EXTENDING PARTIAL ISOMORPHISMS

BY

KOSTYANTYN SLUTSKYY

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2012

Urbana, Illinois

Doctoral Committee:

Professor Lou van den Dries, Chair
Associate Professor Christian Rosendal, UIC, Director of Research
Professor Sławomir Solecki
Associate Professor Ilya Kapovich

Abstract

There are two main topics in the thesis. In the second chapter we study two-dimensional classes of topological similarity in the groups of automorphisms of some linearly ordered Fraïssé classes: the rationals, the linearly ordered random graph and the linearly ordered Urysohn space. The main theorem establishes meagerness of two-dimensional similarity classes in these groups. As a byproduct we get some results about the group of isometries of the Urysohn space.

The third chapter is devoted to the metrics on the free products and HNN extensions of groups with two-sided invariant metrics. Using the approach of Graev to metrics on the free groups we show the existence of the coproducts in the category of groups with two-sided invariant metrics and Lipschitz homomorphisms. We then apply this theory to formulate a criterion when two topologically similar elements in a SIN Polish group are conjugate inside a bigger SIN Polish group.

To Ludmila Slutska and Alexandr Slutsky.

Acknowledgments

I would like to express sincere and deep gratitude to my adviser Christian Rosendal, who always helped me in all possible ways during my study. I would also like to thank Lou van den Dries, Ward C. Henson and Sławomir Solecki for their fight with my mathematical ignorance.

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Chapter 1

Introduction

The topic of this thesis lies on the boundary of several areas. A descriptive name for the subject would be *"Extending partial isomorphisms"*. This is both an old and a very recent area: some central results and constructions have been around for decades but it was not until recently that these results were realized to be so closely connected as to form a separate field of study.

Let me start with a typical question. Take a mathematical structure. The notion of a structure is very vague for the moment, but one can formalize it in the sense of the first order logic. A notion of structure comes with appropriate notions of substructure and isomorphism between structures. For example, we can look at sets and subsets, groups and subgroups, metric spaces and subspaces, graphs and induced subgraphs, etc. By a *partial isomorphism* we mean an isomorphism between substructures.

Let us say we have a structure $\mathbf{A}$ and two substructures $\mathbf{B}, \mathbf{C} < \mathbf{A}$ and assume, moreover, that $\mathbf{B}$ and $\mathbf{C}$ are isomorphic with $\phi : \mathbf{B} \to \mathbf{C}$ being a specific isomorphism. *Can we embed* $\mathbf{A}$ *into a bigger structure* $\mathbf{D}$ *that will admit an automorphism* $\psi : \mathbf{D} \to \mathbf{D}$ *such that* $\psi|_{\mathbf{B}} = \phi$*?* In other words, we start with a *partial* isomorphism of $\mathbf{A}$ and try to extend it to a *full* automorphism of a bigger structure $\mathbf{D}$.

1.1 Fraïssé classes

One concrete instance of this general paradigm of extending partial isomorphisms comes from the theory of Fraïssé classes.

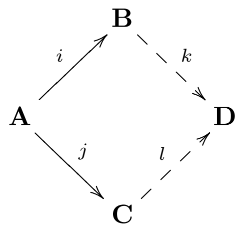
Let L be a relational first order language, and let $\mathcal{K}$ be a collection of finite L-structures. The collection $\mathcal{K}$ is called a *Fraïssé class* if the following conditions are met.

1. Hereditary property: if $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A}$ is a substructure of $\mathbf{B}$, then $\mathbf{A} \in \mathcal{K}$.

2. Joint embedding property: for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that both $\mathbf{A}$ and $\mathbf{B}$ are (up to isomorphism) substructures of $\mathbf{C}$.

3. Amalgamation property: for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and for all embeddings $i : \mathbf{A} \to \mathbf{B}$ and $j : \mathbf{A} \to \mathbf{C}$ there exist $\mathbf{D} \in \mathcal{K}$ and embeddings $k : \mathbf{B} \to \mathbf{D}$,

$l : \mathbf{C} \to \mathbf{D}$ such that $k \circ i = l \circ j$.



4. Up to isomorphism there are only countably many distinct structures in $\mathcal{K}$.

Classical examples of Fraïssé classes include the classes of finite sets, finite graphs, finite linear orderings, and finite metric spaces with rational distances (some restriction on the set of distances is necessary in order to have only countably many different spaces up to isometry).

It was R. Fraïssé who realized that with any such class one can associate what is now called a Fraïssé limit. Here a *Fraïssé limit* of the Fraïssé class $\mathcal{K}$ is a countable L-structure $\mathbb{K}$ such that

1. Finite substructures of $\mathbb{K}$ are (up to isomorphism) exactly the structures in $\mathcal{K}$.
2. If $\mathbf{A}$ and $\mathbf{B}$ are finite substructures of $\mathbb{K}$ and $\phi : \mathbf{A} \to \mathbf{B}$ is an isomorphism, then there exists an automorphism $\psi : \mathbb{K} \to \mathbb{K}$ such that $\psi|_{\mathbf{A}} = \phi$.

It is a theorem of Fraïssé that any $\mathcal{K}$ has a limit and, moreover, this limit is unique up to isomorphism. For the above mentioned examples of Fraïssé classes their limits are: the countably infinite set, the random (or Rado) graph, the set of rationals with the usual ordering, and the rational Urysohn metric space.

Since the work of Fraïssé a lot of work was done to understand the theory of homogeneous structures from both the model-theoretic and combinatorial points of view. An important step was made by G. Cherlin, L. Harrington, and A. H. Lachlan [2], who proved that $\aleph_0$-categorical $\aleph_0$-stable structures can be in a certain sense approximated by finite substructures. Using ideas from this paper E. Hrushovski [10] in 1992 derived the following purely combinatorial result: For any finite graph G there exists a *finite* graph H which contains G as an induced subgraph and such that any partial isomorphism of G extends to a full automorphism of H.

Partially motivated by this theorem of Hrushovski and also by a result on approximate extensions of isometries of metric spaces by V. Pestov [18], C. W. Henson asked the following question: Is it true that for any finite metric space X there is a finite metric space Y such that $X \subseteq Y$ and every partial isometry of X extends to a full isometry of Y?

This problem turned out to be quite difficult and the positive answer was finally obtained by S. Solecki in [24] relying on a deep result of B. Herwig and

D. Lascar [7], which in turn uses the solution of Rhodes' Type II conjecture by C. J. Ash [1] and L. Ribes and P. Zalesskii [20]. Solecki managed to prove the following two important results.

Theorem (Solecki)**.** *Let X be a finite metric space and let p be a partial isometry of X. There exist an $M > 0$, a finite metric space Y, and a full isometry q of Y such that*

1. *X is a subspace of Y.*
2. *q extends p.*
3. *$q^M(X) \cup X$ is the free amalgam of $q^M(X)$ and X over $Z(p)$ — the set of periodic points of p.*

Theorem (Solecki)**.** *Let X be a finite metric space. There exists a finite metric space Y that contains X as a subspace and such that every partial isometry of X extends to a full isometry of Y.*

The first theorem gives a positive answer to the Henson's question for a single partial isometry, but, in fact, contains much more: it gives a construction of an extension to a full isometry that is *as independent as possible.* The second theorem, which gives a complete answer to the question, lacks this property.

Besides having intrinsic beauty these results have a strong connection to classes of topological similarity, which are introduced below.

1.2 Topological similarity

In [21] C. Rosendal gave a short proof of the result (variously attributed to V. A. Rokhlin and A. del Junco) that all the conjugacy classes in the group of measure preserving automorphisms of the standard Lebesgue space are meager. For that he introduced a (one-dimensional) relation of topological similarity.

Let G be a metrizable topological group, and let $e \in G$ be its identity element. Two tuples of elements $(g_1, \ldots, g_n) \in G^n$ and $(f_1, \ldots, f_n) \in G^n$ are said to be *topologically similar* if for all sequences $\{w_k\}_{k=1}^{\infty}$ of group words on n letters

$$\lim_{k\to\infty} w_k(g_1, \ldots, g_n) = e \iff \lim_{k\to\infty} w_k(f_1, \ldots, f_n) = e.$$

For $n = 1$ this simply says that for all sequences of integers $\{m_k\}_{k=1}^{\infty}$

$$\lim_{k\to\infty} g^{m_k} = e \iff \lim_{k\to\infty} f^{m_k} = e.$$

Topological similarity is clearly an equivalence relation on n-tuples, and its equivalence classes are called n-dimensional classes of topological similarity. The relation of topological similarity is easily seen to be coarser than the relation of diagonal conjugacy (two tuples $(g_1, \dots, g_n)$ and $(f_1, \dots, f_n)$ are diagonally conjugate if there is a single $h \in G$ such that $hg_ih^{-1} = f_i$ for all $i = 1, \dots, n$).

The importance of diagonal conjugacy classes in the groups of automorphisms of Fraïssé limits was noted by A. S. Kechris and C. Rosendal in [14], where they extended an earlier work of W. Hodges, I. Hodkinson, D. Lascar, and S. Shelah [9]. They showed that if a Polish group G has ample generics (that is, for any n there is a generic n-dimensional diagonal conjugacy class), then it satisfies the small index property (a group is said to satisfy the small index property if every subgroup of index less than continuum is open) and has automatic continuity: every homomorphism from G into a separable topological group is continuous. Moreover, they gave a criterion when a group of automorphisms of a Fraïssé class has ample generics, and also gave some examples of such groups. This paper motivated different projects that aimed at understanding when a group of automorphisms of a Fraïssé limit has comeager diagonal conjugacy classes.

As topological similarity is an invariant for diagonal conjugacy, by showing that topological similarity classes are meager, one immediately deduces that also diagonal conjugacy classes are meager. As a concrete instance of this general strategy we prove the following theorem (it is a consequence of Theorem 2.2.15 and Theorem 2.4.8 below).

Theorem. *Two-dimensional classes of topological similarity are meager in the groups of automorphisms of the following Fraïssé limits: the rationals, the ordered Rado graph, and the ordered rational Urysohn space.*

The main tool in the proof of the last two cases of this theorem is Theorem 2.3.7, a multidimensional version of the first among Solecki's theorems above.

Theorem. *Let X be a finite metric space, and let p_1 and p_2 be partial isometries of X. Suppose all the periodic points of p_1 and p_2 are fixed points. There exist a finite metric space Y that contains X as a subspace, partial isometries q_1 and q_2 that extend p_1 and p_2 respectively, and a word w on two letters such that $w(q_1, q_2)(X) \cup \mathrm{dom}(q_1)$ is the free amalgam of $w(q_1, q_2)(X)$ and $\mathrm{dom}(q_1)$ over $Z(p_1) \cap Z(q_2)$.*

This theorem also implies some new results about the structure of the group of isometries of the rational Urysohn space. For a metric space X its group of isometries is denoted by $\mathrm{Iso}(X)$, and the subgroup of isometries that fix $A \subseteq X$ pointwise is denoted by $\mathrm{Iso}_A(X)$. Let $\mathbb{QU}$ denote the rational Urysohn space, and let $\mathbb{U}$ be the Urysohn space, which is the metric completion of $\mathbb{QU}$. J. Melleray, answering a question of I. Goldbring, proved in [17] the following

Theorem (Melleray)**.** *For all finite $A, B \subset \mathbb{U}$*

$$\mathrm{Iso}_{A \cap B}(\mathbb{U}) = \overline{\langle \mathrm{Iso}_A(\mathbb{U}), \mathrm{Iso}_B(\mathbb{U}) \rangle}.$$

As a corollary of Theorem 2.3.7 we can get a similar statement for $\mathrm{Iso}(\mathbb{QU})$.

Theorem. *For all finite* $A, B \subset \mathbb{QU}$

$$\mathrm{Iso}_{A\cap B}(\mathbb{QU}) = \overline{\langle \mathrm{Iso}_A(\mathbb{QU}), \mathrm{Iso}_B(\mathbb{QU})\rangle}.$$

Another consequence of Theorem 2.3.7 is an analog of a theorem of Prasad [19], who proved that a generic pair of measure-preserving automorphisms of the standard Lebesgue space (X, λ) generates a dense subgroup of $\mathrm{Aut}(X, \lambda)$.

Theorem. *The set of pairs* $(f, h) \in \mathrm{Iso}(\mathbb{U}) \times \mathrm{Iso}(\mathbb{U})$ *such that* $\overline{\langle f, g\rangle} = \mathrm{Iso}(\mathbb{U})$ *is comeager.*

1.3 HNN-extensions and Graev metrics

It was already mentioned that classes of topological similarity are coarser than conjugacy classes. In fact, if two elements can be made conjugate inside a bigger group, then they are topologically similar. To be more precise, we say that two elements $g_1, g_2 \in G$ are *induced conjugate* if there are a bigger group H that contains G as a topological subgroup and an element $h \in H$ such that $hg_1h^{-1} = g_2$. It is easy to see that if two elements in G are induced conjugate, then they must be topologically similar, and thus it is natural to ask when the inverse is true.

Question. Let G be a Polish group, and let $g, f \in G$ be topologically similar. When are g and f conjugate inside a larger Polish group H?

It turns out that this question is closely connected to another (much more classical) instance of an "Extending partial isomorphisms" theme.

Let us first look at the discrete version of this question. Suppose we have two elements g_1, g_2 in an abstract group G. When can we find a group H such that $G < H$ and $hg_1h^{-1} = g_2$ for some $h \in H$? An obvious necessary condition is that orders of g_1 and g_2 must be the same. It turns out that this condition is also sufficient, as was shown in a beautiful paper [8] by G. Higman, B. H. Neumann and H. Neumann. They gave a canonical construction of such a group H, and this construction is now known as the HNN extension and can be seen to be closely related to free products with amalgamation. One way to define HNN extensions is as follows. Let $G = \langle S|R\rangle$ be a group with the set of generators S and relations R, let A and B be subgroups of G, and let $\phi : A \to B$ be an isomorphism. Let t be a new symbol. The HNN extension of (G, ϕ) is the group H defined by

$$H = \big\langle S, t \big| R, \{tat^{-1} = \phi(a) : a \in A\}\big\rangle.$$

The element $t \in H$ is called the *stable letter* of the HNN extension.

One approach to the Question is to develop a topological version of the HNN extensions. In Chapter 3 we develop a theory of free products (possibly with amalgamation) and HNN extensions for groups with two-sided invariant metrics.

1.3.1 Graev metrics on free products

Let (G, d_G) and (H, d_H) be two topological groups with compatible two-sided invariant metrics d_G and d_H. It is worth mentioning that the class of topological groups that admit compatible two-sided invariant metrics is rather small, but it does include two important subclasses of metrizable groups: compact and abelian. Let now A be a common closed subgroup of G and H and assume that $A = G \cap H$, and that d_G and d_H agree on A, i.e.

$$d_G(a, b) = d_H(a, b) \quad \text{for all } a, b \in A.$$

Put $S = G \cup H$ and let $W(S)$ be the set of non-empty words over the alphabet S. We turn S into a metric space by setting

$$d(s,t) = \begin{cases} d_G(s,t) & \text{if } s,t \in G; \\ d_H(s,t) & \text{if } s,t \in H; \\ \inf\{d_G(s,a) + d_H(a,t) : a \in A\} & \text{if } s \in G \text{ and } t \in H. \end{cases}$$

For a word $u \in W(S)$ its length is denoted by $|u|$, and $u(i)$ denotes its i^{th} letter. For two words $u, v \in W(S)$ of the same length we set

$$\rho(u, v) = \sum_{i=1}^{|u|} d(u(i), v(i)).$$

There is a natural evaluation map from the set of words $W(S)$ to the free product with amalgamation $G *_A H$, which sends a word to the product of its letters. We denote this map by $\hat{\cdot}$.

Finally, for two elements $f_1, f_2 \in G *_A H$ we set

$$\underline{d}(f_1, f_2) = \inf\big\{d(u_1, u_2) : \hat{u}_i = f_i, |u_1| = |u_2|\big\}.$$

One of the main results of Chapter 3 is Theorem 3.5.10

Theorem. *The function $\underline{d}$ defined above is a two-sided invariant metric on $G *_A H$, moreover, it extends the metrics d_G and d_H on G and H respectively. If (G, d_G) and (H, d_H) are separable, then so is $(G *_A H, \underline{d})$.*

This metric $\underline{d}$ is called *the Graev metric* in analogy with a similar construction of M. Graev in [6] for the free groups over metric spaces. The group $G *_A H$

turns out to be a coproduct in the category of groups with two-sided invariant metrics and Lipschitz homomorphisms.

1.3.2 Graev metrics on HNN extensions

As shown by the above theorem, it is always possible to form a free product of groups with two-sided invariant metrics. However, for the HNN extensions the situation is somewhat different. There are counterexamples showing that it is not always possible to extend a two-sided invariant metric from a group to its HNN extension. But, nevertheless, the following is true (see Theorem 3.9.1 below):

Theorem. *Let (G, d) be a group with a two-sided invariant metric, let $A, B < G$ be closed subgroups of G, and let $\phi : A \to B$ be a d-isometric isomorphism. Let H denote the HNN extension of (G, ϕ). If the diameter of A is bounded by some number K, i.e., $diam(A) \leq K$, then there is a two-sided invariant metric $\underline{d}$ on H such that $\underline{d}|_G = d$ and $\underline{d}(e, t) = K$, where t is the stable letter of H, and e is the identity element.*

Recall that a metrizable topological group is called SIN (for "small invariant neighborhoods") if it admits a compatible two-sided invariant metric. Based on the above theorem one can prove Theorem 3.9.4.

Theorem. *Let G be a SIN Polish group, let $A, B < G$ be closed subgroups, and let $\phi : A \to B$ be an isomorphism. There exist a SIN Polish group H and $t \in H$ such that $G < H$ and $tat^{-1} = \phi(a)$ for all $a \in A$ if and only if there is a two-sided invariant metric d on G such that ϕ is a d-isometry.*

1.4 Preliminaries

In this section we would like to mention some of the mathematical preliminaries that are necessary to understand the rest of the text. Proofs and detailed exposition can be found in beautiful textbooks by A. Kechris [13] and S. Gao [4].

1.4.1 Polish Groups

A *Polish space* is a completely metrizable separable topological space. A *Polish group* is a completely metrizable separable topological group. Here are some examples of Polish groups.

(i) Locally compact metrizable groups. For instance: countable groups and Lie groups.

(ii) Abelian metrizable groups. In particular, Banach spaces.

(iii) S_∞ — the group of all permutations of the countably infinite set (say, $\mathbb{N}$) with the topology of pointwise convergence. Basic neighborhoods are indexed by finite subsets $\{a_i\}_{i=1}^n \subset \mathbb{N}$:

$$U(f; a_1, \ldots, a_n) = \{g \in S_\infty : g(a_i) = f(a_i)\}.$$

(iv) The group $U(H)$ of unitary operators of a separable Hilbert space H with the weak (equivalently, strong) operator topology.

This list can, of course, be continued ad infinitum. Polish groups are virtually everywhere. Even though there are so many different Polish groups there is a surprisingly rich theory of Polish groups and their actions. One of the basic tools is the famous Baire Theorem:

Theorem 1.4.1 (Baire Theorem)**.** *Let X be a Polish space. If $\{U_i\}$ is a countable family of dense open subsets of X, then $\bigcap_i U_i$ is dense in X.*

We say that a subset $A \subseteq X$ of a Polish space X is *comeager* or *generic* if there is a dense G_δ set B such that $B \subseteq A$.

We say that a Polish space is *perfect* if it has no isolated points.

Let G be an abstract group and d be a metric on G. We say that the metric d is *left-invariant* if for all $g, f_1, f_2 \in G$ we have $d(gf_1, gf_2) = d(f_1, f_2)$. The metric d is called *two-sided invariant* (or tsi for short) if additionally $d(f_1 g, f_2 g) = d(f_1, f_2)$ for all $f_1, f_2, g \in G$. If G is a topological group and d is a metric on G, then d is called *compatible* if the topology given by the metric d coincides with the topology on G. A metrizable topological group G is called *SIN* (for "small invariant neighborhoods") if it admits a compatible two-sided invariant metric.

By definition any Polish group admits a compatible *complete* metric. One can prove also that any Polish group admits a compatible *left-invariant* metric. But we would like to emphasize that not every Polish group admits a compatible complete left-invariant metric.

1.4.2 Descriptive complexity

Let X be a Polish space. Recall that the σ-algebra of Borel subsets of X is the σ-algebra generated by the open subsets of X. There is an hierarchy on Borel sets, which is known as the *Borel hierarchy.* It is given as follows. Let $\mathbf{\Sigma}^0_1$ be the set of open subsets of X and $\mathbf{\Pi}^0_1$ denote the set of closed subsets. By transfinite induction for $1 \leq \xi < \omega_1$ we construct:

$$\mathbf{\Sigma}^0_{\xi+1} = \{\text{countable unions of sets from } \mathbf{\Pi}^0_\xi\},$$

$$\mathbf{\Pi}^0_{\xi+1} = \{\text{complements of the sets from } \mathbf{\Sigma}^0_{\xi+1}\},$$

$$\mathbf{\Sigma}^0_\lambda = \bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi, \text{ when } \lambda \text{ is a limit ordinal.}$$

For any ξ we have $\mathbf{\Sigma}^0_\xi \subset \mathbf{\Pi}^0_{\xi+1}$, $\mathbf{\Sigma}^0_\xi \subset \mathbf{\Sigma}^0_{\xi+1}$ and for any Borel subset $A \subseteq X$ there is ξ such that $A \in \mathbf{\Sigma}^0_\xi$. We also define the so called *ambiguous sets* $\mathbf{\Delta}^0_\xi$ by $\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$.

We also need a notion of the Wadge reduction. Let X and Y be topological spaces, $A \subseteq X$, $B \subseteq Y$. We say that A is *Wadge reducible* to B if there is a continuous map $f : X \to Y$ such that $x \in A \iff f(x) \in B$. Let now X be a Polish space and $A \subseteq X$. We say that A is $\mathbf{\Sigma}^0_\xi$-hard if any $\mathbf{\Sigma}^0_\xi$ set $B \subseteq 2^{\mathbb{N}}$ is Wadge reducible to A. We say that A is a complete $\mathbf{\Sigma}^0_\xi$ set if it is $\mathbf{\Sigma}^0_\xi$-hard and is itself a $\mathbf{\Sigma}^0_\xi$ set. The definitions of $\mathbf{\Pi}^0_\xi$-hard and complete $\mathbf{\Pi}^0_\xi$ sets are analogous.

Theorem 1.4.2 (Wadge)**.** *Let X be a zero-dimensional Polish space. A set $A \subset X$ is $\mathbf{\Sigma}^0_\xi$-complete if and only if A is in $\mathbf{\Sigma}^0_\xi \setminus \mathbf{\Pi}^0_\xi$. Similarly interchanging $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Sigma}^0_\xi$.*

We use notation $\forall^\infty$ as a shortcut for "for all but finitely many" and $\exists^\infty$ for "exists infinitely many". We will later need two examples of complete sets: the set $A = \{x \in 2^{\mathbb{N}} : \forall^\infty n \; x(n) = 1\}$ is $\mathbf{\Sigma}^0_2$-complete, and its complement $B = \{x \in 2^{\mathbb{N}} : \exists^\infty n \; x(n) = 0\}$ is $\mathbf{\Pi}^0_2$-complete.

