps f_{a_i} are contracting by β^{-1}. Note that if $(a_1, ..., a_n) \in \Sigma^*$ then

$$\pi([a_1, ..., a_n]) = f_{a_1} \circ ... \circ f_{a_n}(\text{Domain}(f_{a_n})).$$

Remark: If $(a_1, ..., a_n) \in \Sigma^*$ then

$$t \in \pi([a_1, ..., a_n]) \Leftrightarrow \forall i \in \{1, ..., n\} : T^{i-1}(t) \in f_{a_i}(\text{Domain}(f_{a_i})).$$

Note that

$$\mu_\beta(\pi([a_1, ..., a_n])) = d(a_1)\beta^{n-1}\lambda(f_{a_n}(\text{Domain}(f_{a_n})))$$

where

$$d(a) = \begin{cases} \frac{1+3\beta}{5\beta} & a \in \{0', 0''\} \\ \frac{2+\beta}{5\beta} & a = 1 \end{cases} .$$

or equivalently

$$\begin{aligned} \mu_\beta(\pi([a_1, ..., a_n])) &= \mu_\beta(\pi([a_1]))\frac{\mu_\beta(\pi([a_1, a_2]))}{\mu_\beta(\pi([a_1]))} \cdots \frac{\mu_\beta(\pi([a_1, ..., a_n]))}{\mu_\beta(\pi([a_1, ..., a_{n-1}]))} \\ &= \mu_\beta(\pi([a_1]))P(a_1, a_2) \cdot ... \cdot P(a_{n-1}, a_n) \end{aligned}$$

where

$$P = \begin{bmatrix} 1/\beta & 1/\beta^2 & 0 \\ 0 & 0 & 1 \\ 1/\beta & 1/\beta^2 & 0 \end{bmatrix} .$$

Definition 6.2.2. *If μ is a non-atomic measure on $[0, 1]$ we will denote be $\pi^{-1}\mu$ the measure on Σ defined by*

$$\pi^{-1}\mu(A) = \mu(\pi(A)).$$

The measure $\pi^{-1}\mu$ is well defined since π, restricted outside a countable set, is injective.

Remark: Subintervals of $[0,1]$ are continuity sets for any non-atomic measure. Hence, if μ is a non-atomic measure on $[0,1]$ and $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of measures on $[0,1]$ then

$$\mu_n \overset{\text{weak}^*}{\longrightarrow} \mu \Leftrightarrow \pi^{-1}\mu_n \overset{\text{weak}^*}{\longrightarrow} \pi^{-1}\mu.$$

Definition 6.2.3. *We will say that $a = (a_1, ..., a_n) \in \Sigma^*$ is regular if it is not of one of the following forms:*

$$(0', 0', ..., 0'),$$

$$(0', 0'', 0', 0'', ..., 0', 0''),$$

or

$$(0'', 0', 0'', 0', ..., 0'', 0').$$

Lemma 6.2.4. *Let $(a_1, ..., a_n) \in \Sigma^*$ be regular. Then*

$$\lim_{n\to\infty} \frac{||[1 \quad 0 \quad 0](A_{a_1} \cdot ... \cdot A_{a_n})^{n+1}||}{||[1 \quad 0 \quad 0](A_{a_1} \cdot ... \cdot A_{a_n})^{n}||} = \rho(A_{a_1} \cdot ... \cdot A_{a_n})$$

Proof. A set $S \subseteq \{0', 0'', 1\}$ will be called essential class of 3×3, indexed by $\{0', 0'', 1\}$, matrix if for each $i \in \{0', 0'', 1\}$ either

$$A_{i,j} \neq 0 \Leftrightarrow j \in S$$

or

$$A_{i,j} = 0 \quad \text{for all } j \in \{0', 0'', 1\}.$$

In that case we set

$$A^{\text{es}}_{i,j} = \begin{cases} A_{i,j}, & i, j \in S \\ 0, & \textit{otherwise} \end{cases}.$$

Notice that $A^n = A \cdot (A^{\text{es}})^{n-1}$. Also A^{es}, after permutation, consists of a strictly positive square block and the rest of the entries are zero. From these two observations we see that

$$\lim_{n\to\infty} \frac{||vA^{n+1}||}{||vA^n||} = \rho(A)$$

for any v that contains non-zero entries in S. So it is enough to prove that $A_{a_1} \cdot ... \cdot A_{a_n}$ has an essential class containing $0'$.

Let $a = (a_1, ..., a_n) \in \Sigma^*$ be regular. Since a is regular, one can see by exhaustion, that if κ is the least natural number for which $(a_1, ..., a_\kappa)$ is a non-regular element of Σ^*, then S is the essential class of $A_{a_1} \cdot ... \cdot A_{a_\kappa}$ for some $S \subseteq \{0', 0'', 1\}$ containing $0'$. Observe that for any two matrices A, B indexed by $\{0', 0'', 1\}$, if A has an essential class containing $0'$ and $B_{0',0'} = 1$ then AB also has an essential class containing $0'$. Thus $A_{a_1} \cdot ... \cdot A_{a_n}$ has an essential class that contains $0'$ as needed, which completes the proof.

