TWO NEW EXTENSIONS TO $\mathcal{L}_1$ ADAPTIVE CONTROL THEORY

BY

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THESIS

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ABSTRACT

This thesis introduces two new extensions to $\mathcal{L}_1$ adaptive control theory. The first is an $\mathcal{L}_1$ adaptive state feedback controller with generalized proportional adaptation law for a class of linear systems with input–gain uncertainties and unmatched nonlinear disturbances. The proportional adaptation law provides an adaptive estimate that is directly proportional to the error between the output of the system and the state predictor. One advantage of the new adaptive law is the additional phase margin in the estimation loop, allowing for accommodation of first order sensor dynamics in the state predictor. An additional benefit is the reduction of the required computational resources, since the error bounds reduce at a rate directly proportional to the adaptation gain as compared to the square root of the adaptation gain achieved by the $\mathcal{L}_1$ adaptive controllers using gradient descent adaptation laws. In addition, an $\mathcal{L}_1$ adaptive funnel controller and variable dependent adaptation law are provided as particular cases for the generalized proportional framework. Also presented is the connection between the generalized proportional feedback law and previous $\mathcal{L}_1$ switching controller. The second extension is an $\mathcal{L}_1$ adaptive controller for a class of uncertain systems in the presence of time and output dependent unknown nonlinearities and uncertain input matrix with performance specifications defined via a time–varying reference system using output feedback. It is shown that both extensions exhibit the standard characteristics of the $\mathcal{L}_1$ adaptive control theory: scaling of transient responses, a guaranteed time–delay margin at high adaptation rates, and the trade off between robustness and performance is determined by the design of a low pass filter.

To my parents, for their endless love and continual support.

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CHAPTER 1

INTRODUCTION

Control systems are often developed with the use of a system model; however, model errors are inevitable due to several factors such as imprecise parameter values, linearization, unmodeled disturbances, and more. From this fundamental problem, the idea of adaptive controllers arose. Adaptive controls is the concept of estimating the system's uncertainty through monitoring the system's behavior, then using the acquired information to produce a control input that can better achieve the desired performance. Examples of such adaptive controllers are presented in [1] and [2]. In order to adapt quickly to parameters and disturbances, fast estimation is required. Fast estimation, on the other hand, can lead to high frequencies in the control channel, ultimately reducing the robustness of the system. Investigation into robustness and proposed modifications to prevent instability are given by Ioannou and Kokotović [3–5], Peterson and Narendra [6], Kresselmeier and Narendra [7], and Narendra and Annaswamy [8]. However, these modifications lack analytical quantification of how robustness margins, adaptation rate, and transient response are related. The result was a need within the adaptive controls community to find a new adaptive architecture with quantifiable decoupling of adaptation from robustness.

The $\mathcal{L}_1$ adaptive controller, first developed by Cao and Hovakimyan [9], describes such an architecture. The core concept is to have the adaptive controller attempt to control the plant only within the bandwidth of the control channel. By doing so, the system can achieve fast adaptation without allowing high frequencies to enter the control channel, resulting in a more robust system. In [9], multiple $\mathcal{L}_1$ adaptive controllers are presented for various classes of uncertain systems. For all $\mathcal{L}_1$ adaptive architectures, the transient performance of the closed–loop adaptive system is quantified both for the system input and output by uniform performance bounds with respect to an $\mathcal{L}_1$ reference system, which incorporates a lowpass filter. The performance bounds can be arbitrarily improved by increasing the adaptation gain. The analytical lower bound on the stability margins of the $\mathcal{L}_1$ adaptive controller is derived in [10], where it is proven that in the presence of arbitrarily large adaptation gains, the $\mathcal{L}_1$ adaptive controller preserves robustness and has guaranteed, bounded away from zero stability margins. For applications of the $\mathcal{L}_1$ adaptive control theory we refer the reader to [11] and references therein. This thesis presents two new extensions to the $\mathcal{L}_1$ adaptive control theory.

Chapter 2 presents the first extension which generalizes the adaptive estimation laws to a proportional type with time–varying gain. In [9], two different estimation laws have been considered in Chapter 3 (Sections 3.2 and 3.3), leading to performance bounds of identical structure. In [12], switching estimation laws are considered, leading to *identical* bounds. The proportional estimation law presented in this thesis again leads to bounds of the same structure, and it is shown that the switching law falls within the generalized proportional framework. As compared to the gradient minimization type adaptation laws for which the performance bounds were inverse proportional to the square root of the adaptation gain, the performance bounds derived here are inverse proportional to the adaptation gain itself and therefore require less computational effort to achieve similar bounds. Three examples of specific adaptation laws that fall under the generalized time–varying proportional adaptive framework are also presented. These examples include a switching adaptation law [12, 13], funnel–control–law [14–16], and a variable dependent adaptation law.

In section 2.4, an analysis of the performance and stability margins is conducted. The proposed $\mathcal{L}_1$ adaptive control architecture results in a linear closed–loop adaptive system which allows for the use of standard frequency domain analysis tools. The results of the proportional controller are compared to the performance and robustness results obtained by $\mathcal{L}_1$ adaptive controllers with gradient minimization type adaptive laws [10].

In section 2.5.3, the variable dependent control law is proposed as method to adjust the proportional adaptation gain between a minimum and maximum value based on the value of a provided variable. We present two examples that benefit from this type of variable based adaptive gain structure. First, we investigate the peaking phenomenon [17–19], which occurs in adaptive control systems due to the initialization errors. The variable dependent control law uses the value of the error to adjust the gain in effort to reduce peaking. We compare this method with other methods used to reduce peaking. Secondly, as mentioned earlier, the $\mathcal{L}_1$ adaptation schemes involve a fast estimation loop that demands a high CPU rate. The proposed variable dependent adaptation gain allows for the system to change adaptation rate in response to CPU processing availability.

Chapter 3 presents the second extension of the thesis. This extension considers the class of uncertain systems with time and output dependent unknown nonlinearities and uncertainty in the input matrix. The presented $\mathcal{L}_1$ adaptive control architecture achieves performance specifications defined by a linear time-varying (LTV) reference system, which is critical in applications covering a wide range of operating conditions. A typical example of time-varying reference system would be the one resulting from a gain-scheduled baseline controller over an entire flight envelope. This extension integrates and extends the results from [20–22] to perform using output feedback for LTV reference systems. It is important to emphasize the relevance of robust output feedback control since full state measurement of a system is often unavailable. Several solutions for this problem have been proposed in [23–28].

The extension presented in this thesis expands upon the $\mathcal{L}_1$ adaptive controller for time-varying reference systems proposed in [20], which yields semiglobal performance results for the original uncertain time-varying system with state feedback. The piecewise constant adaptation law for output-feedback systems, presented in [22], is modified to preform with

LTV reference systems. Like all other $\mathcal{L}_1$ adaptive controllers, the proposed controller retains the property that the $\mathcal{L}_1$ norms for the error signals between the reference system and the closed-loop adaptive system can be systematically reduced by increasing the adaptation rate.

Chapter 4 concludes the thesis by providing a summary of the two new $\mathcal{L}_1$ adaptive control theory extensions presented. Please note that the variable names in Chapter 2 may be reused and redefined in Chapter 3 due to limitation of available variable names.

CHAPTER 2

PROPORTIONAL ADAPTATION LAW

2.1 Problem Formulation

Consider the system given by

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b_m \omega u(t) + f\left(t, x(t)\right)\,, \\ y(t) &= c^\top x(t)\,, \quad x(0) = x_0\,, \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector; $u(t) \in \mathbb{R}$ is the control signal; $b_m \in \mathbb{R}^n$ is a known constant vector; A_m is a known Hurwitz n x n matrix specifying the desired poles of the closed-loop dynamics; $\omega \in \mathbb{R}$ is an unknown constant with known sign; $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is an unknown map; and the initial condition x_0 is inside an arbitrarily large known set, i.e., $\|x_0\|_\infty \leq \rho_0 < \infty$ with known $\rho_0 > 0$. Let $d_r \in \mathbb{R}^+$ be the relative degree of the system. The system above is subject to the following assumptions:

Assumption 1 (*Lipschitz continuity*). There exist constants $L > 0$ and $B > 0$, such that

$$\begin{aligned} \|f(t, x_1) - f(t, x_2)\|_\infty &\leq L\,\|x_1 - x_2\|_\infty\,, \\ \|f(t, x)\|_\infty &\leq L\,\|x\|_\infty + B\,, \end{aligned}$$

hold uniformly for $t \geq 0$, where the numbers L and B can be arbitrarily large.

Assumption 2. Let $\omega \in \Omega \triangleq [\omega_l, \omega_u]$, where $0 < \omega_l < \omega_u$ are given known upper and lower

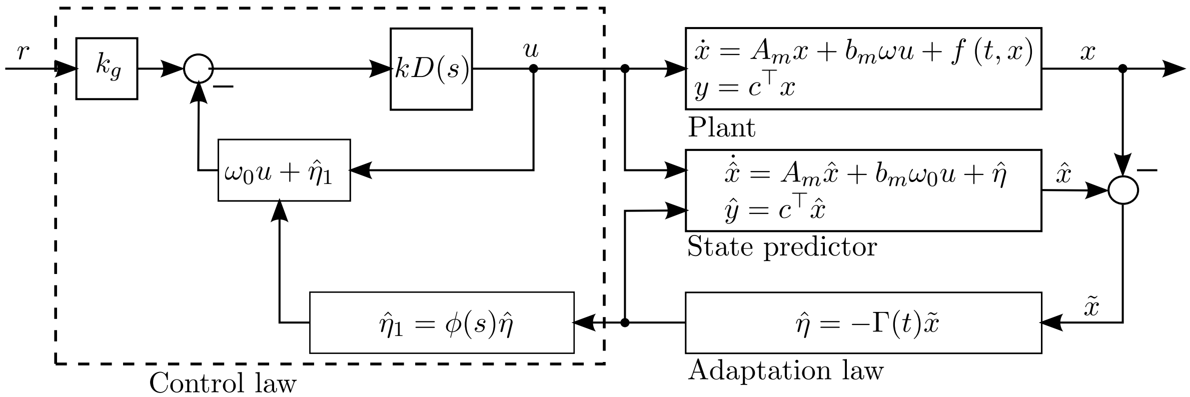


Figure 2.1: Block diagram of the $\mathcal{L}_1$ adaptive control system with proportional adaptation law.

bounds on ω.

The control objective is to design a full-state feedback adaptive controller with a generalized proportional adaptation law that will ensure that $y(t)$ tracks a given bounded piecewise-continuous reference signal $r(t)$ with quantifiable performance bound according to the ideal system defined as

$$\begin{aligned} \dot{x}_{\mathrm{id}}(t) &= A_m x_{\mathrm{id}}(t) + b_m k_g r(t)\,, \\ y_{\mathrm{id}}(t) &= c^\top x_{\mathrm{id}}(t)\,, \quad x_{\mathrm{id}}(0) = x_0\,, \end{aligned} \tag{2.2}$$

where $k_g \triangleq -1/(c^\top A_m^{-1} b_m)$, while allowing for the consideration of external factors such as time CPU performance, initialization errors, or other situations requiring time varying performance bounds.

2.2 $\mathcal{L}_1$ Adaptive Control Architecture

The $\mathcal{L}_1$ adaptive controller presented in this paper, similar to all other $\mathcal{L}_1$ architectures, is comprised of a state predictor, adaptive law, and control law, arranged as shown in Figure 2.1.

2.2.1 State Predictor

We consider the following state predictor:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m \hat{x}(t) + b_m \omega_0 u(t) + \hat{\eta}(t)\,, \\ \hat{y}(t) &= c^\top \hat{x}(t)\,, \quad \hat{x}(0) = x_0\,, \end{aligned} \tag{2.3}$$

which has a similar structure as (2.1), except ω is replaced by its best available guess $\omega_0 \in \Omega$, and the unknown estimates of ω and $f(t, x(t))$ are grouped together to form the estimated parameter $\hat{\eta}(t)$.

2.2.2 Adaptation Laws

The adaptive process is governed by the following generalized adaptation law:

$$\hat{\eta}(t) = -\Gamma(t)\tilde{x}(t)\ , \tag{2.4}$$

where $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$, and $\Gamma(t)$ is diagonal n x n matrix with the only requirement being that $\lambda_{\min}(\Gamma(t)) > 0$ for all $t \geq 0$. This loose condition on the adaptive gain provides a generalized framework for the analysis of stability and performance, under which several adaptive gain generating methods are presented.

2.2.3 Control Law

The $\mathcal{L}_1$ adaptive control law is generated as the output of the following feedback system:

$$u(s) = -kD(s)\left(\omega_0 u(s) + \hat{\eta}_1(s) - k_g r(s)\right)\ , \tag{2.5}$$

with $\hat{\eta}_1(s)$ defined as

$$\hat{\eta}_1(s) \triangleq \phi(s)\hat{\eta}(s)\,, \quad \phi(s) \triangleq \frac{c^\top \left(s\mathbb{I} - A_m\right)^{-1}}{c^\top \left(s\mathbb{I} - A_m\right)^{-1} b_m}$$

where $r(s)$ and $\hat{\eta}(s)$ are the Laplace transforms of $r(t)$ and $\hat{\eta}(t)$ respectively; and $k > 0$ and $D(s)$ are a feedback gain and a strictly proper transfer function respectively, which lead to a strictly proper stable transfer function

$$C(s) \triangleq \frac{\omega k D(s)}{1 + \omega k D(s)}\,, \quad \forall \omega \in \Omega\,, \tag{2.6}$$

with DC gain $C(0) = 1$ and relative degree at least d_r.

The $\mathcal{L}_1$ adaptive controller is defined via (2.3), (2.4) and (2.5), subject to the following $\mathcal{L}_1$-norm condition:

$$\|G(s)\|_{\mathcal{L}_1} L < 1\,, \tag{2.7}$$

given the following definitions:

$$G(s) \triangleq (\mathbb{I}_n - H_{xm}(s)C(s)H_m^{-1}c^\top)H_{xum}\,,$$
$$H_{xum}(s) \triangleq (s\mathbb{I} - A_m)^{-1}, \;\; H_{xm}(s) \triangleq (s\mathbb{I} - A_m)^{-1}b_m,$$
$$H_m(s) \triangleq c^\top(s\mathbb{I} - A_m)^{-1}b_m\,.$$

In [21], a filtered integral adaptation law is presented. Here a filtered proportional adaptation law is presented that adds ninety degrees of phase margin to the adaptation loop by avoiding integration. This extra margin can be used to perform structural modifications of the estimation loop of the $\mathcal{L}_1$ adaptive controller. More specifically, first order sensor dynamics may be added to the state predictor which can help to improve the performance of the closed–loop control system as illustrated later in the paper in Fig. 2.8. Theoretical analysis of such modification can be pursued similar to the prior proofs in $\mathcal{L}_1$ adaptive control theory [9].

2.3 Analysis of the $\mathcal{L}_1$ Adaptive Controller

2.3.1 Closed–loop Reference System

Similar to all $\mathcal{L}_1$ adaptive controllers, we consider the following closed–loop reference system

$$\dot{x}_{\text{ref}}(t) = A_m x_{\text{ref}}(t) + b_m \omega u_{\text{ref}}(t) + \eta_{\text{ref}}(t), \quad x_{\text{ref}}(0) = x_0, \tag{2.8}$$

$$y_{\text{ref}}(t) = c^\top x_{\text{ref}}(t)\,, \tag{2.9}$$

$$\eta_{\text{ref}}(t) \triangleq f(t, x_{\text{ref}}(t))\,, \tag{2.10}$$

$$u_{\text{ref}}(s) = -\frac{C(s)}{\omega}\left(\phi(s)\eta_{\text{ref}}(s) - k_g r(s)\right). \tag{2.11}$$

The stability of the closed–loop system in (2.8)-(2.11) is provided by the following lemma.

Lemma 1. If k and $D(s)$ verify the $\mathcal{L}_1$-norm condition in (2.7), then the closed-loop reference system in (2.8)-(2.11) is BIBS stable with respect to $r(t)$ and x_0.

Proof : It follows from (2.8)–(2.11) that

$$x_{\text{ref}}\left(s\right) = G\left(s\right)\eta_{\text{ref}}\left(s\right) + H_{xm}\left(s\right)C\left(s\right)k_g r\left(s\right) + x_{\text{in}}\left(s\right)\,,$$

where $x_{in}(s) \triangleq H_{xum}x_0$. Since A_m is Hurwitz, $x_{\text{in}}(t)$ is uniformly bounded. Next, given that the system is BIBO-stable and LTI and that $\eta_{\text{ref}}(t)$ and $r(t)$ are uniformly bounded, the following bound may be obtained

$$\|x_{\text{ref}_\tau}\|_{\mathcal{L}_\infty} \le \|G\left(s\right)\|_{\mathcal{L}_1}\|\eta_{\text{ref}_\tau}\|_{\mathcal{L}_\infty} + \|H_{xm}\left(s\right)C\left(s\right)k_g\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|x_{\text{in}}\|_{\mathcal{L}_\infty}\,.$$

Applying bounds to Assumption 1 we have $\|\eta_{\text{ref}_\tau}\|_{\mathcal{L}_\infty} \leq L\|x_{\text{ref}_\tau}\|_{\mathcal{L}_\infty} + B$ for some $0 \leq \tau \leq t$. Substituting and solving for $\|x_{\text{ref}_\tau}\|_{\mathcal{L}_\infty}$, one obtains

$$\|x_{\text{ref}_\tau}\|_{\mathcal{L}_\infty} \leq \rho_r \,, \tag{2.12}$$

where ρ_r is defined as follows

$$\rho_r \triangleq \frac{\|H_{xm}(s)\,C(s)\,k_g\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} B + \|x_{in}\|_{\mathcal{L}_\infty}}{1 - L\|G(s)\|_{\mathcal{L}_1}} . \tag{2.13}$$

Then, because k and $D(s)$ are chosen to verify the condition in (2.7), $\|x_{\text{ref}_\tau}\|_{\mathcal{L}_\infty}$ is uniformly bounded for all $\tau \geq 0$. Hence, the closed-loop reference system in (2.8) is BIBS stable. Similarly, u_{ref} can be bounded as follows

$$\|u_{\text{ref}_\tau}\|_{\mathcal{L}_\infty} \leq \rho_{ur} \,, \tag{2.14}$$

$$\rho_{ur} \triangleq \left\|\frac{LC(s)\phi(s)}{\omega}\right\|_{\mathcal{L}_1} \rho_r + \left\|\frac{BC(s)\phi(s)}{\omega}\right\|_{\mathcal{L}_1} + \left\|\frac{C(s)k_g}{\omega}\right\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} . \tag{2.15}$$

□

It is important to note that the closed–loop reference system assumes partial compensation of uncertainties within the bandwidth of the control channel. It therefore depends on the unknown system parameters and disturbances and cannot be implemented directly. The closed–loop reference system is used solely for the purpose of analysis.