1.4.3 Metric spaces

Here we fix some notions and notations for metric spaces. Let $(\mathbf{A}, d)$ be a finite metric space with at least two elements. The *density* of $\mathbf{A}$ is denoted by $\mathcal{D}(\mathbf{A})$ and is the minimal distance between two distinct points in $\mathbf{A}$:

$$\mathcal{D}(\mathbf{A}) = \min\{d(x, y) : x, y \in \mathbf{A}, x \neq y\}.$$

For a metric space $\mathbf{A}$ its density character, i.e., the smallest cardinality of a dense subset, is denoted by $\chi(\mathbf{A})$. An ordered metric space is a triple $(\mathbf{A}, d, <)$, where d is a metric on $\mathbf{A}$ and $<$ is a linear ordering on $\mathbf{A}$. A *partial isometry* or *partial isomorphism* of a metric space $\mathbf{C}$ is an isometry $p : \mathbf{A} \to \mathbf{B}$ between *finite* subspaces $\mathbf{A}$, $\mathbf{B} \subseteq \mathbf{C}$. A partial isomorphism of an ordered metric space is a partial isometry of the metric space that also preserves the ordering on its domain.

Let p be a partial isometry of a metric space. Then we let $\mathrm{dom}(p)$ denote the domain of p and $\mathrm{ran}(p)$ denote its range. A point $x \in \mathrm{dom}(p)$ is called *periodic* if there is a natural number $n > 0$ such that

$$x, p(x), \ldots, p^n(x) \in \mathrm{dom}(p) \quad \text{and} \quad p^n(x) = x.$$

The set of periodic points is denoted by $\mathrm{Z}(p)$. A point $x \in \mathrm{dom}(p)$ is called *fixed* if $p(x) = x$ and the set of fixed points is denoted by $\mathrm{F}(p)$.

Let $\mathbf{A} = (\mathbf{A}, d_{\mathbf{A}})$, $\mathbf{B} = (\mathbf{B}, d_{\mathbf{B}})$, and $\mathbf{C} = (\mathbf{C}, d_{\mathbf{C}})$ be metric spaces and $i : \mathbf{A} \to \mathbf{B}$, $j : \mathbf{A} \to \mathbf{C}$ be isometries. Suppose that $i(\mathbf{A})$ is a closed subset of

$\mathbf{B}$ and $j(\mathbf{A})$ is closed in $\mathbf{C}$. We define the *free amalgam* $\mathbf{D} = \mathbf{B} *_{\mathbf{A}} \mathbf{C}$ of metric spaces as follows: substituting $\mathbf{B}$ and $\mathbf{C}$ by isomorphic copies we may assume that $\mathbf{B} \cap \mathbf{C} = \mathbf{A}$. Set $\mathbf{D} = \mathbf{B} \cup \mathbf{C}$ and define the metric $d_{\mathbf{D}}$ by:

$$d_{\mathbf{D}}(x,y) = \begin{cases} d_{\mathbf{B}}(x,y) & \text{if } x,\, y \in \mathbf{B}, \\ d_{\mathbf{C}}(x,y) & \text{if } x,\, y \in \mathbf{C}, \\ \inf\limits_{z \in \mathbf{A}} \{d_{\mathbf{B}}(x,z) + d_{\mathbf{C}}(z,y)\} & \text{if } x \in \mathbf{B} \text{ and } y \in \mathbf{C}. \end{cases}$$

Note that the first and the second clauses agree for $x,\, y \in \mathbf{A}$. If $\mathbf{A}$ is finite, then the inf in the last clause can be substituted with a min. If $\mathbf{A}$ is empty and $\mathbf{B}, \mathbf{C}$ have finite diameters, then we set $R = \operatorname{diam}(\mathbf{B}) + \operatorname{diam}(\mathbf{C})$, $\mathbf{D} = \mathbf{B} \sqcup \mathbf{C}$ and

$$d_{\mathbf{D}}(x,y) = \begin{cases} d_{\mathbf{B}}(x,y) & \text{if } x,\, y \in \mathbf{B}, \\ d_{\mathbf{C}}(x,y) & \text{if } x,\, y \in \mathbf{C}, \\ R & \text{otherwise.} \end{cases}$$

Shortly before his death P. S. Urysohn constructed a very interesting metric space that now bears his name. This space will be of central interest for us. The Urysohn space $\mathbb{U}$ is a complete separable metric space, that is uniquely characterized by the following properties:

- Every finite metric space can be isometrically embedded into $\mathbb{U}$;
- $\mathbb{U}$ is ultrahomogeneous, that is, each partial isometry between finite subsets of $\mathbb{U}$ extends to a full isometry of $\mathbb{U}$.

There is a rational counterpart $\mathbb{QU}$ of the Urysohn space. It is called the rational Urysohn space. This is a countable metric space with rational distances, characterized by similar properties:

- Every finite metric space with rational distances can be isometrically embedded into $\mathbb{QU}$;
- $\mathbb{QU}$ is ultrahomogeneous.

The rational Urysohn space is also the Fraïssé limit of the Fraïssé class of finite metric spaces with rational distances.

The groups of isometries $\operatorname{Iso}(\mathbb{U})$ and $\operatorname{Iso}(\mathbb{QU})$ of these spaces are Polish groups when endowed with the topology of pointwise convergence (for this $\mathbb{QU}$ is viewed as a discrete topological space).

Chapter 2

Classes of Topological Similarity

2.1 Definition and Basic Properties

Let G be a metrizable topological group. Following Rosendal [21, Section 4] we say that two elements $g, f \in G$ are *topologically similar* if for all sequences $\{n_k\}_{k=1}^{\infty}$ of integers

$$g^{n_k} \to e \iff f^{n_k} \to e.$$

More generally, we say that a tuple $(g_1, \ldots, g_n) \in G^n$ is topologically similar to a tuple $(f_1, \ldots, f_n) \in G^n$ if for all sequences of group words $\{w_m\}$ in the alphabet with n letters we have

$$w_m(g_1, \ldots, g_n) \to e \iff w_m(f_1, \ldots, f_n) \to e.$$

If g and f are topologically similar, then their orders are the same and, moreover, the groups $\langle g \rangle$ and $\langle f \rangle$ are isomorphic as topological groups. And vice versa, if $\langle g \rangle$ and $\langle f \rangle$ are isomorphic as topological groups, then f and g are topologically similar. Note that this condition is in general strictly stronger than saying that $\overline{\langle g \rangle}$ is topologically isomorphic to $\overline{\langle f \rangle}$.

Proposition 2.1.1. *Let G be a topological group.*

(i) For any n the relation of topological similarity on the n-tuples is an equivalence relation on G^n.

(ii) If $(g_1, \ldots, g_n) \in G^n$ and $(f_1, \ldots, f_n) \in G^n$ are diagonally conjugate (that is $f_i = hg_ih^{-1}$ for some $h \in G$ and all $i = 1, \ldots, n$), then $(g_1, \ldots, g_n)$ and $(f_1, \ldots, f_n)$ are topologically similar. Moreover, if $(g_1, \ldots, g_n) \in G^n$ and $(f_1, \ldots, f_n) \in G^n$ are diagonally conjugate in a bigger topological group (i.e., if there is a topological group H such that $G < H$ is a topological subgroup and $g_i = hf_ih^{-1}$ for some $h \in H$ and all $i = 1, \ldots, b$), then $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ are topologically similar.

Proof. (i) This is immediate from the definition of the topological similarity.

(ii) Conjugations are automorphisms of topological groups, therefore we have $w_m(f_1, \ldots, f_n) \to e$ if and only if $w_m(hf_1h^{-1}, \ldots, hf_nh^{-1}) \to e$ for all sequences of words $\{w_m\}$. □

Equivalence classes of topological similarity on the n-tuples are called *n-dimensional classes of topological similarity.*

Example 2.1.2. Let G be any topological group. Any $g \in G$ is always topologically similar to g^{-1}. If $g, f \in G$ have equal *finite* orders, then g and f are necessarily topologically similar. Indeed, both $\langle g \rangle$ and $\langle f \rangle$ are isomorphic to $\mathbb{Z}/n\mathbb{Z}$ with the discrete topology, where n is the order of g and f (topological groups are assumed to be Hausdorff).

From this example we see that two elements of finite order are topologically similar if and only if they have the same order. For elements of infinite order this relation is much more complicated. If $g \in G$ and $f \in G$ have infinite orders, then both $\langle g \rangle$ and $\langle f \rangle$ *as abstract groups* are isomorphic to $\mathbb{Z}$. The elements f and g are topologically similar if and only if they give the same *topologies* on $\mathbb{Z}$.

Example 2.1.3. Let $\mathbb{T}$ be a circle, viewed as a compact abelian group. Take $\alpha, \beta \in \mathbb{T}$ in the circle. When α is topologically similar to β? The interesting case is, of course, when α and β are of infinite order (if we take $\mathbb{T}$ to be $\mathbb{R}/\mathbb{Z}$, then this means that α and β are irrational). We claim that α is topologically similar to β if and only if $\alpha = \pm\beta$. In other words we claim that in the circle only elements from Example 2.1.2 are topologically similar.

Let $\phi : \langle \alpha \rangle \to \langle \beta \rangle$ be an isomorphism of topological groups. Since $\mathbb{T}$ is compact, we can extend ϕ to an isomorphism $\phi : \overline{\langle \alpha \rangle} \to \overline{\langle \beta \rangle}$. But for irrational α and β we have $\overline{\langle \alpha \rangle} = \mathbb{T} = \overline{\langle \beta \rangle}$. Thus ϕ is an automorphism of the circle, therefore either $\phi = id$ or $\phi = -id$.

2.1.1 Descriptive Complexity. General groups.

As already noted by Rosendal [21], two elements $f, g \in G$ of a Polish group G are topologically similar if and only if

$$\forall i \exists j \forall n \Big[\big(f^n \notin U_j \vee g^n \in \overline{U}_i \big) \& \big(g^n \notin U_j \vee f^n \in \overline{U}_i \big) \Big],$$

where $\{U_i\}$ is a countable basis of the identity element in G. This shows that topological similarity is a $\mathbf{\Pi}^0_3$ subset of $G \times G$. In this subsection we show that one cannot do better: for some [compact, abelian] groups this relation is, in fact, a complete $\mathbf{\Pi}^0_3$ set.

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote a circle and let d be the factor metric on $\mathbb{T}$: for $\alpha, \beta \in \mathbb{T}$

$$d(\alpha, \beta) = \min\{ |\alpha_0 - \beta_0 - n| : n \in \mathbb{Z} \},$$

where $\alpha_0, \beta_0 \in \mathbb{R}$ are such that $\alpha = \alpha_0 + \mathbb{Z}$, $\beta = \beta_0 + \mathbb{Z}$. In other words, d is just the arc-length metric scaled in such a way that the diameter of the circle is $1/2$. For a sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers we say that $\{n_k\}$ is *eventually divisible by m* if there is N such that $m | n_k$ for all $k > N$. We say that $\{n_k\}$ is *eventually divisible* if it is eventually divisible by all m.

Lemma 2.1.4. *Let $\alpha \in \mathbb{T}$ be given. If for any eventually divisible sequence $\{n_k\}$*

$$n_k\alpha \to 0,$$

then α is rational.

Proof. Suppose towards a contradiction that α is irrational and hence $n\alpha \neq 0$ for any positive n. Let $\{n_k\}$ be an eventually divisible sequence. Then $n_k\alpha \to 0$ by assumption. Passing to a subsequence we can assume that

$$d(n_k\alpha, 0) < \frac{1}{k}$$

for all k. Let m_k be such that $d(m_kn_k\alpha, \frac{1}{2}) < \frac{1}{k}$. Then $\{m_kn_k\}$ is also eventually divisible, but $m_kn_k\alpha \to 1/2$ by construction. Contradiction. □

Remark 2.1.5. Note that for any rational $\alpha \in \mathbb{T}$ and for any eventually divisible $\{n_k\}$ we indeed have $n_k\alpha \to 0$.

Set $A = \{x \in \mathbb{T}^{\mathbb{N}} : \forall n\ x(n) \in \mathbb{Q}\}$.

The following two propositions are very simple and well known. See [13, Chapter 23] for details.

Proposition 2.1.6. *Let X be a perfect Polish space and $Q \subseteq X$ be a countable dense subset. Then Q is $\mathbf{\Sigma}^0_2$-complete.*

Proposition 2.1.7. *Let $\{X_n\}$ be a sequence of Polish spaces, $A_n \subseteq X_n$ be $\mathbf{\Sigma}^0_\xi$-complete. Then $\prod A_n$ is a $\mathbf{\Pi}^0_{\xi+1}$-complete subset of $\prod X_n$.*

It now follows immediately that A is $\mathbf{\Pi}^0_3$-complete. Let $z \in \mathbb{T}^{\mathbb{N}}$ be given by $z(n) = \frac{1}{n}$. The following lemma is obvious.

Lemma 2.1.8. *Let $\{n_k\}$ be a sequence of positive integers. Then $n_kz \to 0$ if and only if $\{n_k\}$ is eventually divisible.*

Proposition 2.1.9. *The relation of topological similarity in the group $\mathbb{T}^{\mathbb{N}}$ is $\mathbf{\Pi}^0_3$-complete.*

Proof. The group $\mathbb{T}^{\mathbb{N}}$ is, of course, topologically isomorphic to the group $\mathbb{T}^{\mathbb{N}\times\mathbb{N}}$. Consider the map $\phi : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^{\mathbb{N}\times\mathbb{N}}$ given by $x \mapsto (x, z)$. This is a continuous map. We claim that $\phi(x)$ is topologically similar to (z, z) if and only if $x \in A$.

If $x \in A$, then (x, z) is topologically similar to (z, z) by Remark 2.1.5. If, on the other hand, $\phi(x)$ is topologically similar to (z, z) then, by Lemma 2.1.4, $x(n)$ has to be in $\mathbb{Q}$ for all n. Therefore $x \in A$. This proves the claim.

Finally, since A is $\mathbf{\Pi}^0_3$-hard so is the similarity class of z, and therefore the relation of topological similarity on $\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}$ is $\mathbf{\Pi}^0_3$-hard. □

2.1.2 Descriptive Complexity. The group S_∞.

According to the previous section, the relation of topological similarity can be a complete $\mathbf{\Pi}^0_3$ set. On the other hand, as we show in this subsection, this is never the case for subgroups of S_∞.

Any $f \in S_\infty$ can be written as a product of cycles. For $f \in S_\infty$ let $L_f \subseteq \mathbb{N} \cup \{\infty\}$ denote the set of lengths of cycles in f. Set

$$P_f = \{p^k : p \text{ is prime}, \, k \geq 1, \, p^k | n \text{ for some } n \in L_f \cap \mathbb{N}\}.$$

Since the topology of S_∞ is just the topology of pointwise convergence, $\langle f \rangle$ is infinite discrete if and only if $\infty \in L_f$. Also note that if $\langle f \rangle$ is not discrete, then $f^{n_i} \to e$ if and only if for any $p^k \in P_f$ we have $p^k | n_i$ for i sufficiently large.

Lemma 2.1.10. *Let $f, g \in S_\infty$ and suppose that $\infty \notin L_f \cup L_g$. The elements f and g are topologically similar if and only if $P_f = P_g$.*

Proof. Assume that f and g are topologically similar. If f and g have finite orders, then there is some $N \in \mathbb{N}$ such that $f^N = e = g^N$ and $f^M \neq e$, $g^M \neq e$ for any $0 \leq M < N$. Therefore $P_f = \{p^k : p^k | N\} = P_g$.

Suppose f and g have infinite orders. Since $\infty \notin L_f \cup L_g$ all cycles in the decompositions of f and g are finite. Suppose towards the contradiction that for some prime p and $k \in \mathbb{N}$ one can find $p^k \in P_f \setminus P_g$. Let $\{n_i\}$ be an increasing sequence such that $f^{n_i} \to e$. Write $n_i = p^{k_i} m_i$ with p not dividing m_i. Since $p^k \in P_f$, $p^k | n_i$ for sufficiently large i. Assume without loss of generality that $p^k | n_i$ for all i (by taking a subsequence). For $\bar{n}_i = p^{k-1} m_i$ we have $f^{\bar{n}_i} \not\to e$. Since g and f are topologically similar, we have $g^{n_k} \to e$, but since $p^k \notin P_g$ we also have $g^{\bar{n}_i} \to e$, contradicting the topological similarity of f and g.

The reverse implication is obvious. □

Let D be the set of elements $f \in S_\infty$ such that $\langle f \rangle$ is infinite and discrete:

$$D = \{f \in S_\infty : \langle f \rangle \text{ is a discrete copy of } \mathbb{Z}\}.$$

Note that this set is a single class of topological similarity.

Proposition 2.1.11. *The set D is a complete $\mathbf{\Sigma}^0_2$ subset of S_∞.*

Proof. An element $f \in S_\infty$ generates an infinite discrete group if and only if

$$\exists U_i \forall n \geq 1 \; (g^n \notin U_i),$$

where U_i ranges over a sequence of basic clopen neighborhoods of the identity. Therefore

$$\{g \in S_\infty : \langle g \rangle \text{ is infinite discrete}\}$$

is $\mathbf{\Sigma}^0_2$. We would like to note that there is nothing special about S_∞ here: in any Polish group the class of topological similarity of elements that generate infinite discrete group is a $\mathbf{\Sigma}^0_2$ set.

We show that in S_∞ this set is $\mathbf{\Sigma}^0_2$-hard. Recall that

$$A = \{x \in 2^{\mathbb{N}} : \forall^\infty n \ (x(n) = 1)\}$$

is a complete $\mathbf{\Sigma}^0_2$ set.

Let $m \in \mathbb{N}$. We describe a particular cycle π_m on $\{1, \ldots, m\}$. If m is odd, then

$$\pi_m = (1\ 3\ 5\ \ldots m\ m-1\ m-3\ m-5\ \ldots 2),$$

if m is even, then

$$\pi_m = (1\ 3\ 5\ \ldots m-1\ m\ m-2\ m-4\ \ldots 2).$$

We also define π_∞ by

$$\pi_\infty = (\ldots 2k\ 2(k-1)\ 2(k-2) \ldots 4\ 2\ 1\ 3\ 5\ \ldots 2k+1\ 2k+3\ \ldots).$$

We now construct $\alpha : 2^{\mathbb{N}} \to S_\infty$. Fix $x \in 2^{\mathbb{N}}$ and let $\{m_i\}$ be a sequence such that x starts with m_0 many ones, continues with m_1 many zeroes, then m_2 ones and so on. If from some point on x has ones or zeroes on all coordinates, then $\{m_i\}$ is finite and the last element $m_N = \infty$. If x has infinitely many zeroes and infinitely many ones, then the sequence $\{m_i\}$ is infinite and consists of natural numbers. We define permutations τ_n as follows. If n is even, then τ_n is a copy of π_{m_n} supported on the interval of natural numbers starting from $\sum_{i=0}^{n-1} m_i$. If n is odd, then τ_n is the trivial permutation. Note that the supports of τ_n are disjoint. Define $\alpha(x)$ to be $\prod \tau_n$.

The map α is continuous and, moreover, $x \in A$ if and only if $\langle \alpha(x) \rangle$ is infinite discrete. Thus α is a continuous reduction of A into D, hence D is $\mathbf{\Sigma}^0_2$-hard. □

Proposition 2.1.12. *Let $f \in S_\infty$ be an element of infinite order. If $\langle f \rangle$ is non-discrete, then the class of topological similarity of f is a complete $\mathbf{\Pi}^0_2$ set.*

Proof. One can decompose f into disjoint cycles $\{\sigma_n\}_{n=1}^\infty$. Let $\{\tau_n\}_{n=0}^\infty$ be disjointly supported permutations on $\mathbb{N}$ such that τ_n is a copy of $\sigma_1 \sigma_2 \ldots \sigma_n$. Now for $x \in 2^{\mathbb{N}}$, let

$$\beta(x) = \prod_{x(n)=1} \tau_n.$$

If $\forall^\infty n\ x(n) = 0$, then $\beta(x)$ is not topologically similar to f. If $\exists^\infty n\ x(n) = 1$, then $\beta(x)$ is topologically similar to f. Therefore β is a continuous reduction of the complete $\mathbf{\Pi}^0_2$ set B,

$$B = \{x \in 2^{\mathbb{N}} : \exists^\infty n\ x(n) = 1\},$$

to the topological similarity class of f. □

This describes the descriptive complexity of individual classes of topological similarity in S_∞, but what about the relation of topological similarity itself, when viewed as a subset of $S_\infty \times S_\infty$?

Proposition 2.1.13. *The relation of topological similarity on the group S_∞ is strictly $\mathbf{\Delta}^0_3$.*

Proof. The set of pairs $(f, g) \in S_\infty \times S_\infty$ such that both $\langle f \rangle$ and $\langle g \rangle$ are infinite discrete is $\mathbf{\Sigma}^0_2$. Thus f is topologically similar to g if and only if

$$\begin{aligned}
&\text{either both } f \text{ and } g \text{ generate infinite discrete groups} \quad \textit{or}\\
&\Big((\langle f \rangle \text{ is either finite or non-discrete}) \quad \textit{and}\\
&(\langle g \rangle \text{ is either finite or non-discrete}) \quad \textit{and}\\
&(\text{for any } n \text{ and any } p^k | n \text{ if } f \text{ has a cycle of length } n,\\
&\text{then } g \text{ has a cycle of length } m \text{ and } p^k | m) \quad \textit{and}\\
&(\text{for any } n \text{ and any } p^k | n \text{ if } g \text{ has a cycle of length } n,\\
&\text{then } f \text{ has a cycle of length } m \text{ and } p^k | m)\Big).
\end{aligned}$$

The first condition is $\mathbf{\Sigma}^0_2$ and the second one is $\mathbf{\Pi}^0_2$. The relation is their union, which is $\mathbf{\Delta}^0_3$. It is strictly $\mathbf{\Delta}^0_3$, because by Proposition 2.1.11 and Proposition 2.1.12 it has both $\mathbf{\Sigma}^0_2$-hard and $\mathbf{\Pi}^0_2$-hard classes. □

2.2 Topological Similarity Classes in Aut($\mathbb{Q}$) and Homeo$^+$([0, 1])

2.2.1 Introduction and Basic Definitions

Let $\mathbb{Q}$ be the set of rationals viewed as a dense linear order without endpoints. Let $G = \mathrm{Aut}(\mathbb{Q})$ denote the group of order-preserving bijections of the rationals and *id* be the identity element of this group. The group G is naturally a Polish group when endowed with a topology of pointwise stabilization (i.e., the topology of pointwise convergence, when $\mathbb{Q}$ is endowed with the discrete topology). In other words we naturally view $\mathrm{Aut}(\mathbb{Q})$ as a subgroup of S_∞ with the induced topology. It is known that G has a generic conjugacy class: there is a single $g \in G$ such that its conjugacy class is a comeager subset of G. Here is a description of such a $g \in G$.