□

Proposition 6.2.1. *Let* $a = (a_1, ..., a_n) \in \Sigma^*$ *be a regular and* $x = \pi(a_1, ..., a_n, a_1, ..., a_n, ...)$. *Then*

$$n\log(\rho(M(\beta,x))) = \log\left(\rho\left(A_{a_1}...A_{a_n}\right)\right).$$

Proof. Combining 6.2.1 and 6.2.3 we get

$$\begin{aligned} n\log(\rho(M(\beta,x))) &= \lim_{\kappa\to\infty}\frac{1}{\kappa}\log\left(\#\{(\varepsilon_{n\kappa},...,\varepsilon_1)\in\{0,1\}^n : T_{\varepsilon_{n\kappa}}\circ...\circ T_{\varepsilon_1}(x)\in(0,\beta)\}\right) \\ &= \lim_{\kappa\to\infty}\frac{1}{\kappa}\log\left(||[1 \quad 0 \quad 0]\left(A_{a_1}...A_{a_n}\right)^\kappa||\right). \end{aligned}$$

Now with a similar calculation as in the proof of 6.2.1 we get from 6.2.4

$$\lim_{\kappa\to\infty}\frac{1}{\kappa}\log\left(||[1 \quad 0 \quad 0]\left(A_{a_1}...A_{a_n}\right)^\kappa||\right) = \rho(A_{a_1}\cdot...\cdot A_{a_n}).$$

□

Proposition 6.2.1 suggests that in order to capture the typical behaviour of $\rho(M(\beta,x))$ for $x\in\Sigma_{\mathcal{P}}$ we can define LE, which appears in 6.1.1, to be the Lyapunov exponent of $\{A_{-1},A_0,A_1\}$ driven by $\pi^{-1}\mu_\beta$.

Definition 6.2.4.

$$LE = \lim_{n\to\infty}\frac{1}{n}\int_\Sigma \log\left(||A_{x_1}...A_{x_n}||\right)d\pi^{-1}\mu_\beta(x).$$

The limit exists by sub-additivety.

We also need to define the metrics below.

Definition 6.2.5. *Let I_n be the set of Σ^* elements of length n. We define a metric on probability measures on Σ by*

$$d_S(\mu,\nu) := \sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{i\in I_n}|\mu([i])-\nu([i])|.$$

If μ, ν are probability measures on [0,1] then we set

$$d(\mu, \nu) = d_S(\pi^{-1}\mu, \pi^{-1}\nu)$$

It is easy to observe that the topology given by d is the weak* topology.

Proposition 6.2.2. *Let $a = (a_1, ..., a_n, a_1, ..., a_n, ...) \in \Sigma_{\mathcal{P}}$ and denote by μ_a the normalised counting measure on the orbit of a. For each $\varepsilon > 0$ there is $\delta > 0$ such that if $d_S(\pi^{-1}\mu_\beta, \mu_a) < \delta$ then $|\rho(M(\beta, \pi(a))) - LE| < \varepsilon$.*

Proof. Without loss of generality, by making δ small enough, we can assume that $(a_1, ..., a_n)$ is regular. Define $f_l : \Sigma \to \mathbb{R}$ by

$$f_l(x) = \frac{1}{l} \log \left(||A_{x_1} ... A_{x_l}|| \right).$$

By Gelfand's formula and $\rho(AB) = \rho(BA)$, for any two $\kappa \times \kappa$ matrices A, B, we can choose l large enough such that

$$\left| f_l(\sigma^\kappa a) - \frac{\log(\rho(A_{a_1} ... A_{a_n}))}{n} \right| < \varepsilon/4 \quad \text{for all} \quad \kappa \in \mathbb{N} \tag{6.6}$$

and

$$\left| \int f_l d\pi^{-1}\mu_\beta - LE \right| < \varepsilon/4.$$

There is $m \in \mathbb{N}$ such that

$$\left| \frac{1}{m}(f_l(a) + ... + f_l(\sigma^m(a)) - \int f_l d\mu_a \right| < \varepsilon/4.$$

Since convergence in the metric d_S is equivalent to the weak* convergence there is a $\delta > 0$ such that

$$\left| \int f_l d\pi^{-1}\mu_\beta - \int f_l d\mu_a \right| < \varepsilon/4$$

if $d_S(\pi^{-1}\mu_\beta, \mu_a) < \delta$. Now from 6.6 we also get that

$$\left| \frac{1}{m}(f_l(a) + ... + f_l(\sigma^m(a)) - \frac{\log(\rho(A_{a_1}...A_{a_n}))}{n} \right| < \varepsilon/4.$$

Now the last four inequalities above give that

$$\left| \frac{\log(\rho(A_{a_1}...A_{a_n}))}{n} - LE \right| < \varepsilon$$

so from proposition 6.2.1 we have

$$|\log(\rho(M(\beta, \pi(a)))) - LE| < \varepsilon.$$

□

Proof of Theorem 6.1.1. It is an immediate consequence of proposition 6.2.2 since every periodic point x of T can be written as $x = \pi(a)$ where $a \in \Sigma_{\mathcal{P}}$. □

6.3 Equidistribution of $V(\phi, x)$

We keep the notations of the last section. For $v \in R^3$ denote by $\tau(v)$ the vector that rises by replacing each non zero entry of v by 1. We define

$$\mathcal{O} = \{(1,0,0), (1,0,1), (1,1,0)\}$$

also, without confusion, the symbols $0', 0''$ and 1 will stand for functions from $\mathcal{O}$ to itself as follows

$$i(v) = \tau(vA_i)$$

where $i \in \{0', 0'', 1\}$ and $v \in \mathcal{O}$. Now we are ready to define a dynamical system $(\mathcal{S}, F)$ by setting $\mathcal{S} = \mathcal{O} \times \Sigma$ and

$$F(v, (x_1, x_2 ...)) = (x_1(v), (x_2, ...))$$

where $(x_1, ...) \in \Sigma$ and $v \in \mathcal{O}$. We also set p_1, p_2 to be the first and second coordinate projections of $\mathcal{S}$ respectively. By the construction of $(\mathcal{S}, F)$ and lemma 6.2.3 we have the following lemma.