2.3.2 Error Dynamics

The system dynamics in (2.1) and the state predictor in (2.3) lead to the following prediction-error dynamics:

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + b_m(\omega_0 - \omega)u(t) + \hat{\eta}(t) - \eta(t) \,, \tag{2.16}$$

with $\tilde{x}(0) = 0$, where $\eta(t) = f(t, x(t))$. Rewriting the error dynamics (2.16) in frequency domain, we obtain

$$c^\top \tilde{x}(s) = H_m(s)\tilde{\eta}(s)\,, \tag{2.17}$$

where

$$\tilde{\eta}(s) \triangleq (\omega_0 - \omega)u(s) + \hat{\eta}_1(s) - \eta_1(s)\,, \tag{2.18}$$

and $\eta_1(s) = \phi(s)\eta(s)$.

Lemma 2. Given the system in (2.1) and the $\mathcal{L}_1$ adaptive controller defined via (2.3), (2.4), and (2.5), if $\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho$ and $\|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u$ we have:

$$\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} \leq \gamma_0 \triangleq \frac{1}{\nu}\,\|P\|_2\,\Delta_\eta\sqrt{\frac{\lambda_{\max}(P)}{\lambda^3_{\min}(P)}}\,,$$

where $P = P^\top > 0$ is the solution of the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$ for arbitrary $Q = Q^\top > 0$, Δ_η and ν defined as

$$\Delta_\eta \triangleq (\omega_u - \omega_l)\|b_m\|\rho_u + \sqrt{n}(L\rho + B)$$
$$\nu \triangleq \inf_{t\geq 0} \lambda_{\min}\Gamma(t)\,,$$

and ρ and ρ_u are defined as

$$\rho \triangleq \rho_r + \bar{\gamma}_1\,, \quad \rho_u \triangleq \rho_{ur} + \gamma_2\,,$$

where

$$\rho_r \triangleq \frac{\|H_{xm}(s)C(s)k_g\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} B + \|x_{\text{in}}\|_{\mathcal{L}_\infty}}{1 - L\,\|G(s)\|_{\mathcal{L}_1}}\,, \tag{2.19}$$

$$\rho_{ur} \triangleq \left\| \frac{LC(s)\phi(s)}{\omega} \right\|_{\mathcal{L}_1} \rho_r + \left\| \frac{BC(s)\phi(s)}{\omega} \right\|_{\mathcal{L}_1} + \left\| \frac{C(s)k_g}{\omega} \right\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} , \tag{2.20}$$

$$\gamma_1 \triangleq \frac{\left\| H_{xm}(s)\, C(s)\, H_m^{-1}(s)\, c^\top \right\|_{\mathcal{L}_1}}{1 - L \left\| G(s) \right\|_{\mathcal{L}_1}} \gamma_0 + \beta , \tag{2.21}$$

$$\gamma_2 \triangleq \left\| \frac{C(s)}{\omega} \phi(s) \right\|_{\mathcal{L}_1} L\gamma_1 + \left\| \frac{C(s)\, H_m^{-1}(s) c^\top}{\omega} \right\|_{\mathcal{L}_1} \gamma_0 , \tag{2.22}$$

with β being some arbitrary (small) constant and $\bar{\gamma}_1$ defined such that $\gamma_1 \leq \bar{\gamma}_1$ and $x_{\text{in}}(s) \triangleq (s\mathbb{I} - A_m)^{-1} x_0$.

Proof :

Consider the Lyapunov function candidate:

$$V(t) = \frac{1}{2} \tilde{x}^\top(t) P \tilde{x}(t). \tag{2.23}$$

Since $\tilde{x}(0) = 0$, we have $V(0) = 0$. Taking the time derivative of (2.23), we obtain

$$\dot{V}(t) = \frac{1}{2} \left(-\tilde{x}^\top(t) Q \tilde{x}(t) - 2\tilde{x}^\top(t) P (b_m(\omega_0 - \omega) u(t) - \eta(t)) + 2\tilde{x}^\top(t) P \hat{\eta}(t) \right) .$$

Substituting the adaptation law (2.4) in for $\hat{\eta}$ we get

$$\dot{V}(t) = -\frac{1}{2} \tilde{x}^\top(t) Q \tilde{x}(t) - \tilde{x}^\top(t) P (b_m(\omega_0 - \omega) u(t) - \eta(t)) - \tilde{x}^\top(t) P \Gamma(t) \tilde{x}(t) ,$$

which can be upper bounded for $t \in [0, \tau]$ to achieve the following

$$\dot{V}(t) \leq -\tilde{x}^\top(t) P \Gamma(t) \tilde{x}(t) + \|\tilde{x}(t)\| \, \|P\|_2 \, \Delta_\eta$$

where the value Δ_η is the following bound:

$$\begin{aligned}\|(b_m(\omega_0-\omega)u(t)-\eta(t))\| &\le (\omega_u-\omega_l)\|b_m\|\,\|u_\tau\|_{\mathcal{L}_\infty}+\sqrt{n}(L\,\|x_\tau\|_{\mathcal{L}_\infty}+B)\\ &\le (\omega_u-\omega_l)\|b_m\|\rho_u+\sqrt{n}(L\rho+B)=\Delta_\eta.\end{aligned}$$

Given that $\Gamma(t)$ is diagonal and P is a positive definite and symmetric we obtain

$$\lambda_{\min}(P\Gamma(t))=\lambda_{\min}(P)\lambda_{\min}(\Gamma(t))\ge\lambda_{\min}(P)\nu\,.$$

Then, since ν and $\lambda_{\min}(P)$ are greater than zero, for all $t\in[0,\tau]$ we have

$$\|\tilde{x}(t)\|>\frac{\|P\|_2\,\Delta_\eta}{\lambda_{\min}(P)\,\nu}\Rightarrow\dot{V}(t)<0.$$

If at any time $t_1\in[0,\tau]$, one has

$$V(t_1)>\frac{1}{2}\lambda_{\max}(P)\left(\frac{\|P\|_2\,\Delta_\eta}{\lambda_{\min}(P)\nu}\right)^2,$$

then

$$\frac{1}{2}\lambda_{\max}(P)\left(\frac{\|P\|_2\,\Delta_\eta}{\lambda_{\min}(P)\nu}\right)^2<V(t_1)=\frac{1}{2}\tilde{x}(t)^\top P\tilde{x}(t)\le\frac{1}{2}\lambda_{\max}(P)\,\|\tilde{x}(t)\|^2\,,$$

which results in $\|\tilde{x}(t)\|>\frac{\|P\|_2\Delta_\eta}{\lambda_{\min}(P)\nu}$, causing $\dot{V}(t_1)<0$. Then it follows that

$$V(t)\le\frac{1}{2}\lambda_{\max}(P)\left(\frac{\|P\|_2\,\Delta_\eta}{\lambda_{\min}(P)\nu}\right)^2,\quad\forall t\in[0,\tau].$$

Since $\frac{1}{2}\lambda_{\min}(P)\,\|\tilde{x}(t)\|^2 \le V(t)$, it follows that

$$\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} \le \frac{1}{\nu}\,\|P\|_2\,\Delta_\eta\sqrt{\frac{\lambda_{\max}(P)}{\lambda^3_{\min}(P)}}\,,$$

and the proof is complete. □

2.3.3 Performance Bounds of Closed–Loop Adaptive System

The error bounds between the system's states and the reference states and the system's input and the reference input are given in the next theorem.

Theorem 1. Given the system in (2.1) and the $\mathcal{L}_1$ adaptive controller defined via (2.3), (2.4), and (2.5), subject to the $\mathcal{L}_1$-norm condition in (2.7), if we have $\|x_0\|_\infty \le \rho_0$, then

$$\|x\|_{\mathcal{L}_\infty} \le \rho\,, \quad \|u\|_{\mathcal{L}_\infty} \le \rho_u\,, \quad \|\tilde{x}\|_{\mathcal{L}_\infty} \le \gamma_0\,, \tag{2.24}$$

$$\|x_{\text{ref}} - x\|_{\mathcal{L}_\infty} \le \gamma_1\,, \quad \|u_{\text{ref}} - u\|_{\mathcal{L}_\infty} \le \gamma_2\,. \tag{2.25}$$

Proof :

Assume that the bounds in (2.25) do not hold. Then, since $\|x_{\text{ref}}(0) - x(0)\|_\infty = 0 \le \gamma_1$, $\|u_{\text{ref}}(0) - u(0)\|_\infty = 0 \le \gamma_2$, and $x(t)$, $x_{\text{ref}}(t)$, $u(t)$, and $u_{\text{ref}}(t)$ are continuous, there exists $\tau > 0$ such that

$$\|x_{\text{ref}}(\tau) - x(\tau)\|_\infty = \gamma_1 \quad \text{or} \quad \|u_{\text{ref}}(\tau) - u(\tau)\|_\infty = \gamma_2\,,$$

while

$$\|x_{\text{ref}}(t) - x(t)\|_\infty < \gamma_1\,, \quad \|u_{\text{ref}}(t) - u(t)\|_\infty < \gamma_2\,,$$

for all $t \in [0, \tau)$. This implies that at least one of the following equalities holds:

$$\|(x_{\text{ref}} - x)_\tau\|_{\mathcal{L}_\infty} = \gamma_1\,, \quad \|(u_{\text{ref}} - u)_\tau\|_{\mathcal{L}_\infty} = \gamma_2. \tag{2.26}$$

Then, using the fact that $\|x_{\text{ref}}\|_{\mathcal{L}_\infty} \leq \rho_r$ and $\|u_{\text{ref}}\|_{\mathcal{L}_\infty} \leq \rho_{ur}$ from Lemma 1, along with definitions of ρ and ρ_u, it follows from the bounds in (2.26) that

$$\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_r + \gamma_1 \leq \rho\,,$$
$$\|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_{ur} + \gamma_2 \leq \rho_u.$$

This validates the assumptions used in Lemma 2, which in turn implies that $\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} < \gamma_0$.

Next, it follows from (2.5) that

$$u\,(s) = -KD(s)\,(\omega u(s) + \eta_1(s) - k_g r(s) - \tilde{\eta}(s))\ ,$$

where $\tilde{\eta}\,(s)$ is defined in (2.18). Solving for $u(s)$ gives

$$u\,(s) = \frac{-KD\,(s)}{1 + \omega KD\,(s)}\,(\eta_1(s) - k_g r\,(s) - \tilde{\eta}\,(s))\,.$$

Using the definition of $C(s)$ from (2.6), we can write:

$$u\,(s) = \frac{-C\,(s)}{\omega}\,(\eta_1(s) - k_g r\,(s) - \tilde{\eta}\,(s))\ , \tag{2.27}$$

and the system in (2.1) takes the form

$$x(s) = G(s)\eta(s) + H_{xm}(s)C(s)k_g r(s) + H_{xm}(s)C(s)\tilde{\eta}(s) + x_{\text{in}}(s).$$

Similarly, it follows from (2.8) that

$$x_{\rm ref}(s) = G(s)\eta_{\rm ref}(s) + H_{xm}(s)C(s)k_g r(s) + x_{\rm in}(s).$$

Then

$$x_{\rm ref}(s) - x(s) = G(s)(\eta_{\rm ref}(s) - \eta(s)) - H_{xm}(s)C(s)\tilde{\eta}(s)\,. \tag{2.28}$$

Pre-multiplying both sides of (2.17) by $H_m^{-1}(s)$ leads to the following equation:

$$\tilde{\eta}(s) = H_m^{-1}(s)c^\top \tilde{x}(s). \tag{2.29}$$

Substituting (2.29) into (2.28) leads to

$$x_{\rm ref}(s) - x(s) = G(s)(\eta_{\rm ref}(s) - \eta(s)) - H_{xm}(s)C(s)H_m^{-1}(s)c^\top \tilde{x}(s)\,,$$

from which, with the use of Assumption 1, we obtain the following bound

$$\|(x_{\rm ref} - x)_\tau\|_{\mathcal{L}_\infty} \leq L\,\|G(s)\|_{\mathcal{L}_1}\,\|(x_{\rm ref} - x)_\tau\|_{\mathcal{L}_\infty} + \left\|H_{xm}(s)C(s)H_m^{-1}(s)c^\top\right\|_{\mathcal{L}_1}\|\tilde{x}_\tau\|_{\mathcal{L}_\infty}\,.$$

Then, solving for $\|(x_{\rm ref} - x)_\tau\|_{\mathcal{L}_\infty}$, we obtain

$$\|(x_{\rm ref} - x)_\tau\|_{\mathcal{L}_\infty} \leq \frac{\left\|H_{xm}\,(s)\,C\,(s)\,H_m^{-1}\,(s)\,c^\top\right\|_{\mathcal{L}_1}}{1 - L\,\|G\,(s)\|_{\mathcal{L}_1}}\,\gamma_0\,,$$

which along with the definition of γ_1 in (2.21) leads to

$$\|(x_{\rm ref} - x)_\tau\|_{\mathcal{L}_\infty} \leq \gamma_1 - \beta < \gamma_1. \tag{2.30}$$

On the other hand, from (2.11) and (2.27) one can derive

$$u_{\text{ref}}(s) - u(s) = -\frac{C(s)}{\omega}\phi(s)(\eta_{\text{ref}}(s) - \eta(s)) - \frac{C(s)}{\omega}\tilde{\eta}(s).$$

Substituting (2.29) into the previous we obtain the following bound

$$\|(u_{\text{ref}} - u)_\tau\|_{\mathcal{L}_\infty} \leq \left\|\frac{C(s)}{\omega}\phi(s)\right\|_{\mathcal{L}_1} L\,\|(x_{\text{ref}} - x)_\tau\|_{\mathcal{L}_\infty} + \left\|\frac{C(s)H_m^{-1}(s)c^\top}{\omega}\right\|_{\mathcal{L}_1} \|\tilde{x}_\tau\|_{\mathcal{L}_\infty}.$$

Substituting the bounds for $\|(x_{\text{ref}} - x)_\tau\|_{\mathcal{L}_\infty}$ and $\|\tilde{x}_\tau\|_{\mathcal{L}_\infty}$ leads to

$$\|(u_{\text{ref}} - u)_\tau\|_{\mathcal{L}_\infty} \leq \left\|\frac{C(s)}{\omega}\phi(s)\right\|_{\mathcal{L}_1} L(\gamma_1 - \beta) + \left\|\frac{C(s)H_m^{-1}(s)c^\top}{\omega}\right\|_{\mathcal{L}_1} \gamma_0 < \gamma_2. \tag{2.31}$$

Finally, we note that the upper bounds in (2.30) and (2.31) contradict the equalities in (2.26), which proves the bounds in (2.25). The results in (2.24) follow directly from combining the bounds derived in Lemma 1 with the results from Lemma 2. □

Notice that one can achieve arbitrarily small performance bounds γ_0, γ_1, and γ_2 by increasing the value of ν. Also notice that the performance bounds are inverse proportional to ν itself, whereas for the integral (gradient descent) adaptation laws, the performance bounds are inverse proportional to $\sqrt{\nu}$. Therefore the adaptation law presented here requires smaller adaptation gain to achieve similar performance bounds as the previous integral controller. Finally, the structure of the performance bounds (2.24)–(2.25) is identical to the performance bounds of the $\mathcal{L}_1$ adaptive controller given in Section 2.2.1 of [9]. This similarity is a consequence of the decoupling of the estimation loop from the control loop, which is achieved by the $\mathcal{L}_1$ adaptive control architectures [29].

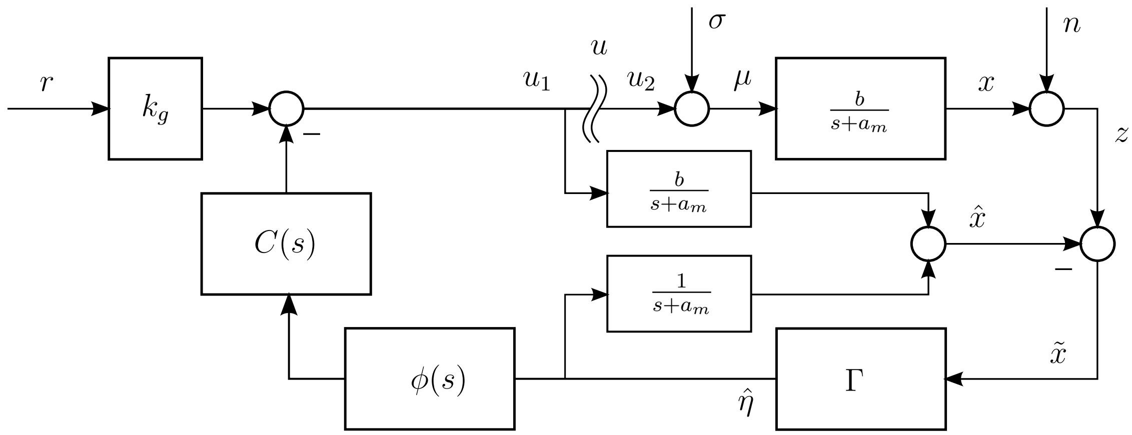


Figure 2.2: Closed–loop system with $\mathcal{L}_1$ adaptive controller

2.4 Analysis of Performance and Stability Margins

In [29], the authors compare the performance and the robustness of several *linear adaptive* controllers using a linear system with time–varying disturbance in the presence of measurement noise. In order to make the closed loop adaptive system linear in [29], the integral adaptation laws were used while omitting the projection operator. The proportional adaptive system is linear as presented. For fair comparison, we consider a first order system of the same structure, namely we use (2.1) with $A_m = -a_m$, $\omega = 1$, $f(t, x(t)) = 0$, and use a static adaptation gain Γ. In the absence of input uncertainty, the control law can be rewritten as follows

$$u(s) = C(s)\hat{\eta}_1(t) - k_g r(s)\,,$$

noting that the reference signal is not multiplied by the filter $C(s)$ in order to mimic [29]. The resulting block diagram of the system described above is given in Figure 2.2.

2.4.1 Performance Analysis

The *Gang of Six* for the closed–loop system with proportional $\mathcal{L}_1$ adaptive controller is given in Table 2.1. Similar to MRAC and conventional $\mathcal{L}_1$ adaptive control, in the absence of the

$H_{xr}(s) = H_{zr}(s) = \frac{bk_g}{s+a_m}$
$H_{x\sigma}(s) = H_{z\sigma}(s) = \frac{b}{s+a_m}\left(1 - \frac{\Gamma C(s)}{s+a_m+\Gamma}\right)$
$H_{xn}(s) = H_{u\sigma}(s) = -\frac{\Gamma}{s+a_m+\Gamma}C(s)$
$H_{ur}(s) = H_{\mu r}(s) = k_g$
$H_{zn}(s) = H_{\mu\sigma}(s) = 1 - \frac{\Gamma C(s)}{s+a_m+\Gamma}$
$H_{un}(s) = H_{\mu n}(s) = \frac{-\Gamma(s+a_m)}{b(s+a_m+\Gamma)}C(s)$

Table 2.1: *Gang of Six* for the system with proportional $\mathcal{L}_1$ adaptive controller

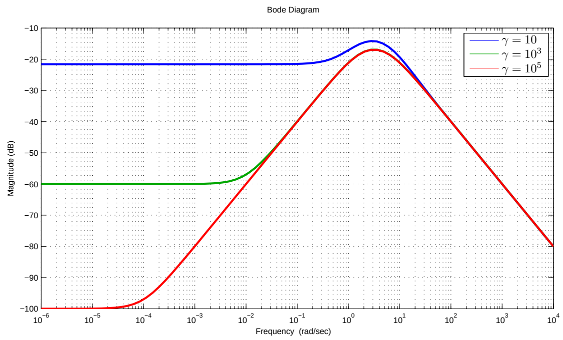
disturbance (σ) and the measurement noise (n), the closed loop system is identical to the ideal system in (2.2). Similar to [29], the remaining transfer functions contain the filter $C(s)$. Next we investigate the performance and robustness of this closed–loop control system and compare it to the conventional $\mathcal{L}_1$ adaptive controller with gradient descent adaptation laws.