Let $f \in G$ be given and let $a, b \in \mathbb{Q}$ be two rational numbers. We say that a and b are f-equivalent if there are $m, n \in \mathbb{Z}$ such that $f^m(a) \leq b \leq f^n(a)$. It is easy to see that this is, indeed, an equivalence relations. We say that f is *increasing* at $a \in \mathbb{Q}$ if $f(a) > a$, *decreasing* if $f(a) < a$, and *fixed* if $f(a) = a$. An element $g \in G$ lies in the comeager conjugacy class if and only if every g-equivalence class is a bounded subset of $\mathbb{Q}$ and for all $a < b$ if a and b are not

g-equivalent, then there are c_1, c_2, c_3 such that $a < c_i < b$ and f is increasing at c_1, decreasing at c_2, and fixed at c_3.

It is now natural to ask whether there is a generic two-dimensional conjugacy class in G. I. Hodkinson (see J. K. Truss [25]) showed that this is not the case, he proved that every two-dimensional conjugacy class in G is meager. The goal of this section is to generalize his result and show that every two-dimensional class of topological similarity in G is meager.

By an open interval $I = (a, b) \subset \mathbb{Q}$ we mean the set of rational numbers $\{c : a < c < b\} \subset \mathbb{Q}$. A closed interval $[a, b]$ also includes endpoints a and b. If I is a bounded interval (open or closed) $L(I)$ will denote its left endpoint and $R(I)$ will be its right endpoint. We will use this notation only when both $L(I)$ and $R(I)$ are in $\mathbb{Q}$. If $\mathbf{A} \subset \mathbb{Q}$ is a finite subset, $\min(\mathbf{A})$ and $\max(\mathbf{A})$ will denote its minimal and maximal elements respectively.

Definition 2.2.1. A partial isomorphism of $\mathbb{Q}$ is an order preserving bijection p between *finite* subsets $\mathbf{A}$ and $\mathbf{B}$ of $\mathbb{Q}$.

It is a basic property of the rationals (and, as mentioned earlier, of a Fraïssé limit in general) that each partial isomorphism can be extended (certainly, not uniquely) to a full automorphism.

Letters p and q (with possible sub- or superscripts) will denote partial isomorphisms; let $\mathrm{dom}(p)$ be the domain of p, and $\mathrm{ran}(p)$ be its range. If $I \subseteq \mathbb{Q}$ then $p|_I$ denotes the restriction of p to $I \cap \mathrm{dom}(p)$; $\mathrm{F}(p)$ will be the set of fixed points in the domain of p, i.e.,

$$\mathrm{F}(p) = \{c \in \mathrm{dom}(p) : p(c) = c\}.$$

As we mentioned earlier G is a Polish group (i.e., a separable completely metrizable topological group) in the topology given by the basic open sets

$$U(p) = \{g \in G : g \text{ extends } p\},$$

where p is a partial isomorphism of $\mathbb{Q}$. Note that if p and q are two partial isomorphisms and q extends p then $U(q) \subseteq U(p)$; we will use this observation frequently.

Let $F(s, t)$ denote the free group on two generators: s and t; elements of $F(s, t)$ are reduced words on the alphabet $\{s, t, s^{-1}, t^{-1}\}$. Every element $w \in F(s, t)$ has certain length associated to it, namely the length of the reduced word w. This length is denoted by $|w|$. If $u, v \in F(s, t)$ are words, we say that the word $uv \in F(s, t)$ is reduced if $|uv| = |u| + |v|$, that is there is no cancellation between u and v.

If $w \in F(s, t)$ is a reduced word, $w = t^{n_k} s^{m_k} \cdots t^{n_1} s^{m_1}$, and p, q are partial isomorphisms, then we can define a partial isomorphism $w(p, q)$ by $w(p, q)(c) = q^{n_k} p^{m_k} \cdots q^{n_1} p^{m_1}(c)$, whenever the right-hand side is defined. The *orbit* of c

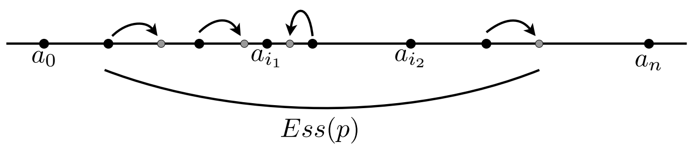


Figure 2.1: Informative partial isomorphism.

under $w(p,q)$ is by definition

$$\mathrm{Orb}_{w(p,q)}(c) = \bigcup_{l=1}^{k} \{p^{i\,\mathrm{sign}(m_l)} q^{n_{l-1}} \cdots p^{m_1}(c),\ q^{j\,\mathrm{sign}(n_l)} p^{m_l} \cdots p^{m_1}(c) : \\ i = 0, \ldots, |m_l|,\ j = 0, \ldots, |n_l|\}.$$

We say that a word w starts from the word v if w can be written as $w = vu$ for some word u, where vu is reduced. Similarly, we say that w ends in v if there is a word u such that $w = uv$, where uv is reduced. On the one hand this is consistent with the intuitive understanding of these notions for, say, left-to-right languages. On the other hand, we consider left actions, and then *the end of the word act first*, i.e., if $w = st$ then $w(p,q)(c) = p(q(c))$. This may be a bit confusing, we apologize for that and emphasize this possible confusion.

Definition 2.2.2. Let p be a partial isomorphism of $\mathbb{Q}$. An interval $(a,b) \subset \mathbb{Q}$ is called *p-increasing* if $a, b \in \mathrm{dom}(p)$, $p(a) = a$, $p(b) = b$ and $p(c) > c$ for any $c \in \mathrm{dom}(p) \cap (a,b)$. The definition of *$p$-decreasing* interval is analogous. Note that if $[a,b] \cap \mathrm{dom}(p) = \{a,b\}$ and $p(a) = a$, $p(b) = b$ then the interval (a,b) is both p-increasing and p-decreasing. An interval is *p-monotone* if it is either p-increasing or p-decreasing.

Definition 2.2.3. Let p be a partial isomorphism. Let $\mathrm{dom}(p) = \{a_0, \ldots, a_n\}$ and assume that $a_0 < \ldots < a_n$. We say that p is *informative* if $p(a_0) = a_0$, $p(a_n) = a_n$ and there is a list $\{i_0, \ldots, i_r\}$ of indices such that

(i) $i_0 = 0$, $i_r = n$;

(ii) $a_{i_k} = p(a_{i_k})$ for $0 \leq k \leq r$;

(iii) for any $0 \leq k < r$ the interval $(a_{i_k}, a_{i_{k+1}})$ is p-monotone.

If p is an informative partial isomorphism and $\mathrm{dom}(p) = \{a_0, \ldots, a_n\}$ as above then we set

$$\mathrm{Ess(p)} = \big(\mathrm{dom}(p) \cup \mathrm{ran}(p)\big) \setminus \{a_0, a_n\}$$

and refer to it as to *the set of essential points of p*.

Definition 2.2.4. A pair (p,q) of partial isomorphisms is called *piecewise elementary* if the following holds

(i) p and q are informative;

(ii) $\min(\mathrm{dom}(p)) = \min(\mathrm{dom}(q))$,

(iii) $\max(\mathrm{dom}(p)) = \max(\mathrm{dom}(q))$.

If additionally $\mathrm{F}(p) \cap \mathrm{F}(q)$ has cardinality at most 2 (i.e., consists of the above minimum and maximum) then the pair (p,q) is called *elementary*.

Let (p,q) be a piecewise elementary pair, and $\mathrm{F}(p) \cap \mathrm{F}(q) = \{a_0, \ldots, a_n\}$ with $a_i < a_j$ for $i < j$. Set $I_j = [a_j, a_{j+1}]$, then $(p|_{I_j}, q|_{I_j})$ is elementary for any $0 \leq j < n$. Thus every piecewise elementary pair (p,q) can be decomposed into finitely many elementary pairs.

The following obvious lemma partially explains the importance of piecewise elementary pairs.

Lemma 2.2.5. *For any non-empty open $V \subseteq G \times G$ there is a piecewise elementary pair (p,q) such that $U(p) \times U(q) \subseteq V$.*

2.2.2 Liberation of Elementary Pairs

Now we come to a somewhat technical, but extremely important notion of liberation.

Definition 2.2.6. Let (p,q) be an elementary pair. We say that a triple (p', q', w) *liberates p in (p,q)*, where p' and q' are partial isomorphisms that extend p and q respectively, and $w \in F(s,t)$ is a reduced word, if the following holds

(i) p' and q' are informative;

(ii) $\min(\mathrm{dom}(p')) = \min(\mathrm{dom}(p)) = \min(\mathrm{dom}(q')) = \min(\mathrm{dom}(q))$;

(iii) $\max(\mathrm{dom}(p')) = \max(\mathrm{dom}(p)) = \max(\mathrm{dom}(q')) = \max(\mathrm{dom}(q))$;

(iv) the word w starts from a non-zero power of t, $w = t^n v$ for $n \neq 0$;

(v) $w(p', q')(c)$ is defined for any $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$ and

$$\max(\mathrm{dom}(q)) \geq w(p', q')(\min(\mathrm{Ess(p)} \cup \mathrm{Ess(q)})) > \max(\mathrm{Ess(p')}),$$

(vi) there is an open interval J such that $R(J) = \max(\mathrm{dom}(q))$, q' is monotone on J and $w(p', q')(c) \in J$ for any $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$; moreover, if $n > 0$ in the item (iv), then J is q'-increasing, and it is q'-decreasing otherwise.

Similarly, we say that a triple (p', q', w) *liberates* q *in* (p, q) if the above holds with roles of p and q, s and t interchanged.

For a piecewise elementary pair (p, q), we say that a triple (p', q', w) liberates p [liberates q] in (p, q) if

(i) $\min(\mathrm{dom}(p')) = \min(\mathrm{dom}(p)) = \min(\mathrm{dom}(q')) = \min(\mathrm{dom}(q))$;

(ii) $\max(\mathrm{dom}(p')) = \max(\mathrm{dom}(p)) = \max(\mathrm{dom}(q')) = \max(\mathrm{dom}(q))$;

(iii) for any interval I, such that $(p|_I, q|_I)$ is elementary, the triple $(p'|_I, q'|_I, w)$ liberates $p|_I$ [liberates $q|_I$] in $(p|_I, q|_I)$.

Lemma 2.2.7. *For any elementary pair* (p, q) *there is a triple* (p', q', w) *that liberates* p *[liberates* q*] in* (p, q)*.*

Proof. We show the existence of a triple that liberates p, and the second clause then follows by symmetry.

Extending p and q if necessary, we may assume that

(i) $\mathrm{Ess(p)} \neq \emptyset$, $\mathrm{Ess(q)} \neq \emptyset$;

Let $I_1, \ldots, I_k$ be the list of the (open) intervals of monotonicity for p and $J_1, \ldots, J_l$ be the list of the (open) intervals of monotonicity for q; we list intervals in increasing order, i.e., $I_i < I_{i+1}$, $J_j < J_{j+1}$; then also

(ii) $I_1 \cap \mathrm{dom}(p) \neq \emptyset$ and $J_1 \cap \mathrm{dom}(q) \neq \emptyset$;

(iii) $I_k \cap \mathrm{dom}(p) = \emptyset$ and $J_l \cap \mathrm{dom}(q) = \emptyset$;

(iv) $L(I_k) > L(J_l)$.

Let $\alpha = \min(\mathrm{Ess(p)} \cup \mathrm{Ess(q)})$. Then $\alpha \in I_1 \cap J_1$ by (ii) (and in particular α is not a fixed point of p or q). We first find an informative extension p_1 of p that has the same intervals of monotonicity as p and $m_1 \in \mathbb{Z}$ (the sign of m_1 depends on whether p is increasing or decreasing) such that $p_1^{m_1}(\alpha)$ is defined and is "close enough" to the right endpoint of I_1. "Close enough" exactly means the following. Since by assumptions $R(I_1)$ is not fixed by q (because (p, q) is elementary), there is some j_1 such that $R(I_1) \in J_{j_1}$ and we want $p_1^{m_1}(\alpha) \in J_{j_1}$. At the second step we find an informative extension q_1 of q (also with the same intervals of monotonicity) and $n_1 \in \mathbb{Z}$ (similarly the sign of n_1 depends on whether q is increasing or decreasing) such that $q_1^{n_1} p_1^{m_1}(\alpha)$ is defined and is "close enough" in the above sense to the right endpoint of J_{j_1}. We proceed in this way and stop as soon as the image of α reaches J_l, i.e., we obtain extensions $\bar{p}$, $\bar{q}$ of p and q and a word $u = s^{m_{N+1}} v$, where $v = t^{n_N} s^{m_N} \cdots t^{n_1} s^{m_1}$ such that $u(\bar{p}, \bar{q})(\alpha)$ is defined, lies in J_l and $v(\bar{p}, \bar{q})(\alpha) \notin J_l$. Note that since we added to the domain of q only points of the orbit of α under u, this implies $\mathrm{dom}(\bar{q}) \cap J_l = \emptyset$. Also by induction $\bar{p}$ and $\bar{q}$ are informative with the same decomposition into intervals of monotonicity as for p and q.

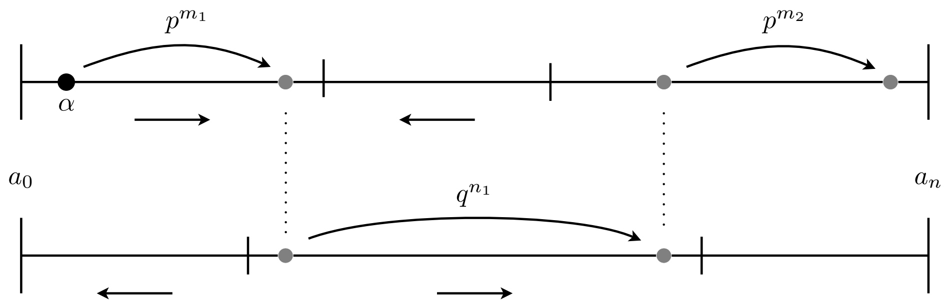


Figure 2.2: Construction of the liberating triple. Horizontal arrows indicate monotonicity of partial isomorphisms, bars stand for fixed points, the black dot is the minimal element α, and gray dots are its images under w.

We now take extensions $\bar{p}'$, $\bar{q}'$ of $\bar{p}$ and $\bar{q}$ such that

(i) $u(\bar{p}', \bar{q}')(c)$ is defined for every $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$;

(ii) $\bar{p}'$ and $\bar{q}'$ are informative with the same decomposition into intervals of monotonicity as for $\bar{p}$ and $\bar{q}$;

(iii) the minimum and maximum of the domains of $\bar{p}'$ and $\bar{q}'$ are equal to the minimum and maximum of the domains of $\bar{p}$ and $\bar{q}$;

(iv) $\bar{q}'$ is monotone on J_l (this is possible since $J_l \cap \mathrm{dom}(\bar{q}) = \emptyset$).

Set $p' = \bar{p}'$. Finally extending $\bar{q}'$ to q' we can find $M \in \mathbb{Z} \setminus \{0\}$ such that

$$q'^M u(p', q')(\alpha) > \max(\mathrm{Ess(p')}).$$

And so let $w = t^M u$, then (p', q', w) liberates p in (p, q). □

Remark 2.2.8. In the lemma above we started our construction by applying a power of p, but we likewise could start it by applying a power of q.

Remark 2.2.9. We view rationals as a dense linear ordering without endpoints. But note that if we have the usual metric on $\mathbb{Q}$ then the above construction gives us p', q', and w such that $w(p', q')(\alpha)$ is as close in this metric to the endpoint $\max(\mathrm{dom}(p))$ as one wants. We will use this observation later.

Lemma 2.2.10. *For any elementary pair (p, q) and any word u there are a word v, and partial isomorphisms p' and q' such that the triple (p', q', vu) liberates p [liberates q] in (p, q) and $|vu| = |v| + |u|$ (i.e., no cancellation between v and u happens).*

Proof. First we take extensions p_1 and q_1 of p and q respectively such that $u(p_1, q_1)(c)$ is defined for any $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$, (p_1, q_1) is elementary and

$$\min(\mathrm{dom}(p_1)) = \min(\mathrm{dom}(p)) = \min(\mathrm{dom}(q)) = \min(\mathrm{dom}(q_1)),$$
$$\max(\mathrm{dom}(p_1)) = \max(\mathrm{dom}(p)) = \max(\mathrm{dom}(q)) = \max(\mathrm{dom}(q_1)).$$

By Lemma 2.2.7 one can find a word v and extensions p', q' of p_1, q_1 such that (p', q', v) liberates p_1 in (p_1, q_1). By Remark 2.2.8 we may also assume that there is no cancellation in vu. We claim that (p', q', vu) liberates p in (p, q). Items (i-iv) from the definition of liberation are obvious.

For item (v) note that by construction $u(p_1, q_1)(c)$ for all $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$ is defined. Since p' and q' extend p_1 and q_1 we get that $u(p', q')(c)$ is defined for all $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$ and since (p', q', v) liberates p_1 in (p_1, q_1) we have that for all $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$ the expression $v(p', q')(u(p_1, q_1)(c))$ is defined (just because $u(p_1, q_1)(c) \in \mathrm{Ess(p_1)} \cup \mathrm{Ess(q_1)}$). This shows that $vu(p', q')(c)$ is defined for $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$. Also we have

$$v\big(p', q'\big)\big(\min(\mathrm{Ess(p_1)} \cup \mathrm{Ess(q_1)})\big) > \max(\mathrm{Ess(p')}).$$

Finally $u(p_1, q_1)(\mathrm{Ess(p)} \cup \mathrm{Ess(q)}) \subseteq \mathrm{Ess(p_1)} \cup \mathrm{Ess(q_1)}$ implies

$$vu\big(p', q'\big)\big(\min(\mathrm{Ess(p)} \cup \mathrm{Ess(q)})\big) > \max(\mathrm{Ess(p')}).$$

Item (vi) follows immediately from the fact that (p', q', v) liberates p_1 in (p_1, q_1) and from the observation that

$$u\big(p_1, q_1\big)\big(\,\mathrm{Ess(p)} \cup \mathrm{Ess(q)}\,\big) \subseteq \mathrm{Ess(p_1)} \cup \mathrm{Ess(q_1)}\,. \qquad \square$$

Lemma 2.2.11. *Let (p, q) be a piecewise elementary pair and assume a triple (p', q', w) liberates p [liberates q] in (p, q). Let $u = t^n v$ [$u = s^m v$] be a reduced word such that uw is irreducible. Then there is a triple (p'', q'', uw) that liberates p [liberates q] in (p, q). Moreover, one can take p'' to be an extension of p' and q'' to be an extension of q'.*

Proof. By the definition of liberation for piecewise elementary pairs it is enough to prove the statement for elementary triples only. So assume (p, q) is elementary. Since w liberates p in (p, q) it has to start with a non-zero power l of t, i.e, $w = t^l *$. We prove the statement by induction on $|u|$. If u is empty the statement is trivial. Now consider the inductive step. Either $u = *t^k$ and the sign of k matches the sign of l (because uw has to be reduced by assumptions) or $u = *s^k$ with $k \neq 0$. In the former case extend q' to q'_1 in such a way that $(t^k w)(p', q'_1)(c)$ is defined for $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$, then $(p', q'_1, t^k w)$ will be a p-liberating tuple by the item (vi) of the definition of liberation. In the second case we can find p'_1 such that $(p'_1, q', s^k w)$ liberates q' in (p', q') by taking p'_1 such that $(s^k w)(p'_1, q')(c) > \max(\mathrm{Ess(p')} \cup \mathrm{Ess(q')})$ for any $c \in \mathrm{Ess(p')} \cup \mathrm{Ess(q')}$. This proves the induction step and the lemma. $\square$

Lemma 2.2.12. *Let (p, q) be a piecewise elementary pair and $u \in F(s, t)$. Then there is a triple (p', q', w) that liberates p [liberates q] in (p, q) and such that $w = vu$ is reduced.*

Proof. We prove the statement by induction on the number of elementary components of (p,q). Lemma 2.2.10 covers the base of induction. Assume we have proved the lemma for r-many elementary components and inductively constructed a triple $(\bar{p}_r, \bar{q}_r, w_r)$ that liberates p_r in (p_r, q_r), where p_r and q_r are restrictions of p and q to the first r-many elementary components. Consider the restrictions $\tilde{p}_{r+1}$, $\tilde{q}_{r+1}$ of p and q to the $r+1$ elementary component. By the base of induction (i.e., Lemma 2.2.10) we can find extensions $\tilde{p}'_{r+1}$, $\tilde{q}'_{r+1}$ of $\tilde{p}_{r+1}$ and $\tilde{q}_{r+1}$ and a word v_{r+1} such that $(\tilde{p}'_{r+1}, \tilde{q}'_{r+1}, v_{r+1}w_r)$ liberates $\tilde{p}_{r+1}$ in $(\tilde{p}_{r+1}, \tilde{q}_{r+1})$ and $v_{r+1}w_r$ is irreducible. By Lemma 2.2.11 we can also extend p_r and q_r to p'_r, q'_r in such a way that $(p'_r, q'_r, v_{r+1}w_r)$ liberates p_r in (p_r, q_r). Now set $\bar{p}_{r+1}$ to coincide with p'_r on the first r-many elementary components and with $\tilde{p}'_{r+1}$ on the $r+1$ component. Define $\bar{q}_{r+1}$ similarly. Then $(\bar{p}_{r+1}, \bar{q}_{r+1}, w_{r+1})$ liberates p_{r+1} in (p_{r+1}, q_{r+1}). This proves the induction step and the lemma. □

2.2.3 Two-dimensional similarity classes are meager

Lemma 2.2.13. *For any pair (p,q) of partial isomorphisms and any word $u \in F(s,t)$ there are extensions p' and q' of p and q respectively and a reduced word $w = vu$ such that $w(p', q')(c) = c$ for any $c \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$.*

Proof. By Lemma 2.2.5 it is enough to prove the statement for a piecewise elementary pair (p,q). By Lemma 2.2.12 we can find extensions $\bar{p}$, $\bar{q}$ and a word v such that $(\bar{p}, \bar{q}, vu)$ liberates p in (p,q). By the definition of liberation we can now extend $\bar{p}$ to p' by declaring

$$p'(c) = c, \text{ for any } c \in vu(\bar{p}, \bar{q})(\mathrm{Ess(p)} \cup \mathrm{Ess(q)}).$$

Now set $q' = \bar{q}$ and $w = u^{-1}v^{-1}svu$. Then $w(p', q')(c) = c$ for any $c \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$. □

Lemma 2.2.14. *Fix a sequence $\{u_k\}$ of reduced words. For a generic $(f,g) \in G \times G$ there is a sequence of reduced words $w_k = v_k u_k$ such that $w_k(f,g) \to id$.*

Proof. Take an enumeration $\{c_i\} = \mathbb{Q}$ of the rationals. Let

$$B^k_n = \{(f,g) \in G \times G : \exists w = vu_k \text{ reduced and } w(f,g)(c_i) = c_i \text{ for } 0 \le i \le n\}.$$

We claim that each B^k_n is dense and open. Indeed, assume for a certain n one has $(f,g) \in B^k_n$. This is witnessed by a word w. Set

$$D = \cup_{i=0}^n \mathrm{Orb}_{w(f,g)}(c_i)$$

and let $p = f|_D$, $q = g|_D$. Then $(f,g) \in U(p) \times U(q) \subseteq B^k_n$ and so B^k_n is open. Density follows from Lemma 2.2.13.