Lemma 6.3.1. *Let* $x = (x_1, ...) \in \Sigma_{\mathcal{P}}$ *then*

$$\{T_{\varepsilon_n} \circ ... \circ T_{\varepsilon_1}(\pi(x)) : (\varepsilon_n, ..., \varepsilon_1) \in \{0,1\}^n\} \cap (0, \beta)$$
$$= \{T^n \circ \pi(x), (T^n \circ \pi(x) + 1/\beta)p_1(F^n((1,0,0), x))(2), (T^n \circ \pi(x) + 1)p_1(F^n((1,0,0), x))(3)\}.$$

We also need to define the geometric analogue of $(\mathcal{S}, F)$. That is $(\mathcal{S}', F')$ where $\mathcal{S}' = \mathcal{O} \times [0,1]$ and

$$F'(v, x) = ((1, \chi_{Y(v,x)}(T(x) + 1/\beta), \chi_{Y(v,x)}(T(x) + 1)), T(x)).$$

where

$$Y(v, x) = \{T_i(y) : i \in \{0,1\} \quad \text{and} \quad y \in \{x, v(2)(x + 1/\beta), v(3)(x + 1)\}\} \cap (0, \beta).$$

We will denote the coordinate projections of $\mathcal{S}'$ with the same symbols p_1, p_2 as before. The sets of the form $\{v\} \times \pi([i])$ for $v \in \mathcal{O}$ and $i \in \{0', 0'', 1\}$ are a

Markov partition P of $(\mathcal{S}', F')$. Heuristically the respective transfer operator of the zero potential is expressed by the following matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the elements of P are lexicographically ordered according to $(1,0,0) < (1,0,1) < (1,1,0)$ and $0' < 0'' < 1$. Let

$$\{\{v\} \times \pi([i]) : v \in \mathcal{O}, i \in \{0', 0'', 1\}\} = \{S'_1, ..., S'_9\}$$

where $i \mapsto S'_i$ respects the order just mentioned.

The leading eigenvector of the normal form of the above matrix is

$$\overrightarrow{v} := \left(0, 0, 1 - \frac{2}{\sqrt{5}}, \frac{1}{10}(5 - \sqrt{5}), \frac{1}{10}(-5 + 3\sqrt{5}), 0, \frac{1}{10}(-5 + 3\sqrt{5}), 1 - \frac{2}{\sqrt{5}}, \frac{1}{10}(-5 + 3\sqrt{5})\right)$$

describing a piecewise uniform measure ν on $\mathcal{S}'$ satisfying

$$\bar{\nu}(S'_i) = \overrightarrow{v}(i).$$

One can directly verify that this is indeed an invariant measure of F'. The uniqueness follows from the fact that F' is ergodic in respect to $\bar{\nu}$ and that for each $y \in S'$ outside the support of $\bar{\nu}$ there is $n_y \in \mathbb{N}$ such that F'^{n_y} is in the support of $\bar{\nu}$.

Definition 6.3.1. *Let $a \in \Sigma_{\mathcal{P}}$. The orbit of $((1,0,0),a)$ under F is finite as a subset of $\mathcal{O} \times \{\sigma^n(a) : n \in \mathbb{N}\}$ so it is pre-periodic. We will denote by $M_s(a)$ the periodic part of that orbit. Similarly $M_g(a)$ will be the periodic part of the F'-orbit of $((1,0,0),\pi(a))$.*

Lemma 6.3.2. *Let $\{a_n\}_{n\mathbb{N}}$ be a sequence of elements of $\Sigma_{\mathcal{P}}$. Let μ_{a_n} be the normalised counting measure on the T-orbit of $\pi(a_n)$ and ν_{a_n} be the normalised counting measure on $M_g(a_n)$. If $\mu_a \stackrel{weak^*}{\longrightarrow} \mu_\beta$ then $\nu_a \stackrel{weak^*}{\longrightarrow} \bar{\nu}$.*

Proof. Let M be the set of invariant probability measures of (S', F'). The weak* topology makes M a metrizable compact space. Assume, aiming at a contradiction, that there is a weak*-open subset B of M containing $\bar{\nu}$ such that $\{n \in \mathbb{N} : \nu_{a_n} \notin B\}$ is infinite. Since M is a weak*-compact we can assume, taking a sub-sequence if necessary, that $\nu_a \stackrel{\text{weak}^*}{\longrightarrow} \nu'$ where $\nu' \neq \bar{\nu}$. It is clear that $p_2(\nu_{a_n}) = \mu_{a_n}$ so $p_2(\nu') = \lim_n p_2(\nu_{a_n}) = \lim_n \mu_{a_n} = \mu_\beta$, where the limits are weak*. From $p_2(\nu') = \mu_\beta$ and that μ_β is absolutely continuous we get that so is ν' which contradicts the uniqueness of the absolutely continuous invariant probability measure $\bar{\nu}$. □

Lemma 6.3.3. *Let $a = (a_1, ...) \in \Sigma_{\mathcal{P}}$ with period l. Then*

$$V(\beta, \pi(a)) = \bigcup \{\{\pi(x), (\pi(x)+1/\beta)v(2), (\pi(x)+1)v(3)\} : (v,x) \in M_s(a)\}.$$

In addition the size of $M_s(a)$ is l.