We do so by considering a simple, first order filter of the following form:

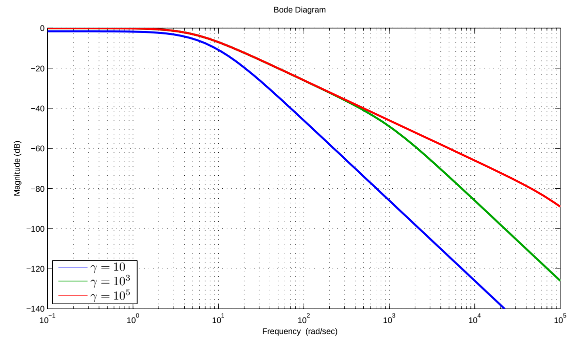
$$C(s) = \frac{\omega_c}{s + \omega_c}\,, \quad \omega_c = 5\,,$$

where $\omega_c \in \mathbb{R}^+$ is a filter parameter. Figures 2.3 shows the Bode plots for the closed–loop system with the proportional $\mathcal{L}_1$ controller for different values of the adaptation gain. Notice that increasing the adaptation gain leads to better attenuation of the input disturbances. However, unlike in [29], the high frequency content of the measurement noise does not get amplified in the control channel with large adaptation gains and the system exhibits no resonance peaks, similar to the results obtained by modifying the state predictor or using high order filters with conventional $\mathcal{L}_1$ adaptive control.

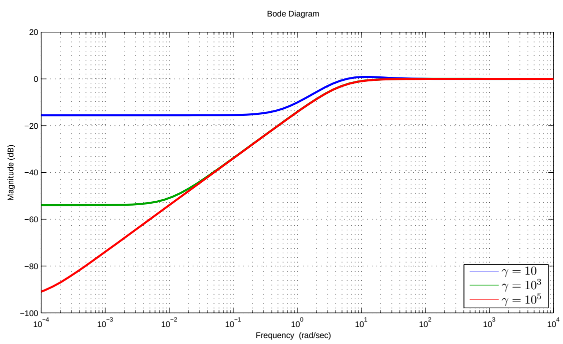
Also, unlike the $\mathcal{L}_1$ adaptive controller with gradient descent adaptation laws, the the magnitudes of $H_{x\sigma}(s)$, $H_{z\sigma}(s)$, $H_{zn}(s)$, and $H_{\mu\sigma}(s)$ at low frequencies are dependent on the adaptation gain and do not become arbitrarily small. Therefore the adaptation gain value



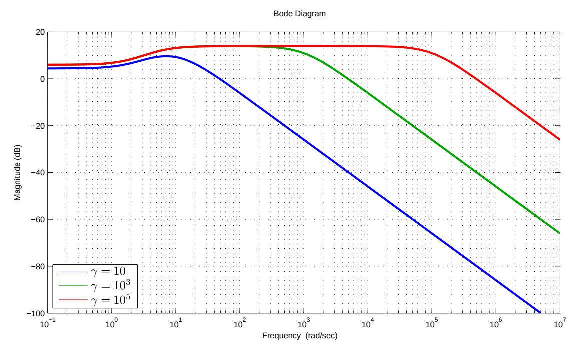
(a) $H_{x\sigma}(s)$, $H_{z\sigma}(s)$



(b) $H_{xn}(s)$, $H_{u\sigma}(s)$



(c) $H_{zn}(s)$, $H_{\mu\sigma}(s)$



(d) $H_{un}(s)$, $H_{\mu n}(s)$

Figure 2.3: Bode plots for teh system with proportional $\mathcal{L}_1$ adaptive controller for different values of the adaptation gain

can be easily chosen to obtain desired disturbance rejection at low frequencies for a given filter $C(s)$.

2.4.2 Analysis of Stability Margins

Using Figure 2.2 we compute the input loop transfer function:

$$\begin{aligned} L_{u_1 u_2} &= -\frac{\Gamma}{s + a_m + (1 - C(s))\Gamma} C(s) \\ &= -\frac{\Gamma w_c}{s^2 + (\Gamma + a_m + w_c)\, s + w_c a_m}\,. \end{aligned}$$

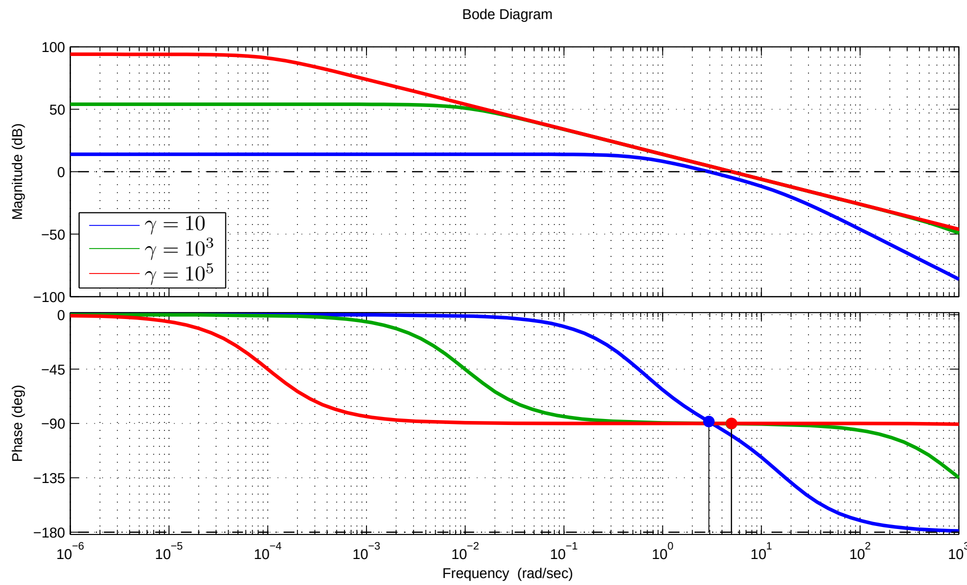
The Nyquist plot in Figure 2.4, computed with the same values as earlier ($a_m = 2$ and $w_c = 5$), shows that the phase and gain margins of the proportional $\mathcal{L}_1$ adaptive controller are not affected by large values of Γ. The figure shows that the proportional $\mathcal{L}_1$ adaptive controller has guaranteed bounded–away–from–zero phase and gain margins in the presence of *fast adaptation*.

Standard $\mathcal{L}_1$ adaptation laws with a first order filter exhibit resonance peaks in the input loop transfer function that increase in magnitude and frequency with increase in adaptation gain [29]. These resonance peaks can cause amplification of noise which reduces robustness and can lead to implementation problems in the control channel. The peaks can be removed using higher order filter or state predictor modifications. However, with the proposed proportional adaptation law, the Bode plot of the transfer function in Figure 2.4 does not exhibit resonance peaks, thus improving stability margins of the system without the need for any modification or high order filter.

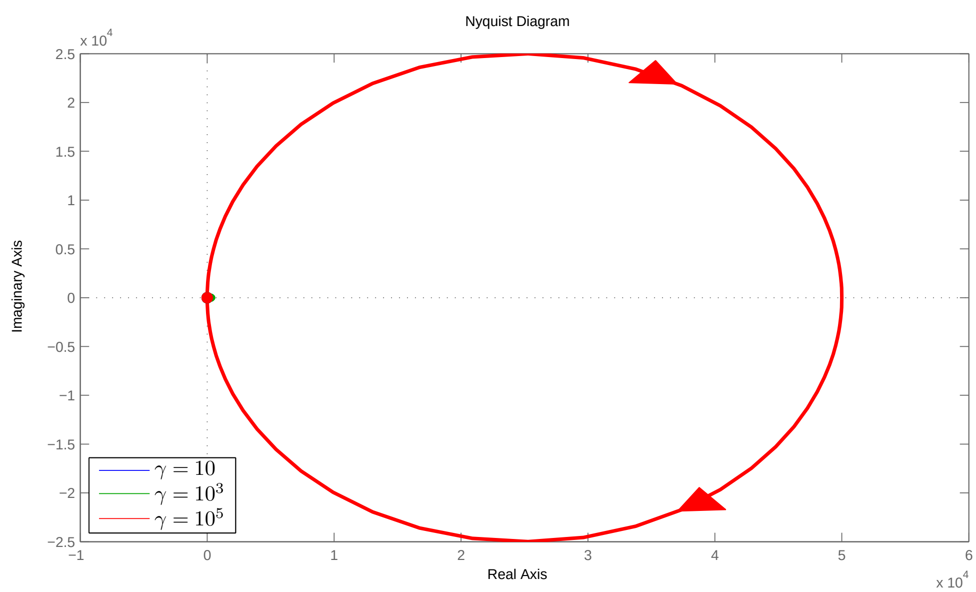
This result can be explained by calculating the damping coefficient for each type of adaptation law. The values are as follows

$$\xi_c = \frac{w_c + a_m}{2\sqrt{a_m w_c + \Gamma b}}\,, \quad \xi_p = \frac{\Gamma + a_m + w_c}{2\sqrt{a_m w_c}}\,,$$

where ξ_c and ξ_p are the damping coefficients for the conventional and proportional adaptation laws, respectively. As shown, an increase in the adaptation gain reduces the damping for the conventional adaptation law and increases damping for the proportional adaptation law. These results display the stability margin benefits gained by the use of the proposed proportional adaptation law.



(a) Bode plots



(b) Nyquist Plots

Figure 2.4: Loop transfer function of of system with Proportional Adaptation Law

2.5 Adaptation Gain Generation Methods

This section examines three specific methods for adaptive gain generation that fall under the generalized proportional $\mathcal{L}_1$ adaptive controller.

2.5.1 Switching Adaptation Law

We begin by presenting the switching–adaptation law presented in [12], which is given by

$$\hat{\eta}(t) = -G\text{sgn}(dz\left[(Pb)^\top \tilde{x}(t)\right]),$$

where $dz[\cdot]$ is the dead zone function, the size of which is set arbitrarily small. The adaptive architecture presented in [12] has the same performance bounds structure standard for the $\mathcal{L}_1$ adaptive control architecture. The above adaptation law can be presented as a time–varying adaptive gain proposed above by approximating it as

$$\Gamma_{i,i}(t) = \epsilon + \frac{\left[(Pb)^\top\right]_i}{\epsilon + |\tilde{x}_i(t)|},$$

where ϵ is an arbitrarily small positive constant used to avoid singularity and $[\,\cdot\,]_i$ denotes the i^{th} component of the vector while i, i denotes the i^{th} component of the diagonal gain matrix.

In the subsequent sections we now consider examples of adaptive gain generation techniques which fit the architecture presented in this paper, but have not been considered for $\mathcal{L}_1$ adaptive control thus far.

2.5.2 Funnel Adaptation Law

One specific application of time–varying proportional gain is funnel control. In funnel control, the time varying gain is defined as follows

$$\Gamma(t) = \frac{\mathbb{I}}{\mathcal{F}(t) - \|\tilde{x}(t)\|_p}, \tag{2.32}$$

where p can be any vector norm, $\tilde{x}(t) \in \mathbb{R}^n$ is the prediction error and $\mathcal{F}(t) \in \mathbb{R}$ is a continuous function of time such that $\|\tilde{x}(0)\| < \mathcal{F}(0)$ and $\min_{t>0} \mathcal{F}(t) \geq F_\infty > 0$, and represents a time varying funnel in which the error remains as shown in Figure 2.5. For example, the Exponential Funnel Boundary is given by

$$\mathcal{F}_{exp}(t) = F_0 \exp\left(-\frac{t}{T}\right) + F_\infty; \tag{2.33}$$

where T is the time constant and F_0 and F_∞ are positive constants such that $\mathcal{F}(0) = F_0 + F_\infty$ and $\lim_{t\to\infty} \mathcal{F}_{exp}(t) = F_\infty$.

Notice that $\|\tilde{x}(t)\|$ must be less than $\mathcal{F}(0)$ in order for the initial gain to be positive, which corresponds to the requirement that the error must be initialized within the funnel. Then, as the value of the error approaches the funnel boundary, the denominator in (2.32) decreases resulting in an increased gain and forces the error towards zero. Infinite gain occurs as the error approaches the funnel boundary. Therefore, funnel control can exhibit very large adaptation gains which can lead to robustness issues, such as small time delay margins.

The funnel control concept can be applied to the adaptive gain in the proportional $\mathcal{L}_1$ adaptive control theory mentioned above. The key difference is that the adaptive gain in the $\mathcal{L}_1$ architecture considers the error between the plant states and the state predictor instead of the error between the output and the reference signal. However, due to the decoupling of robustness and adaptation in $\mathcal{L}_1$ theory [11], the high gain resulting from the funnel control

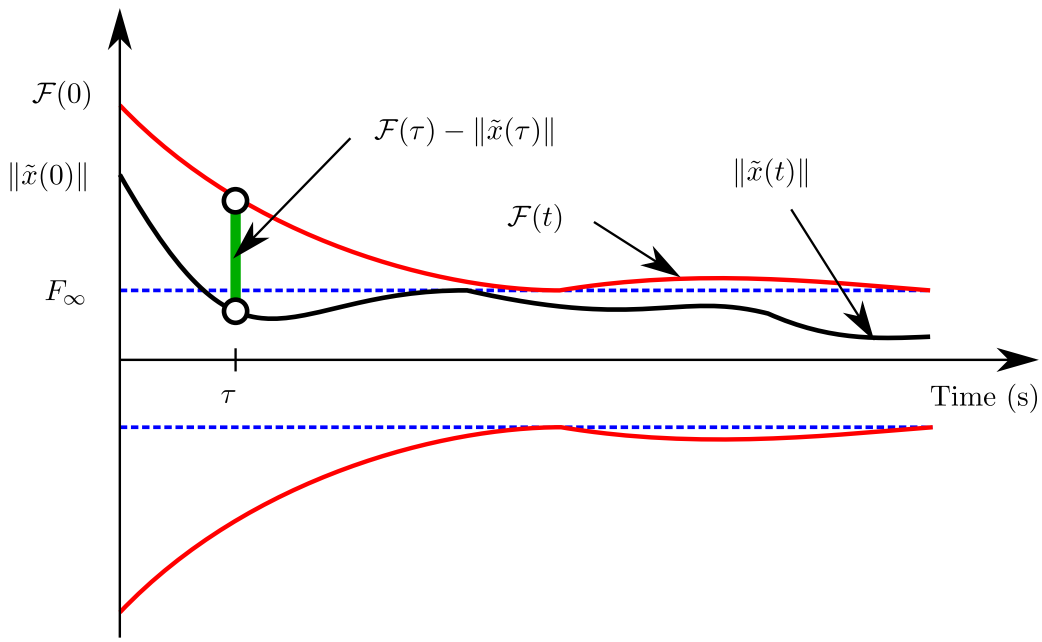


Figure 2.5: Basic Concept of Funnel Control

acts to improve performance without decreasing robustness. The trade off between robustness and performance instead manifests itself in the choice of the low pass filter. Funnel control is used to prescribe a bound on the error between the actual system and reference system, while the choice of filter determines how close the reference system matches the ideal system, with the cost of performance being the reduction in robustness.

2.5.3 Variable Adaptation Law

Finally, we present an adaptive gain generating method dependent on a vector $\psi(t) \in \mathbb{R}^q$, with q being finite, of variables by which the value of the adaptation gain is adjusted. The gain adjustment equation is given as

$$\Gamma(t) = \left(G_L + \frac{G_U}{1 + \alpha \left\| \psi(t) \right\|_p} \right) \mathbb{I} \, . \tag{2.34}$$

with G_L, G_U, $\alpha \in \mathbb{R}^+$ and p representing any vector norm. The adaptation gain has a lower bound G_L and upper bound $G_U + G_L$, while the rate of variation is dependent on the design parameter α. Thus, as the norm of the vector $\psi(t)$ increases, the adaptation gain decreases. Note that the lower bound can be set to ensure given performance bounds. Next, we suggest two variables that can be used to vary the adaptation gain.

First, the variable adaptation law may use the error $\tilde{x}(t)$ for $\psi(t)$ in (2.34) as a method to reduce the effects of the peaking phenomenon experienced in adaptive controls [17–19]. Peaking refers to the large spike, or peak, in control effort that occurs as a result of an error between the initial conditions of the plant and state predictor. When using the above equation, the adaptation gain is reduced when there is a large error between the state predictor and plant, reducing the magnitude of peaking.

The second variable considered is CPU rate. The main limiting factor for achieving fast adaptation is the availability of the CPU since it requires a high CPU demand. Varying the adaptation gain according to CPU demand can be used to regulate the CPU usage, and may lead to improved performance, since in many cases the adaptation gain is selected based on conservative CPU demand estimations.

We illustrate some of the results presented in this section in the simulations below.