Now by the Baire theorem $\cap_{n,k} B^k_n$ is a dense G_δ. The lemma follows. □

Theorem 2.2.15. *Each two-dimensional topological similarity class in G is meager.*

Proof. Assume towards a contradiction that there is a pair $(f_1, g_1) \in G \times G$ that has a non-meager class of topological similarity. Then by Lemma 2.2.14 there must be a sequence $w_n = v_n t^n s^n$ of reduced words such that (f_1, g_1) converges to the identity along this sequence (we apply Lemma 2.2.14 with the sequence $u_k = t^k s^k$).

Take and fix $a \in \mathbb{Q}$. Set

$$F_a = \{(f, g) \in G \times G : f(a) = a = g(a)\}.$$

Let

$$C_n = \{(x, y) \in G \times G : \exists m > n \; w_m(x, y)(a) \neq a\}.$$

Then C_n is open and dense in $(G \times G) \setminus F_a$. To see density take a basic open set $U(p) \times U(q) \subseteq (G \times G) \setminus F_a$ and assume $p(a) \neq a$ (the case when $p(a) = a$, but $q(a) \neq a$ is similar). For some $k > n$ $p^k(a)$ is not in the domain of p. Thus the set

$$\{b \in \mathbb{Q} : \exists f \in U(p) \; f^{k+1}(a) = b\}$$

is infinite, and so (by induction) there are infinitely many values that $w_{k+1}(f, g)(a)$ may attain for a pair $(f, g) \in U(p) \times U(q)$. Hence $w_{k+1}(f, g)(a) \neq a$ for some (f, g). And so C_n is dense in $G \times G \setminus F_a$. An application of the Baire theorem shows that $\cap C_n$ is a dense G_δ and so for a generic $(f, g) \in (G \times G) \setminus F_a$ one has $w_n(f, g)(a) \not\to a$ in the discrete topology. Since $\cup_a (G \times G) \setminus F_a = (G \times G) \setminus \{id \times id\}$ we get a contradiction with the assumption that $w_n(f_1, g_1) \to id$ and that the class of topological similarity of (f_1, g_1) is non-meager. □

2.2.4 Homeomorphisms of the unit interval.

We now turn to the group of homeomorphisms of the unit interval. This is a Polish group in the natural topology, given by the basic open sets:

$$U(f; a_1, \ldots, a_n; \varepsilon) = \{g \in \mathrm{Homeo}([0, 1]) : |g(a_i) - f(a_i)| < \varepsilon\}.$$

We may write this neighborhood as $U(p; \varepsilon)$, where $p = f|_{\{a_1, \ldots, a_n\}}$ is a partial isomorphism. Since $\mathbb{Q}$ is dense in $[0, 1]$, we may assume that p is a partial isomorphism of the rationals: this will give us a base of open sets.

This group Homeo$([0, 1])$ has a normal subgroup of index 2, namely the subgroup Homeo$^+([0, 1])$ of order preserving homeomorphisms. If $H = \mathrm{Homeo}^+([0, 1])$, then $\mathrm{Aut}(\mathbb{Q}) = G$ naturally embeds into H (this embedding is a continuous injective homomorphism, its inverse, though, is not continuous), and the image of G under this embedding is dense in H.

Theorem 2.2.16. *Every two-dimensional class of topological similarity in H*

is meager.

Proof. We imitate the proof of Theorem 2.2.15. If $\{x_m\}$ is an enumeration of the rationals $\mathbb{Q} \cap [0,1]$, then $\{x_m\}$ is dense in $[0,1]$. Set

$$A_{m,n} = \{f \in H : |f(x_m) - x_m| > 1/n \text{ and } |f^{-1}(x_m) - x_m| > 1/n\},$$
$$B_{m,n} = \{(f,g) \in H \times H : f \in A_{m,n} \text{ or } g \in A_{m,n}\}.$$

Note that $B_{m,n}$ is open for every m and n. Then $\cup_{m,n} B_{m,n} = H \times H \setminus \{(id, id)\}$ and so it is enough to prove that each two-dimensional class of topological similarity is meager in each of $B_{m,n}$.

Let u_k be a sequence of words such that for every piecewise elementary pair (p,q) (here p and q are partial isomorphisms of the rationals, as before) there are infinitely many k such that for some p'_k, q'_k, (p'_k, q'_k, u_k) liberates p in (p,q). Then by Lemma 2.2.14 for a generic pair $(f,g) \in G \times G$ there is a sequence of reduced words $w_k = v_k u_k$ such that $w_k(f,g) \to id$. This implies that for a generic pair $(f,g) \in H \times H$ there is a sequence w_k as above (because the topology in H is coarser than in G). If there is a non-meager two-dimensional class of topological similarity then there is a sequence of reduced words $\{w_k\} = \{v_k u_k\}$ (for some $\{v_k\}$) such that the set of pairs $(f_1, g_1) \in H \times H$ that converges to the identity along w_k is non-meager.

Fix now m, n and a sequence of reduced words $w_k = v_k u_k$. Set

$$C_k = \{(f,g) \in H \times H : \exists K > k \; |w_K(f,g)(x_m) - x_m| > 1/2n\}.$$

Each C_k is open, and we claim that it is also dense in $B_{m,n}$. Let $V \subseteq B_{m,n}$ be an open set. Without loss of generality we may assume that $V = U(p; \varepsilon_1) \times U(q; \varepsilon_2)$, where p and q are partial isomorphisms of the rationals. Let

$$\delta = \min\{|x_m - c| : c \in \mathrm{F}(p) \cap \mathrm{F}(q)\} > 1/n.$$

Then there is $K > k$ and p', q' such that (p', q', u_K) liberates p in (p,q). Now repeat the proof of Lemma 2.2.11 and use Remark 2.2.9 to get p'', q'' that extend p' and q' and such that $|w_K(p'', q'')(x_m) - x_m| \geq 1/2\delta$. Hence each C_k is dense in $B_{m,n}$. Now by the Baire theorem the intersection $\cap_k C_k$ is a dense G_δ in $B_{m,n}$ and thus for any specific sequence w_k the set of elements $(f_1, g_1) \in H \times H$ that converges to the identity along this sequence is meager in $B_{m,n}$. Finally we showed that each two-dimensional topological similarity class is meager in $B_{m,n}$ for any m, n and so is in $H \times H$. □

2.3 Extensions of Partial Isometries

In this section we prove several results, that will be used later, when dealing with the ordered Urysohn space. But we believe that some of the theorems below are of independent interest for understanding the group of isometries of the Urysohn space. We mostly work with the classical Urysohn space, but some of the results will be later applied to the ordered rational Urysohn space. The following proposition will let us do that.

Proposition 2.3.1. *Let* $\mathbf{A}$ *be a finite ordered metric space, and let* p *be a partial isomorphism of* $\mathbf{A}$*. Let* $\mathbf{B}$ *be a finite metric space (with no ordering) and let* q *be a partial isometry of* $\mathbf{B}$ *with* $\mathrm{Z}(q) = \mathrm{F}(q)$*. Suppose that* $\mathbf{A} \subseteq \mathbf{B}$ *as metric spaces and* q *extends* p*. If*

$$\forall x \in \mathrm{dom}(q)\ q(x) \in \mathbf{A} \iff x \in \mathrm{dom}(p),$$

then there is a linear ordering on $\mathbf{B}$ *that extends an ordering on* $\mathbf{A}$ *and such that* q *becomes a partial isomorphism of an ordered metric space* $\mathbf{B}$*.*

Proof. We prove the statement by induction on $|\mathbf{B} \setminus \mathbf{A}|$. If $\mathbf{A} = \mathbf{B}$ the statement is obvious. For the inductive step we consider two cases.

Case 1. There is some $x \in \mathbf{A}$ such that $x \in \mathrm{dom}(q)$ but $x \notin \mathrm{dom}(p)$. Then by the assumption, $q(x) \in \mathbf{B} \setminus \mathbf{A}$. Now extend the linear ordering on $\mathbf{A}$ to a partial ordering on $\mathbf{A} \cup \{q(x)\}$ by declaring for $y \in \mathbf{A}$

$$\begin{aligned} q(x) < y &\iff \exists z \in \mathrm{dom}(p)\ (p(z) \leq y) \& (x < z), \\ y < q(x) &\iff \exists z \in \mathrm{dom}(p)\ (y \leq p(z)) \& (z < x). \end{aligned}$$

It is straightforward to check that this relation is indeed a partial ordering on $\mathbf{A} \cup \{q(x)\}$. Extend this partial ordering to a linear ordering on $\mathbf{A} \cup \{q(x)\}$ in any way. Then q is a partial isomorphism of $\mathbf{A} \cup \{q(x)\}$ and we apply the induction.

Case 2. Assume the opposite to the first case happens. Then $q|_{\mathbf{A}} = p$. Take any $x \in \mathrm{dom}(q) \setminus \mathbf{A}$ (if there is no such x then $\mathrm{dom}(p) = \mathrm{dom}(q)$ and the statement is obvious). Assume first that x is not a fixed point of q. Then define a linear ordering on $\mathbf{A} \cup \{x, q(x)\}$ by declaring

$$\forall y \in \mathbf{A}\ (y < x) \& (y < q(x)) \& (x < q(x)).$$

Then q is a partial isomorphism of $\mathbf{A} \cup \{x, q(x)\}$ and we can apply the induction hypothesis. If x was a fixed point then we declare

$$\forall y \in \mathbf{A}\ (y < x),$$

and, again, induction does the rest. □

The core of our arguments will be the following seminal result due to Sławomir Solecki established in 2005, see [24]. The second item is slightly modified compared to the original statement, but the modification follows from the proof in [24] without any additional work.

Theorem 2.3.2 (Solecki)**.** *Let a finite metric space* $\mathbf{A}$ *and a partial isometry* p *of* $\mathbf{A}$ *be given. There exist a finite metric space* $\mathbf{B}$ *with* $\mathbf{A} \subseteq \mathbf{B}$ *as metric spaces, an isometry* $\bar{p}$ *of* $\mathbf{B}$ *extending* p*, and a natural number* M *such that*

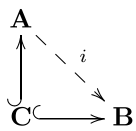
(i) $\bar{p}^{2M} = id_B$*;*

(ii) if $a \in \mathbf{A}$ *is aperiodic then* $\bar{p}^j(a) \neq a$ *for* $0 < j < 2M$*, and moreover for any* j *such that* $0 < j < 2M$ $\bar{p}^j(a) \in \mathbf{A}$ *iff* $\bar{p}^{j-1}(a) \in \mathrm{dom}(p)$*;*

(iii) $\mathbf{A} \cup \bar{p}^M(\mathbf{A})$ *is the free amalgam of* $\mathbf{A}$ *and* $\bar{p}^M(\mathbf{A})$ *over* $(\mathrm{Z}(p), id_{\mathrm{Z}(p)}, \bar{p}^M|_{\mathrm{Z}(p)})$*.*

Moreover, the distances in $\mathbf{B}$ *may be taken from the additive semigroup generated by the distances in* $\mathbf{A}$*.*

Definition 2.3.3. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be metric spaces and let $\mathbf{C}$ be embedded into $\mathbf{A}$ and $\mathbf{B}$. We say that $\mathbf{B}$ *extends* $\mathbf{A}$ *over* $\mathbf{C}$ if there exists an embedding $i : \mathbf{A} \to \mathbf{B}$ such that the following diagram commutes:



We say that $\mathbf{A}$ and $\mathbf{B}$ are *disjoint over* $\mathbf{C}$ if neither $\mathbf{B}$ extends $\mathbf{A}$ over $\mathbf{C}$ nor $\mathbf{A}$ extends $\mathbf{B}$ over $\mathbf{C}$.

Lemma 2.3.4. *Let* $\mathbf{A}$ *be a finite metric space, let* p *be a partial isometry of* $\mathbf{A}$*, and* $x \in \mathrm{dom}(p)$ *be a non-periodic point* $x \notin \mathrm{Z}(p)$ *such that and* $x \notin \mathrm{ran}(p)$ *(i.e.,* $p^{-1}(x)$ *is undefined). Then there are metric spaces* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *that both extend* $\mathbf{A}$*:* $\mathbf{A} \subset \mathbf{A}_1$ *and* $\mathbf{A} \subset \mathbf{A}_2$*, and partial isometries* p_1 *of* $\mathbf{A}_1$ *and* p_2 *of* $\mathbf{A}_2$ *that both extend* p *and such that* $x \notin \mathrm{ran}(p_1) \cup \mathrm{ran}(p_2)$ *and* $\mathrm{Orb}_{\mathrm{p}_1}(x)$ *and* $\mathrm{Orb}_{\mathrm{p}_2}(x)$ *are disjoint over* $\mathrm{Orb}_{\mathrm{p}}(x)$*.*

Moreover, one can assume that

$$\begin{aligned}
&\mathrm{Z}(p_1) = \mathrm{Z}(p) = \mathrm{Z}(p_2), \\
&\forall x \in \mathrm{dom}(p_1)\ p_1(x) \in \mathbf{A} \iff x \in \mathrm{dom}(p), \\
&\forall x \in \mathrm{dom}(p_2)\ p_2(x) \in \mathbf{A} \iff x \in \mathrm{dom}(p).
\end{aligned}$$

Proof. Apply Theorem 2.3.2 to get a full isometry $\bar{p}$ of a finite metric space $\mathbf{B}$ that extends p and a natural number M. Set

$$\bar{\mathbf{A}} = \mathbf{A} \cup \bar{p}(\mathbf{A}) \cup \ldots \cup \bar{p}^{2M-1}(\mathbf{A}) \cup \{y\},$$

where y is a new point, i.e., a point not in $\mathbf{B}$. Let $\delta = \mathcal{D}(\mathbf{A})$ denote the density of $\mathbf{A}$ and fix an $\varepsilon > 0$ such that $\varepsilon \leq 2\delta$. We turn $\bar{\mathbf{A}}$ into a metric space by defining the distance between $a, b \in \bar{\mathbf{A}}$, $a \neq b$ as follows.

$$d_{\bar{\mathbf{A}}}(a,b) = d_{\mathbf{B}}(a,b) \quad \text{when } a, b \neq y;$$
$$d_{\bar{\mathbf{A}}}(a,y) = d_{\mathbf{B}}(a,x) \quad \text{when } a \neq x, y;$$
$$d_{\bar{\mathbf{A}}}(x,y) = \varepsilon.$$

We claim that $(\bar{\mathbf{A}}, d_{\bar{\mathbf{A}}})$ is a metric space. We have to check the triangle inequality (other conditions are obviously fulfilled). For this note that both $\bar{\mathbf{A}} \setminus \{y\}$ and $\bar{\mathbf{A}} \setminus \{x\}$ are isometrically embeddable into $\mathbf{B}$, where the triangle inequality is known to be satisfied. So one needs to prove two claims.

Claim 1. For any $z \in \bar{\mathbf{A}}$

$$d_{\bar{\mathbf{A}}}(x,y) \leq d_{\bar{\mathbf{A}}}(x,z) + d_{\bar{\mathbf{A}}}(z,y).$$

If $z \in \{x,y\}$ then the statement is obvious. If $z \notin \{x,y\}$ then $d_{\bar{\mathbf{A}}}(x,z) + d_{\bar{\mathbf{A}}}(z,y) \geq 2\delta$ and $d_{\bar{\mathbf{A}}(x,y)} = \varepsilon \leq 2\delta$ and Claim 1 follows.

Claim 2. For any $z \in \bar{\mathbf{A}}$

$$d_{\bar{\mathbf{A}}}(x,z) \leq d_{\bar{\mathbf{A}}}(x,y) + d_{\bar{\mathbf{A}}}(y,z),$$
$$d_{\bar{\mathbf{A}}}(z,y) \leq d_{\bar{\mathbf{A}}}(z,x) + d_{\bar{\mathbf{A}}}(x,y).$$

Note that for $z \notin \{x,y\}$ one has $d_{\bar{\mathbf{A}}}(y,z) = d_{\bar{\mathbf{A}}}(x,z)$. From this both inequalities follow immediately.

So $\bar{\mathbf{A}}$ is a metric space. We denote it by $\bar{\mathbf{A}}(\varepsilon)$ to signify the dependence on epsilon. Define a partial isometry $\hat{p}$ on $\bar{\mathbf{A}}(\varepsilon)$ by

$$\hat{p}(z) = \bar{p}(z),$$

whenever $z \in \bar{\mathbf{A}}$ and $\bar{p}(z) \in \bar{\mathbf{A}}$; and $\hat{p}(\bar{p}^{2M-1}(x)) = y$. Using $\bar{p}^{2M} = id_{\mathbf{B}}$ it is straightforward to check that $\hat{p}$ is indeed a partial isometry. Now the construction of two extensions that are disjoint over $\mathrm{Orb_p}(x)$ is easy. Take, for example, two different $\varepsilon_1 \leq 2\delta$, $\varepsilon_2 \leq 2\delta$, $\varepsilon_1 \neq \varepsilon_2$ such that

$$\varepsilon_i \notin \{d_{\mathbf{B}}(x_1, x_2) : x_1, x_2 \in \mathbf{B}\},$$

let $(\mathbf{A}_i, p_i) = (\bar{\mathbf{A}}(\varepsilon_i), \hat{p})$. Then $\mathrm{Orb_{p_1}}(x)$ and $\mathrm{Orb_{p_2}}(x)$ are disjoint over $\mathrm{Orb_p}(x)$. $\square$

The main power of Theorem 2.3.2 is the explicit construction of an extension of a partial isometry to a full isometry of a finite metric space. Moreover, this extension is as independent as possible. For our purposes we only need an extension to a partial isomorphism, but we want to keep the independence. Let us state explicitly a corollary of the theorem that gives everything that we need.

Corollary 2.3.5. *For any finite metric space* $\mathbf{A}$ *and a partial isometry* p *there is finite metric space* $\mathbf{C}$*, a partial isometry* p_1 *of* $\mathbf{C}$*, which is an extension of* p*, and a natural number* M *such that*

(i) $\mathrm{Z}(p) = \mathrm{Z}(p_1)$*;*

(ii) $\mathbf{A} \cup p_1^M(\mathbf{A})$ *is the amalgam of* $\mathbf{A}$ *and* $p_1^M(\mathbf{A})$ *over* $(\mathrm{Z}(p), id_{\mathrm{Z}(p)}, p_1^M|_{\mathrm{Z}(p)})$*.*

(iii) for any $x \in \mathrm{dom}(p_1)$

$$p_1(x) \in \mathbf{A} \iff x \in \mathrm{dom}(p).$$

Moreover, the distances in $\mathbf{C}$ *are taken from the additive semigroup generated by the distances in* $\mathbf{A}$*, and hence the density is preserved:* $\mathcal{D}(\mathbf{C}) = \mathcal{D}(\mathbf{A})$*.*

Proof. Apply Theorem 2.3.2 to $\mathbf{A}$ and p to get a metric space $\mathbf{B}$, a full isometry $\bar{p}$ of $\mathbf{B}$ and a natural number M. Now set

$$\mathbf{C} = \mathbf{A} \cup \bar{p}(\mathbf{A}) \cup \ldots \cup \bar{p}^M(\mathbf{A}),$$

and $p_1 = \bar{p}|_{\mathbf{A}\cup\bar{p}(\mathbf{A})\cup\ldots\cup\bar{p}^{M-1}(\mathbf{A})}$. It is trivial to check that such a $\mathbf{C}$ and p_1 satisfy the conditions. □

Definition 2.3.6. Let (M, d) be a metric space, and let $x, y \in M$. We say that the distance $d(x, y)$ *passes through a point* $z \in M$ if

$$d(x, y) = d(x, z) + d(z, y).$$

We are going to apply Corollary 2.3.5 to partial isometries that also preserve an ordering. That is why we impose an additional assumption: all periodic points are fixed points, i.e., $\mathrm{Z}(p) = \mathrm{F}(p)$.

Theorem 2.3.7. *Let* $\mathbf{A}$ *be a finite metric space. Let* p *and* q *be two partial isometries of* $\mathbf{A}$ *such that* $\mathrm{Z}(p) = \mathrm{F}(p)$ *and* $\mathrm{Z}(q) = \mathrm{F}(q)$*. Suppose* $\mathrm{F}(p) \cap \mathrm{F}(q) \neq \emptyset$*. Then there are finite metric space* $\mathbf{B}$*, extensions* $\bar{p}$*,* $\bar{q}$ *of* p *and* q *respectively (these extensions are partial isometries of* $\mathbf{B}$*) and an element* $w = t^K v \in F(s, t)$*,* $K \neq 0$ *such that*

(i) $\mathrm{Z}(\bar{p}) = \mathrm{Z}(p)\ (= \mathrm{F}(p))$*,* $\mathrm{Z}(\bar{q}) = \mathrm{Z}(q)\ (= \mathrm{F}(q))$*;*

(ii) $\mathrm{dom}(\bar{p}) \cup w(\bar{p}, \bar{q})(\mathbf{A})$ *is the free amalgam of* $\mathrm{dom}(\bar{p})$ *and* $w(\bar{p}, \bar{q})(\mathbf{A})$ *over* $F(p) \cap F(q)$*.*

Moreover, the distances in $\mathbf{B}$ *are taken from the additive semigroup generated by the distances in* $\mathbf{A}$*, and hence* $\mathcal{D}(\mathrm{dom}(\bar{p}) \cup \mathbf{A}) = \mathcal{D}(\mathbf{A})$*,* $\mathcal{D}(\mathrm{dom}(\bar{q}) \cup \mathbf{A}) = \mathcal{D}(\mathbf{A})$*.*

Proof. Let

$$N = \left\lceil \frac{2\mathrm{diam}(\mathbf{A})}{\mathcal{D}(\mathbf{A})} \right\rceil.$$

Define inductively the sequence of elements $w_k \in F(s,t)$, extensions $\bar{p}_k$, $\bar{q}_k$ and metric spaces $\mathbf{A}_k$ as follows:

Step 0: Let $\bar{p}_0 = p$, $\bar{q}_0 = q$, $w_0 =$ empty word, $\mathbf{A}_0 = \mathbf{A}$;

Step k: If k is odd then apply Corollary 2.3.5 to $\bar{p}_{k-1}$ and $\mathbf{A}_{k-1}$ to get $\bar{p}_k$ and M_k; set $\bar{q}_k = \bar{q}_{k-1}$, $w_k = s^{M_k} w_{k-1}$, $\mathbf{A}_k = \mathbf{A}_{k-1} \cup \mathrm{dom}(\bar{p}_k) \cup \mathrm{ran}(\bar{p}_k)$.

If k is even do the same thing with the roles of p and q interchanged.