Proof. By lemma 6.3.1 we have that

$$V(\beta, \pi(a)) \supseteq \bigcup \{\{\pi(x), (\pi(x)+1/\beta)v(2), (\pi(x)+1)v(3)\} : (v,x) \in M_s(a)\}.$$

Observe that $\mathcal{O}\backslash((\{(1,0,0)\}\times([0']\cup[0'']))\cup(\{(1,0,1)\}\times[1]))$ is invariant under F. Let ξ be least natural number such that $F^{\xi}(((1,0,0),a)) \notin \{(1,0,0)\}\times([0']\cup[0''])$ Of course there is no repetition in $(((1,0,0),a),...,F^{l+\xi-1}(((1,0,0),a)))$. On the other hand $F^{l+\xi}(((1,0,0),a)), F^{\xi}(((1,0,0),a)) \in \mathcal{O}\times[1]$ hence

$$F(F^{l+\xi}(((1,0,0),a))) = F(F^{\xi}(((1,0,0),a))) = ((1,0,1),\sigma^{\xi+1}(a))$$

which implies

$$F^{l}(((1,0,1),\sigma^{\xi+1}(a))) = ((1,0,1),\sigma^{\xi+1}(a)).$$

By the above the periodic part of the orbit of $((1,0,0),a)$ is either $\{F^n(((1,0,0),a)) : n\in\mathbb{N}\}\backslash\{F^n(((1,0,0),a)) : n\in\mathbb{N} \quad \text{and} \quad n<\xi\}$ or $\{F^n(((1,0,0),a)) : n\in\mathbb{N}\}\backslash\{F^n(((1,0,0),a)) : n\in\mathbb{N} \quad \text{and} \quad n\leqslant\xi\}$ and the size of the periodic part is l.

Now assume that $m\in V(\beta,\pi(a))$. By lemma 6.3.1 we know that there exists (u,x) in the orbit of $((1,0,0),a)$ such that

$$m\in\{\pi(x),(\pi(x)+1/\beta)v(2),(\pi(x)+1)v(3)\}.$$

By the discussion above if $v\in\{(1,1,0),(1,0,1)\}$ then $(v,x)\in M_s(a)$. If $v=(1,0,0)$ then $m=\pi(x)$ and since $p_2(M_s(a))$ is equal to the σ-orbit a which contains x there is $(u,x)\in M_s(a)$ such that

$$m\in\{\pi(x),(\pi(x)+1/\beta)u(2),(\pi(x)+1)u(3)\}$$

which completes the proof.

□

Lemma 6.3.4. *Let $a = (a_1, ...) \in \Sigma_{\mathcal{P}}$ with period l. If $(v, x) \neq (u, y)$ are in $M_g(a)$ then*

$$\{x, (x + 1/\beta)v(2), (x + 1)v(3)\} \bigcap \{y, (y + 1/\beta)u(2), (y + 1)u(3)\} = \emptyset.$$

In order to prove the lemma above we first need to extent the symbolic dynamics of T to $[0, \phi]$. We give the definitions again including an extra symbol. The new definitions are compatible with the existing ones.

Now let $f_s : [1/\beta, \beta] \to [0, 1]$ be defined as $f_1(t) = \beta^{-1}(t + 1)$. Also f_0', f_0'' and f_1 are defined as before. A sequence $\{a_i\}_{i\in\mathbb{N}} \in \{0', 0'', 1, s\}^{\mathbb{N}}$ is called admissible if or all $i \in \mathbb{N}$

$$A^s_{a_i,a_{i+1}} = 1$$

where A^s is the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

indexed by $(0', 0'', 1, s)$ and similarly, for any $n \in \mathbb{N}$, $(a_1, ..., a_n) \in \{0', 0'', 1, s\}^n$ is called admissible if $A^s_{a_i,a_{i+1}} = 1$ for all $i \in \{1, ..., n - 1\}$. Let Σ_c be the set of infinite admissible sequences and $\sigma : \Sigma_c \to \Sigma_c$ be the left shift map, without confusion. Also we set Σ_c^* to be the set of all finite admissible words with letters in $\{0', 0'', 1, s\}$. We define the cylinder set notation as $[a_1, ..., a_n] = \{(x_1, ...) \in \Sigma | \forall i \in \{1, ..., n\} : x_i = a_i\}$ for $(a_1, ..., a_n) \in \Sigma_c^*$. Finally we define the function $\pi : \Sigma \to [0, 1/(\beta - 1)]$ by

$$\pi(a_1, ...) = \bigcap_{n\in\mathbb{N}} f_{a_1} \circ ... \circ f_{a_n}(\text{Domain}(f_{a_n})).$$

Note that if $(a_1, ..., a_n) \in \Sigma_c^*$ then

$$\pi([a_1, ..., a_n]) = f_{a_1} \circ ... \circ f_{a_n}(\text{Domain}(f_{a_n})).$$

Proof of lemma 6.3.4. Set $x = \pi(a)$. By lemma 6.3.3 it is enough to prove that there if m, n are different $\mod l$ then

$$\{T^n(x), T^n(x) + 1/\beta, T^n(x) + 1\} \bigcap \{T^m(x), T^m(x) + 1/\beta, T^m(x) + 1\} = \emptyset.$$

Aiming towards a contradiction assume there is counterexample pair of m and n for the statement above. Since $T^n(x) \neq T^m(x)$ and by symmetry the only cases that we need to consider are $T^n(x) + 1/\beta = T^m(x)$, $T^n(x) + 1/\beta = T^m(x) + 1$ and $T^n(x) + 1 = T^m(x)$. The last one gives a contradiction since $T^m(x) < 1$ and $T^n(x) + 1 > 1$.