2.6 Simulations

Consider the system in (2.1) with the following values

$$A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}, \quad b_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c^\top = [1, 0]\,. \tag{2.35}$$

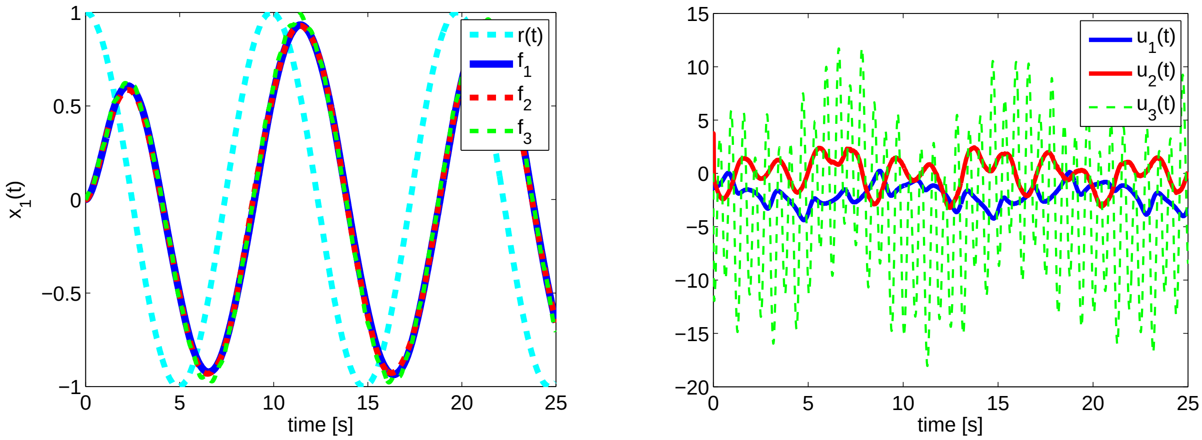
Consider the following three cases of parametric uncertainties,

$$
\begin{aligned}
\omega_1 =& 1\,, \quad \omega_2 = 1.5\,, \quad \omega_3 = 0.8\,,\\
f_1(t) =& \begin{bmatrix} 0.5\cos(\pi t)\sin(x_2(t)) \\ 2 + 0.3\sin(\pi t)x_1(t) + 0.2\cos(\pi t)x_2 \end{bmatrix}\,,\\
f_2(t) =& \begin{bmatrix} \sin(0.5\pi t) + 2\sin(x_1(t)) - .1 \\ 0.1\cos(3\pi t)x_2(t) + \sin(x_2(t)) + 2\sin(\pi t) \end{bmatrix}\,,\\
f_3(t) =& \begin{bmatrix} 0.5x_2(t) + \sin(x_1(t)) + \sin\left(\frac{16\pi}{5}t\right) \\ 3x_1(t) - \sin(t) + 3\cos(x_1(t)) + 2\sin(2\pi t) \end{bmatrix}\,.
\end{aligned}
$$

We implement the $\mathcal{L}_1$ adaptive controller according to (2.3), (2.4), and (2.5), subject to the $\mathcal{L}_1$-norm condition in (2.7). Since the system has a relative degree 2, we select $D(s) = \frac{200}{s(s+200)}$. We set $k = 60$ and let $\Gamma = 100,000$.

Figure 2.6 depicts the response of the system to a reference trajectory $r(t) = \cos(2t/\pi)$ for three sets of nonlinear disturbances and input uncertainties above. The three output responses are nearly identical although the control responses are different. This implies that to achieve uniform performance, the compensation for the different time varying disturbances requires different control signals to cancel their effects. Figure 2.7 shows the response of the system to step reference inputs of 1, 5, and 10 with $f_1(t, x(t))$ and ω_1 uncertainties. As shown, both the system response and the control input scale uniformly.

In [30] it was shown that the fast estimation loop of $\mathcal{L}_1$ adaptive controller can be systematically modified to improve the performance of the closed–loop system by accommodating different types of real world system components such as actuator and sensor dynamics. Previous work has illustrated the benefits of modifying the control signal of the state predictor to take into account the saturation magnitudes and the known input delays [30]. Without going into technical details and providing theoretical analysis, here we show that the sensor



(a) Transient performance

(b) Control history

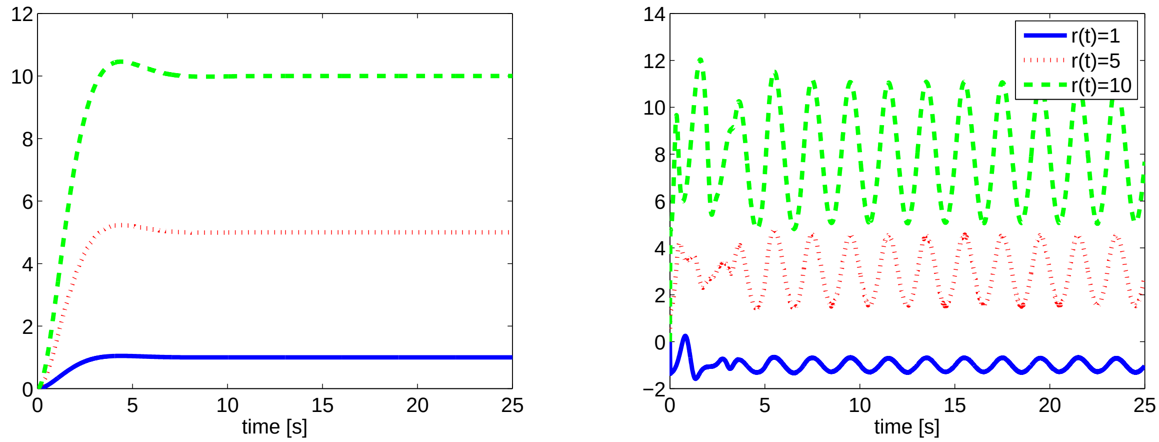
Figure 2.6: Closed–loop system response for the three sets of parametric uncertainties and disturbances.

dynamics cannot be compensated for by modifications of the state predictor when using gradient descent adaptation laws, while the proposed proportional adaptive law allows us to address this problem. Toward this end, consider the same plant given in (2.35) with the sensor dynamics given by

$$S(s) = \frac{20}{s+20}.$$

Since the sensor dynamics are assumed to be known, we modify the output of the state predictor according to the sensor dynamics. The simulation results for both gradient descent and proportional adaption laws are given in Figure 2.8. One can see that the gradient descent adaptation law becomes unstable, while the proportional adaptation law is able to track the step reference command.

Next, consider the funnel control algorithm in (2.32) for the time–varying adaptive gain with the use of the infinity norm. The Exponential Funnel Boundary from (2.33) is used with $F_0 = 0.3$, $F_\infty = 0.05$, and $T = 2$. The plant states are initialized as $x_0 = [0.1, 0.2]^\top$, while the state predictor is initialized with $\hat{x}_0 = [0, 0]^\top$. A small initialization error is presented



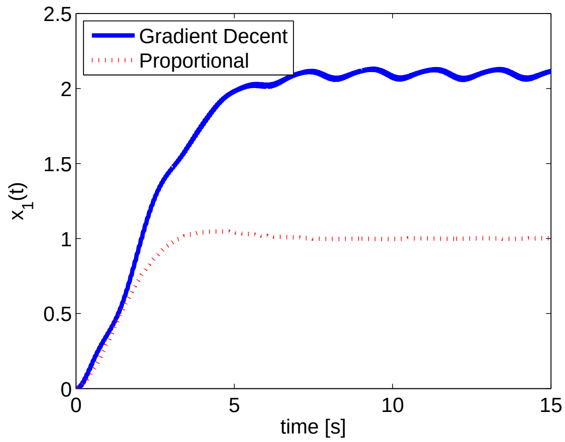
(a) Transient performance (b) Control history

Figure 2.7: Closed–loop system response to step signals of various amplitudes showing scalable response.

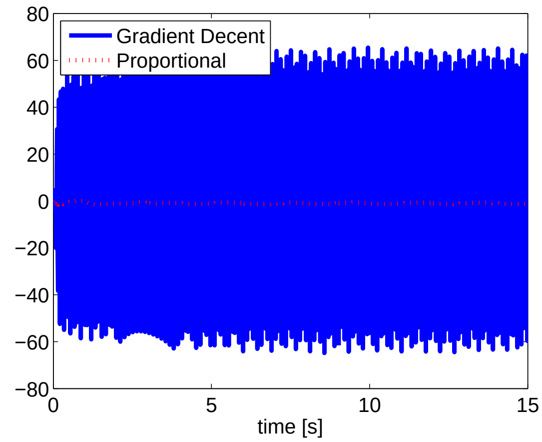
so that the initial error does not start at the orginin. Figure 2.9 shows the state response and the control input. There is an initial error between the output of the system and the reference system, which eventually converges as the funnel narrows. The adaptive gain varies in order to keep the error within the funnel as shown in Figure 2.10.

Next, simulation results are shown that demonstrate the use of the variable adaptation gain to decrease the peaking phenomenon. The results are compared to the results of two other methods commonly used to reduce peaking: funnel control and input saturation. We also show the case where a large static gain is used to depict the severity of peaking in the absence of any peaking reduction method. The plant model is initialized to $x_0 = [3, -1]^\top$, while the state predictor is initialized with $\hat{x}_0 = [0, 0]^\top$. For the constant adaptive gain, a value of $\Gamma = 10,000\mathbb{I}$ is used. The funnel boundary parameters are set to $F_0 = 5$, $F_\infty = 0.001$, and $T = 2$, while the parameters $G_L = 10$, $G_U = 10,000$, and $\alpha = 2,000$ are used for the state dependent gain described in (2.34). Control saturation is set to ± 15. The infinity norm was used for both the funnel control and variable gain laws.

Figure 2.11 shows the output response and the control input value for the four different



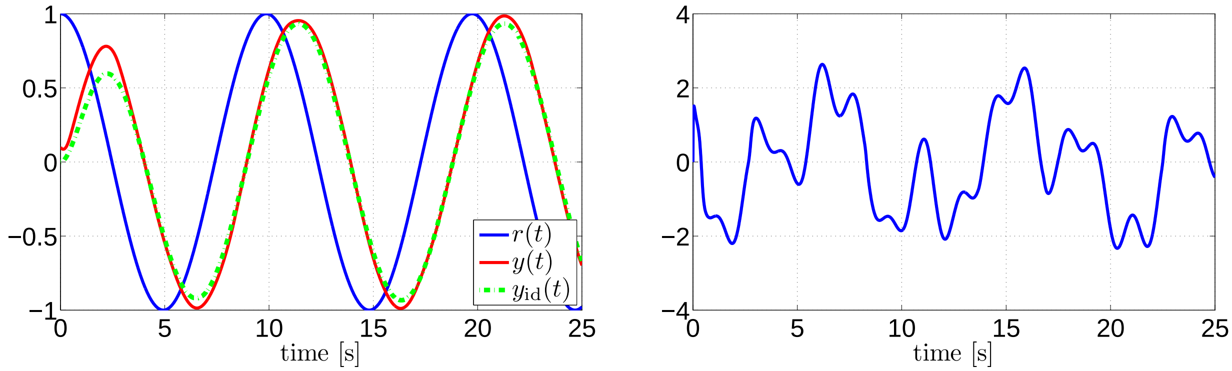
(a) Transient performance



(b) Control history

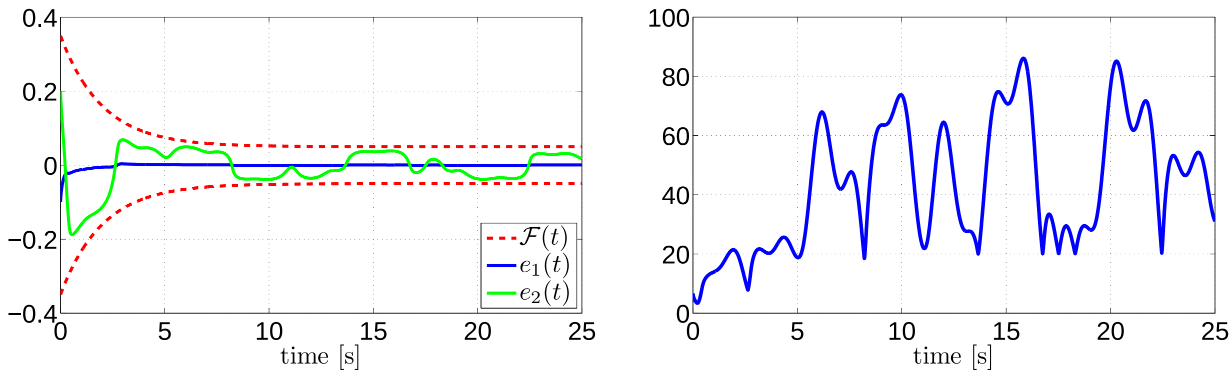
Figure 2.8: Comparison of performance to step reference command in the presence of known sensor dynamics.

methods. The control input $u_c(t)$ for the constant gain exhibits the peaking phenomenon, discussed in Section 2.5.3. The funnel control, variable gain, and saturation control inputs, denoted as $u_f(t)$, $u_v(t)$, and $u_s(t)$ respectively, avoid large peaks due to initialization. However, the output using the funnel control significantly deviates from the desired reference value before eventually converging. This is due to the fact that the gain remains small as shown in Figure 2.12. The output of the state dependent gain closely matches the output achieved using the constant gain but avoids peaking in the control channel. The saturation and variable methods avoid high gains and do not provide any significant change in response, although both methods have some downsides. For saturation, the saturation level must be chosen carefully in order to ensure no loss of stability or performance. In the variable gain generation method, stability is guaranteed with by choice of positive G_L, however, the value of α is a design parameter that must be tuned to achieve desired performance.



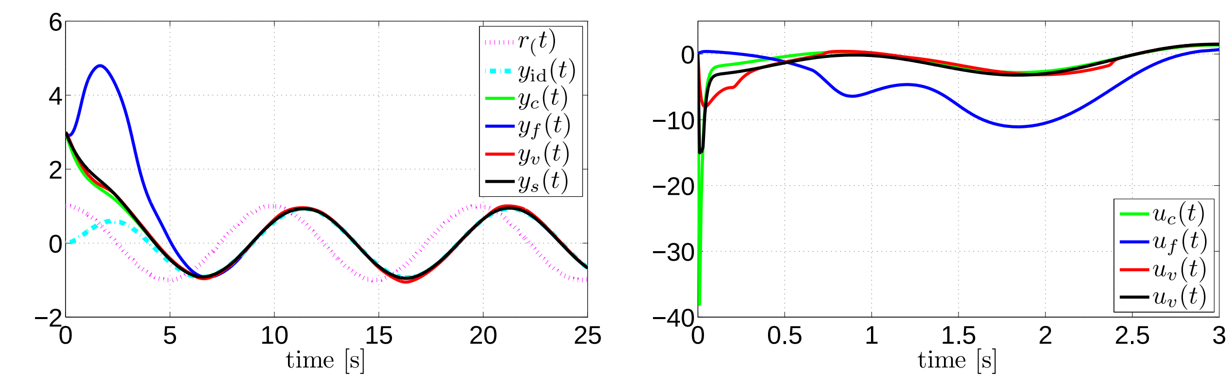
(a) $r(t)$, $y(t)$, and $y_{\rm id}(t)$
(b) Time history of $u(t)$

Figure 2.9: Performance of proportional $\mathcal{L}_1$ adaptive controller using funnel control gain.



(a) Funnel
(b) Time history of $\Gamma(t)$

Figure 2.10: Funnel control performance for proportional $\mathcal{L}_1$ adaptive controller.



(a) $r(t)$, $y(t)$, and $y_{\rm id}(t)$
(b) Time history of $u(t)$

Figure 2.11: Comparison of performance of proportional $\mathcal{L}_1$ adaptive controller using constant, funnel control, and state dependent gains.

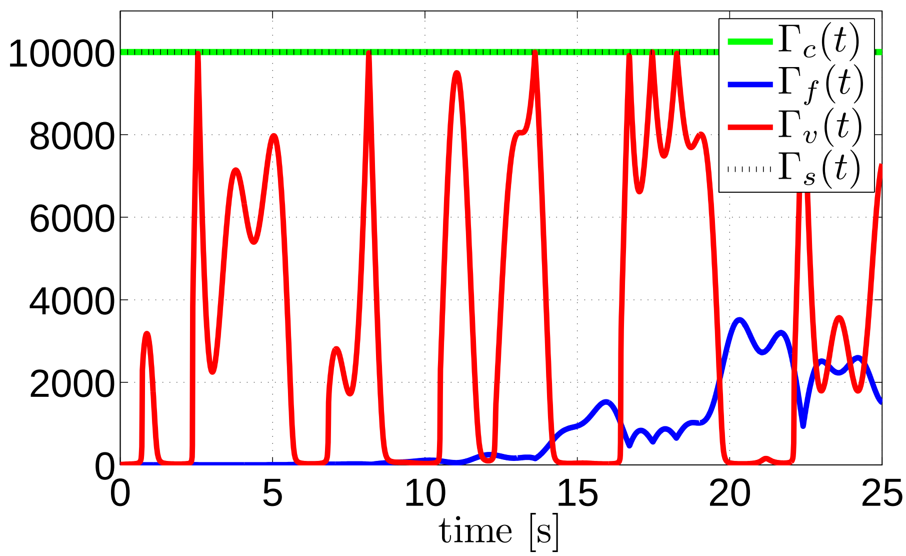


Figure 2.12: Time–varying gains values for constant, funnel control, and state dependent gain schemes.

CHAPTER 3

LINEAR TIME VARYING OUTPUT FEEDBACK

Please note that the variable names used in this chapter are independent from the previous chapter; that is the variable names may be reused and redefined in without any connection to Chapter 2.

3.1 Problem Formulation

Consider the following class of nonlinear systems:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + b(t)u(t) + f(t, x(t))\,, \quad x(0) = x_0\,, \\ y(t) &= c_m^\top(t)x(t)\,, \end{aligned} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ are unmeasured system state and the measured system output, respectively; $A(t) \in \mathbb{R}^{n\times n}$ and $b(t) \in \mathbb{R}^n$ are unknown time–varying matrix and a vector, respectively; $c_m(t) \in \mathbb{R}^n$ is a known time–varying vector; $u(t) \in \mathbb{R}$ is the control input. The initial condition x_0 is unknown, however it is assumed to belong to a known set such that $\|x_0\|_\infty \leq \rho_0 < \infty$, for a given $\rho_0 \in \mathbb{R}^+$; and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is an unknown map subject to the following assumption:

Assumption 3 (*Semi–global Lipschitz continuity and boundedness*). For all $\delta \in \mathbb{R}^+$ there exist constants $L_0(\delta) \in \mathbb{R}^+$ and $B(\delta) \in \mathbb{R}^+$, such that for all x, x_1 and x_2 with $\|x\|_\infty \leq \delta$,

$\|x_1\|_\infty \leq \delta$, $\|x_2\|_\infty \leq \delta$ the following bounds

$$\|f(t,x_1) - f(t,x_2)\|_\infty \leq L_0(\delta)\|x_1 - x_2\|_\infty \,,$$
$$\|f(t,x)\|_\infty \leq L_0(\delta)\|x\|_\infty + B(\delta)$$

hold uniformly for $t \geq 0$.

Let $r(t) \in \mathbb{R}$ be a given bounded reference input signal. The control objective is to design an adaptive output–feedback controller, which ensures that the system output $y(t)$ tracks the reference input $r(t)$ according to a desired model given by

$$\dot{x}_{\mathrm{id}}(t) = A_m(t)x_{\mathrm{id}}(t) + b_m(t)r_g(t)\,, \quad x_{\mathrm{id}}(0) = x_0\,,$$
$$y_{\mathrm{id}}(t) = c_m^\top(t)x_{\mathrm{id}}(t)\,,$$

where $A_m(t) \in \mathbb{R}^{n\times n}$ is a known Hurwitz time–varying matrix; $b_m(t) \in \mathbb{R}^n$ is a known time–varying vector such that the relative degree of the desired model is less or equal to $d_r \geq 1$ for all $t \geq 0$; and

$$r_g(t) \triangleq k_g(t)r(t)\,, \quad k_g(t) \triangleq -1/\left(c_m^\top(t)A_m^{-1}(t)b_m(t)\right)\,.$$

Let the systems above verify the following assumptions:

Assumption 4. There exit constants Δ_1 and Δ_2 such that

$$\|(A(t) - A_m(t))\|_\infty < \Delta_1\,, \quad \|(b(t) - b_m(t))\|_1 < \Delta_2 \quad \forall t \geq 0\,.$$

Assumption 5. There exist constants μ_b, μ_{bm}, $\mu_c \in \mathbb{R}^+$ such that $\|b(t)\|_\infty \leq \mu_b$, $\|b_m(t)\|_\infty \leq \mu_{bm}$, and $\|c_m(t)\|_1 \leq \mu_c$.