We claim that $\bar{p} = \bar{p}_{2N+2}$, $\bar{q} = \bar{q}_{2N+2}$, $\mathbf{B} = \mathbf{A}_{2N+2}$, and $w = w_{2N+2}$ fulfill the requirements of the statement. Let d denote the metric on $\mathbf{B}$. It is obvious that $\mathrm{F}(\bar{p}) = \mathrm{F}(p)$ and $\mathrm{F}(\bar{q}) = \mathrm{F}(q)$ (this is given by Corollary 2.3.5 at each stage). The moreover part is also obvious, since it is fulfilled at every step of the construction. It remains to show that for any $x \in w(\bar{p}, \bar{q})(\mathbf{A})$ and any $y \in \mathrm{dom}(\bar{p})$ one has

$$d(x,y) = \min\{d(x,z) + d(z,y) : z \in \mathrm{F}(p) \cap \mathrm{F}(q)\}. \tag{1}$$

Note that by the last step of the construction for any $x \in w(\bar{p}, \bar{q})(\mathbf{A})$ and $y \in \mathrm{dom}(\bar{p})$ we have

$$d(x,y) = \min\{d(x,z) + d(z,y) : z \in \mathrm{F}(q)\}.$$

We first prove several claims.

Claim 1. It is enough to show that (1) holds for all $x \in w(\bar{p}, \bar{q})(\mathbf{A})$ and $y \in \mathrm{F}(q)$.

Proof of Claim 1. Assume (1) holds for all $x \in w(\bar{p}, \bar{q})(\mathbf{A})$ and $y \in \mathrm{F}(q)$. If $y' \in \mathrm{dom}(\bar{p})$, then for some $c \in \mathrm{F}(q)$

$$d(x,y') = d(x,c) + d(c,y') = \min\{d(x,e) + d(e,y') : e \in \mathrm{F}(q)\}. \tag{2}$$

By the assumptions of the claim we get

$$d(x,y') = d(x,z) + d(z,c) + d(c,y') \geq d(x,z) + d(z,y'),$$

for some $z \in \mathrm{F}(p) \cap \mathrm{F}(q)$; and so, by (2),

$$d(x,y') = d(x,z) + d(z,y').$$

This proves the claim.

Let $w_i(c)$ denote $w_i(\bar{p}_i, \bar{q}_i)(c)$.

Claim 2. Let $x \in \mathrm{F}(p) \cup \mathrm{F}(q)$, $c \in \mathbf{A}$ and suppose that for some $z \in \mathrm{F}(p) \cap \mathrm{F}(q)$ and for some i the distance between $w_i(c)$ and x passes through z. Then for any $j \geq i$ the distance between $w_j(c)$ and x passes through the same point z.

Proof of Claim 2. This follows by induction. Here is an inductive step. Assume for definiteness that $j+1$ is odd (the case when $j+1$ is even, is similar).

The distance between x and $w_{j+1}(c)$ passes through a point $z' \in \mathrm{F}(p)$ ($z' \in \mathrm{F}(q)$ if $j+1$ is even). Then

$$d(w_j(c), x) = (w_j(c), z) + d(z, x) \le d(w_j(c), z') + d(z', x),$$
$$d(w_{j+1}(c), x) = d(w_{j+1}(c), z') + d(z', x),$$

but $d(w_{j+1}(c), z') = d(w_j(c), z')$ (this is because $w_{j+1} = s^m w_j$ and z' is fixed by p). Hence

$$d(w_j(c), x) \le d(w_{j+1}(c), x),$$

but also

$$d(w_{j+1}(c), x) \le d(w_{j+1}(c), z) + d(z, x) = d(w_j(c), z) + d(z, x) = d(w_j(c), x),$$

and so $d(w_{j+1}(c), x) = d(w_j(c), x)$. This proves the claim.

Claim 3. Let $x \in \mathrm{F}(p) \triangle \mathrm{F}(q)$ (here $\triangle$ is symmetric difference of sets), $c \in \mathbf{A}$. Suppose that the distance between $w_i(c)$ and x does not pass through a point in $\mathrm{F}(p) \cap \mathrm{F}(q)$. Then $d(w_i(c), x) \ge \lfloor i/2 \rfloor \mathcal{D}(\mathbf{A})$.

Proof of Claim 3. Suppose first that $x \in \mathrm{F}(p) \setminus \mathrm{F}(q)$. We prove the statement by induction on i. The base of the induction is trivial, so we show the inductive step: assume the statement is true for i and we need to show it for $i+1$. If i is even then, since $\lfloor i/2 \rfloor = \lfloor (i+1)/2 \rfloor$ and because $d(w_{i+1}(c), x) = d(w_i(c), x)$ (this is since i is even and $x \in \mathrm{F}(p)$) the statement follows immediately. So, assume i is odd. Then the distance between $w_{i+1}(c)$ and x passes through a point $z \in \mathrm{F}(q)$. Now two things can happen. Suppose first for some $j \le i$ the distance between $w_j(c)$ and z passes through a point $z' \in \mathrm{F}(p) \cap \mathrm{F}(q)$. Then by Claim 2, the distance between z and $w_{i+1}(c)$ must pass through z'. Now

$$d(w_{i+1}(c), x) = d(w_{i+1}(c), z) + d(z, x) =$$
$$d(w_{i+1}(c), z') + d(z', z) + d(z, x) \ge d(w_{i+1}(c), z') + d(z', x).$$

And so the distance between $w_{i+1}(c)$ and x passes through a point $z' \in \mathrm{F}(p) \cap \mathrm{F}(q)$. This contradicts the assumptions of the claim. So, for no $j \le i$ does the distance between $w_j(c)$ and x pass through a point in $\mathrm{F}(p) \cap \mathrm{F}(q)$. Then, applying induction to $w_i(c)$ and z, we get $d(w_i(c), z) \ge \lfloor i/2 \rfloor \mathcal{D}(\mathbf{A})$. But since $d(w_{i+1}(c), z) = d(w_i(c), z)$ and since $d(x, z) \ge \mathcal{D}(\mathbf{A})$ we get

$$d(w_{i+1}(c), x) \ge \lfloor i/2 \rfloor \mathcal{D}(\mathbf{A}) + \mathcal{D}(\mathbf{A}) \ge \lfloor (i+1)/2 \rfloor \mathcal{D}(\mathbf{A}).$$

In the case when $x \in \mathrm{F}(q) \setminus \mathrm{F}(q)$, the distance increases by $\mathcal{D}(\mathbf{A})$ at even stages of the construction, and the rest of the argument for this case is similar. The claim is proved.

Now fix $c \in \mathbf{A}$ and $y \in \mathrm{F}(q)$. It remains to show that

$$d(w_{2N+2}(c), y) = \min\{d(w_{2N+2}(c), z) + d(z, y) : z \in \mathrm{F}(p) \cap \mathrm{F}(q)\}.$$

We have two cases (we will show, though, that Case 2 is impossible).

Case 1. For some $i \leq 2N+2$ the distance between $w_i(c)$ and y passes through a point $z \in \mathrm{F}(p) \cap \mathrm{F}(q)$. Then

$$d(y, w_i(c)) = \min\{d(y, z) + d(z, w_i(c)) : z \in \mathrm{F}(p) \cap \mathrm{F}(q)\}.$$

Applying Claim 2 for $j = 2N+2$, we get

$$d(y, w_{2N+2}(c)) = \min\{d(y, z) + d(z, w_{2N+2}(c)) : z \in \mathrm{F}(p) \cap \mathrm{F}(q)\}.$$

And the theorem is proved for this case.

Case 2. For no $i \leq 2N+2$ does the distance between $w_i(c)$ and y pass through a point in $\mathrm{F}(p) \cap \mathrm{F}(q)$. Then by Claim 3

$$d(w_{2N+2}(c), y) \geq (N+1)\mathcal{D}(\mathbf{A}) > 2diam(\mathbf{A}).$$

On the other hand, let $z \in \mathrm{F}(p) \cap \mathrm{F}(q)$ be any common fixed point. Then $d(w_{2N+2}(c), y) \leq d(w_{2N+2}(c), z) + d(z, y) = d(c, z) + d(z, y) \leq 2diam(\mathbf{A})$. This gives a contradiction. So this case never happens. □

Remark 2.3.8. Note that the same result is also true for ordered metric spaces. For this one just has to apply Proposition 2.3.1 at each step of the construction of $\bar{p}$ and $\bar{q}$.

Before we apply this result to the classes of topological similarity let us mention another application. For a subset $\mathbf{A} \subseteq \mathbb{U}$ ($\mathbf{A} \subseteq \mathbb{QU}$) let $\mathrm{Iso}_{\mathbf{A}}(\mathbb{U})$ ($\mathrm{Iso}_{\mathbf{A}}(\mathbb{QU})$, respectively) denote the subgroup of isometries that pointwise fix $\mathbf{A}$. Recall a theorem of Julien Melleray from [17].

Theorem 2.3.9 (Melleray)**.** *Let $\mathbb{U}$ be the Urysohn space, and let $\mathbf{A}, \mathbf{B} \subset \mathbb{U}$ be two finite subsets. Then*

$$\mathrm{Iso}_{\mathbf{A}\cap\mathbf{B}}(\mathbb{U}) = \overline{\langle \mathrm{Iso}_{\mathbf{A}}(\mathbb{U}), \mathrm{Iso}_{\mathbf{B}}(\mathbb{U}) \rangle}.$$

Let us give an equivalent reformulation of the above result.

Theorem 2.3.10 (Melleray)**.** *Let $\mathbb{U}$ be the Urysohn space, and let $\mathbf{A}, \mathbf{B} \subset \mathbb{U}$ be two finite subsets. Then for any $\varepsilon > 0$, for any $p \in \mathrm{Iso}_{\mathbf{A}\cap\mathbf{B}}(\mathbb{U})$, and for any finite $\mathbf{C} \subseteq \mathbb{U}$ there is $q \in \langle \mathrm{Iso}_{\mathbf{A}}(\mathbb{U}), \mathrm{Iso}_{\mathbf{B}}(\mathbb{U}) \rangle$ such that*

$$\forall x \in \mathbf{C}\ d(p(x), q(x)) < \varepsilon.$$

We show that one can actually eliminate the epsilon in the above reformulation.

Theorem 2.3.11. *Let* $\mathbb{U}$ *be the Urysohn space, and let* $\mathbf{A}, \mathbf{B} \subset \mathbb{U}$ *be two finite subsets. Then for any* $p \in \mathrm{Iso}_{\mathbf{A}\cap\mathbf{B}}(\mathbb{U})$ *and for any finite* $\mathbf{C} \subseteq \mathbb{U}$ *there is* $q \in \langle \mathrm{Iso}_{\mathbf{A}}(\mathbb{U}), \mathrm{Iso}_{\mathbf{B}}(\mathbb{U})\rangle$ *such that*

$$\forall x \in \mathbf{C}\ p(x) = q(x).$$

Proof. Without loss of generality we may assume that $\mathbf{A} \subseteq \mathbf{C}$ and $\mathbf{B} \subseteq \mathbf{C}$. If $\mathbf{D} = \mathbf{C} \cup p(\mathbf{C})$, then $p|_{\mathbf{C}}$ is a partial isometry of $\mathbf{D}$. Define two partial isometries p_1 and p_2 of $\mathbf{D}$ by

$$\forall x \in \mathbf{A}\ p_1(x) = x,$$
$$\forall x \in \mathbf{B}\ p_2(x) = x.$$

Now apply Theorem 2.3.7 to p_1, p_2 and $\mathbf{D}$ to get a metric space $\mathbf{D}'$ and extension q_1 of p_1 and q_2 of p_2, and a word $w \in F_2$. Extend q_1 to q_1' by setting

$$\forall x \in \mathbf{C}\ q_1'(w(q_1, q_2)(x)) = w(q_1, q_2)(p(x)).$$

Such a q_1' is then a partial isometry of $\mathbf{D}'$. This follows from the fact that

$$w(q_1, q_2)(\mathbf{C}) \cup \mathrm{dom}(q_1)$$

is an amalgam of $w(q_1, q_2)(\mathbf{C})$ and $\mathrm{dom}(q_1)$ over $F(p_1) \cap F(p_2) = \mathbf{A} \cap \mathbf{B} \subseteq F(p)$. Indeed, if $y \in \mathrm{dom}(q_1)$ and $x \in \mathbf{C}$ then

$$\begin{aligned}
&d\big(q_1(y), w(q_1, q_2)(p(x))\big) = \\
&\min\Big\{d\big(q_1(y), z\big) + d\big(z, w(q_1, q_2)(p(x))\big) : z \in F(p_1) \cap F(p_2)\Big\} = \\
&\min\Big\{d\big(q_1(y), q_1(z)\big) + d\big(w(q_1, q_2)(p(z)), w(q_1, q_2)(p(x))\big) : z \in F(p_1) \cap F(p_2)\Big\} = \\
&\min\Big\{d(y, z) + d(z, x) : z \in F(p_1) \cap F(p_2)\Big\} = \\
&\min\Big\{d(y, z) + d(z, w(q_1, q_2)(x)) : z \in F(p_1) \cap F(p_2)\Big\} = d(y, w(q_1, q_2)(x)).
\end{aligned}$$

Now extend q_1' and q_2 to full isometries (we still denote them by the same symbols) and set

$$q = w^{-1}(q_1', q_2) q_1' w(q_1', q_2).$$

Then for any $x \in \mathbf{C}$, $p(x) = q(x)$, and $q_1' \in \mathrm{Iso}_{\mathbf{A}}(\mathbb{U})$, $q_2 \in \mathrm{Iso}_{\mathbf{B}}(\mathbb{U})$. $\square$

Note that if we start from metric spaces with rational distances, then the space $\mathbf{D}'$, constructed in the proof, would also have rational distances. And we arrive at the

Corollary 2.3.12. *Let* $\mathbb{QU}$ *be the rational Urysohn space, and let* $\mathbf{A}, \mathbf{B} \subset \mathbb{QU}$ *be two finite subsets. Then*

$$\mathrm{Iso}_{\mathbf{A}\cap\mathbf{B}}(\mathbb{QU}) = \overline{\langle \mathrm{Iso}_{\mathbf{A}}(\mathbb{QU}), \mathrm{Iso}_{\mathbf{B}}(\mathbb{QU})\rangle}.$$

Before showing another application of our extension result we need the following easy observation.

Lemma 2.3.13. *Let p, q be partial isometries of the Urysohn space $\mathbb{U}$ such that* $\mathrm{dom}(p) = \mathrm{dom}(q)$*, and let*

$$\{c_i\}_{i=1}^n = \mathrm{dom}(p).$$

For any $\varepsilon > 0$ there are partial isometries $\bar{p}, \bar{q}$ of $\mathbb{U}$ such that

$$\begin{gathered} \mathrm{dom}(\bar{p}) = \mathrm{dom}(p) = \mathrm{dom}(q) = \mathrm{dom}(\bar{q}), \\ \forall i \quad d(\bar{p}(c_i), p(c_i)) < \varepsilon, \quad d(\bar{q}(c_i), q(c_i)) < \varepsilon, \end{gathered}$$

and the sets $\mathrm{dom}(p), \bar{p}(\mathrm{dom}(p)), \bar{q}(\mathrm{dom}(p))$ *are pairwise disjoint.*

Proof. Set $\mathbf{A} = \mathrm{dom}(p) \cup p(\mathrm{dom}(p)) \cup q(\mathrm{dom}(p))$. Let $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ be new symbols, disjoint from all other data. Set

$$\mathbf{B} = \{c_i\} \cup \{p(c_i)\} \cup \{q(c_i)\} \cup \{a_i\} \cup \{b_i\}.$$

Let $\varepsilon > 0$ be given. We may decrease it to ensure that $\varepsilon < \mathcal{D}(\mathbf{A})$. Now define the metric on $\mathbf{B}$ as follows. The metric on $\mathbf{A}$ is the one inherited from $\mathbb{U}$. For $x \in \mathbf{A}$ set

$$d(a_i, x) = \begin{cases} d(p(c_i), x) & \text{if } x \neq p(c_i); \\ \varepsilon & \text{if } x = p(c_i); \end{cases}$$

$$d(b_i, x) = \begin{cases} d(q(c_i), x) & \text{if } x \neq q(c_i); \\ \varepsilon & \text{if } x = q(c_i); \end{cases}$$

$$d(a_i, b_j) = \begin{cases} d(p(c_i), q(c_j)) & \text{if } p(c_i) \neq q(c_j); \\ \varepsilon & \text{if } p(c_i) = q(c_j). \end{cases}$$

It is routine to check that d is indeed a metric on $\mathbf{A}$, and we leave this to the reader. Finally, set

$$\bar{p}(c_i) = a_i \quad \bar{q}(c_i) = b_i.$$

Then $\bar{p}$ and $\bar{q}$ satisfy the conclusions of the lemma. ☐

One of the corollaries from the results in [24] is that the group $\mathrm{Aut}(\mathbb{U})$ is topologically 2-generated, in other words there is a pair of isometries (f, g) such that the group $\langle f, g\rangle$ is dense in $\mathrm{Aut}(\mathbb{U})$. If Λ is the set of pairs that generate a dense subgroup, then

$$\Lambda = \{(f, g) \in \mathrm{Aut}(\mathbb{U}) \times \mathrm{Aut}(\mathbb{U}) : \forall \varepsilon > 0\ \forall h \\ \forall n\ \forall \{c_i\}_{i=1}^n\ \exists w\ \forall i \quad d(w(f, g)(c_i), h(c_i)) < \varepsilon\}.$$

We show that, in fact, a generic pair of isometries generates the whole group.

Theorem 2.3.14. *Λ is a dense G_δ-subset of* $\mathrm{Aut}(\mathbb{U}) \times \mathrm{Aut}(\mathbb{U})$.

Proof. Let $\{h_j\}_{j=1}^{\infty}$ be a dense subset of $\mathrm{Aut}(\mathbb{U})$, and $\{c_i\}_{i=1}^{\infty}$ be a dense set of points in $\mathbb{U}$. Set

$$B(n,m,j) = \{(f,g) \in \mathrm{Aut}(\mathbb{U}) \times \mathrm{Aut}(\mathbb{U}) : \\ \exists w\ d(w(f,g)(c_i), h_j(c_i)) < 1/n \text{ for } 1 \leq i \leq m\}.$$

Each $B(n,m,j)$ is open and

$$\Lambda = \bigcap_{n,m,j} B(n,m,j),$$

hence Λ is G_δ. It remains to check that all of the $B(n,m,j)$ are dense. Fix m,n, and j. Let p,q be partial isometries of $\mathbb{U}$, $\varepsilon > 0$, and without loss of generality we assume that $\mathrm{dom}(p) = \mathrm{dom}(q)$ and that $\{c_i\}_{i=1}^{m} \subseteq \mathrm{dom}(p)$. Let $\tilde{h}_j$ be the partial isometry given by the restriction of h_j onto $\{c_i\}$. By ultrahomogeneity of $\mathbb{U}$ it is enough to show that there are partial isometries $\tilde{p}, \tilde{q}$ such that

$$d(\tilde{p}(c), p(c)) < \varepsilon, \quad d(\tilde{q}(c), q(c)) < \varepsilon \quad \text{for all } c \in \mathrm{dom}(p)$$

and a word w such that

$$d(w(\tilde{p},\tilde{q})(c_i), \tilde{h}_j(c_i)) < 1/n$$

for all $i \in \{1, \ldots, m\}$. By Lemma 2.3.13 we may find $\bar{p}$, $\bar{q}$ such that

$$\mathrm{dom}(\bar{p}) = \mathrm{dom}(p) = \mathrm{dom}(q) = \mathrm{dom}(\bar{q}),$$

$$d(\bar{p}(c), p(c)) < \varepsilon, \quad d(\bar{q}(c), q(c)) < \varepsilon \quad \text{for all } c \in \mathrm{dom}(p)$$

and

$$\mathrm{dom}(p),\ \bar{p}(\mathrm{dom}(p)),\ \bar{q}(\mathrm{dom}(p))$$

are pairwise disjoint. Now add a common fixed point z to $\bar{p}$, $\bar{q}$ and $\tilde{h}_j$ (and denote the new partial isometries still by $\bar{p}$, $\bar{q}$ and $\tilde{h}_j$.)

We can now apply Theorem 2.3.7 to the partial isometries $\bar{p}, \bar{q}$ and the set

$$\mathbf{A} = \mathrm{dom}(\bar{p}) \cup \bar{p}(\mathrm{dom}(\bar{p})) \cup \bar{q}(\mathrm{dom}(\bar{p})) \cup \tilde{h}_j(\mathrm{dom}(\bar{p})).$$

This gives us partial isometries p', q' that extend $\bar{p}$ and $\bar{q}$ and a word w_1.

The next step is to extend p' to $\tilde{p}$ by setting

$$\tilde{p}\Big((w_1(p',q'))(c_i)\Big) = w_1(p',q')(\tilde{h}_j(c_i)).$$

We claim that $\tilde{p}$ is still a partial isometry. The argument is similar to the one in the proof of Theorem 2.3.11. We have $\{z\} = F(p') \cap F(q') \cap F(\tilde{h}_j)$. Then for any $y \in \mathrm{dom}(p')$ and any c_i

$$\begin{aligned}
&d\big(p'(y), w_1(p',q')(\tilde{h}_j(c_i))\big) = d(p'(y), z) + d\big(z, w_1(p',q')(\tilde{h}_j(c_i))\big) = \\
&d(p'(y), p'(z)) + d\big(w_1(p',q')(\tilde{h}_j(z)), w_1(p',q')(\tilde{h}_j(c_i))\big) = \\
&d(y,z) + d(z,c_i) = d(y,z) + d(w_1(p',q')(z), w_1(p',q')(c_i)) = \\
&d(y,z) + d(z, w_1(p',q')(c_i)) = d(y, w_1(p',q')(c_i)),
\end{aligned}$$

and hence $d\big(\tilde{p}(y), w_1(p',q')(\tilde{h}_j(c_i))\big) = d(y, w_1(p',q')(c_i))$.

Finally set $w = w_1^{-1} s w_1$ then for $\tilde{q} = q'$

$$w(\tilde{p}, \tilde{q})(c_i) = \tilde{h}_j(c_i) = h_j(c_i) \quad \text{for all } i.$$

So $B(n,m,j)$ is dense and by Baire Theorem Λ is dense G_δ. □

2.4 Isometries of the Ordered Urysohn Space

There is a rich variety of linearly ordered Fraïssé limits, of which the countable dense linear ordering without endpoints is the simplest example. In fact, as proved in [11], if the group of automorphisms of a particular Fraïssé class $\mathcal{K}$ is extremely amenable, then there is a linear ordering on the Fraïssé limit of $\mathcal{K}$ that is preserved by all automorphisms. Moreover, the ordered limit is still Fraïssé , i.e., is a Fraïssé limit of a Fraïssé class.

We consider another example of a linearly ordered Fraïssé limit: the ordered rational Urysohn space $\mathbb{QU}_\prec$.