We focus on the case where $T^n(x)+1/\beta = T^m(x)+1$, the case of $T^n(x)+1/\beta = T^m(x)$ can be treated similarly and is quite simpler. We have that $T^n(x) > 1-1/\beta$ and that $T^m(x) < \beta - 1 = 1/\beta$ which means that the first symbol of $\sigma^n(a)$ belongs to $\{0'', 1\}$ while the fist symbol of $\sigma^m(a)$ belongs to $\{0', 0''\}$. The following rules for adding $1/\beta$ can be verified by trivial calculations

$$\pi((0'', b_2, b_3, ...)) + 1/\beta = \pi((s, b_2, b_3, ...)),$$

$$\pi((1, 0'', 1, 0'', 1, ..., 0'', 1, 0', 0'', b_\kappa, b_{\kappa+1}, ...)) + 1/\beta = \pi((s, s, ..., b_\kappa, b_{\kappa+1}, ...)),$$

where the number of 's' appearances on the right hand side is $2r+3$ where r is the number of $0'', 1$ successive repetitions after the first symbol in the left hand side and

$$\pi((1,0'',1,0'',1,...,0'',1,0',0',b_\kappa,b_{\kappa+1},...))+1/\beta=\pi((s,s,...,1,b_\kappa,b_{\kappa+1},...)),$$

where the number of 's' appearances on the right hand side is $2r+2$ where r is the number of $0'', 1$ successive repetitions after the first symbol in the left hand side. For adding 1 we have

$$\pi((0',0',b_3,b_4,...))+1=\pi((s,1,b_3,b_3,...)),$$

$$\pi((0',0'',b_3,b_4,...))+1=\pi((s,s,b_3,b_3,...)),$$

$$\pi((0'',1,0'',1,0'',...1,0'',0'',b_\kappa,b_{\kappa+1},...))+1/\beta=\pi((s,s,...,b_\kappa,b_{\kappa+1},...)),$$

where the number of 's' appearances on the right hand side is $2r+2$ where r is the number of $1, 0''$ successive repetitions after the first symbol in the left hand side,

$$\pi((0'',1,0'',1,0'',...1,0'',0',b_\kappa,b_{\kappa+1},...))+1/\beta=\pi((s,s,...,1,b_\kappa,b_{\kappa+1},...)),$$

where the number of 's' appearances on the right hand side is $2r+1$ where r is the number of $1, 0''$ successive repetitions after the first symbol in the left hand side. Applying the rules above to a we can conclude that there are finally periodic elements $b, c \in \Sigma_s$ such that $\pi(b) = T^n(x) + 1/\beta$ and $\pi(c) = T^m(x) + 1$ with different values for arbitrary large natural number. From that it is implied that $T^n(x) + 1/\beta \neq T^m(x) + 1$ giving the required contradiction.

$\square$

Proof of theorem 6.1.2. By lemmata 6.3.3 and 6.3.4 we have that for any subinterval J of $[0, \phi]$,

$$
\begin{aligned}
\#V(\beta, x) \cap J = & \sum_{(v,x) \in M_g(a)} \# \{x, (x + 1/\beta)v(2), (x + 1)v(3)\} \cap J \\
& = \sum_{((1,0,0),x) \in M_g(a)} \# \{x, (x + 1/\beta)v(2), (x + 1)v(3)\} \cap J \\
& + \sum_{((1,0,1),x) \in M_g(a)} \# \{x, (x + 1/\beta)v(2), (x + 1)v(3)\} \cap J \\
& + \sum_{((1,1,0),x) \in M_g(a)} \# \{x, (x + 1/\beta)v(2), (x + 1)v(3)\} \cap J \\
& = \#(\{(1, 0, 0)\} \times J) \cap M_g(a) + \#(\{(1, 0, 1)\} \times J) \cap M_g(a) \\
& + \#(\{(1, 0, 1)\} \times (J - 1)) \cap M_g(a) + \#(\{(1, 1, 0)\} \times J) \cap M_g(a) \\
& + \#(\{(1, 1, 0)\} \times (J - 1/\beta)) \cap M_g(a) \\
& = \#M_g(a)[\nu_a(\{(1, 0, 0)\} \times J) + \nu_a(\{(1, 0, 1)\} \times J) \\
& + \nu_a(\{(1, 0, 1)\} \times (J - 1)) + \nu_a(\{(1, 1, 0)\} \times J) \\
& + \nu_a(\{(1, 1, 0)\} \times (J - 1/\beta))].
\end{aligned}
$$

For convenience set for any subinterval J of $[0, \phi]$,

$$Z_{J,1} = \{(1,0,0)\} \times J$$

$$Z_{J,2} = \{(1,0,1)\} \times J$$

$$Z_{J,3} = \{(1,0,1)\} \times (J-1)$$

$$Z_{J,4} = \{(1,1,0)\} \times J$$

$$Z_{J,5} = \{(1,1,0)\} \times (J-1/\beta)$$

so the equation above can be written as

$$\#V(\beta,x) \cap J = M_g(a) \sum_{i=1}^{5} \nu_a(Z_{J,i}).$$

From this, by making δ small enough, lemma 6.3.2 gives

$$\left| \frac{\#V(\beta,x) \cap I}{\#V(\beta,x)} - \frac{M_g(a) \sum_{i=1}^{5} \bar{\nu}(Z_{I,i})}{M_g(a) \sum_{i=1}^{5} \bar{\nu}(Z_{[0,\phi],i})} \right| < \varepsilon. \tag{6.7}$$

Since we have earlier computed $\bar{\nu}$, a straightforward calculation gives us that there exists $c > 0$ such that

$$M_g(a) \sum_{i=1}^{5} \bar{\nu}(Z_{J,i}) = c\lambda(J),$$

for any subinterval J of $[0, \phi]$. Hence equation 6.7 gives

$$\left| \frac{\#V(\beta,x) \cap I}{\#V(\beta,x)} - \lambda(I) \right| < \varepsilon.$$

□

6.4 The Lyapunov exponent LE and local dimension

This aim of this section is to prove theorem 6.4.1 below.