Assumption 6. $A_m(t)$, $b_m(t)$, and $c_m(t)$ are at least d_r times continuously differentiable.

Assumption 7. The pairs $(A_m(t),\, b_m(t))$ and $(A(t),\, b(t))$ are strongly controllable, and the pairs $(A_m(t),\, c_m^\top(t))$ and $(A(t),\, c_m^\top(t))$ are strongly observable.

Assumption 8 (*Stability of the desired system*). The matrix $A_m(t)$ is continuously differentiable and there exist positive constants $\mu_A > 0$, $d_A > 0$, and $\mu_\lambda > 0$, such that for all $t \geq 0$, $\|A_m(t)\|_\infty \leq \mu_A$, $\|\dot{A}_m(t)\|_\infty \leq d_A$, and $\mathrm{Re}[\lambda_i(A_m(t))] \leq -\mu_\lambda$, $\forall\, i = 1, \ldots, n$, where $\lambda_i(A_m(t))$ is a pointwise eigenvalue of $A_m(t)$. Further, for all $t \geq 0$, the equilibrium of the state equation

$$\dot{x} = A_m(t)x(t)\,,$$

is exponentially stable, and the solution of

$$A_m^\top(t)P(t) + P(t)A_m(t) = -\mathbb{I}$$

satisfies $P(t) = P^\top(t) > 0$ and $\|\dot{P}(t)\|_\infty \leq \epsilon_P < 1$.

Assumption 8 is standard for LTV control theory. Sometimes it is referred as stability of slowly varying systems [17].

3.2 $\mathcal{L}_1$ Adaptive Control Architecture

3.2.1 Definitions and $\mathcal{L}_1$–norm Sufficient Condition for Stability

We can rewrite the system in (3.1) as

$$\begin{aligned} \dot{x}(t) &= A_m(t)x(t) + b_m(t)u(t) + \sigma(t)\,, \quad x(0) = x_0\,, \\ y(t) &= c_m^\top(t)x(t)\,, \end{aligned} \tag{3.2}$$

where

$$\sigma(t) \triangleq (A(t) - A_m(t))x(t) + (b(t) - b_m(t))u(t) + f(t, x(t))\,. \tag{3.3}$$

Let $x_{\text{in}}(t)$ and $\tilde{y}_{\text{in}}(t)$ be the initial condition response of the systems

$$\begin{aligned}
\dot{x}_{\text{in}}(t) &= A_m(t)x_{\text{in}}(t)\,, \quad x_{\text{in}}(0) = x_0\,,\\
\dot{\tilde{x}}_{\text{in}}(t) &= A_m(t)\tilde{x}_{\text{in}}(t)\,, \quad \tilde{x}_{\text{in}}(0) = \hat{x}_0 - x_0\,,\\
\tilde{y}_{\text{in}}(t) &= c_m^\top(t)\tilde{x}_{\text{in}}(t)\,,
\end{aligned}$$

where $\hat{x}_0$ is our guess of the initial condition x_0 such that $c_m^\top(0)\hat{x}_0 = y(0)$. Then let ρ_{in} and $\tilde{\rho}_{\text{in}}$ be defined as

$$\rho_{\text{in}} \triangleq \max_{\|x_0\|_\infty \in \rho_0} \|x_{\text{in}}\|_{\mathcal{L}_\infty}\,, \quad \tilde{\rho}_{\text{in}} \triangleq \max_{\|x_0\|_\infty \in \rho_0,\ \|\hat{x}_0\|_\infty \in \rho_0} \|\tilde{y}_{\text{in}}\|_{\mathcal{L}_\infty}\,.$$

Next, let $\mathcal{H}_m$ and $\mathcal{H}_{xm}$ be the maps from $u(t)$ to $y(t)$ and $u(t)$ to $x(t)$, respectively, and $\mathcal{H}_{xum}$ be a map from $\sigma(t)$ to $x(t)$, of the system in (3.2) with initial conditions equal to zero. Also, let $\mathcal{H}_{yum}$ be the map from $\sigma(t)$ to $y(t)$ in (3.2) with initial conditions set to zero as well. The design of the $\mathcal{L}_1$ adaptive controller proceeds by considering a strictly proper system $C(s)$ with relative degree greater or equal to d_r and $C(0) = 1$. Further, the selection of $C(s)$ must ensure that there exists a constant $\rho_r \in \mathbb{R}^+$ such that the following $\mathcal{L}_1$–norm condition is satisfied:

$$\|\mathcal{G}_{um}\|_{\mathcal{L}_1} < \frac{\rho_r - \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\tilde{\rho}_{\text{in}} - \|\mathcal{H}_{xm}\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} - \rho_{\text{in}}}{L_\rho\rho_r + \Delta_2\|\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \Delta_2\|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\tilde{\rho}_{\text{in}} + B(\rho_r)}\,, \tag{3.4}$$

where $\mathcal{C}$ is the input–output map of the lowpass filter transfer function $C(s)$, $\mathcal{F}$ is the input–output map of

$$F(s) = \frac{1}{\sum_{i=0}^{d_r} a_i s^{(i)}}\,, \quad a_0 = 1\,, \tag{3.5}$$

which has real poles, and

$$\mathcal{G}_{um} \triangleq \left(\mathbb{I} - \mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}c_m^\top\right)\mathcal{H}_{xum}\left(\mathbb{I} + (b-b_m)\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}\right)^{-1}, \tag{3.6}$$

and L_ρ is defined as

$$L_\rho \triangleq \Delta_1 + L_0(\rho), \tag{3.7}$$

where

$$\rho \triangleq \rho_r + \bar{\gamma}_1, \tag{3.8}$$

with $\bar{\gamma}_1$ an arbitrary (small) positive constant. The definition and the procedure of computing the $\mathcal{L}_1$–norms for LTV systems can be found in [9].

In many state feedback architectures the $\mathcal{L}_1$–norm condition can always be satisfied by choosing the filter with sufficiently large bandwidth. However we notice that the $\mathcal{L}_1$-norm stability condition (3.4) cannot be satisfied simply by increasing the bandwidth of the lowpass filter $C(s)$, since $\|\mathcal{G}_{um}\|_{\mathcal{L}_1}$ does not necessarily decrease if the bandwidth of the filter is increased. Similar to [22], the condition (3.4) requires appropriate filter tuning. Filter design can be done using some of the methods presented in [31].

Next, define

$$\begin{aligned}\rho_{ur} \triangleq \|\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \|\mathcal{C}\mathcal{H}_m^{-1}\|_{\mathcal{L}_1}\Big(\|\mathcal{H}_{yum}\mathcal{H}_\omega\|_{\mathcal{L}_1}\big(\Delta_2(\|\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} \\ + \|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\tilde{\rho}_{\text{in}}) + L_\rho\rho_r + B(\rho_r)\big) + \|\mathcal{F}\|_{\mathcal{L}_1}\tilde{\rho}_{\text{in}}\Big),\end{aligned} \tag{3.9}$$

where

$$\mathcal{H}_\omega \triangleq \left(\mathbb{I} + (b-b_m)\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}\right)^{-1}, \tag{3.10}$$

and let

$$\rho_u \triangleq \rho_{ur} + \bar{\gamma}_2, \tag{3.11}$$

where $\bar{\gamma}_2 \in \mathbb{R}^+$ is an arbitrary (small) constant. Further, define Δ as

$$\Delta \triangleq L_\rho \rho + \Delta_2 \rho_u + B(\rho)\,. \tag{3.12}$$

Further, let

$$\begin{aligned}\gamma_1 \triangleq \frac{1}{1-\|\mathcal{G}_{um}\|_{\mathcal{L}_1} L_\rho}\Big(&\|\mathcal{G}_{um}(b-b_m)\mathcal{C}\mathcal{H}_m^{-1}\|_{\mathcal{L}_1}(\|\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 \\ &+ \|(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta) + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 \\ &+ \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta\Big) + \beta\,,\end{aligned} \tag{3.13}$$

where the values of $\bar{\gamma}_0 \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$ are arbitrarily small positive constants. Note that the denominators in (3.13) are nonsingular due to (3.4). We also define

$$\begin{aligned}\gamma_2 \triangleq \|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}\mathcal{H}_\omega\|_{\mathcal{L}_1}\big(&L_\rho\gamma_1 + \|(b-b_m)\mathcal{C}\mathcal{H}_m^{-1}\|_{\mathcal{L}_1} \\ &\cdot (\|\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 + \|(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta)\big) \\ &+ \|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 + \|\mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta + \beta\,.\end{aligned} \tag{3.14}$$

The choice of $\bar{\gamma}_0$, β, and $F(s)$ must ensure that

$$\gamma_1 < \bar{\gamma}_1\,, \quad \gamma_2 < \bar{\gamma}_2\,.$$

The $\mathcal{L}_1$–norm given by $\|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}$ exists since $\mathcal{H}_{xm}$ and $\mathcal{H}_{yum}$ are stable maps and the relative degree of $C(s)$ is greater than or equal to the relative degree of $\mathcal{H}_m$. Further, the value of this $\mathcal{L}_1$–norm can be made arbitrarily small by increasing the bandwidth of $F(s)$ in a similar manner as in Lemma 2.1.5 in [9].

Using Assumption 8 and the properties of $P(t)$, it follows that there exits a nonsingular

$\sqrt{P(t)}$ for all $t \geq 0$ such that

$$P(t) = \left(\sqrt{P(t)}\right)^\top \sqrt{P(t)}\,.$$

Let $D(t) \in \mathbb{R}^{n-1\times n}$ contain the basis of the null–space of $c_m^\top(t)\left(\sqrt{P(t)}\right)^{-1}$, that is

$$D(t)\left(c_m^\top(t)\left(\sqrt{P(t)}\right)^{-1}\right)^\top = 0\,, \tag{3.15}$$

for all $t \geq 0$; and further let

$$\Lambda(t) \triangleq \begin{bmatrix} c_m^\top(t) \\ D(t)\sqrt{P(t)} \end{bmatrix}. \tag{3.16}$$

Notice that

$$\Lambda(t)\left(\sqrt{P(t)}\right)^{-1} = \begin{bmatrix} c_m^\top(t)\left(\sqrt{P(t)}\right)^{-1} \\ D(t) \end{bmatrix}$$

is full rank, and hence $\Lambda^{-1}(t)$ exists $\forall t \in [0,\infty)$.

Lemma 3. For arbitrary $\xi(t) \triangleq [y(t)\ z(t)]^\top \in \mathbb{R}^n$, where $y(t) \in \mathbb{R}$ and $z(t) \in \mathbb{R}^{n-1}$, there exist $p_1(t) \in \mathbb{R}^+$ and a positive definite $P_2(t) \in \mathbb{R}^{n-1\times n-1}$ such that

$$\xi^\top(t)\Lambda^{-\top}(t)P(t)\Lambda^{-1}(t)\xi(t) = p_1(t)y^2(t) + z^\top(t)P_2(t)z(t)$$

for all $t \geq 0$.

The proof of the lemma is similar to the proof of Lemma 4.2.1 in [9] and therefore is omitted.

Further, let $T_s \in \mathbb{R}^+$ be an arbitrary constant that can be associated with the sampling rate of the available CPU, and let $\phi_{\tilde{\xi}}(i, T_s) \in \mathbb{R}^{n\times n}$ be given by

$$\phi_{\tilde{\xi}}(i, T_s) \triangleq \int_{iT_s}^{(i+1)T_s} \Phi_{\tilde{\xi}}((i+1)T_s, \tau)\Lambda(\tau)d\tau\,, \tag{3.17}$$

where $\Phi_{\tilde{\xi}}(\cdot,\cdot)$ represents the state transition matrix for the autonomous system with state matrix given by $\left(\Lambda(t)A_m(t) - \frac{d}{dt}\Lambda(t)\right)\Lambda^{-1}(t)$.

Next, define $\mathbf{1}_1 = [1,\ 0,\ \ldots,\ 0]^\top \in \mathbb{R}^n$, and let

$$\mathbf{1}_1^\top \Phi_{\tilde{\xi}}(iT_s + t, iT_s) = [\eta_1(i, T_s, t),\ \eta_2^\top(i, T_s, t)]\,, \tag{3.18}$$

where $\eta_1(i, T_s, t) \in \mathbb{R}$ and $\eta_2(i, T_s, t) \in \mathbb{R}^{n-1}$ contain the first and the 2–to–n elements of the row vector $\mathbf{1}_1^\top \Phi_{\tilde{\xi}}(iT_s + t, iT_s)$. Next let

$$\kappa(i, T_s) \triangleq \int_{iT_s}^{(i+1)T_s} \|\mathbf{1}_1^\top \Phi_{\tilde{\xi}}((i+1)T_s, \tau)\Lambda(\tau)\|_1 d\tau\,. \tag{3.19}$$

Also, let $\varsigma(i, T_s)$ and α be defined as $\varsigma(0, T_s) \triangleq 0$, and for $i \in \mathbb{N}$:

$$\varsigma(i, T_s) \triangleq \|\eta_2(i-1, T_s, T_s)\| \sqrt{\frac{\alpha}{\lambda_{P_{2\max}}}} + \kappa(i-1, T_s)\Delta\,, \tag{3.20}$$

$$\alpha \triangleq \max\left\{\lambda_{P_{\max}}\left(\frac{2\lambda_{P_{\max}}\Delta\mu_P\sqrt{n}}{\lambda_{P_{\min}}(1-\epsilon_P)}\right)^2,\ 4n\lambda_{\max}(P_2(0))\|\Lambda(0)\|_\infty^2\rho_0^2\right\}\,, \tag{3.21}$$

where

$$\lambda_{P_{2\max}} \triangleq \sup_{\substack{t\in[0,\infty),\\ i=1\ldots n-1}} \lambda_i(P_2(t))\,,\quad \lambda_{P_{\max}} \triangleq \sup_{\substack{t\in[0,\infty),\\ i=1\ldots n}} \lambda_i\left(P(t)\right)\,,\quad \lambda_{P_{\min}} \triangleq \inf_{\substack{t\in[0,\infty),\\ i=1\ldots n}} \lambda_i\left(P(t)\right)\,,$$

and

$$\mu_P \triangleq \sup_{t\in[0,\infty)} \|P(t)\|_\infty\,. \tag{3.22}$$

Next, we introduce the following constants

$$\beta_1(i,T_s) \triangleq \max_{t\in[iT_s,\ (i+1)T_s]} |\eta_1(i,T_s,t)|\,, \quad \beta_2(i,T_s) \triangleq \max_{t\in[iT_s,\ (i+1)T_s]} \|\eta_2(i,T_s,t)\|\,, \tag{3.23}$$

$$\beta_3(i,T_s) \triangleq \max_{t\in[iT_s,\ (i+1)T_s]} \eta_3(i,T_s,t)\,, \quad \beta_4(i,T_s) \triangleq \max_{t\in[iT_s,\ (i+1)T_s]} \eta_4(i,T_s,t)\,, \tag{3.24}$$

where

$$\eta_3(i,T_s,t) \triangleq \int_{iT_s}^{iT_s+t} |\mathbf{1}_1^\top \Phi_{\tilde{\xi}}(iT_s+t,\tau)\Lambda(\tau)\phi_{\tilde{\xi}}^{-1}(i,T_s)\Phi_{\tilde{\xi}}((i+1)T_s,iT_s)\,\mathbf{1}_1|d\tau\,,$$

$$\eta_4(i,T_s,t) \triangleq \int_{iT_s}^{iT_s+t} \|\mathbf{1}_1^\top \Phi_{\tilde{\xi}}(iT_s+t,\tau)\Lambda(\tau)\|_1 d\tau\,.$$

Finally, let

$$\gamma_0(i,T_s) \triangleq \beta_1(i,T_s)\varsigma(i,T_s) + \beta_2(i,T_s)\sqrt{\frac{\alpha}{\lambda_{P_{2\max}}}} + \beta_3(i,T_s)\varsigma(i,T_s) + \beta_4(i,T_s)\Delta\,, \tag{3.25}$$

and

$$\gamma_{\tilde{x}}(T_s) \triangleq \sup_{i\in\mathbb{N}\cup\{0\}} \gamma_0(i,T_s)\,. \tag{3.26}$$

The following lemma shows that the value of $\gamma_{\tilde{x}}(T_s)$ can be made arbitrarily small by reducing the value of T_s.

Lemma 4. The following limiting relationship is true:

$$\lim_{T_s\to 0} \gamma_{\tilde{x}}(T_s) = 0\,.$$

The proof of the lemma follows from the proof of Lemma 4.2.2 in [9], given the property $\Phi_{\tilde{\xi}}(iT_s, iT_s) = 0$.

3.2.2 $\mathcal{L}_1$ Adaptive Control Architecture

The $\mathcal{L}_1$ adaptive controller is comprised of the following elements:

Output Predictor:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m(t)\hat{x}(t) + b_m(t)u(t) + \hat{\sigma}(t)\,, \quad \hat{x}(0) = \hat{x}_0\,, \\ \hat{y}(t) &= c_m^\top(t)\hat{x}(t)\,, \end{aligned} \tag{3.27}$$

where $\hat{\sigma}(t) \in \mathbb{R}^n$ is the vector of adaptive parameters updated by the following piecewise–constant adaptation laws:

Adaptation Laws:

$$\hat{\sigma}(t) = -\mu(i, T_s)\tilde{y}(iT_s)\,, \quad t \in [iT_s,\ (i+1)T_s)\,, \quad i = 0, 1, 2, \dots\,, \tag{3.28}$$

where $\tilde{y}(t) \triangleq \hat{y}(t) - y(t)$, and

$$\mu(i, T_s) = \phi_{\tilde{\xi}}^{-1}(i, T_s)\Phi_{\tilde{\xi}}((i+1)T_s, iT_s)\mathbf{1}_1\,,$$

with $\phi_{\tilde{\xi}}(i, T_s)$ and $\Phi_{\tilde{\xi}}((i+1)T_s, iT_s)$ are defined in (3.17). The matrix $\mu(i, T_s)$ consists of time dependent gains that may be computed off–line. For the numerical computation of the state transition matrix for time varying systems can one can use the Peano–Baker Series [32].

Control Law:

$$u = \mathcal{C}r_g - \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\mathcal{H}_{yum}\hat{\sigma}\,. \tag{3.29}$$

Next we present an algorithm for on–line computation of $\mathcal{H}_m^{-1}\mathcal{F}$.