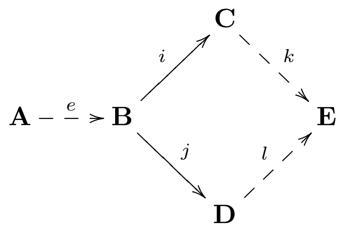
Let us briefly recall the definition of this structure. Formally speaking, one has to consider the Fraïssé class $\mathcal{M}$ of finite ordered metric spaces with rational distances. Then $\mathbb{QU}_\prec$ is, by definition, the Fraïssé limit of $\mathcal{M}$. Intuitively one can think of this structure as a classical rational Urysohn space with a linear ordering on top (such that ordering is isomorphic to the ordering of the rationals) and such that this ordering is independent of the metric structure.

Our goal is to prove that every two-dimensional class of topological similarity in the group of automorphisms of $\mathbb{QU}_\prec$ is meager. We would like to emphasize that the structure of conjugacy classes in $\mathrm{Aut}(\mathbb{Q})$ and $\mathrm{Aut}(\mathbb{QU}_\prec)$ is substantially different. As was mentioned earlier there is a generic conjugacy class in $\mathrm{Aut}(\mathbb{Q})$, while it is not hard to derive from results in [14], that each conjugacy class in $\mathrm{Aut}(\mathbb{QU}_\prec)$ is meager.

Recall (see [14], Definition 3.3)

Definition 2.4.1. A class $\mathcal{K}$ of finite structures satisfies the *weak amalgamation property* (WAP for short) if for every $\mathbf{A} \in \mathcal{K}$ there are $\mathbf{B} \in \mathcal{K}$ and an embedding $e : \mathbf{A} \to \mathbf{B}$ such that for all $\mathbf{C} \in \mathcal{K}$, $\mathbf{D} \in \mathcal{K}$ and all embeddings $i : \mathbf{B} \to \mathbf{C}$,

$j : \mathbf{B} \to \mathbf{D}$ there are $\mathbf{E} \in \mathcal{K}$ and embeddings $k : \mathbf{C} \to \mathbf{E}$, $l : \mathbf{D} \to \mathbf{E}$ such that $k \circ i \circ e = l \circ j \circ e$, i.e. in the following diagram the paths from $\mathbf{A}$ to $\mathbf{E}$ commute (but not necessarily paths from $\mathbf{B}$ to $\mathbf{E}$).



A class $\mathcal{K}$ satisfies the *local weak amalgamation property* if for some $\mathbf{A} \in \mathcal{K}$ weak amalgamation holds for the class of structures $\mathbf{B} \in \mathcal{K}$ that extend $\mathbf{A}$.

Definition 2.4.2. Let $\mathcal{K}$ be a Fraïssé class. We associate with it a class of structures $\mathcal{K}_p$. Elements of $\mathcal{K}_p$ are partial isomorphisms of $\mathcal{K}$, more precisely tuples

$$(\mathbf{A}; p : \mathbf{A}' \to \mathbf{A}''),$$

where $\mathbf{A}$, $\mathbf{A}'$ and $\mathbf{A}'' \in \mathcal{K}$, $\mathbf{A}'$, $\mathbf{A}'' \subseteq \mathbf{A}$ and p is an isomorphism.

Theorem 2.4.3 (Kechris–Rosendal, see [14], Theorem 3.7)**.** *The group of automorphisms of a Fraïssé class $\mathcal{K}$ has a non-meager conjugacy class if and only if class $\mathcal{K}_p$ satisfies the local weak amalgamation property.*

Proposition 2.4.4. *Every conjugacy class in* $\mathrm{Aut}(\mathbb{QU}_{\prec})$ *is meager.*

Proof. By Theorem 2.4.3 it is enough to show that the class $\mathcal{M}_p$ does not have the local WAP. Let $\bar{\mathbf{A}} = (\mathbf{A},\ \phi : \mathbf{A}' \to \mathbf{A}'') \in \mathcal{M}_p$, and assume without loss of generality that ϕ has at least one non-fixed point (otherwise take an extension of ϕ). We claim that the class of structures that extend $\mathbf{A}$ does not have WAP.

Fix $\bar{\mathbf{B}} = (\mathbf{B}, \psi : \mathbf{B}' \to \mathbf{B}'')$ that extends $\mathbf{A}$ and assume for notational simplicity that $\mathbf{A} \subseteq \mathbf{B}$. Let $z \in \mathbf{A}'$ be such that $\phi(z) \neq z$ and let $\mathrm{Orb}_\phi(z)$ be the orbit of z under ϕ. Then $\mathrm{Orb}_\psi(z) \supseteq \mathrm{Orb}_\phi(z)$. Since we have ordering $\phi(z) \neq z$ implies that z is not a periodic point of ψ, because for ordered structures periodic points coincide with fixed points. Let $x \in \mathbf{B}'$ be "the beginning of the orbit of z", that is $x \in \mathrm{Orb}_\psi(z)$ and $x \notin \mathrm{ran}(\psi)$. Such an x exists and is unique. Let $m_0 \in \mathbb{N}$ be such that $x = \psi^{-m_0}(z)$. Now take (by Lemma 2.3.4 and Proposition 2.3.1) two structures $\bar{\mathbf{C}} = (\mathbf{C},\ \sigma : \mathbf{C}' \to \mathbf{C}'') \in \mathcal{M}_p$, $\bar{\mathbf{D}} = (\mathbf{D},\ \tau : \mathbf{D}' \to \mathbf{D}'') \in \mathcal{M}_p$ such that $\bar{\mathbf{B}} \subseteq \bar{\mathbf{C}}$ and $\bar{\mathbf{B}} \subseteq \bar{\mathbf{D}}$ such that $x \notin \mathrm{ran}(\sigma)$, $x \notin \mathrm{ran}(\tau)$ and $\mathrm{Orb}_\sigma(x)$ and $\mathrm{Orb}_\tau(x)$ are disjoint over $\mathrm{Orb}_\phi(x)$. We claim that there is no weak amalgamation of $\bar{\mathbf{C}}$ and $\bar{\mathbf{D}}$ over $\bar{\mathbf{B}}$ and $\bar{\mathbf{A}}$. Indeed, suppose there is a structure $(\mathbf{E},\ \xi : \mathbf{E}' \to \mathbf{E}'')$ together with two embeddings $k : \mathbf{C} \to \mathbf{E}$ and $l : \mathbf{D} \to \mathbf{E}$ such that $k(a) = l(a)$ for all $a \in \mathbf{A}'$. In particular, $k(z) = l(z)$. But the maps k, l are not only isometries but also preserve partial isometries ϕ, ψ, σ, τ. Hence

$$k(\sigma^m(z)) = l(\tau^m(z))$$

for any $m \in \mathbb{Z}$ whenever both sides are defined. And thus $k(x) = k(\sigma^{-m_0}(z)) = l(\tau^{-m_0}(z)) = l(x)$. Suppose, for definiteness, that $|\operatorname{Orb}_\sigma(x)| \geq |\operatorname{Orb}_\tau(x)|$ or, in other words, there is $m_1 \in \mathbb{N}$ such that $\sigma^{m_1}(x)$ is defined but $\tau^{m_1+1}(x)$ is not (i.e., $\tau^{m_1}(x) \notin \operatorname{dom}(\tau)$). Then $\operatorname{Orb}_\sigma(x)$ extends $\operatorname{Orb}_\tau(x)$ over $\operatorname{Orb}_\psi(x)$. This is because

$$k^{-1}(l(\tau^m(z))) = \sigma^m(z) \quad \forall m \in \{0, \ldots, m_1\}.$$

This contradicts the choice of $\operatorname{Orb}_\sigma(x)$ and $\operatorname{Orb}_\tau(x)$. □

For classes of topological similarity the situation is rather different. All non-trivial elements in $\operatorname{Aut}(\mathbb{QU}_\prec)$ fall into a single class of topological similarity. And more generally, if $\mathbb{K}$ is any countable linearly ordered structure and $\operatorname{Aut}(\mathbb{K})$ is endowed with the topology of pointwise convergence ($\mathbb{K}$ is discrete here), then $\operatorname{Aut}(\mathbb{K})$ has exactly two classes of topological similarity (unless $\operatorname{Aut}(\mathbb{K}) = \{id\}$, then, of course, there is only one): all non-trivial automorphisms generate a discrete copy of $\mathbb{Z}$ and hence fall into a single class. Thus, in spite of the previous proposition, it makes sense to ask if there is a non-meager two-dimensional similarity class in $\operatorname{Aut}(\mathbb{QU}_\prec)$.

We define the notions of elementary and piecewise elementary pairs of partial isomorphisms of $\mathbb{QU}_\prec$ and the notion of liberation exactly as for the partial isomorphisms of the rationals.

It turns out that the analog of Theorem 2.2.15 for the ordered Urysohn space holds. Let us first briefly sketch the idea of the proof before diving into the details. We will prove that, again, for a generic pair there is a sequence of reduced words, such that this pair converges along it. One can repeat all the arguments up to Lemma 2.2.12 (only obvious changes are necessary). So one gets for a piecewise elementary pair (p, q) a triple (p', q', w) that liberates p in (p, q). But now, contrary to the case of the rationals, one cannot in general declare that $p'(w(p', q')(c)) = c$ for $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$, since such a p' may be not an isometry. At this moment we have to take further extensions of p' and q'. But once an analog of Lemma 2.2.13 is proved for the Urysohn case, the rest of Theorem 2.2.15 goes unchanged.

If p is a partial isometry, we can use amalgamation of its domain with a one point metric space over the empty set to add a fixed point for p. Using this observation the following two lemmata, which are analogs of Lemma 2.2.12 and Lemma 2.2.11, are proved as for the rationals, and we omit the details.

Lemma 2.4.5. *Let (p, q) be a piecewise elementary pair of partial isomorphisms of $\mathbb{QU}_\prec$ and assume a triple (p', q', w) liberates p [liberates q] in (p, q). Let $u = t^n v$ [$u = s^m v$] be a reduced word such that uw is irreducible. Then there is a triple (p'', q'', uw) that liberates p [liberates q] in (p, q). Moreover, one can take p'' to be an extension of p' and q'' to be an extension of q'.*

Lemma 2.4.6. *Let (p, q) be a piecewise elementary pair of partial isomorphisms of $\mathbb{QU}_\prec$ and $u \in F(s, t)$ be a reduced word. Then there is a triple (p', q', vu) that* *liberates p in (p,q) [liberates q] and such that $|vu| = |v| + |u|$.*

Lemma 2.4.7. *For any pair (p,q) of partial isomorphisms of the $\mathbb{QU}_{\prec}$ and any word $u \in F(s,t)$ there are extensions p' and q' of p and q respectively and a reduced word $w = *u$ such that $w(p',q')(c) = c$ for any $c \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$.*

Proof. We can assume that (p,q) is piecewise elementary. By Lemma 2.4.6 there are extensions $\tilde{p}$, $\tilde{q}$ of p and q and a reduced word vu such that $(\tilde{p}, \tilde{q}, vu)$ liberates p in (p,q). Now apply Theorem 2.3.7 (with Remark 2.3.8 and Lemma 2.2.11) to $\tilde{p}$, $\tilde{q}$ and

$$\mathbf{A} = \mathrm{dom}(\bar{p}) \cup \mathrm{ran}(\bar{p}) \cup \mathrm{dom}(\bar{q}) \cup \mathrm{ran}(\bar{q})$$

to get extensions $\bar{p}$ and $\bar{q}$ and a reduced word v'. Note that $v'vu$ is reduced, because v starts from a power of t and v' by construction ends in a power of s. By the item (ii) of Theorem 2.3.7 we can extend $\bar{p}$ to p' by declaring

$$p'|_{v'vu(\mathrm{dom}(p) \cup \mathrm{dom}(q))} = id.$$

Set $q' = \bar{q}$ and $w = (v'uv)^{-1}s(v'uv)$. It is easy to see that $w(p',q')(c) = c$ holds for any $c \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$. □

Theorem 2.4.8. *Every two-dimensional class of topological similarity in* $\mathrm{Aut}(\mathbb{QU}_{\prec})$ *is meager.*

Proof. Repeat the proofs of Lemma 2.2.14 and Theorem 2.2.15 using Lemma 2.4.7 instead of Lemma 2.2.13. □

Remark 2.4.9. All the results in this section can be proved for the ordered random graph in the same way, as they were proved for the ordered rational Urysohn space. One can also formally deduce this case from the above results viewing graphs as metric spaces with all the distances in $\{0, 1, 2\}$.

Chapter 3

Graev Metrics

3.1 Introduction

3.1.1 History

Back in the 40's in his seminal papers [15, 16] A. Markov came up with a notion of the free topological group over a completely regular (Tychonoff) space. This notion gave birth to a deep and important area in the general theory of topological groups. We highly recommend an excellent overview of free topological groups by O. Sipacheva [22]. Later M. Graev [6] gave another proof of the existence of free topological groups over completely regular spaces. In his approach Graev starts with a pointed metric space (X, x_0, d) and defines in a canonical way a two-sided invariant metric on $F\big(X \setminus \{x_0\}\big)$ — the free group with bases $X \setminus \{x_0\}$. Moreover, this metric extends the metric d on $X \setminus \{x_0\}$. In modern terms, Graev constructed a functor from the category of pointed metric spaces with Lipschitz maps to the category of groups with two-sided invariant metrics and Lipschitz homomorphisms.

The topology given by the Graev metric on the free group $F(X \setminus \{x_0\})$ is, of course, much weaker than the free topology on $F\big(X \setminus \{x_0\}\big)$. Since the early 40's a lot of work was done to understand the free topology on free groups, and some of this work shed light onto properties of the Graev metrics.

Graev metrics were used to construct exotic examples of Polish groups (see [3, 12, 26]). For example, the group completion of the free group $F(\mathbb{N}^{\mathbb{N}})$ over the Baire space with the topology given by the Graev metric is an example of a surjectively universal group in the class of Polish groups that admit compatible two-sided invariant metrics (see [12]).

Once the notion of a free topological group is available, the next step is to construct free products. It was made by Graev himself in [5], where he proves the existence of free products in the category of topological groups. For this he uses, in a clever and unexpected way, Graev metrics on free groups. But this time his approach does not produce a canonical metric on the free product out of metrics on factors.

In this chapter we would like to try to push Graev's method from free groups to free products of groups with and without amalgamation. As will be evident from the construction, the natural realm for this approach is the category of groups with two-sided invariant metrics. To be precise, a basic object for us will be an abstract group G with a two-sided invariant metric d on it. We recall that G will then automatically be a topological group in the topology given by d. Topological groups that admit a compatible two-sided invariant metric form a very restrictive subclass of the class of all the metrizable topological groups, but it includes compact metrizable and abelian metrizable groups.

3.1.2 Notations

In this chapter we use the following conventions. By an interval we mean an interval of natural numbers. An interval $\{m, m+1, \ldots, n\}$ is denoted by $[m, n]$. For a finite set F of natural numbers $\min(F)$ and $\max(F)$ denote its minimal and maximal elements respectively. For two sets F_1 and F_2 if $\max(F_1) < \min(F_2)$, then we say that F_1 is less than F_2 and denote this by $F_1 < F_2$.

A finite set F of natural numbers can be represented uniquely as a union of its maximal sub-intervals, i.e., there are intervals $\{I_k\}_{k=1}^n$ such that

(i) $F = \bigcup_k I_k$;

(ii) $\max(I_k) + 1 < \min(I_{k+1})$ for all $k \in [1, n-1]$.

We refer to such a decomposition of F as to the *family of maximal sub-intervals*.

By a *tree* we mean a connected directed graph without undirected cycles and with a distinguished vertex, which is called the *root* of the tree. For any tree T its root will be denoted by $\emptyset$. The *height* on a tree T is a function H_T that assigns to a vertex of the tree its graph-theoretic distance to the root. For example $H_T(\emptyset) = 0$ and $H_T(t) = 1$ for all $t \in T \setminus \{\emptyset\}$ such that $(\emptyset.t) \in E(T)$, where $E(T)$ is the set of directed edges of T. We use the word *node* as a synonym for the phrase *vertex of a tree*. We say that a node $s \in T$ is a *predecessor* of $t \in T$, and denote this by $s \prec t$, if there are nodes $s_0, \ldots, s_m \in T$ such that $s_0 = t, s_m = s$ and $(s_i, s_{i+1}) \in E(T)$.

3.2 Graev metric groups

Before going into the details of the construction of Graev metrics on free products we would like to recall the definition of the Graev metrics on free groups. The reader may consult [6], [3], [4] or [12] for the details and proofs.

Classically one starts with a pointed metric space (X, e, d), where d is a metric and $e \in X$ is a distinguished point. Take another copy of this space, denote it by (X^{-1}, d), and its elements are the formal inverses of the elements in X with the agreement $e^{-1} = e$ and $X \cap X^{-1} = \{e\}$. Then X^{-1} is also a metric space and we can amalgamate (X, d) and (X^{-1}, d) over the point e. Denote the resulting space by $(\overline{X}, e, d)$. Equivalently, $\overline{X} = X \cup X^{-1}$, and for all $x, y \in X$

$$d(x^{-1}, y^{-1}) = d(x, y), \quad d(x, y^{-1}) = d(x, e) + d(e, y).$$

With the set $\overline{X}$ we associate two objects: the set of *nonempty* words $\mathrm{Words}(\overline{X})$ over the alphabet $\overline{X}$ and the free group $F(X)$ over the basis X. There is a small issue with the second object. We want e to be the identity element of this group rather than an element of the basis. In other words, we formally have to write $F(X \setminus \{e\})$, but we adopt the convention that given a pointed metric space (X, e, d), in $F(X)$ the letter $e \in X$ is interpreted as the identity element. The inverse operation in $F(X)$ naturally extends the inverse operation on $\overline{X}$. We have a natural map

$$\widehat{\ } : \mathrm{Words}(\overline{X}) \to F(X),$$

for $u \in \mathrm{Words}(\overline{X})$ its image $\hat{u}$ is just the reduced form of u. For a word $u \in \mathrm{Words}(\overline{X})$ its length is denoted by $|u|$ and its i^{th} letter is denoted by $u(i)$. For two words $u, v \in \mathrm{Words}(\overline{X})$ of the same length n we define a function

$$\rho(u, v) = \sum_{i=1}^{n} d(u(i), v(i)).$$

And finally, we define a metric $\underline{d}$ by

$$\underline{d}(f, g) = \inf\{\rho(u, v) : |u| = |v| \text{ and } \hat{u} = f, \hat{v} = g\}.$$

A theorem of Graev [6] states that $\underline{d}$ is indeed a two-sided invariant metric on $F(X)$, and moreover, it extends the metric d on the amalgam $\overline{X}$. It is straightforward to see that $\underline{d}$ is a two-sided invariant *pseudo*-metric and the hard part of the Graev's theorem is to show that it assigns a non-zero distance to distinct elements. Graev showed this by proving some restrictions on u and v in the infimum in the definition of d. The effective formula for the Graev metric was first suggested by Sipacheva and Uspenskyy in [23] and later, but independently, a similar result was obtained in [3] by L. Ding and S. Gao. In our presentation we follow the latter.

Definition 3.2.1. Let I be an interval of natural numbers. A bijection $\theta : I \to I$ is called a *match* if

(i) $\theta \circ \theta = id$;

(ii) there are no $i, j \in I$ such that $i < j < \theta(i) < \theta(j)$.

Definition 3.2.2. Let $w \in \mathrm{Words}(\overline{X})$ be a word of length n, let θ be a match on $[1, n]$. A word w^θ has length n and is defined as

$$w^\theta(i) = \begin{cases} e & \text{if } \theta(i) = i; \\ w(i) & \text{if } \theta(i) > i; \\ w\big(\theta(i)\big)^{-1} & \text{if } \theta(i) < i. \end{cases}$$

It is not hard to check that for any word w and any match θ on $[1, |w|]$ the word w^θ is trivial, i.e. $\widehat{w^\theta} = e$.

Theorem 3.2.3 (Ding–Gao)**.** *If $f \in F(X)$ and $w \in \mathrm{Words}(\overline{X})$ is the reduced form of f, then*

$$\underline{d}(f, e) = \min\big\{\rho\big(w, w^{\theta}\big) : \theta \textit{ is a match on } [1, |w|]\big\}.$$

Here are some of the properties of the Graev metrics. They are easy consequences of the definition of the Graev metric and Theorem 3.2.3.

Proposition 3.2.4. *Let (X, e, d) be a pointed metric space, and let $\underline{d}$ be the Graev metric on $F(X)$.*

(i) If (T, d_T) is a tsi group and $\phi : X \to T$ is a K-Lipschitz map such that $\phi(e) = e$, then this map extends uniquely to a K-Lipschitz homomorphism $\phi : F(X) \to T$.

(ii) If $Y \subseteq X$, $e \in Y$ is a pointed subspace of X with the induced metric, then the natural embedding $i : Y \to X$ extends uniquely to an isometric embedding

$$i : F(Y) \to F(X).$$

Moreover, if Y is closed in X, then $F(Y)$ is closed in $F(X)$.

(iii) If δ is any tsi metric $F(X)$ that extends d, i.e., if $d(x_1, x_2) = \delta(x_1, x_2)$ for all $x_1, x_2 \in X$, then $\delta(u_1, u_2) \leq \underline{d}(u_1, u_2)$ for all $u_1, u_2 \in F(X)$. In other words, $\underline{d}$ is maximal among all the tsi metrics that extend d.

(iv) If $X \neq \{e\}$, then

$$\chi(F(X)) = \max\{\aleph_0, \chi(X)\}.$$

In particular, if X is separable, then so is $F(X)$.

3.2.1 Free groups over metric groups

In this subsection we prove a technical result that will be used later in Section 3.6.

Suppose X is itself a group and $e \in X$ is the identity element of that group. Let $\circ$ denote the multiplication operation on X, and let $x^{\dagger}$ denote the group inverse of an element $x \in X$. Suppose also that d is a two sided invariant metric on X. For $u \in \mathrm{Words}(\overline{X})$ define a word $u^{\sharp}$ by

$$u^{\sharp}(i) = \begin{cases} u(i) & \text{if } u(i) \in X; \\ (u(i)^{-1})^{\dagger} & \text{if } u(i) \in X^{-1}. \end{cases}$$

For $h \in F(X)$ let $h^{\sharp} = \widehat{w^{\sharp}}$, where w is the reduced form of h.