Lemma 6.4.1. *For every positive integer* n *we have*

$$\bigcup\{\{\sup\pi([a_1,...,a_n]),\inf\pi([a_1,...,a_n])\}:(a_1,...,a_n)\in\{0',0'',1\}^n\}$$
$$\supseteq\left\{T_{\varepsilon_n}^{-1}\circ...\circ T_{\varepsilon_1}^{-1}(x):(\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n,\ x\in\{0,\phi\}\right\}\cap[0,1].$$

Proof. Let n be a positive integer. Assume, aiming at a contradiction, that there is

$$x\in\left\{T_{\varepsilon_n}^{-1}\circ...\circ T_{\varepsilon_1}^{-1}(x):(\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n,\ x\in\{0,\phi\}\right\}\cap[0,1]$$

and $(a_1,...,a_n)\in\Sigma^*$ such that x belongs to the interior of $\pi([a_1,...,a_n])$. Let $\varepsilon_n,...,\varepsilon_1\in\{0,1\}$ be such that $x=T_{\xi_n}^{-1}\circ...\circ T_{\xi_1}^{-1}(\phi)$. The case of $x=T_{\xi_n}^{-1}\circ...\circ T_{\xi_1}^{-1}(0)$ is similar. Choose some $x'\in(\inf\pi([a_1,...,a_n]),x)$. Notice that

$$T^n(x)-T^n(x')=f_{a_n}^{-1}\circ...\circ f_{a_1}^{-1}(x)-f_{a_n}^{-1}\circ...\circ f_{a_1}^{-1}(x')=(x-x')\phi^n.$$

Let $b_1,...,b_n\in\{0,1\}$ such that $T_{b_n}\circ...\circ T_{b_1}(x)\in(0,\phi]$. Then

$$\phi\geqslant T_{b_n}\circ...\circ T_{b_1}(x)>T_{b_n}\circ...\circ T_{b_1}(x')=T_{b_n}\circ...\circ T_{b_1}(x)-(x'-x)\phi^n$$
$$\geqslant T^n(x)-(x'-x)\phi^n=T^n(x)-T^n(x)+T^n(x')>0$$

hence

$$T_{b_n}\circ...\circ T_{b_1}(x)-(x'-x)\phi^n\in\{T_{\varepsilon_n}\circ...\circ T_{\varepsilon_1}(x'):(\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n\}\cap(0,\phi).$$

The above implies

$$\{T_{\varepsilon_n}\circ ...\circ T_{\varepsilon_1}(x'):(\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n\}\cap(0,\phi)$$
$$\supseteq\{T_{\varepsilon_n}\circ ...\circ T_{\varepsilon_1}(x):(\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n\}\cap(0,\phi]-(x'-x)\phi^n$$

which given $x=T_{\xi_n}^{-1}\circ ...\circ T_{\xi_1}^{-1}(\phi)$ it gives us

$$\#\{T_{\varepsilon_n}\circ ...\circ T_{\varepsilon_1}(x'):(\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n\}\cap(0,\phi)$$
$$>\#\{T_{\varepsilon_n}\circ ...\circ T_{\varepsilon_1}(x):(\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n\}\cap(0,\phi)$$

which contradicts

$$\#\{T_{\varepsilon_n}\circ ...\circ T_{\varepsilon_1}(x'):(\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n\}\cap(0,\phi)$$
$$=\#\Big\{i\in\{1,2,3\}:[1\quad 0\quad 0]A_{a_1}...A_{a_n}(i)\neq 0\Big\}$$
$$=\#\{T_{\varepsilon_n}\circ ...\circ T_{\varepsilon_1}(x):(\varepsilon_n,...,\varepsilon_1)\in\{0,1\}^n\}\cap(0,\phi).$$

□

Definition 6.4.1. *For every positive integer n we define*

$$\mathcal{F}_n:=\{\pi([a_1,...,a_n]):(a_1,...,a_n)\in\Sigma^*\}$$

and for $x\in(0,1)$ we set

$$P_n(x)=\#\left\{(\varepsilon_1,...,\varepsilon_n)\in\{0,1\}^n:x\in T_{\varepsilon_1}^{-1}\circ ...\circ T_{\varepsilon_n}^{-1}((0,\phi))\right\}$$
$$=\#\left\{(\varepsilon_1,...,\varepsilon_n)\in\{0,1\}^n:T_{\varepsilon_n}\circ ...\circ T_{\varepsilon_1}(x)\in(0,\phi)\right\}.$$

Also we will denote by Σ_I the set of all elements in Σ that are not terminally constant.

Notice that, from lemma 6.2.3, if $x \in \pi([a_1, ..., a_n])^\circ$, for $(a_1, ..., a_n) \in \Sigma^*$, then

$$P_n(x) = \left|\left|[1 \quad 0 \quad 0]A_{a_1}...A_{a_n}\right|\right|.$$

Lemma 6.4.2. *Let $\Delta \in \mathcal{F}_n$ and $x \in \Delta^\circ$, then*

$$P_n(x) = \#\{(\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n : T_{\varepsilon_1}^{-1} \circ ... \circ T_{\varepsilon_n}^{-1}([0, \phi]) \supseteq \Delta\}.$$