3.2.3 Computation of $\mathcal{H}_m^{-1}\mathcal{F}$

Let

$$\nu_{\text{out}} = \mathcal{H}_m^{-1}\mathcal{F}\nu_{\text{in}}\,,$$

where $\nu_{\text{out}}(t) \in \mathbb{R}$ and $\nu_{\text{in}}(t) \in \mathbb{R}$ are the output and the input of $\mathcal{H}_m^{-1}\mathcal{F}$ in (3.29) respectively. The mapping for $\mathcal{H}_m$ may be represented in Byrnes–Isidori form [33] through a coordinate transformation $U(t) \in \mathbb{R}^{n\times n}$ leading to

$$\begin{aligned}
\dot{\psi}(t) &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ R_1(t) & R_2(t) & \cdots & R_{d_r-1}(t) & R_{d_r}(t) \end{bmatrix} \psi(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ S(t) \end{bmatrix} \theta(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ J(t) \end{bmatrix} \nu_{\text{out}}(t)\,, \\
\dot{\theta}(t) &= \begin{bmatrix} E(t) & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \psi(t) \quad + G(t)\theta(t)\,, \\
\nu_{\text{in}F}(t) &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \psi(t) \\ \theta(t) \end{bmatrix},
\end{aligned} \tag{3.30}$$

where $\psi(t) \in \mathbb{R}^{d_r}$ is the system state, $\theta(t) \in \mathbb{R}^{n-d_r}$ are the zero dynamics state, $\nu_{\text{in}F} = \mathcal{F}\nu_{\text{in}}$, and $R(t) \in \mathbb{R}^{d_r\times d_r}$, $S(t) \in \mathbb{R}^{1\times n-d_r}$, $J(t) \in \mathbb{R}$, $E(t) \in \mathbb{R}^{n-d_r\times 1}$, and $G(t) \in \mathbb{R}^{n-d_r\times n-d_r}$ are defined in [33]. Notice that

$$\psi(t) \triangleq \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \\ \psi_{d_r}(t) \end{bmatrix} = \begin{bmatrix} \nu_{\text{in}F}(t) \\ \nu_{\text{in}F}^{(1)}(t) \\ \vdots \\ \nu_{\text{in}F}^{(d_r-1)}(t) \end{bmatrix}.$$

This implies that

$$\nu_{\text{in}F}^{(d_r)}(t) = S(t)\theta(t) + J(t)\nu_{\text{out}}(t) + R_1(t)\nu_{\text{in}F}(t) + R_2(t)\dot{\nu}_{\text{in}F}(t) + \ldots + R_{d_r}(t)\nu_{\text{in}F}^{(d_r-1)}(t)\,.$$

Therefore, $\nu_{\text{out}}(t)$ may be computed as follows

$$\nu_{\text{out}}(t) = \frac{\nu_{\text{in}F}^{(d_r)}(t) - S(t)\theta(t) - R_1(t)\nu_{\text{in}F}(t) - R_2(t)\dot{\nu}_{\text{in}F}(t) - \ldots - R_{d_r}(t)\nu_{\text{in}F}^{(d_r-1)}(t)}{J(t)}\,,$$

where $J(t) \neq 0$ due to Assumption 7. The derivatives $\dot{\nu}_{\text{in}F}$ to $\nu_{\text{in}F}^{(d_r)}$ can be computed using a fast filter $F(s)$ defined in (3.5), which gives us

$$\nu_{\text{in}F}^{(i)}(s) = \frac{s^i}{a_{d_r}s^{d_r} + \ldots + a_1 s + 1}\nu_{\text{in}}(s) = F(s)s^i\nu_{\text{in}}(s)\,, \quad i = 1\ldots d_r\,,$$

where the relative degree of $F(s)$ is d_r. The value of $\theta(t)$ can be computed by (3.30). Notice that the lowpass filter $\mathcal{C}$ in (3.29) cuts out high–frequency content produced by differentiation.

3.3 Analysis of the $\mathcal{L}_1$ Adaptive Controller

3.3.1 $\mathcal{L}_1$ Reference System

The $\mathcal{L}_1$ reference system is given by

$$\dot{x}_{\text{ref}}(t) = A_m(t)x_{\text{ref}}(t) + b_m(t)u_{\text{ref}}(t) + \sigma_{\text{ref}}(t)\,, \quad x_{\text{ref}}(0) = x_0\,, \tag{3.31}$$

$$y_{\text{ref}}(t) = c_m^\top(t)x_{\text{ref}}(t)\,, \tag{3.32}$$

$$u_{\text{ref}} = \mathcal{C}r_g - \mathcal{C}\mathcal{H}_m^{-1}\left(\mathcal{H}_{yum}\sigma_{\text{ref}} - \mathcal{F}\tilde{y}_{\text{in}}\right)\,, \tag{3.33}$$

$$\sigma_{\text{ref}}(t) \triangleq \left(A(t) - A_m(t)\right)x_{\text{ref}}(t) + (b(t) - b_m(t))u_{\text{ref}}(t) + f(t, x_{\text{ref}}(t))\,, \tag{3.34}$$

where $\tilde{y}_{\text{in}}(t)$ is the output of the following system

$$\dot{\tilde{x}}_{\text{in}}(t) = A_m(t)\tilde{x}_{\text{in}}(t)\,, \quad x_{\text{in}}(0) = \hat{x}_0 - x_0\,,$$
$$\tilde{y}_{\text{in}}(t) = c_m^\top(t)\tilde{x}_{\text{in}}(t)\,.$$

We notice that the $\mathcal{L}_1$ reference system contains the system uncertainties and the unknown initial condition x_0. Therefore it is not implementable and is used only for the analysis purposes.

Lemma 5. For the $\mathcal{L}_1$ reference system in (3.31)–(3.33), subject to the $\mathcal{L}_1$–norm condition (3.4), we have

$$\|x_{\text{ref}}\|_{\mathcal{L}_\infty} < \rho_r\,, \tag{3.35}$$

$$\|u_{\text{ref}}\|_{\mathcal{L}_\infty} \leq \rho_{ur}\,, \tag{3.36}$$

$$\|y_{\text{ref}}\|_{\mathcal{L}_\infty} < \mu_c\rho_r\,. \tag{3.37}$$

Proof :

Let

$$\vartheta_{\text{ref}}(t) \triangleq (A(t) - A_m(t))\, x_{\text{ref}}(t) + f(t, x_{\text{ref}}(t))\,, \tag{3.38}$$

then from (3.34) it follows that $\sigma_{\text{ref}}(t) = (b(t) - b_m(t))u_{\text{ref}}(t) + \vartheta_{\text{ref}}(t)$. Substituting the reference control law (3.33) and taking into account the definition in (3.10), we obtain

$$\begin{aligned}
\sigma_{\text{ref}} &= (b - b_m)\left(\mathcal{C}r_g - \mathcal{C}\mathcal{H}_m^{-1}\left(\mathcal{H}_{yum}\sigma_{\text{ref}} - \mathcal{F}\tilde{y}_{\text{in}}\right)\right) + \vartheta_{\text{ref}} \\
&= -(b - b_m)\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}\sigma_{\text{ref}} + (b - b_m)\left(\mathcal{C}r_g + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y}_{\text{in}}\right) + \vartheta_{\text{ref}} \\
&= \left(\mathbb{I} + (b - b_m)\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}\right)^{-1}\left((b - b_m)\left(\mathcal{C}r_g + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y}_{\text{in}}\right) + \vartheta_{\text{ref}}\right) \\
&= \mathcal{H}_\omega\left((b - b_m)\left(\mathcal{C}r_g + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y}_{\text{in}}\right) + \vartheta_{\text{ref}}\right)\,.
\end{aligned} \tag{3.39}$$

The system in (3.31) can be written as

$$x_{\mathrm{ref}} = \mathcal{H}_{xm} u_{\mathrm{ref}} + \mathcal{H}_{xum} \sigma_{\mathrm{ref}} + x_{\mathrm{in}}\,.$$

Then, by substituting $u_{\mathrm{ref}}(t)$ (3.33), we obtain

$$x_{\mathrm{ref}} = \mathcal{H}_{xm} \left(\mathcal{C} r_g - \mathcal{C} \mathcal{H}_m^{-1} (\mathcal{H}_{yum} \sigma_{\mathrm{ref}} - \mathcal{F} \tilde{y}_{\mathrm{in}}) \right) + \mathcal{H}_{xum} \sigma_{\mathrm{ref}} + x_{\mathrm{in}}\,.$$

The above equation can be rearranged to obtain the following

$$x_{\mathrm{ref}} = \left(\mathcal{H}_{xum} - \mathcal{H}_{xm} \mathcal{C} \mathcal{H}_m^{-1} \mathcal{H}_{yum} \right) \sigma_{\mathrm{ref}} + \mathcal{H}_{xm} \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \tilde{y}_{\mathrm{in}} + \mathcal{H}_{xm} \mathcal{C} r_g + x_{\mathrm{in}}\,.$$

Next, using the fact that $\mathcal{H}_{yum} = c_m^\top \mathcal{H}_{xum}$, (3.39), and the definition in (3.6), we obtain

$$\begin{aligned} x_{\mathrm{ref}} &= \left(\mathbb{I} - \mathcal{H}_{xm} \mathcal{C} \mathcal{H}_m^{-1} c_m^\top \right) \mathcal{H}_{xum} \sigma_{\mathrm{ref}} + \mathcal{H}_{xm} \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \tilde{y}_{\mathrm{in}} + \mathcal{H}_{xm} \mathcal{C} r_g + x_{\mathrm{in}} \\ &= \mathcal{G}_{um} \left((b - b_m) \left(\mathcal{C} r_g + \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \tilde{y}_{\mathrm{in}} \right) + \vartheta_{\mathrm{ref}} \right) \\ &\quad + \mathcal{H}_{xm} \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \tilde{y}_{\mathrm{in}} + \mathcal{H}_{xm} \mathcal{C} r_g + x_{\mathrm{in}}\,. \end{aligned} \tag{3.40}$$

Next, we use a contradictive argument to prove the bound in (3.35). For this we assume that (3.35) does not hold. Since $x_{\mathrm{ref}}(t)$ is continuous and $\|x_{\mathrm{ref}}(0)\|_\infty = \|x_0\|_\infty \le \rho_0 < \rho_r$, then there exists time $\tau > 0$ such that

$$\|x_{\mathrm{ref}}(t)\|_\infty < \rho_r\,, \quad \forall t \in [0,\ \tau)\,, \tag{3.41}$$

$$\|x_{\mathrm{ref}}(\tau)\|_\infty = \rho_r\,, \tag{3.42}$$

which implies $\|x_{\text{ref}\tau}\|_{\mathcal{L}_\infty} = \rho_r$. Using Assumptions 3 and 4 and (3.7), we obtain the following bound from (3.38):

$$\|\vartheta_{\text{ref}\tau}\|_{\mathcal{L}_\infty} \leq \Delta_1 \rho_r + L_0(\rho_r)\rho_r + B(\rho_r) \leq L_\rho \rho_r + B(\rho_r)\,, \tag{3.43}$$

where we use the fact that $L_\rho \geq L_{\rho_r}$. This allows us, using (3.40) and Assumption 4, obtain the following bound:

$$\begin{aligned}
\|x_{\text{ref}\tau}\|_{\mathcal{L}_\infty} \leq{}& \|\mathcal{G}_{um}\|_{\mathcal{L}_1} \left(\Delta_2 \|\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \Delta_2 \|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\|\tilde{y}_{\text{in}}\|_{\mathcal{L}_\infty} + \|\vartheta_{\text{ref}\tau}\|_{\mathcal{L}_\infty}\right) \\
& + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\|\tilde{y}_{\text{in}}\|_{\mathcal{L}_\infty} + \|\mathcal{H}_{xm}\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \rho_{\text{in}} \\
\leq{}& \|\mathcal{G}_{um}\|_{\mathcal{L}_1}\Big(\Delta_2 \|\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \Delta_2 \|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\|\tilde{y}_{\text{in}}\|_{\mathcal{L}_\infty} + L_\rho \rho_r \\
& + B(\rho_r)\Big) + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\|\tilde{y}_{\text{in}}\|_{\mathcal{L}_\infty} + \|\mathcal{H}_{xm}\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \rho_{\text{in}}\,.
\end{aligned} \tag{3.44}$$

Notice that (3.4) can be rewritten as

$$\begin{aligned}
&\|\mathcal{G}_{um}\|_{\mathcal{L}_1}\Big(L_\rho \rho_r + \Delta_2 \|\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \Delta_2 \|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\tilde{\rho}_{\text{in}} + B(\rho_r)\Big) \\
&\qquad + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\tilde{\rho}_{\text{in}} + \|\mathcal{H}_{xm}\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \rho_{\text{in}} < \rho_r
\end{aligned}$$

which along with (3.44), implies

$$\|x_{\text{ref}\tau}\|_{\mathcal{L}_\infty} < \rho_r\,.$$

This fact contradicts to (3.42), and hence the bound in (3.35) is proven. The bound in (3.37) follows immediately from the Assumption 5.

To prove the bound in (3.36), we substitute (3.39) in (3.33) to obtain

$$u_{\text{ref}} = \mathcal{C} r_g - \mathcal{C}\mathcal{H}_m^{-1}\left(\mathcal{H}_{yum}\mathcal{H}_\omega \left((b - b_m)\left(\mathcal{C} r_g + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y}_{\text{in}}\right) + \vartheta_{\text{ref}}\right) - \mathcal{F}\tilde{y}_{\text{in}}\right)\,,$$

which using the bound in (3.43), results in

$$\|u_{\rm ref}\|_{\mathcal{L}_\infty} \leq \|\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \|\mathcal{C}\mathcal{H}_m^{-1}\|_{\mathcal{L}_1}\Big(\|\mathcal{H}_{yum}\mathcal{H}_\omega\|_{\mathcal{L}_1}\big(\Delta_2(\|\mathcal{C}\|_{\mathcal{L}_1}\|r_g\|_{\mathcal{L}_\infty} + \|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\|\tilde{y}_{\rm in}\|_{\mathcal{L}_\infty}) + L_\rho\rho_r + B(\rho_r)\big) + \|\mathcal{F}\|_{\mathcal{L}_1}\|\tilde{y}_{\rm in}\|_{\mathcal{L}_\infty}\Big) .$$

Taking into account the definition of ρ_{ur} in (3.9), we conclude that

$$\|u_{\rm ref}\|_{\mathcal{L}_\infty} \leq \rho_{ur} \,,$$

which completes the proof. □

3.3.2 Transient and Steady–State Performance

We will now proceed with the derivation of the performance bounds. Towards this end, let $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$ and $\tilde{\sigma}(t) \triangleq \hat{\sigma}(t) - \sigma(t)$. Then, the error dynamics between (3.27) and (3.2) are given by

$$\begin{aligned} \dot{\tilde{x}}(t) &= A_m(t)\tilde{x}(t) + \tilde{\sigma}(t), \quad \tilde{x}(0) = \hat{x}_0 - x_0 \,, \\ \tilde{y}(t) &= c_m^\top(t)\tilde{x}(t) \,. \end{aligned} \tag{3.45}$$

Notice that due to initialization of the output predictor, we have $\tilde{y}(0) = 0$.

Next, consider the state transformation

$$\tilde{\xi}(t) = \Lambda(t)\tilde{x}(t) \,.$$

It follows from (3.45) and the definition of $\Lambda(t)$ in (3.16) that

$$\begin{aligned} \dot{\tilde{\xi}}(t) &= \left(\Lambda(t)A_m(t)\Lambda^{-1}(t) - \frac{d}{dt}\left(\Lambda(t)\right)\Lambda^{-1}(t)\right)\tilde{\xi}(t) + \Lambda(t)\tilde{\sigma}(t)\,, \\ \tilde{y}(t) &= \tilde{\xi}_1(t)\,, \quad \tilde{\xi}(0) = \Lambda(0)\tilde{x}_0\,, \end{aligned} \tag{3.46}$$

where $\tilde{\xi}_1(t)$ is the first element of $\tilde{\xi}(t)$ and $\tilde{\xi}_1(0) = 0$. The next lemma derives the bound on the output prediction error.

Lemma 6. Consider the system in (3.1) and the $\mathcal{L}_1$ adaptive controller in (3.27), (3.28), and (3.29) subject to the $\mathcal{L}_1$–norm condition in (3.4). If we choose T_s to ensure

$$\gamma_{\tilde{x}}(T_s) < \bar{\gamma}_0\,, \tag{3.47}$$

where $\bar{\gamma}_0$ is an arbitrary positive constant introduced in (3.13), and if for an arbitrary $\tau \geq 0$ the following bounds hold:

$$\|x_\tau\|_{\mathcal{L}_\infty} < \rho\,, \quad \|u_\tau\|_{\mathcal{L}_\infty} < \rho_u\,,$$

then

$$\|\tilde{y}_\tau\|_{\mathcal{L}_\infty} < \bar{\gamma}_0\,. \tag{3.48}$$

Proof :

We prove the bound in (3.48) by a contradiction argument. Since $\tilde{y}(0) = 0$ and $\tilde{y}(t)$ is continuous, then assuming that (3.48) does not hold, implies that there exists $t' \in (0,\ \tau]$ such that

$$|\tilde{y}(t)| < \bar{\gamma}_0\,, \quad \forall\ t \in [0, t')\,, \tag{3.49}$$

$$|\tilde{y}(t')| = \bar{\gamma}_0\,, \tag{3.50}$$

which leads to

$$\|\tilde{y}_{t'}\|_{\mathcal{L}_\infty} = \bar{\gamma}_0\,. \tag{3.51}$$

The following bound can be produced from (3.3) using Assumptions 3, 4 and definition (3.12):

$$\begin{aligned}\|\sigma_{t'}\|_{\mathcal{L}_\infty} &\le \Delta_1\rho + \Delta_2\rho_u + L_0(\rho)\rho + B(\rho)\\ &= L_\rho\rho + \Delta_2\rho_u + B(\rho) = \Delta\,.\end{aligned} \tag{3.52}$$

It follows from (3.46) that

$$\begin{aligned}\tilde{\xi}(iT_s+t) = \Phi_{\tilde{\xi}}(iT_s+t, iT_s)\tilde{\xi}(iT_s) &+ \int_{iT_s}^{iT_s+t}\Phi_{\tilde{\xi}}(iT_s+t,\tau)\Lambda(\tau)\hat{\sigma}(iT_s)d\tau\\ &- \int_{iT_s}^{iT_s+t}\Phi(iT_s+t,\tau)\Lambda(\tau)\sigma(\tau)d\tau\,.\end{aligned} \tag{3.53}$$

Since

$$\tilde{\xi}(iT_s+t) = \begin{bmatrix}\tilde{y}(iT_s+t)\\ 0\end{bmatrix} + \begin{bmatrix}0\\ \tilde{z}(iT_s+t)\end{bmatrix}, \tag{3.54}$$

where $\tilde{z}(t) \triangleq [\tilde{\xi}_2(t),\ \tilde{\xi}_3(t),\ \ldots,\ \tilde{\xi}_n(t)]^\top$, it follows from (3.53) that $\tilde{\xi}(iT_s+t)$ can be decomposed as

$$\tilde{\xi}(iT_s+t) = \chi(iT_s+t) + \zeta(iT_s+t)\,, \tag{3.55}$$

where

$$\begin{aligned}\chi(iT_s+t) &\triangleq \Phi_{\tilde{\xi}}(iT_s+t, iT_s)\begin{bmatrix}\tilde{y}(iT_s)\\ 0\end{bmatrix} + \int_{iT_s}^{iT_s+t}\Phi_{\tilde{\xi}}(iT_s+t,\tau)\Lambda(\tau)\hat{\sigma}(iT_s)d\tau\,,\\ \zeta(iT_s+t) &\triangleq \Phi_{\tilde{\xi}}(iT_s+t, iT_s)\begin{bmatrix}0\\ \tilde{z}(iT_s)\end{bmatrix} - \int_{iT_s}^{iT_s+t}\Phi_{\tilde{\xi}}(iT_s+t,\tau)\Lambda(\tau)\sigma(\tau)d\tau\,.\end{aligned} \tag{3.56}$$

Next we prove by induction that for all i such that $iT_s \leq t'$ one has

$$|\tilde{y}(iT_s)| \leq \varsigma(i, T_s)\,, \tag{3.57}$$

$$\tilde{z}^\top(iT_s)P_2(iT_s)\tilde{z}(iT_s) \leq \alpha\,, \tag{3.58}$$

where $\varsigma(i, T_s)$ and α were defined in (3.20)–(3.21).