Proposition 3.2.5. *Let $f \in F(X)$, and let w be the reduced form of f. If $w \in \mathrm{Words}(X)$, then for any $h \in F(X)$*

$$\underline{d}(fh, e) \geq \underline{d}(fh^{\sharp}, e).$$

Proof. Suppose $w \in \mathrm{Words}(X)$ and fix an $h \in F(X)$. Let $u \in \mathrm{Words}(\overline{X})$ be the reduced form of h. It is enough to show that

$$\rho\Big(w^\frown u, \big(w^\frown u\big)^\theta\Big) \geq \rho\Big(w^\frown u^\sharp, \big(w^\frown u^\sharp\big)^\theta\Big)$$

for any match θ on $[1, |w| + |u|]$. This follows from the following inequalities:

- if $x, y \in X^{-1}$, then by the two-sided invariance of the metric d

$$d(x, y) = d(x^{-1}, y^{-1}) = d\big((x^{-1})^\dagger, (y^{-1})^\dagger\big);$$

- if $x \in X^{-1}$ and $y \in X$, then by the two-sided invariance of the metric d

$$d(x, y) = d(x, e) + d(e, y) = d(x^{-1}, e) + d(e, y) = \\ d\big((x^{-1})^\dagger, e\big) + d(e, y) \geq d\big((x^{-1})^\dagger, y\big).$$

Thus $\underline{d}(fh, e) \geq \underline{d}(fh^\sharp, e)$. □

3.3 Trivial words in amalgams

Let a family $\{G_\lambda\}_{\lambda\in\Lambda}$ of groups be given, where Λ is an index set. Suppose all of the groups contain a subgroup $A \subseteq G_\lambda$, and assume that $G_{\lambda_1} \cap G_{\lambda_2} = A$ for all $\lambda_1 \neq \lambda_2$. Let $G = \bigcup_{\lambda\in\Lambda} G_\lambda$ denote the union of the groups G_λ. The identity element in any group is denoted by e, the ambient group will be evident from the context. Let 0 be a symbol not in Λ. For $g_1, g_2 \in G$ we set $g_1 \cong g_2$ to denote the existence of $\lambda \in \Lambda$ such that $g_1, g_2 \in G_\lambda$. If $g_1 \cong g_2$, we say that g_1 and g_2 are *congruent.* We also define a congruence relation on $\Lambda \cup \{0\}$ by declaring that $x, y \in \Lambda \cup \{0\}$ are congruent if and only if either $x = y$ or at least one of x, y is 0. This congruence on $\Lambda \cup \{0\}$ is also denoted by $\cong$.

The free product of the groups G_λ with amalgamation over the subgroup A is denoted by $\coprod_A G_\lambda$. We carefully distinguish words over the alphabet G from elements of the amalgam $\coprod_A G_\lambda$. For that we introduce the following notation. $\mathrm{Words}(G)$ denotes the set of finite nonempty words over the alphabet G. The length of a word $\alpha \in \mathrm{Words}(G)$ is denoted by $|\alpha|$, the concatenation of two words α and β is denoted by $\alpha^\frown\beta$, and the i^{th} letter of α is denoted by $\alpha(i)$; in particular, for any $\alpha \in \mathrm{Words}(G)$

$$\alpha = \alpha(1)^\frown\alpha(2)^\frown \cdots {}^\frown\alpha(|\alpha|).$$

Two words $\alpha, \beta \in \mathrm{Words}(G)$ are said to be *congruent* if $|\alpha| = |\beta|$ and $\alpha(i) \cong \beta(i)$ for all $i \in [1, |\alpha|]$. For technical reasons (to be concrete, for the induction argument in Proposition 3.3.11) we need the following notion of a labeled word. A *labeled word* is a pair (α, l_α), where α is a word of length n, and $l_\alpha : [1, n] \to$

$\Lambda \cup \{0\}$ is a function, called the label of α, such that

$$\alpha(i) \in G_\lambda \setminus A \implies l_\alpha(i) = \lambda$$

for all $i \in [1, n]$.

Example 3.3.1. Let $\alpha \in \mathrm{Words}(G)$ be any word. There is a canonical label for α given by

$$l_\alpha(i) = \begin{cases} 0 & \text{if } \alpha(i) \in A; \\ \lambda & \text{if } \alpha(i) \in G_\lambda \setminus A. \end{cases}$$

In fact, everywhere, except for the proof of Proposition 3.3.11, we use this canonical labeling only.

Let α be a word of length n. For a subset $F \subseteq [1, n]$, with $F = \{i_k\}_{k=1}^m$, where $i_1 < i_2 < \ldots < i_m$, set

$$\alpha[F] = \alpha(i_1)^\frown\alpha(i_2)^\frown \cdots {}^\frown\alpha(i_m).$$

We say that a subset $F \subseteq [1, n]$ is *α-congruent* if $\alpha(i) \cong \alpha(j)$ for all $i, j \in F$.

There is a natural evaluation map from the set of words $\mathrm{Words}(G)$ over the alphabet G to the amalgam $\coprod_A G_\lambda$ given by the multiplication of letters in the group $\coprod_A G_\lambda$:

$$\alpha \mapsto \alpha(1) \cdot \alpha(2) \cdots \alpha(|\alpha|).$$

This map is denoted by a hat

$$\widehat{\ } : \mathrm{Words}(G) \to \coprod_A G_\lambda.$$

Note that this is map is obviously surjective. For a word $\alpha \in \mathrm{Words}(G)$ and a subset $F \subseteq [1, |\alpha|]$ we write $\hat{\alpha}[F]$ instead of $\widehat{\alpha[F]}$. We hope this will not confuse the reader too much. A word α is said to be *trivial* if $\hat{\alpha} = e$.

3.3.1 Structure of trivial words

Elements of the group A will be special for us. Let $\alpha \in \mathrm{Words}(G)$ be a word of length n. We say that its i^{th} letter is *outside of A* if, as the name suggests, $\alpha(i) \notin A$. The *list of external letters* of α is a, possibly empty, sequence $\{i_k\}_{k=1}^m$ such that

(i) $i_k < i_{k+1}$ for all $k \in [1, m-1]$;

(ii) $\alpha(i_k) \notin A$ for all $k \in [1, m]$;

(iii) $\alpha(i) \notin A$ implies $i = i_k$ for some $k \in [1, m]$.

In other words, this is just the increasing list of all the letters in α that are outside of A.

Definition 3.3.2. Let $\alpha \in \text{Words}(G)$ be a word with the list of external letters $\{i_k\}_{k=1}^m$. The word α is called *alternating* if $\alpha(i_k) \ncong \alpha(i_{k+1})$ for all $k \in [1, m-1]$. Note that a word is always alternating if $m \leq 1$. The word α is said to be *reduced* if $\alpha(i) \ncong \alpha(i+1)$ for all $i \in [1, |\alpha| - 1]$, and it is called a *reduced form of* $f \in \coprod_A G_\lambda$ if additionally $\hat{\alpha} = f$.

The following is a basic fact about free products with amalgamation.

Lemma 3.3.3. *Let* $\alpha \in \text{Words}(G)$ *be a reduced word. If* $\alpha \neq e$, *then* $\hat{\alpha} \neq e$.

It is worth mentioning that if $A \neq \{e\}$, then an element $f \in \coprod_A G_\lambda$ has many different reduced forms (unless $f \in G$, then it has only one). But all these reduced forms have the same length, therefore it is legitimate to talk about the length of an element f itself.

Lemma 3.3.4. *Any element* $f \in \coprod_A G_\lambda$ *has a reduced form* $\alpha \in \text{Words}(G)$. *Moreover, if* $\beta \in \text{Words}(G)$ *is another reduced form of* f, *then* $|\alpha| = |\beta|$ *and* $A\alpha(i)A = A\beta(i)A$ *for all* $i \in [1, |\alpha|]$.

Proof. The existence of a reduced form of $f \in \coprod_A G_\lambda$ is obvious. Suppose α and β are both reduced forms of f. Set

$$\zeta = \alpha(|\alpha|)^{-1\frown} \cdots {}^\frown\alpha(1)^{-1\frown}\beta(1)^\frown \cdots {}^\frown\beta(|\beta|).$$

Since $\hat{\zeta} = e$ and $\zeta \neq e$, by Lemma 3.3.3 ζ is not reduced. By assumption, α and β were reduced, therefore $\alpha(1) \cong \beta(1)$. We claim that $\alpha(1)^{-1}\beta(1) \in A$. Indeed, if $\alpha(1)^{-1}\beta(1) \notin A$, then the word

$$\xi = \alpha(|\alpha|)^{-1\frown} \cdots {}^\frown\alpha(1)^{-1} \cdot \beta(1)^\frown \cdots {}^\frown\beta(|\beta|)$$

is reduced, $\hat{\xi} = e$, and $\xi \neq e$, contradicting Lemma 3.3.3. So $\alpha(1)^{-1}\beta(1) \in A$, and therefore $\beta(1) = \alpha(1)a_1$ for some $a_1 \in A$ and $A\alpha(1)A = A\beta(1)A$. Now set

$$\alpha_1 = \alpha(2)^\frown \cdots {}^\frown\alpha(|\alpha|), \quad \beta_2 = a_1 \cdot \beta(2)^\frown \cdots {}^\frown\beta(|\beta|).$$

Since $\hat{\alpha}_1 = \hat{\beta}_1$ and α_1, β_1 are reduced, we can apply the same argument to get $\alpha_1(1) = \beta_1(1)a_2$ for some $a_2 \in A$, whence

$$A\alpha(2)A = A\alpha_1(1)A = A\beta_1(1)A = A\beta(2)A.$$

And we proceed by induction on $|\alpha| + |\beta|$. □

Lemma 3.3.5. *Let* $f \in \coprod_A G_\lambda$ *and* $\alpha, \beta \in \text{Words}(G)$ *be given. If* α *is a reduced form of* f, $|\alpha| = |\beta|$ *and* $\hat{\alpha} = \hat{\beta}$, *then* β *is a reduced form of* f.

Proof. If β is not a reduced form of f, we perform cancellations in β and get a reduced word β_1 such that $\hat{\beta}_1 = f$ and $|\beta_1| < |\beta|$. By Lemma 3.3.4 we have $|\beta_1| = |\alpha|$, contradicting $|\beta| = |\alpha|$. Hence β is reduced. □

Lemma 3.3.6. *If α is an alternating word with a nonempty list of external letters, then $\hat{\alpha} \neq e$.*

Proof. Let $\{i_k\}_{k=1}^m$ be the list of external letters of α. For $k \in [2, m-1]$ set

$$\xi_1 = \alpha(1) \cdots \alpha(i_2 - 1),$$

$$\xi_k = \alpha(i_k) \cdot \alpha(i_k + 1) \cdots \alpha(i_{k+1} - 1),$$

$$\xi_m = \alpha(i_m) \cdot \alpha(i_m + 1) \cdots \alpha(n),$$

and put

$$\xi = \xi_1{}^\frown \cdots {}^\frown \xi_m.$$

Then $\hat{\xi} = \hat{\alpha}$, $\xi \neq e$ (since $\xi_i \neq e$ for all $i \in [1, m]$), and, as one easily checks, ξ is reduced. An application of Lemma 3.3.3 finishes the proof. □

Lemma 3.3.7. *If ζ is a trivial word of length n with a nonempty list of external letters, then there is an interval $I \subseteq [1, n]$ such that*

(i) $\hat{\zeta}[I] \in A$;

(ii) I *is* ζ*-congruent;*

(iii) $\zeta\big(\min(I)\big), \zeta\big(\max(I)\big) \notin A$.

Proof. Let $\{i_k\}_{k=1}^m$ be the list of external letters. For all $k \in [1, m]$ define m_k and M_k by

$$m_k = \min\{j \in [1, k] : [i_j, i_k] \text{ is } \zeta\text{-congruent}\},$$

$$M_k = \max\{j \in [k, m] : [i_k, i_j] \text{ is } \zeta\text{-congruent}\}.$$

Set $I_k = [m_k, M_k]$, and note that for $k, l \in [1, m]$

$$I_k \cap I_l \neq \emptyset \implies I_l = I_k.$$

Let $I_{k_1}, \ldots, I_{k_p}$ be a list of all the distinct intervals I_{k_i}. Then $\{I_{k_i}\}_{i=1}^p$ are pairwise disjoint. Note that each of I_{k_i} satisfies items (ii) and (iii). To prove the lemma it is enough to show that for some $i \in [1, p]$ the corresponding I_{k_i} satisfies also item (i). Suppose this is false and $\hat{\zeta}[I_{k_i}] \notin A$ for all $i \in [1, p]$. Set $\xi_i = \hat{\zeta}[I_{k_i}]$ and

$$\begin{aligned}\xi = \zeta(1)^\frown \cdots {}^\frown \zeta(\min(I_{k_1}) - 1)^\frown \xi_1{}^\frown \zeta(\max(I_{k_1}) + 1)^\frown \cdots \\ \cdots {}^\frown \zeta(\min(I_{k_2}) - 1)^\frown \xi_2{}^\frown \zeta(\max(I_{k_2}) + 1)^\frown \cdots \\ \cdots {}^\frown \zeta(\min(I_{k_p}) - 1)^\frown \xi_p{}^\frown \zeta(\max(I_{k_p}) + 1)^\frown \cdots {}^\frown \zeta(n).\end{aligned}$$

Then, of course, $\hat{\xi} = \hat{\zeta} = e$ and ξ is alternating by the choice of $\{I_{k_i}\}$. By Lemma 3.3.6 the word ξ is non-trivial, which is a contradiction. □

Lemma 3.3.8. *If (ζ, l_ζ) is a trivial labeled word of length n with a nonempty list of external letters, then there is an interval $I \subseteq [1, n]$ such that*

(i) $\hat{\zeta}[I] \in A$*;*

(ii) I *is* ζ*-congruent;*

(iii) $\zeta(i) \notin A$ *for some* $i \in I$*;*

(iv) if $\min(I) > 1$*, then* $l_\zeta(\min(I) - 1) \neq 0$*; if* $\max(I) < n$*, then* $l_\zeta(\max(I) + 1) \neq 0$*;*

(v) if $\zeta(\min(I)) \in A$*, then* $l_\zeta(\min(I)) = 0$*; if* $\zeta(\max(I)) \in A$*, then* $l_\zeta(\max(I)) = 0$.

Proof. We start by applying Lemma 3.3.7 to the word ζ. This Lemma gives as an output an interval $J \subseteq [1, n]$. We will now enlarge this interval as follows. If $l_\zeta(i) = 0$ for all $i \in [1, \min(J) - 1]$, then set $j_l = 1$. If there is some $i < \min(J)$ such that $l_\zeta(i) \neq 0$, then let $j \in [1, \min(J) - 1]$ be maximal such that $l_\zeta(j) \neq 0$ and set $j_l = j + 1$. Similarly, if $l_\zeta(i) = 0$ for all $i \in [\max(J) + 1, n]$, then set $j_r = n$. If there is some $i > \max(J)$ such that $l_\zeta(i) \neq 0$, then let $j \in [\max(J) + 1, n]$ be minimal such that $l_\zeta(j) \neq 0$ and set $j_r = j - 1$. Define

$$I = J \cup [j_l, \min(J)] \cup [\max(J), j_r] = [j_l, j_r].$$

We claim that I satisfies the assumptions. Note that $J \subseteq I$ and $I \setminus J \subseteq A$, so (i), (ii) and (iii) follow from items (i), (ii) and (iii) of Lemma 3.3.7. Items (iv) and (v) follow from the choice of j_l and j_r and from item (iii) of Lemma 3.3.7. □

Definition 3.3.9. Let (ζ, l_ζ) be a trivial labeled word of length n, and let T be a tree. Suppose that to each node $t \in T$ an interval $I_t \subseteq [1, n]$ is assigned. Set $R_t = I_t \setminus \bigcup_{t' \prec t} I_{t'}$. The tree T together with the assignment $t \mapsto I_t$ is called *an evaluation tree for* (ζ, l_ζ) if for all $s, t \in T$ the following holds:

(i) $I_\emptyset = [1, n]$;

(ii) $\hat{\zeta}[I_t] \in A$;

(iii) if $t \neq \emptyset$ and $\min(I_t) \in A$, then $l_\zeta(\min(I_t)) = 0$; if $t \neq \emptyset$ and $\max(I_t) \in A$, then $l_\zeta(\max(I_t)) = 0$;

(iv) if $H(t) \leq H(s)$ and $I_s \cap I_t \neq \emptyset$, then $s \prec t$ or $s = t$;

(v) if $s \prec t$ and $t \neq \emptyset$, then

$$\min(I_t) < \min(I_s) \leq \max(I_s) < \max(I_t);$$

(vi) $\zeta(i) \cong \zeta(j)$ for all $i, j \in R_t$;

An evaluation tree T is called *balanced* if additionally the following two conditions hold:

(vii) if $T_\zeta \neq \{\emptyset\}$, then for any $t \in T_\zeta$ if R_t is written as a disjoint union of maximal sub-intervals $\{\mathcal{I}_j\}_{j=1}^k$, then for any j there is $i \in \mathcal{I}_j$ such that $l_\zeta(i) \neq 0$;

(viii) if $s \prec t$, then

$$\min(I_s) - 1 \in R_t \implies l_\zeta(\min(I_s) - 1) \neq 0;$$

$$\max(I_s) + 1 \in R_t \implies l_\zeta(\max(I_s) + 1) \neq 0.$$

Remark 3.3.10. Note that if $\zeta \in \mathrm{Words}(G)$ is a trivial word with the canonical label as in Example 3.3.1, then item (iii) in the definition of an evaluation tree is vacuous.

Proposition 3.3.11. *Any trivial labeled word (ζ, l_ζ) has a balanced evaluation tree.*

Proof. We prove the proposition by induction on the cardinality of the list of external letters of ζ. Suppose first that the list is empty, and $\zeta(i) \in A$ for all $i \in [1, n]$. Set $T_\zeta = \{\emptyset\}$ and $I_\emptyset = [1, n]$. It is easy to check that all the conditions are satisfied, and T_ζ is a balanced evaluation tree for (ζ, l_ζ).

From now on we assume there is $i \in [1, n]$ such that $\zeta(i) \notin A$. Apply Lemma 3.3.8 to (ζ, l_ζ) and let I be the interval granted by this lemma. Set $\lambda_0 = l_\zeta(i)$ for some (equivalently, any) $i \in I$ such that $\zeta(i) \notin A$. Note that $\lambda_0 \neq 0$. Let $m = |I|$ be the length of I. If $m = n$, then we set $T_\zeta = \{\emptyset\}$ and $I_\emptyset = [1, n]$. Similarly to the base of induction this tree is a balanced evaluation tree for (ζ, l_ζ). From now on we assume that $m < n$. We define the word ξ of length $n - m + 1$ as follows. Set

$$\xi(i) = \begin{cases} \zeta(i) & \text{if } i < \min(I) \\ \hat{\zeta}[I] & \text{if } i = \min(I) \\ \zeta(i + m - 1) & \text{if } i > \min(I). \end{cases}$$

Define the label for ξ to be

$$l_\xi(i) = \begin{cases} l_\zeta(i) & \text{if } i < \min(I) \\ \lambda_0 & \text{if } i = \min(I) \\ l_\zeta(i + m - 1) & \text{if } i > \min(I). \end{cases}$$

We claim that

$$\big|\{i \in [1, |\xi|] : \xi(i) \notin A\}\big| < \big|\{i \in [1, n] : \zeta(i) \notin A\}\big|.$$

Indeed, by the construction $\zeta[I]$ has at least one letter (in fact, at least two letters) not from A.

By inductive assumption applied to the labeled word (ξ, l_ξ), there is a balanced evaluation tree T_ξ with intervals $J_t \subseteq [1, |\xi|]$ for $t \in T_\xi$. Since $J_\emptyset = [1, |\xi|]$, there is at least one $t \in T_\xi$ (namely $t = \emptyset$) such that the interval J_t contains $\min(I)$. By item (iv) there is the smallest node $t_0 \in T_\xi$ such that $\min(I) \in J_{t_0}$.

We define T_ζ to be $T_\xi \cup \{s_0\}$, where s_0 is a new predecessor of t_0, i. e. , $s_0 \prec t_0$. For $t \in T_\xi$ set

$$I_t = \begin{cases} [\min(J_t), \max(J_t)] & \text{if } \max(J_t) < \min(I); \\ [\min(J_t), \max(J_t) + m - 1] & \text{if } \min(J_t) \leq \min(I) \leq \max(J_t); \\ [\min(J_t) + m - 1, \max(J_t) + m - 1] & \text{if } \min(I) < \min(J_t); \end{cases}$$

and

$$I_{s_0} = [\min(I), \max(I)].$$

We claim that such a tree T_ζ with such an assignment of intervals I_t is a balanced evaluation tree for (ζ, l_ζ).

(i) Since $J_\emptyset = [1, |\xi|]$, it follows that $I_\emptyset = [1, n]$.

(ii) For any $t \in T_\xi$ one has $\hat{\xi}[J_t] = \hat{\zeta}[I_t]$. Also, $\hat{\zeta}[I_{s_0}] \in A$ by item (i) of Lemma 3.3.8.

(iii) Since $\xi(\min(I)) \in A$ and $l_\xi(\min(I)) = \lambda_0 \neq 0$, by inductive hypothesis $\min(I_t) \neq \min(I)$ and $\max(I_t) \neq \min(I)$ for all $t \in T_\xi \setminus \{\emptyset\}$. Therefore $l_\xi(\min(J_t)) = l_\zeta(\min(I_t))$, $l_\xi(\max(J_t)) = l_\zeta(\max(I_t))$ for all $t \in T_\xi \setminus \{\emptyset\}$. Thus for $t \neq s_0$ the item follows from the inductive hypothesis, and for $t = s_0$ it follows from item (v) of Lemma 3.3.8.

(iv) Follows from the inductive hypothesis and the definition of s_0.

(v) It follows from the inductive hypothesis that this item is satisfied for all $s, t \in T_\xi$. We need to consider the case $s = s_0$, $t = t_0$ only. By item (iii) of the definition of an evaluation tree, and since $l_\xi(\min(I)) = \lambda_0 \neq 0$, it follows that if $t_0 \neq \emptyset$, then $\min(I_{t_0}) < \min(I_{s_0})$ and $\max(I_{s_0}) < \max(I_{t_0})$.

(vi) Follows easily from the inductive hypothesis and item (ii) of Lemma 3.3.8.

Thus T_ζ is an evaluation tree for (ζ, l_ζ). It remains to check that it is balanced.

(vii) For $t \in T_\xi \setminus \{t_0\}$ the maximal sub-intervals of $J_t \setminus \cup_{s \prec t} J_s$ naturally correspond to the maximal sub-intervals of $I_t \setminus \cup_{s \prec t} I_s$, and hence for such a t the item follows from the inductive hypothesis. For $t = s_0$ the item follows from item (iii) of Lemma 3.3.8. The remaining case $t = t_0$ follows from item (iv) of Lemma 3.3.8.

(viii) Again, for $s \neq s_0$ this item follows from the inductive hypothesis and for $s = s_0$, $t = t_0$ follows from item (iv) of Lemma 3.3.8. □

If ζ is just a word with no labeling, then we canonically associate a label to it by declaring $l_\zeta(i) = 0$ if and only if $\zeta(i) \in A$ (as in Example 3.3.1).

From now on we view all trivial words as labeled words with the canonical labeling.

Definition 3.3.12. A trivial word $\zeta \in \mathrm{Words}(G)$ of length n is called *slim* if there exists an evaluation tree T_ζ such that $\hat{\zeta}[I_t] = e$ for all $t \in T_\zeta$; such a tree is then called a *slim* evaluation tree. We say that ζ is *simple* if it is slim and $\zeta(i) \in A$ implies $\zeta(i) = e$ for all $i \in [1, n]$.