Proof. Suppose that $\Delta = \pi([a_1, ..., a_n])$ for $(a_1, ..., a_n) \in \Sigma^*$. Lemma 6.4.1 and $x \in \pi([a_1, ..., a_n])^\circ$ gives us

$$\begin{aligned} x \in T_{\varepsilon_n}^{-1} \circ ... \circ T_{\varepsilon_1}^{-1}((0, \phi)) &\Leftrightarrow \pi([a_1, ..., a_n])^\circ \subset T_{\varepsilon_n}^{-1} \circ ... \circ T_{\varepsilon_1}^{-1}((0, \phi)) \\ &\Leftrightarrow \pi([a_1, ..., a_n]) \subset T_{\varepsilon_n}^{-1} \circ ... \circ T_{\varepsilon_1}^{-1}([0, \phi]). \end{aligned}$$

Definition 6.4.2. *For $\Delta \in \mathcal{F}_n$ we set $P_n(\Delta) = P_n(x)$ where x is any element of Δ°.*

□

Lemma 6.4.1 tells us that the our sets $\mathcal{F}_n$ are a finer version of the net intervals $\mathcal{F}_n$, restricted to $[0, 1] \subseteq [0, \phi]$, of the sets $\mathcal{F}_n$ defined in the section 2.3 of [40] and for the case of the Bernoulli convolution ν_ϕ. Also lemma 6.4.2 shows that P_n is the equivalent, for our partitions, of the quantities P_n defined in notation 3.3 of the same paper. We should mention that the Bernoulli convolution ν_ϕ is of finite type as it mentioned on page 2 of the same paper. That gives us, by the same arguments they used, an version of their corollary 3.7. For completeness we include their arguments bellow.

Lemma 6.4.3. *There is $C > 1$ such that for all positive integers n, $\Delta \in \mathcal{F}_n$ and $x \in \Delta^\circ$ we have*

$$C^{-1}\nu_\beta(\Delta) < 2^{-n}P_n(x) < C\nu_\beta(\Delta).$$

Proof. Let $\Delta = \pi([a_1, ..., a_n])$ for $(a_1, ..., a_n) \in \Sigma^*$. Recall that the Bernoulli convolution ν_β satisfies

$$\nu_\beta = \frac{1}{2}T_0^{-1}(\nu_\beta) + \frac{1}{2}T_1^{-1}(\nu_\beta),$$

which also implies

$$\nu_\beta = \frac{1}{2^{-n}} \sum_{x_1,...,x_n \in \{0,1\}} T_{x_1}^{-1} \circ ... \circ T_{x_n}^{-1}(\nu_\beta). \tag{6.8}$$

Now suppose that

$$T_{\varepsilon_1}^{-1} \circ ... \circ T_{\varepsilon_n}^{-1}([0, \phi]) \supseteq \Delta.$$

Then $f_{a_1}^{-1} \circ ... \circ f_{a_n}^{-1}(\Delta) \in \{[0, 1], [0, 1/\beta\}$ so from lemma 6.2.2

$$T_{\varepsilon_n} \circ ... \circ T_{\varepsilon_1}(\Delta) \in \{[0, 1/\beta], [1/\beta, 2/\beta], [1, \phi], [0, 1], [1/\beta, \phi]\}$$

so there is a finite set of possible values for

$$\nu_\beta(T_{\varepsilon_n} \circ ... \circ T_{\varepsilon_1}(\Delta)) = T_{\varepsilon_1}^{-1} \circ ... \circ T_{\varepsilon_n}^{-1}(\nu_\beta)(\Delta).$$

We conclude that there is $C > 1$ such that for all positive integers n, $\varepsilon_1, ..., \varepsilon_n \in \{0, 1\}$ and $\Delta \in \mathcal{F}_n$,

$$C^{-1} < T_{\varepsilon_1}^{-1} \circ ... \circ T_{\varepsilon_n}^{-1}(\nu_\beta)(\pi([a_1, ..., a_n])) < C,$$

which combined with lemma 6.4.2 and equation 6.8 completes the proof. □

Lemma 6.4.4. *There is $C_1 > 1$ such that all positive integers n and adjacent intervals $\Delta_1, \Delta_2 \in \mathcal{F}_n$ we have*

$$C_1^{-1}\frac{1}{n}P_n(\Delta_2)P_n(\Delta_1) \leqslant C_1 n P_n(\Delta_2).$$

Proof. We will do induction on n. The base case is trivial so we assume that the result holds for $n-1$ and prove the inequality for n. Notice that if $\Delta \in \mathcal{F}_n$, $\Delta' \in \mathcal{F}_{n-1}$ and $\Delta \subseteq \Delta'$ then if $T_{\varepsilon_1}^{-1} \circ ... \circ T_{\varepsilon_{n-1}}^{-1}([0,\phi]) \supseteq \Delta'$ there is $\varepsilon_n \in \{0,1\}$ such that $T_{\varepsilon_1}^{-1} \circ ... \circ T_{\varepsilon_n}^{-1}([0,\phi]) \supseteq \Delta$. This observation and lemma 6.4.2 implies that $P_n(\Delta) \geqslant P_{n-1}(\Delta')$. Now if there is $\widehat{\Delta} \in \mathcal{F}_{n-1}$ containing Δ_1, Δ_2 then, for $i \in \{1,2\}$,

$$P_{n-1}(\widehat{\Delta}) \leqslant P_n(\Delta_i) \leqslant 2P_{n-1}(\widehat{\Delta})$$

so the result holds. Next we assume that there are adjacent $\widehat{\Delta_1}, \widehat{\Delta_2} \in \mathcal{F}_{n-1}$ such that Δ_1, Δ_2 are contained in $\widehat{\Delta_1}, \widehat{\Delta_1}$ respectively. Define