We start by noting that, since $\tilde{y}(0) = 0$, we have $|\tilde{y}(0)| \leq \varsigma(0, T_s)$. We can also show the following, given the definition of α in (3.21) we have

$$\begin{aligned}\tilde{z}^\top(0)P_2(0)\tilde{z}(0) &\leq \lambda_{\max}(P_2(0))\|\tilde{z}(0)\|^2 = \lambda_{\max}(P_2(0))\|\tilde{\xi}(0)\|^2 \\ &\leq \lambda_{\max}(P_2(0))\|\Lambda(0)\tilde{x}_0\|^2 \leq 4n\lambda_{\max}(P_2(0))\|\Lambda(0)\|_\infty^2\rho_0^2 \leq \alpha\,.\end{aligned}$$

Next, we prove that if (3.57)–(3.58) hold for arbitrary i such that $(i+1)T_s \leq t'$, then

$$|\tilde{y}((i+1)T_s)| \leq \varsigma(i+1, T_s)\,, \tag{3.59}$$

$$\tilde{z}^\top((i+1)T_s)P_2((i+1)T_s)\tilde{z}((i+1)T_s) \leq \alpha\,, \tag{3.60}$$

hold as well. To this end, assume that (3.57)–(3.58) hold for i, and in addition, that $(i+1)T_s \leq t'$. Then, it follows from (3.55) that

$$\tilde{\xi}((i+1)T_s) = \chi((i+1)T_s) + \zeta((i+1)T_s)\,, \tag{3.61}$$

where

$$\chi((i+1)T_s) = \Phi_{\tilde{\xi}}((i+1)T_s, iT_s)\begin{bmatrix} \tilde{y}(iT_s) \\ 0 \end{bmatrix} + \int_{iT_s}^{(i+1)T_s} \Phi_{\tilde{\xi}}((i+1)T_s, \tau)\Lambda(\tau)\hat{\sigma}(iT_s)d\tau\,, \tag{3.62}$$

$$\zeta((i+1)T_s) = \Phi_{\tilde{\xi}}((i+1)T_s, iT_s)\begin{bmatrix} 0 \\ \tilde{z}(iT_s) \end{bmatrix} - \int_{iT_s}^{(i+1)T_s} \Phi_{\tilde{\xi}}((i+1)T_s, \tau)\Lambda(\tau)\sigma(\tau)d\tau\,. \tag{3.63}$$

Substituting the adaptive law from (3.28) in (3.62), we have

$$\chi((i+1)T_s) = 0\,. \tag{3.64}$$

On the other hand, it follows from (3.56) that $\zeta(t)$ is the solution to the following dynamics:

$$\begin{aligned} \dot{\zeta}(t) &= \left(\Lambda(t)A_m(t) - \frac{d}{dt}\Lambda(t)\right)\Lambda^{-1}(t)\zeta(t) - \Lambda(t)\sigma(t)\,, \\ \zeta(iT_s) &= \begin{bmatrix} 0 \\ \tilde{z}(iT_s) \end{bmatrix}, \quad t \in [iT_s,\ (i+1)T_s]\,. \end{aligned} \tag{3.65}$$

Consider now the following function

$$V(\zeta(t)) = \zeta^\top(t)\Lambda^{-\top}(t)P(t)\Lambda^{-1}(t)\zeta(t)\,, \tag{3.66}$$

over $t \in [iT_s,\ (i+1)T_s]$. Lemma 3 implies that $\Lambda^{-\top}(t)P(t)\Lambda^{-1}(t)$ is positive definite and, hence, $V(\zeta)$ is a positive definite function. Further, it follows from (3.56), Lemma 3 and the fact (3.64) that

$$V(\zeta(iT_s)) = \tilde{z}^\top(iT_s)P_2(iT_s)\tilde{z}(iT_s)\,,$$

which, along with the upper bound in (3.58), leads to

$$V(\zeta(iT_s)) \leq \alpha \,. \tag{3.67}$$

Next we perform a reverse state transformation for the system in (3.65) with the state transition matrix $\bar{\zeta}(t) \triangleq \Lambda^{-1}(t)\zeta(t)$ to obtain the following system

$$\dot{\bar{\zeta}}(t) = A_m(t)\bar{\zeta}(t) - \sigma(t) \,. \tag{3.68}$$

The function $V(\zeta)$ in (3.66) now takes the form

$$V(\bar{\zeta}(t)) = \bar{\zeta}^\top(t)P(t)\bar{\zeta}(t) \,. \tag{3.69}$$

Taking the time derivative of (3.69) along the trajectories (3.68) over $t \in [iT_s,\ (i+1)T_s]$, and using Assumption 8, we obtain

$$\begin{aligned}
\dot{V}(t) &= \dot{\bar{\zeta}}^\top(t)P(t)\bar{\zeta}(t) + \bar{\zeta}^\top(t)\dot{P}(t)\bar{\zeta}(t) + \bar{\zeta}^\top(t)P(t)\dot{\bar{\zeta}}(t) \\
&= \left(\bar{\zeta}^\top(t)A_m^\top(t) - \sigma^\top(t)\right) P(t)\bar{\zeta}(t) + \bar{\zeta}^\top(t)\dot{P}(t)\bar{\zeta}(t) + \bar{\zeta}^\top(t)P(t)\left(A_m(t)\bar{\zeta}(t) - \sigma(t)\right) \\
&= \bar{\zeta}^\top(t)\left(A_m^\top(t)P(t) + P(t)A_m(t) + \dot{P}(t)\right)\bar{\zeta}(t) - \sigma^\top(t)P(t)\bar{\zeta}(t) - \bar{\zeta}^\top(t)P(t)\sigma(t) \\
&= -\bar{\zeta}^\top(t)\left(\mathbb{I} - \dot{P}(t)\right)\bar{\zeta}(t) - 2\bar{\zeta}^\top(t)P(t)\sigma(t) \,,
\end{aligned}$$

which, using the the facts that $\|\sigma(t)\|_\infty \leq \|\sigma_{t'}\|_{\mathcal{L}_\infty}$, $\|\cdot\|_1 \leq \sqrt{n}\|\cdot\|_2$, the definition (3.22), and the bound in (3.52), can be bounded as follows

$$\begin{aligned}\dot{V}(t) &\leq -\bar{\zeta}^\top(t)\left(\mathbb{I}-\dot{P}(t)\right)\bar{\zeta}(t) + 2\|\bar{\zeta}^\top(t)P(t)\sigma(t)\|_\infty \\ &\leq -\bar{\zeta}^\top(t)\left(\mathbb{I}-\dot{P}(t)\right)\bar{\zeta}(t) + 2\|\bar{\zeta}(t)\|_1\|P(t)\|_\infty\|\sigma(t)\|_\infty \\ &\leq -\bar{\zeta}^\top(t)\left(\mathbb{I}-\dot{P}(t)\right)\bar{\zeta}(t) + 2\sqrt{n}\|\bar{\zeta}(t)\|\mu_P\Delta\,.\end{aligned}$$

Assumption 8 along with the fact that $|\bar{\zeta}^\top(t)\dot{P}(t)\bar{\zeta}(t)| \leq \|\bar{\zeta}(t)\|^2\rho(\dot{P}(t)) \leq \|\bar{\zeta}(t)\|^2\|\dot{P}(t)\|_\infty$ implies

$$\bar{\zeta}^\top(t)\left(\mathbb{I}-\dot{P}(t)\right)\bar{\zeta}(t) \geq \frac{1-\epsilon_P}{\lambda_{P_{\max}}}\bar{\zeta}^\top(t)P(t)\bar{\zeta}(t) \geq \frac{1-\epsilon_P}{\lambda_{P_{\max}}}\lambda_{P_{\min}}\|\bar{\zeta}(t)\|^2\,.$$

This results in

$$\dot{V}(t) \leq -\frac{1-\epsilon_P}{\lambda_{P_{\max}}}\lambda_{P_{\min}}\|\bar{\zeta}(t)\|^2 + 2\sqrt{n}\|\bar{\zeta}(t)\|\mu_P\Delta\,. \tag{3.70}$$

Notice that for any $t \in [iT_s,\ (i+1)T_s]$, if

$$V(t) > \alpha\,,$$

we have

$$\|\bar{\zeta}(t)\| > \sqrt{\frac{\alpha}{\lambda_{P_{\max}}}} \geq \frac{2\lambda_{P_{\max}}\Delta\mu_P\sqrt{n}}{\lambda_{P_{\min}}(1-\epsilon_P)}\,,$$

and the upper bound in (3.70) yields

$$\dot{V}(t) < 0\,. \tag{3.71}$$

Thust, it follows from (3.67) that

$$V(t) \leq \alpha\,, \quad \forall\ t \in [iT_s,\ (i+1)T_s]\,.$$

Taking into account the relationship (3.61) along with (3.64), we can rewrite (3.66) as

$$V((i+1)T_s) = \tilde{\xi}^\top((i+1)T_s)(\Lambda^{-\top}((i+1)T_s)P((i+1)T_s)\Lambda^{-1}((i+1)T_s))\tilde{\xi}((i+1)T_s) \le \alpha\,.$$

Using the result of Lemma 3, one can derive

$$\begin{aligned}&\tilde{z}^\top((i+1)T_s)P_2(t)\tilde{z}((i+1)T_s) \le \\ &\tilde{\xi}^\top((i+1)T_s)(\Lambda^{-\top}((i+1)T_s)P((i+1)T_s)\Lambda^{-1}((i+1)T_s))\tilde{\xi}((i+1)T_s) \le \alpha\,,\end{aligned}$$

which implies that the upper bound in (3.60) holds. Next, it follows from (3.61), (3.64), and (3.54) that

$$\tilde{y}((i+1)T_s) = \mathbf{1}_1^\top\zeta((i+1)T_s)\,,$$

and (3.63) leads to the following expression:

$$\begin{aligned}\tilde{y}((i+1)T_s) = \mathbf{1}_1^\top\Phi_{\tilde{\xi}}((i+1)T_s, iT_s)&\begin{bmatrix} 0 \\ \tilde{z}(iT_s)\end{bmatrix} \\ -\,\mathbf{1}_1^\top\int_{iT_s}^{(i+1)T_s}&\Phi_{\tilde{\xi}}((i+1)T_s,\tau)\Lambda(\tau)\sigma(\tau)d\tau\,.\end{aligned}$$

The upper bounds in (3.60) and (3.52) yield the following upper bound:

$$\begin{aligned}|\tilde{y}((i+1)T_s)| &\le \|\eta_2(i,T_s,T_s)\|\|\tilde{z}(iT_s)\| + \int_{iT_s}^{(i+1)T_s}\|\mathbf{1}_1^\top\Phi_{\tilde{\xi}}((i+1)T_s,\tau)\Lambda(\tau)\|_1\|\sigma(\tau)\|_\infty d\tau \\ &\le \|\eta_2(i,T_s,T_s)\|\sqrt{\frac{\alpha}{\lambda_{P_2\max}}} + \kappa(i,T_s)\Delta = \varsigma(i+1,T_s)\,,\end{aligned}$$

where $\eta_2(i,T_s)$ and $\kappa(i,T_s)$ were defined in (3.18) and (3.19), while $\varsigma(i,T_s)$ was defined in (3.20). This confirms the upper bound in (3.59). Hence, (3.57)–(3.58) hold for all i such that $iT_s \le t'$.

For all $iT_s + t \le t'$, where $0 \le t \le T_s$, using the expression from (3.53), we can write that

$$\tilde{y}(iT_s + t) = \mathbf{1}_1^\top \Phi_{\tilde{\xi}}(iT_s + t, iT_s)\tilde{\xi}(iT_s) + \mathbf{1}_1^\top \int_{iT_s}^{iT_s+t} \Phi_{\tilde{\xi}}(iT_s + t, \tau)\Lambda(\tau)\hat{\sigma}(iT_s)d\tau$$
$$- \mathbf{1}_1^\top \int_{iT_s}^{iT_s+t} \Phi_{\tilde{\xi}}(iT_s + t, \tau)\Lambda(\tau)\sigma(\tau)d\tau \,.$$

The upper bound in (3.52) and the definitions of $\eta_1(t)$, $\eta_2(t)$, $\eta_3(t)$ and $\eta_4(t)$ lead to the following upper bound:

$$|\tilde{y}(iT_s + t)| \le |\eta_1(i, T_s, t)||\tilde{y}(iT_s)| + \|\eta_2(i, T_s, t)\|\|\tilde{z}(iT_s)\| + \eta_3(i, T_s, t)|\tilde{y}(iT_s)| + \eta_4(i, T_s, t)\Delta \,.$$

Taking into consideration (3.57)–(3.58), and recalling the definitions of $\beta_1(i, T_s)$, $\beta_2(i, T_s)$, $\beta_3(i, T_s)$, $\beta_4(i, T_s)$ in (3.23)–(3.24), for all $0 \le t \le T_s$ and for arbitrary non–negative integer i subject to $iT_s + t \le t'$, we have

$$|\tilde{y}(iT_s + t)| \le \beta_1(i, T_s)\varsigma(i, T_s) + \beta_2(i, T_s)\sqrt{\frac{\alpha}{\lambda_{P_{2\max}}}} + \beta_3(i, T_s)\varsigma(i, T_s) + \beta_4(i, T_s)\Delta \,.$$

Since the right hand side coincides with the definition of $\gamma_0(i, T_s)$ in (3.25), for all $t \in [0,\ t']$ we have the following bound

$$|\tilde{y}(t)| \le \gamma_0(i, T_s) \,, \quad \forall i \in \mathbb{N} \cup \{0\} \,,$$

which along with (3.47) yields

$$\|\tilde{y}_{t'}\|_{\mathcal{L}_\infty} \le \gamma_{\tilde{x}}(T_s) < \bar{\gamma}_0 \,.$$

This clearly contradicts the statement in (3.50). Therefore (3.48) holds and the proof is completed. □

The next theorem states the main result of the extension presented in this chapter.

Theorem 2. Given the closed–loop system with the $\mathcal{L}_1$ adaptive controller defined via (3.1), (3.27), (3.28), (3.29), subject to the $\mathcal{L}_1$–norm condition in (3.4), and the closed–loop reference system in (3.31)–(3.33), if we choose T_s to ensure

$$\gamma_{\tilde{x}}(T_s) < \bar{\gamma}_0 \,,$$

where $\bar{\gamma}_0$ is an arbitrary positive constant introduced in (3.13), we have

$$\|x\|_{\mathcal{L}_\infty} \leq \rho \,, \tag{3.72}$$

$$\|u\|_{\mathcal{L}_\infty} \leq \rho_u \,, \tag{3.73}$$

$$\|x_{\mathrm{ref}} - x\|_{\mathcal{L}_\infty} < \gamma_1 \,, \tag{3.74}$$

$$\|y_{\mathrm{ref}} - y\|_{\mathcal{L}_\infty} < \mu_c \gamma_1 \,, \tag{3.75}$$

$$\|u_{\mathrm{ref}} - u\|_{\mathcal{L}_\infty} < \gamma_2 \,, \tag{3.76}$$

where γ_1 and γ_2 are defined in (3.13) and (3.14) respectively.

Proof :

To accomplish the proof, we use contradictory argument. Assume that the bounds in (3.74) and (3.76) do not hold (either one of them or both simultaneously). Then, since

$$\|x_{\mathrm{ref}}(0) - x(0)\|_\infty = 0 < \gamma_1 \,, \quad \|u_{\mathrm{ref}}(0) - u(0)\|_\infty = 0 < \gamma_2 \,,$$

and $x(t)$, $x_{\mathrm{ref}}(t)$, $u(t)$, and $u_{\mathrm{ref}}(t)$ are continuous, there exists time $\tau \in \mathbb{R}^+$ such that

$$\|x_{\mathrm{ref}}(\tau) - x(\tau)\|_\infty = \gamma_1 \,, \quad \text{or} \quad \|u_{\mathrm{ref}}(\tau) - u(\tau)\|_\infty = \gamma_2 \,, \tag{3.77}$$

while

$$\|x_{\text{ref}}(t) - x(t)\|_\infty < \gamma_1\,, \quad \text{and} \quad \|u_{\text{ref}}(t) - u(t)\|_\infty < \gamma_2\,,$$

for all $t \in [0, \tau)$. This implies that that the following equalities hold

$$\|(x_{\text{ref}} - x)_\tau\|_{\mathcal{L}_\infty} \leq \gamma_1\,, \quad \|(u_{\text{ref}} - u)_\tau\|_{\mathcal{L}_\infty} \leq \gamma_2\,. \tag{3.78}$$

From Lemma 5 we obtain

$$\|x_{\text{ref}}\|_{\mathcal{L}_\infty} \leq \rho_r\,, \quad \|u_{\text{ref}}\|_{\mathcal{L}_\infty} \leq \rho_{ur}\,, \tag{3.79}$$

which along with the definitions of ρ and ρ_u in (3.8) and (3.11) allows us to derive from (3.78) that

$$\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_r + \gamma_1 < \rho\,, \quad \text{and} \quad \|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_{ur} + \gamma_2 < \rho_u\,. \tag{3.80}$$

Let

$$\vartheta(t) \triangleq (A(t) - A_m(t))\,x(t) + f(t, x(t))\,,$$
$$\vartheta_{\text{ref}}(t) \triangleq (A(t) - A_m(t))\,x_{\text{ref}}(t) + f(t, x_{\text{ref}}(t))\,,$$

then using Assumptions 3, 4 and the bounds (3.78)–(3.80) we obtain

$$\|(\vartheta_{\text{ref}} - \vartheta)_\tau\|_{\mathcal{L}_\infty} \leq \Delta_1\|(x_{\text{ref}} - x)_\tau\|_{\mathcal{L}_\infty} + L_0(\rho)\|(x_{\text{ref}} - x)_\tau\|_{\mathcal{L}_\infty} = L_\rho\|(x_{\text{ref}} - x)_\tau\|_{\mathcal{L}_\infty}\,, \tag{3.81}$$

$$\|\sigma_\tau\|_{\mathcal{L}_\infty} \leq \Delta_1\rho + \Delta_2\rho_u + L(\rho)\rho + B(\rho) = L_\rho\rho + \Delta_2\rho_u + B(\rho) = \Delta\,, \tag{3.82}$$

where L_ρ was defined in (3.7). Notice that from (3.3) and (3.34) it follows that

$$\sigma_{\text{ref}}(t) - \sigma(t) = \vartheta_{\text{ref}}(t) - \vartheta(t) + (b(t) - b_m(t))(u_{\text{ref}}(t) - u(t))\,. \tag{3.83}$$