$$D_1 = \{(\varepsilon_1, ..., \varepsilon_{n-1}) \in \{0,1\}^{n-1} : \sup T_{\varepsilon_1}^{-1} \circ ... \circ T_{\varepsilon_{n-1}}^{-1}([0,\phi]) = \sup \widehat{\Delta_1}\}$$
$$D_2 = \{(\varepsilon_1, ..., \varepsilon_{n-1}) \in \{0,1\}^{n-1} : \inf T_{\varepsilon_1}^{-1} \circ ... \circ T_{\varepsilon_{n-1}}^{-1}([0,\phi]) = \inf \widehat{\Delta_2}\}$$
$$E = \{(\varepsilon_1, ..., \varepsilon_{n-1}) \in \{0,1\}^{n-1} : T_{\varepsilon_1}^{-1} \circ ... \circ T_{\varepsilon_{n-1}}^{-1}([0,\phi]) \supseteq \widehat{\Delta_1} \cup \widehat{\Delta_2}\}$$

and observe that

$$P_n(\Delta_1) \leqslant \#D_1 + 2\#E \leqslant P_{n-1}(\widehat{\Delta_1}) + 2P_{n-1}(\widehat{\Delta_2})$$

so, choosing $C_1 \geqslant 2$ and using the inductive hypothesis we get

$$\begin{aligned} P_n(\Delta_1) &\leqslant P_{n-1}(\widehat{\Delta_1}) + 2P_{n-1}(\widehat{\Delta_2}) \\ &\leqslant C_1(n-1)P_{n-1}(\widehat{\Delta_2}) + 2P_{n-1}(\widehat{\Delta_2}) \\ &= C_1\left((n-1)P_{n-1}(\widehat{\Delta_2}) + \frac{2}{C_1}P_{n-1}(\widehat{\Delta_2})\right) \\ &\leqslant C_1\left((n-1)P_{n-1}(\widehat{\Delta_2}) + P_{n-1}(\widehat{\Delta_2})\right) \\ &= C_1 n P_{n-1}(\widehat{\Delta_2}) \\ &\leqslant C_1 n P_n(\Delta_2). \end{aligned}$$

the other inequality is similar.

□

Lemma 6.4.5. *Let $(a_i)_{i\in\mathbb{N}} \in \Sigma_T$ and $x = \pi(a_1, ...)$ then*

$$\dim_{\text{loc}}(\nu_\phi, x) = \lim_{n\to\infty} \frac{\log(2^{-n}P_n(x))}{\log(\phi^{-n})} = \lim_{n\to\infty} \frac{\log(2^{-n}\left|\left|[1 \quad 0 \quad 0]A_{a_1}...A_{a_n}\right|\right|)}{\log(\phi^{-n})}$$

if the limit exists.

Proof. Note that since $(a_i)_{i\in\mathbb{N}} \in \Sigma_T$ we have $x \in \pi([a_1, ..., a_n])^\circ$ for every positive integer n. Since the length of $\pi([a_1, ..., a_n])$ is at most β^{-n} we have $\pi([a_1, ..., a_n]) \subset [x - \beta^{-n}, x + \beta^{-n}]$ so, from lemma 6.4.3,

$$\begin{aligned} \frac{\log(\nu_\beta([x-\beta^{-n}, x+\beta^{-n}]))}{\log(\beta^{-n})} &\leqslant \frac{\log(\nu_\beta(\pi([a_1, ..., a_n])))}{\log(\beta^{-n})} \\ &\leqslant \frac{\log(C^{-1}2^{-n}P_n(x))}{\log(\beta^{-n})}. \end{aligned}$$

Now for the lower bound we observe that there is a natural number M, which does not depend on n, such that $[x-\beta^{-n}, x+\beta^{-n}]$ can be covered by at most M adjacent elements of $\mathcal{F}_n$ one of which is $\pi([a_1, ..., a_n])$. So by lemmata 6.4.3 and 6.4.4 we have that

$$\frac{\log(\nu_\beta([x-\beta^{-n}, x+\beta^{-n}]))}{\log(\beta^{-n})} \geqslant \frac{\log(MC_1^M n^M C 2^{-n} P_n(x))}{\log(\beta^{-n})}.$$

Combining the two inequalities gives us the result. □

The following theorem shows a connection between the number LE we defined in order to understand how the spectral radius of the matrices $M(\beta, x)$ behaves and the local dimension of the Bernoulli convolution ν_β. See also proposition 1.4 and table 1 in [30] and [60] where similar techniques were used.

Theorem 6.4.1. *For Lebesgue a.e. $x \in (0,1)$*

$$\dim_{\text{loc}}(\nu_\phi, x) = \frac{LE - \log(2)}{\log(\phi)}.$$

Proof. From Kingman's ergodic theorem and dominated convergence for $\pi^{-1}(\mu_\beta)$-a.e. $a \in \Sigma$ we have that

$$\lim_{n\to\infty} \frac{1}{n} \log(||A_{a_1}...A_{a_n}||) = LE. \tag{6.9}$$

But $\pi^{-1}(\mu_\beta)(\Sigma_I) = 1$ since $\pi^{-1}(\mu_\beta)$ is non-atomic. So from lemma 6.4.5, for $\pi^{-1}(\mu_\beta)$-a.e. $a \in \Sigma$, the equation above is equivalent to

$$\dim_{\text{loc}}(\nu_\phi, \pi(a)) = \lim_{n\to\infty} \frac{\frac{1}{n}\log(2^{-n}||A_{a_1}...A_{a_n}||)}{\frac{1}{n}\log(\phi^{-n})} = \frac{LE - \log(2)}{\log(\phi)}$$

giving us that for μ_β-a.e. x we have $\dim_{\text{loc}}(\nu_\phi, x) = (LE - \log(2))/\log(\phi)$. The result follows since μ_β is equivalent to the Lebesgue measure restricted on [0,1]. □

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