Adding and subtracting $\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\mathcal{H}_{yum}\sigma$ and $\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}\sigma$ from (3.29) results in

$$u = \mathcal{C}r_g - \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\mathcal{H}_{yum}\tilde{\sigma} + \mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\sigma - \mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}\sigma\,.$$

Next, subtracting this result from (3.33) yields

$$u_{\text{ref}} - u = -\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}(\sigma_{\text{ref}} - \sigma) + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y}_{\text{in}} + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\mathcal{H}_{yum}\tilde{\sigma} - \mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\sigma\,.$$

Further we rewrite (3.45) as $\tilde{y} = \mathcal{H}_{yum}\tilde{\sigma} + \tilde{y}_{\text{in}}$, which leads to

$$u_{\text{ref}} - u = -\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}(\sigma_{\text{ref}} - \sigma) + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y} - \mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\sigma\,. \tag{3.84}$$

Substituting this in (3.83) and using the definition (3.10) we obtain

$$\begin{aligned} \sigma_{\text{ref}} - \sigma &= \vartheta_{\text{ref}} - \vartheta + (b - b_m)(-\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}(\sigma_{\text{ref}} - \sigma) \\ &\quad + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y} - \mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\sigma) \\ &= \mathcal{H}_\omega(\vartheta_{\text{ref}} - \vartheta) + \mathcal{H}_\omega(b - b_m)\mathcal{C}\mathcal{H}_m^{-1} \\ &\quad \cdot (\mathcal{F}\tilde{y} - (1-\mathcal{F})\mathcal{H}_{yum}\sigma)\,. \end{aligned} \tag{3.85}$$

The systems in (3.2) and (3.31) can be written as

$$\begin{aligned} x &= \mathcal{H}_{xm}u + \mathcal{H}_{xum}\sigma + x_{\text{in}}\,, \\ x_{\text{ref}} &= \mathcal{H}_{xm}u_{\text{ref}} + \mathcal{H}_{xum}\sigma_{\text{ref}} + x_{\text{in}}\,, \end{aligned}$$

which leads to

$$x_{\text{ref}} - x = \mathcal{H}_{xm}(u_{\text{ref}} - u) + \mathcal{H}_{xum}(\sigma_{\text{ref}} - \sigma)\,.$$

Substituting (3.84) yields

$$x_{\text{ref}} - x = \mathcal{H}_{xm}(-\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}(\sigma_{\text{ref}} - \sigma) + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y} - \mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\sigma) + \mathcal{H}_{xum}(\sigma_{\text{ref}} - \sigma)$$
$$= (\mathcal{H}_{xum} - \mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum})(\sigma_{\text{ref}} - \sigma) + \mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y} - \mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\sigma\,,$$

which after substituting (3.85) and using the definition (3.6) takes the following form

$$x_{\text{ref}} - x = \mathcal{G}_{um}(\vartheta_{\text{ref}} - \vartheta) + \mathcal{G}_{um}(b - b_m)\mathcal{C}\mathcal{H}_m^{-1}(\mathcal{F}\tilde{y} - (1-\mathcal{F})\mathcal{H}_{yum}\sigma)$$
$$+ \mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y} - \mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\sigma\,.$$

Notice that due to (3.80), the assumptions of Lemma 6 are satisfyed. Therefore, taking into account (3.81) and (3.82), we obtain the following bound

$$\begin{aligned}
\|(x_{\text{ref}} - x)_\tau\|_{\mathcal{L}_\infty} &\le \|\mathcal{G}_{um}\|_{\mathcal{L}_1}\|(\vartheta_{\text{ref}} - \vartheta)_\tau\|_{\mathcal{L}_\infty} + \|\mathcal{G}_{um}(b - b_m)\mathcal{C}\mathcal{H}_m^{-1}\|_{\mathcal{L}_1}(\|\mathcal{F}\|_{\mathcal{L}_1}\|\tilde{y}_\tau\|_{\mathcal{L}_\infty} \\
&\quad + \|(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\|\sigma_\tau\|_{\mathcal{L}_\infty}) + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\|\tilde{y}_\tau\|_{\mathcal{L}_\infty} \\
&\quad + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\|\sigma_\tau\|_{\mathcal{L}_\infty} \\
&\le \|\mathcal{G}_{um}\|_{\mathcal{L}_1}L_\rho\|(x_{\text{ref}} - x)_\tau\|_{\mathcal{L}_\infty} + \|\mathcal{G}_{um}(b - b_m)\mathcal{C}\mathcal{H}_m^{-1}\|_{\mathcal{L}_1}(\|\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 \\
&\quad + \|(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta) + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 \\
&\quad + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta \\
&\le \frac{1}{1 - \|\mathcal{G}_{um}\|_{\mathcal{L}_1}L_\rho}\Big(\|\mathcal{G}_{um}(b - b_m)\mathcal{C}\mathcal{H}_m^{-1}\|_{\mathcal{L}_1}(\|\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 \\
&\quad + \|(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta) + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 \\
&\quad + \|\mathcal{H}_{xm}\mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta\Big) < \gamma_1\,.
\end{aligned}$$

This contradicts to the first equality in (3.77). It remains to show that the second equality also is no true. Towards this end we substitute (3.85) into (3.84), and taking into account the bounds in (3.81), (3.82), and Lemma 6, we obtain

$$\begin{aligned}\|(u_{\mathrm{ref}}-u)_\tau\|_{\mathcal{L}_\infty} \leq \|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{yum}\mathcal{H}_\omega\|_{\mathcal{L}_1}\big(& L_\rho\gamma_1 + \|(b-b_m)\mathcal{C}\mathcal{H}_m^{-1}\|_{\mathcal{L}_1} \\ & \cdot (\|\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 + \|(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta)\big) \\ & + \|\mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\|_{\mathcal{L}_1}\bar{\gamma}_0 + \|\mathcal{C}\mathcal{H}_m^{-1}(1-\mathcal{F})\mathcal{H}_{yum}\|_{\mathcal{L}_1}\Delta < \gamma_2\,,\end{aligned}$$

which contradicts to the second equation in (3.77). Thus the bounds in (3.74), (3.76) hold. The bound in (3.75) follows from the fact that $y_{\mathrm{ref}}(t) - y(t) = c_m^\top(t)\,(x_{\mathrm{ref}}(t) - x(t))$. The results (3.72) and (3.73) follow directly from the bounds in (3.80). □

Notice that from the definitions of γ_1 and γ_2 are defined in (3.13) and (3.14) and Lemmas 4 and 6, it follows that by reducing the sampling time T_s and the bandwidth of the lowpass filter $F(s)$ one can achieve arbitrarily small performance bounds (3.74)–(3.76).

3.4 Simulations

To verify numerically the results proven, we consider the system in (3.1) with the following values

$$\begin{aligned} A(t) &= \begin{bmatrix} \sin(t) & 1 \\ -\omega_m^2(t) & -4 \end{bmatrix}, \quad b(t) = \begin{bmatrix} 1 \\ \omega_m^2(t) \end{bmatrix}, \\ c_m(t) &= \begin{bmatrix} \omega_c(t) \\ 0 \end{bmatrix}^\top, \end{aligned}$$

where $\omega_m(t) = 1 + 0.4\sin((\pi/40)t)$ and $\omega_c(t) = \;3 - 1.5\cos((\pi/30)t)$. Let the desired state matrix be

$$A_m(t) = \begin{bmatrix} 0 & 1 \\ -\omega_m^2(t) & -2\zeta\omega_m(t) \end{bmatrix},$$

with $\zeta = 0.7$, such that $(A(t) - A_m(t))$ is as follows

$$A(t) - A_m(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & -4 + 2\zeta\omega_m(t) \end{bmatrix},$$

which results in unmatched uncertainties and a bound $\Delta_1 = 3.16$. For the simulation, we assume $b(t) = b_m(t)$ and consider the following nonlinear disturbances:

$$f_1(t, x(t)) = 2x(t) + 2 + \begin{bmatrix} \sin(0.3t) \\ \sin(0.3t) \end{bmatrix}, \quad f_2(t, x(t)) = 0.5x(t) + \begin{bmatrix} \cos(0.5t) \\ \cos(0.5t) \end{bmatrix},$$

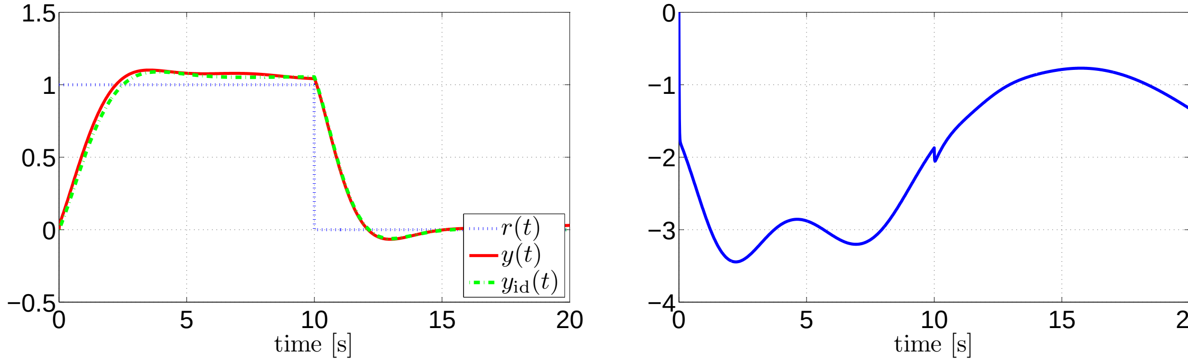
so that Assumption 3 holds with $L_0 = 2$ and $B = 3$ for all values of δ.

We implement the $\mathcal{L}_1$ adaptive controller according to (3.27), (3.28), and (3.29). In the implementation of the control law we use the filter $C(s) = 100/(s+100)$, $F(s) = 1000s/(s+1000)$, and sample rate $Ts = 0.001s$.

One can easily see that Assumptions 5 and 7 are satisfied for the class of uncertainties and disturbances introduced above. Also, Assumption 8 is verified as shown in Section 5.1.4 of [9] with $\epsilon_P = .55 < 1$. To better show the the disturbance rejection and scaling properties of the $\mathcal{L}_1$ adaptive controller, we first consider the initial conditions: $x_0 = \hat{x}_0 = [0, 0]^\top$.

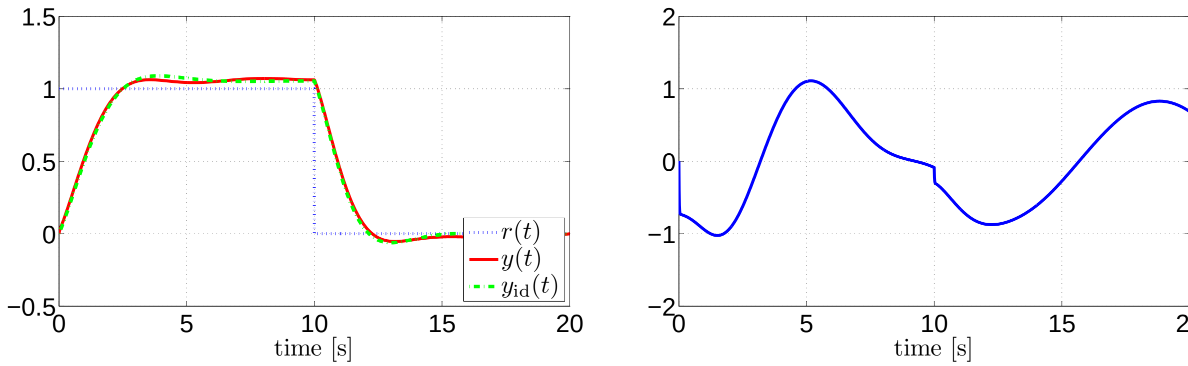
Figures 3.1 and 3.2 show the simulation results for both $f_1(t, x(t))$ and $f_2(t, x(t))$ respectively. From these results one can see that the fast adaptation ability of the $\mathcal{L}_1$ adaptive controller ensures uniform transient performance for different uncertainties and disturbances. We notice that while the system's output remains close to the desired reference signal in the presence of different uncertainties and disturbances, the control signal changes significantly to ensure adequate compensation for the uncertainties and the disturbances.

Figure 3.3 shows the simulation results for step reference signals of different amplitudes. We observe that the system response is close to scaled response, similar to linear systems.



(a) $r(t)$, $y(t)$, and $y_{\rm id}(t)$ (b) Time–history of $u(t)$

Figure 3.1: Performance of the $\mathcal{L}_1$ adaptive controller for $f_1(t, x(t))$

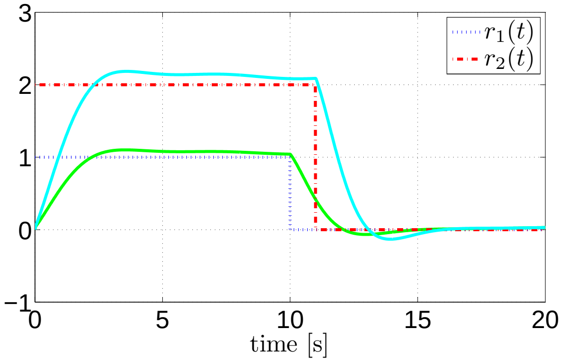


(a) $r(t)$, $y(t)$, and $y_{\rm id}(t)$ (b) Time–history of $u(t)$

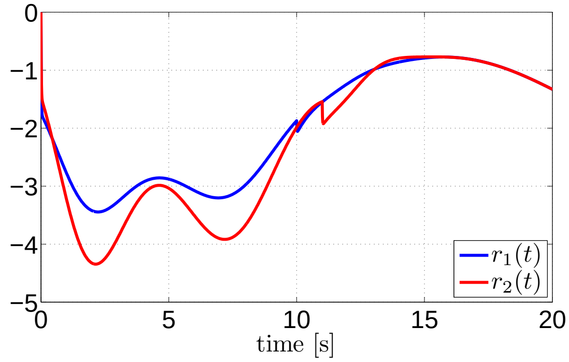
Figure 3.2: Performance of the $\mathcal{L}_1$ adaptive controller for $f_2(t, x(t))$

Next, we test the tracking performance of the closed–loop adaptive system. We set the reference signal to $r(t) = \sin\left(\frac{\pi}{5}t\right)$ and use the disturbance $f_1(t, x(t))$. We now consider initialization error such that $x_0 = [1, 1]^\top$ while we set the state predictor to $\hat{x}_0 = [1, 0]^\top$. Notice that $y(0) = c(0)x_0 = c(0)\hat{x}_0$. The simulation results are shown in Figure 3.4. One can see that the closed–loop adaptive system has satisfactory tracking performance. It compensates for the uncertainties in the system and rejects the disturbance within the bandwidth of the control channel specified via $C(s)$.

It is important to emphasize that in the simulations above there is no retuning of the $\mathcal{L}_1$ adaptive controller from one scenario to another, and the same *constant* control parame-

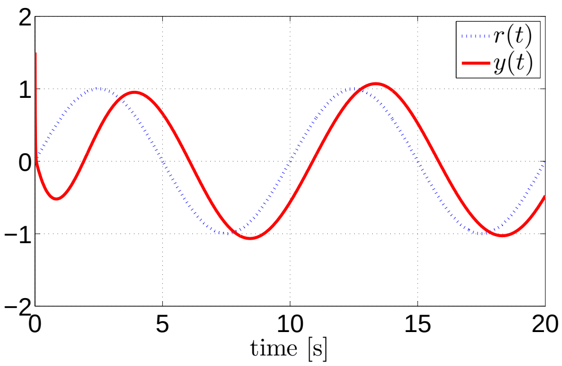


(a) $r(t)$, $y(t)$

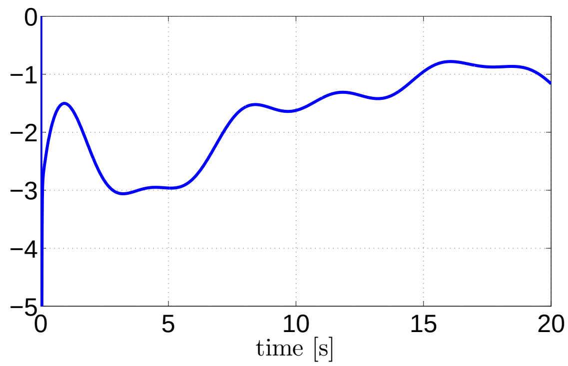


(b) Time–history of $u(t)$

Figure 3.3: Performance of the $\mathcal{L}_1$ adaptive controller for step reference signals



(a) $r(t)$, $y(t)$



(b) Time–history of $u(t)$

Figure 3.4: Performance of the $\mathcal{L}_1$ adaptive controller for $r(t) = \sin(\frac{\pi}{5}t)$

ters are used for every simulation. The time–varying nature of the desired reference system is reflected in the state–predictor, which uses $A_m(t)$, $b_m(t)$, and $c_m^\top(t)$.

CHAPTER 4

CONCLUSIONS

Two new extensions to $\mathcal{L}_1$ adaptive control theory have presented. The first being an $\mathcal{L}_1$ adaptive feedback control architecture using a generalized proportional adaptation law which leads to uniform performance bounds for the system's output and control signal both in transient and steady–state in the presence of fast adaptation, while reducing the computational requirement needed to achieve similar bounds as compared to the projection based adaptation methods. The addition of phase margin in the estimation loop allows for modeling of known first order sensor dynamics at the output of the state predictor. The linearity of the proposed $\mathcal{L}_1$ adaptive architecture helped to compute the performance and stability margins of the closed–loop adaptive system. Three particular cases of the generalized law were presented: an adaptive switching law, an $\mathcal{L}_1$ adaptive funnel–control, and a variable dependent adaptation law. Finally, the variable dependent law was suggested as a method to help reduce peaking or to improve performance by adjusting the adaptive gain the according to CPU demand.

Also presented was an extension of the $\mathcal{L}_1$ adaptive control architecture for the class of systems with time-varying reference systems in the presence of unmatched nonlinear disturbances with unknown input matrix using output feedback. This extension inherits and integrates the performance and robustness properties from the architectures derived in [20–22].

Both extensions follow the standard $\mathcal{L}_1$ adaptive control theory: separation of robustness and adaptation through the use of a low pass filter. In both cases, the bound between the actual system and reference system can be arbitrarily reduced by increasing the adaptation rate, while the bound between the reference system and ideal system depends on the choice of a low pass filter, subject to an $\mathcal{L}_1$ norm condition, with robustness as a trade off.

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