Definition 3.3.13. Let $f \in \coprod_A G_\lambda$. A pair of words (α, ζ) is called an *f-pair* if $|\alpha| = |\zeta|$ and $\hat{\alpha} = f$, $\hat{\zeta} = e$. An f-pair (α, ζ) is said to be a *congruent* f-pair if α is congruent to ζ. An f-pair (α, ζ) is called *slim* if it is congruent and ζ is slim. It is called *simple* if it is congruent and ζ is simple.

For a congruent pair (α, β) of length n we define the notions of right and left transfers. Let $a \in A$ and $i \in [1, n-1]$ be given. The *right (a, i)-transfer* of (α, β) is the pair $\mathrm{RTran}(\alpha, \beta; a, i) = (\gamma, \delta)$ defined as follows:

$$(\gamma(j), \delta(j)) = \begin{cases} (\alpha(j), \beta(j)) & \text{if } j \notin \{i, i+1\}; \\ (\alpha(i)a^{-1}, \beta(i)a^{-1}) & \text{if } j = i; \\ (a\alpha(i+1), a\beta(i+1)) & \text{if } j = i+1. \end{cases}$$

For $a \in A$ and $i \in [2, n]$ the *left (a, i)-transfer* of (α, β) is denoted by $\mathrm{LTran}(\alpha, \beta; a, i) = (\gamma, \delta)$ and is defined as

$$(\gamma(j), \delta(j)) = \begin{cases} (\alpha(j), \beta(j)) & \text{if } j \notin \{i-1, i\}; \\ (a^{-1}\alpha(i), a^{-1}\beta(i)) & \text{if } j = i; \\ (\alpha(i-1)a, \beta(i-1)a) & \text{if } j = i-1. \end{cases}$$

We will typically have specific sequences of transfers, so it is convenient to make the following definition. Let (α, ζ) be a congruent pair of words of length n. In all the applications ζ will be a trivial word. Let $\{I_k\}_{k=1}^m$ be a sequence of intervals such that:

1. $I_k \subseteq [1, n]$;
2. $I_k < I_{k+1}$ for all $k \in [1, m-1]$;
3. $\hat{\zeta}[I_k] \in A$ for all $k \in [1, m]$;
4. $\max(I_m) < n$.

Such a sequence is called *right transfer admissible.* If together with items (1) – (3) the following is satisfied

(4′) $\min(I_1) > 1$,

then the sequence $\{I_k\}_{k=1}^m$ is called *left transfer admissible.*

Let $\{I_k\}_{k=1}^m$ be a right transfer admissible sequence of intervals. Define inductively words (β_k, ξ_k) by setting $(\beta_0, \xi_0) = (\alpha, \zeta)$ and

$$(\beta_{k+1}, \xi_{k+1}) = \mathrm{RTran}(\beta_k, \xi_k; \hat{\xi}_k[I_{k+1}], \max(I_{k+1})).$$

We have to show that the right-hand side is well-defined, i.e., that $\hat{\xi}_k[I_{k+1}] \in A$. For the first step of the construction we have $\hat{\xi}_0[I_1] = \hat{\zeta}[I_1] \in A$, because the sequence is right transfer admissible. Suppose we have proved that $\hat{\xi}_{k-1}[I_k] \in A$. There are two cases: either $\max(I_k) + 1 = \min(I_{k+1})$, and then

$$\hat{\xi}_k[I_{k+1}] = (\hat{\xi}_{k-1}[I_k]) \cdot \hat{\zeta}[I_{k+1}],$$

or $\max(I_k) + 1 < \min(I_{k+1})$, and then $\hat{\xi}_k[I_{k+1}] = \hat{\zeta}[I_{k+1}]$. In both cases we get $\hat{\xi}_k[I_{k+1}] \in A$.

By definition, the right $\{I_k\}$-transfer of (α, ζ) is the pair (β_m, ξ_m).

The left transfer is defined similarly, but with one extra change: we apply left transfers in the decreasing order from I_m to I_1. Here is a formal definition. For a left admissible sequence of intervals $\{I_k\}_{k=1}^m$ set inductively $(\beta_0, \xi_0) = (\alpha, \zeta)$ and

$$(\beta_{k+1}, \xi_{k+1}) = \mathrm{LTran}(\beta_k, \xi_k; \hat{\xi}_k[I_{m-k}], \min(I_{m-k})).$$

Similarly to the case of the right transfer one shows that the right-hand side in the above construction is well-defined. By definition, the left $\{I_k\}$-transfer of (α, ζ) is the pair (β_m, ξ_m).

This notion of transfer, though a bit technical, will be crucial in some reductions in the next section. The following lemma establishes basic properties of the transfer operation with respect to the earlier notion of the evaluation tree.

Lemma 3.3.14. *Let (α, ζ) be a congruent f-pair of length n and let T_ζ be a [balanced] evaluation tree for ζ. Let $\{I_k\}_{k=1}^m$ be a right [left] transfer admissible sequence of intervals. Let (β, ξ) be the right [left] $\{I_k\}$-transfer of (α, ζ). Then*

(i) $|\beta| = n = |\xi|$*;*

(ii) (β, ξ) *is a congruent f-pair;*

(iii) T_ζ *is a [balanced] evaluation tree for* ξ.

(iv) $\xi(i) = \zeta(i)$ *for all* $i \notin \{\max(I_k), \max(I_{k+1}) : k \in [1, m]\}$ *for the right transfer and for all* $i \notin \{\min(I_k), \min(I_{k-1}) - 1 : k \in [1, m]\}$ *in the case of the left transfer;*

(v) $\hat{\xi}[I_k] = e$ *for all* $k \in [1, m]$.

Proof. Items (i), (ii), and (iv) are trivial; item (iii) follows easily from the observation that $\xi(i) \in A$ if and only if $\zeta(i) \in A$. For item (v) let ξ_k be as in the definition of the $\{I_k\}$-transfer. Suppose for definiteness that we are in the case of the right transfer. Then $\hat{\xi}_k[I_k] = e$ by construction and also $\xi_{k+1}[I_j] = \xi_k[I_j]$ for all $j \in [1, k]$. The lemma follows. $\square$

We will later need another operation on words, we call it symmetrization. Here is the definition.

Definition 3.3.15. Let (α, ζ) be a slim f-pair with a slim evaluation tree T_ζ. Let $t \in T_\zeta$ and $\{i_k\}_{k=1}^m \subseteq R_t$ be a list such that

(i) $i_k < i_{k+1}$ for $k \in [1, m-1]$;

(ii) if $\zeta(i) \neq e$ for some $i \in R_t$, then $i = i_k$ for some $k \in [1, m]$;

(iii) $\alpha(i_k) \cong \alpha(i_l)$ for all $k, l \in [1, m]$.

Such a list is called *symmetrization admissible.* For $j_0 \in \{i_k\}_{k=1}^m$ let k_0 be such that $j_0 = i_{k_0}$ and define a *symmetrization* $\mathrm{Sym}(\alpha, \zeta; j_0, \{i_k\}_{k=1}^m)$ of ζ to be the word ξ such that

$$\xi(i) = \begin{cases} \zeta(i) & \text{if } i \neq i_p \text{ for all } p \in [1, m]; \\ \alpha(i) & \text{if } i \in \{i_k\}_{k=1}^m \setminus \{j_0\}; \\ \alpha(i_{k_0-1})^{-1} \ldots \alpha(i_1)^{-1} \cdot \alpha(i_m)^{-1} \ldots \alpha(i_{k_0+1})^{-1} & \text{if } i = j_0. \end{cases}$$

If $m = 1$, the above definition does not make sense, so we set that in this case $\mathrm{Sym}(\alpha, \zeta; i_1, i_1) = \zeta$.

Lemma 3.3.16. *Let (α, ζ) be a slim f-pair with a slim evaluation tree T_ζ. Let $t \in T_\zeta$, and let $\{i_k\}_{k=1}^m \subseteq R_t$ be a symmetrization admissible list. Fix some $j_0 \in \{i_k\}_{k=1}^m$. If ξ is the symmetrization $\mathrm{Sym}(\alpha, \zeta; j_0, \{i_k\}_{k=1}^m)$ of ζ, then (α, ξ) is a slim f-pair and T_ζ is a slim evaluation tree for ξ with the same assignment of intervals $t \mapsto I_t$.*

Proof. The only non-trivial part in the lemma is to show that $\hat{\xi}[I_t] = e$. This follows from the facts that $\hat{\zeta}[I_s] = e$ for all $s \prec t$ (because T_ζ is slim) and that $\zeta(i) = e$ for all $i \in R_t \setminus \{i_1, \ldots, i_m\}$ (by the definition of the symmetrization admissible list). $\square$

3.4 Groups with two-sided invariant metrics

In this section we would like to recall some facts from the theory of groups with two-sided invariant metrics. The reader can consult [4] for the details.

Definition 3.4.1. A metric d on a group G is called two-sided invariant if

$$d(gf_1, gf_2) = d(f_1, f_2) = d(f_1 g, f_2 g)$$

for all $g, f_1, f_2 \in G$. A tsi group is a pair (G, d), where G is a group and d is a two-sided invariant metric on G; tsi stands for two-sided invariant.

Proposition 3.4.2. *If (G, d) be a tsi group, then G is a topological group in the topology of the metric d.*

Proposition 3.4.3. *Let d be a left invariant metric on the group G.*

(i) If for all $g_1, g_2, f_1, f_2 \in G$

$$d(g_1 g_2, f_1 f_2) \leq d(g_1, f_1) + d(g_2, f_2),$$

then d is two-sided invariant;

(ii) If d is two-sided invariant, then for all $g_1, \ldots, g_k, f_1, \ldots, f_k \in G$

$$d(g_1 \cdots g_k, f_1 \cdots f_k) \leq \sum_{i=1}^{k} d(g_i, f_i).$$

Because of Proposition 3.4.2 we choose to speak not about topological groups that admit a compatible two-sided invariant metric, but rather about abstract groups with a two-sided invariant metric. Note that the class of metrizable groups that admit a compatible two-sided invariant metric is very small, but it includes two important subclasses: abelian and compact metrizable groups.

The class of tsi groups is closed under taking factors by closed normal subgroups, and, moreover, there is a canonical metric on the factor.

Proposition 3.4.4. *If (G, d) is a tsi group and $N < G$ is a closed normal subgroup, then the function*

$$d_0(g_1 N, g_2 N) = \inf\{d(g_1 h_1, g_2 h_2) : h_1, h_2 \in N\}$$

is a two-sided invariant metric on the factor group G/N and the factor map $\pi : G \to G/N$ is a 1-Lipschitz surjection from (G, d) onto $(G/N, d_0)$.

The metric d_0 is called the *factor metric.*

Proposition 3.4.5. *Let (G, d) be a tsi group. Let $(\overline{G}, d)$ be the completion of G as a metric space; the extension of the metric d on G to the completion $\overline{G}$ is again denoted by d. There is a unique extension of group operation from G to $\overline{G}$. This extension turns $(\overline{G}, d)$ into a tsi group.*

This proposition states that for tsi groups metric and group completions are the same.

3.5 Metrics on amalgams

3.5.1 Basic set up

Let (G_λ, d_λ) be a family of tsi groups, $A < G_\lambda$ be a common closed subgroup, $G_{\lambda_1} \cap G_{\lambda_2} = A$, and assume additionally that the metrics $\{d_\lambda\}$ agree on A:

$$d_{\lambda_1}(a_1, a_2) = d_{\lambda_2}(a_1, a_2) \quad \text{for all } a_1, a_2 \in A \text{ and all } \lambda_1, \lambda_2 \in \Lambda.$$

Our main goal is to define a metric on the free product of G_λ with amalgamation over A that extends all the metrics d_λ. It will be an analog of the Graev metrics on free groups.

First of all, let d denote the amalgam metric on $G = \bigcup_\lambda G_\lambda$ given by

$$d(f_1, f_2) = \begin{cases} d_\lambda(f_1, f_2) & \text{if } f_1, f_2 \in G_\lambda \text{ for some } \lambda \in \Lambda; \\ \inf\limits_{a \in A} \big\{ d_{\lambda_1}(f_1, a) + d_{\lambda_2}(a, f_2) \big\} & \text{if } f_1 \in G_{\lambda_1},\ f_2 \in G_{\lambda_2} \text{ for } \lambda_1 \neq \lambda_2. \end{cases}$$

If α_1 and α_2 are two words in $\mathrm{Words}(G)$ of the same length n, then the value $\rho(\alpha_1, \alpha_2)$ is defined by

$$\rho(\alpha_1, \alpha_2) = \sum_{i=1}^{n} d\big(\alpha_1(i), \alpha_2(i)\big).$$

Finally, for elements $f_1, f_2 \in \coprod_A G_\lambda$ the Graev metric on the free product with amalgamation $\coprod_A G_\lambda$ is defined as

$$\underline{d}(f) = \inf \big\{ \rho(\alpha_1, \alpha_2) : |\alpha_1| = |\alpha_2| \text{ and } \hat{\alpha}_i = f_i \big\}.$$

Lemma 3.5.1. *$\underline{d}$ is a tsi pseudo-metric.*

Proof. It is obvious that $\underline{d}$ is non-negative, symmetric and attains value zero on the diagonal. We show that it is two-sided invariant. Let $f_1, f_2, h \in \coprod_A G_\lambda$ be given. Let $\gamma \in \mathrm{Words}(G)$ be any word such that $\hat{\gamma} = h$. For any $\alpha_1, \alpha_2 \in \mathrm{Words}(G)$ that have the same length and are such that $\hat{\alpha}_i = f_i$ we get

$$\rho(\alpha_1, \alpha_2) = \rho(\gamma^\frown \alpha_1, \gamma^\frown \alpha_2),$$

and therefore $\underline{d}(hf_1, hf_2) \leq \underline{d}(f_1, f_2)$. But similarly, if β_1, β_2 are of the same length and $\hat{\beta}_i = hf_i$, then

$$\rho(\beta_1, \beta_2) = \rho(\gamma^{-1\frown} \beta_1, \gamma^{-1\frown} \beta_2),$$

where $\gamma^{-1} = \gamma(|\gamma|)^{-1\frown} \ldots {}^\frown \gamma(1)^{-1}$. Hence $\underline{d}(f_1, f_2) = \underline{d}(hf_1, hf_2)$, i.e., $\underline{d}$ is left-invariant. Right invariance is shown similarly.

We also need to check the triangle inequality. By the two-sided invariance triangle inequality is equivalent to

$$\underline{d}(f_1 f_2, e) \leq \underline{d}(f_1, e) + \underline{d}(f_2, e) \quad \text{for all } f_1, f_2 \in \coprod_A G_\lambda.$$

The latter follows immediately from the observation that if $\hat{\alpha}_i = f_i$, $|\alpha_i| = |\zeta_i|$, and $\hat{\zeta}_1 = e = \hat{\zeta}_2$, then $\widehat{\alpha_1 {}^\frown \alpha_2} = f_1 f_2$, $\widehat{\zeta_1 {}^\frown \zeta_2} = e$, and also

$$\rho(\alpha_1 {}^\frown \alpha_2, \zeta_1 {}^\frown \zeta_2) = \rho(\alpha_1, \zeta_1) + \rho(\alpha_2, \zeta_2). \qquad \square$$

We will show eventually that, in fact, $\underline{d}$ is not only a pseudo-metric, but a genuine metric. This will take us a while though.

It will be convenient for us to talk about norms rather than about metrics. For this we set $\mathrm{N}(f) = \underline{d}(f, e)$. Then N is a tsi pseudo-norm on G (again, it will turn out to be a norm). Note that $\underline{d}$ is a metric if and only if N is a norm, i. e., if and only if $\mathrm{N}(f) = 0$ implies $f = e$.

3.5.2 Reductions

We start a series of reductions and will gradually simplify the structure of α in the definition of the pseudo-norm N.

Using the notion of an f-pair the definition of N can be rewritten as

$$\mathrm{N}(f) = \inf\big\{\rho(\alpha, \zeta) : (\alpha, \zeta) \text{ is an } f\text{-pair}\big\}.$$

Lemma 3.5.2. *For all $f \in \coprod_A G_\lambda$*

$$\mathrm{N}(f) = \inf\big\{\rho(\alpha, \zeta) : (\alpha, \zeta) \textit{ is a congruent } f\textit{-pair}\big\}.$$

Proof. Fix an $f \in \coprod_A G_\lambda$. We need to show that for any f-pair (α, ζ) and for any $\epsilon > 0$ there is a congruent f-pair (β, ξ) such that

$$\rho(\beta, \xi) \leq \rho(\alpha, \zeta) + \epsilon.$$

Take an f-pair (α, ζ) and fix an $\epsilon > 0$. Let n be the length of α. For an $i \in [1, n]$ we define a pair of words β_i, ξ_i as follows: if $\alpha(i) \cong \zeta(i)$, then $\beta_i = \alpha(i)$, $\xi_i = \zeta(i)$; if $\alpha(i) \ncong \zeta(i)$, then $\beta_i = \alpha(i) {}^\frown e$ and $\xi_i = a_i {}^\frown a_i^{-1} \zeta(i)$, where $a_i \in A$ is any element such that

$$d\big(\alpha(i), \zeta(i)\big) + \frac{\epsilon}{n} \geq d\big(\alpha(i), a_i\big) + d\big(a_i, \zeta(i)\big),$$

which exists by the definition of the amalgam metric d. Then

$$\rho(\beta_i, \xi_i) \leq \rho\big(\alpha(i), \zeta(i)\big) + \frac{\epsilon}{n} \quad \text{for all } i.$$

Set $\beta = \beta_1 {}^\frown \ldots {}^\frown \beta_n$, $\xi = \xi_1 {}^\frown \ldots {}^\frown \xi_n$. It is now easy to see that (β, ξ) is a congruent f-pair and that indeed

$$\rho(\beta, \xi) \leq \rho(\alpha, \zeta) + \epsilon. \qquad \square$$

The next lemma follows immediately from the two-sided invariance of the metrics d_λ.

Lemma 3.5.3. *Let (α, ζ) be a congruent pair of length n, and let $\{I_k\}_{k=1}^m$ be a right [left] transfer admissible sequence of intervals. If (β, ξ) is the right [left] $\{I_k\}_{k=1}^m$-transfer of the pair (α, ζ), then*

$$\rho(\alpha, \zeta) = \rho(\beta, \xi).$$

Lemma 3.5.4. *Let (α, ζ) be a congruent f-pair, and let T_ζ be an evaluation tree for ζ. There is a slim f-pair (β, ξ) such that*

(i) $|\alpha| = |\beta|$*;*

(ii) $\rho(\alpha, \zeta) = \rho(\beta, \xi)$*;*

(iii) T_ζ *is a slim evaluation tree for* ξ*;*

(iv) if T_ζ *is a balanced evaluation tree for* ζ*, then it is also balanced as an evaluation tree for* ξ*.*

Proof. Let (α, ζ) be a congruent f-pair, let T_ζ be an evaluation tree for ζ, and let H_{T_ζ} denote the height of the tree T_ζ. We do an inductive construction of words (β_k, ξ_k) for $k = 0, \ldots, H_{T_\zeta}$ and claim that $(\beta_{H_{T_\zeta}}, \xi_{H_{T_\zeta}})$ is as desired. We start by setting $(\beta_0, \xi_0) = (\alpha, \zeta)$.

Suppose the pair (β_k, ξ_k) has been constructed. Let $t_1, \ldots, t_m \in T$ be all the nodes at the level $H_{T_\zeta} - k$ listed in the increasing order: $\max(I_{t_i}) < \min(I_{t_{i+1}})$. We define a relation $\sim$ on $[1, m]$ by setting $k \sim l$ if for any $i \in [\min(I_{t_k} \cup I_{t_l}), \max(I_{t_k} \cup I_{t_l})]$ there is $j \in [1, m]$ such that $i \in I_{t_j}$. It is straightforward to check that $\sim$ is an equivalence relation on $[1, m]$. Note that any $\sim$-equivalence class is a sub-interval of $[1, m]$. Let $J_1, \ldots, J_p$ be the increasing list of all the distinct equivalence classes, $J_1 < J_2 < \ldots < J_p$.

Case 1. $p \geq 2$. Set (γ, ω) to be the right $\{I_{t_r}\}_{r=1}^{\max(J_{p-1})}$-transfer of (β_k, ξ_k), and define (β_{k+1}, ξ_{k+1}) to be the left $\{I_{t_r}\}_{r=\min(J_p)}^m$-transfer of (γ, ω).

Case 2. $p = 1$. Suppose there is only one equivalence class. We have a trichotomy:

- if $\max(I_{\max(J_1)}) < n$, then set

$$(\beta_{k+1}, \xi_{k+1}) = \text{the right } \{I_{t_r}\}_{r=1}^m\text{-transfer of } (\beta_k, \xi_k);$$

- if $\max(I_{\max(J_1)}) = n$, but $\min(I_{\min(J_1)}) > 1$, then set

$$(\beta_{k+1}, \xi_{k+1}) = \text{the left } \{I_{t_r}\}_{r=1}^m\text{-transfer of } (\beta_k, \xi_k);$$

- if $\min(I_{\min(J_1)}) = 1$ and $\max(I_{\max(J_1)}) = n$, then set

$$(\beta_{k+1}, \xi_{k+1}) = \text{the right } \{I_{t_r}\}_{r=1}^{m-1}\text{-transfer of } (\beta_k, \xi_k).$$

  Notice the difference from the first case: the last element of the transfer sequence is $r = m - 1$, not m.

Denote $(\beta_{H_{T_\zeta}}, \xi_{H_{T_\zeta}})$ simply by (β, ξ). We claim that this pair satisfies all the requirements. Since (β, ξ) is obtained by the sequence of transfers, items (i) and (iv) follow from Lemma 3.3.14. Item (ii) is a consequence of Lemma 3.5.3.

It remains to check that $\hat{\xi}[I_t] = e$ for all $t \in T_\zeta$. By item (v) of Lemma 3.3.14 $\hat{\xi}_{k+1}[I_t] = e$ for all $t \in T_\zeta$ such that $H_{T_\zeta}(t) = H_{T_\zeta} - k$. Therefore it is enough to show that $\hat{\xi}_{k+1}[I_t] = \hat{\xi}_k[I_t]$ for all $t \in T_\zeta$ such that $H_{T_\zeta}(t) > H_{T_\zeta} - k$. This follows from item (iv) of Lemma 3.3.14 and item (v) of the definition of the evaluation tree. □

Lemma 3.5.5. *Let (α, ζ) be a slim f-pair, and let T_ζ be a slim balanced evaluation tree for ζ. There is a simple f-pair (β, ξ) such that*

(i) $|\alpha| = |\beta|$;

(ii) $\rho(\alpha, \zeta) = \rho(\beta, \xi)$;

(iii) T_ζ *is a slim balanced evaluation tree for* ξ.

Proof. Let (α, ζ) be a slim f-pair of length n, and let T_ζ be a slim evaluation tree for ζ. Sets $\{R_t\}_{t \in T_\zeta}$ form a partition of $[1, n]$. For $t \in T$ let $J_1^t, \ldots, J_{q_t}^t$ be the maximal sub-intervals of R_t. Let $\{i_k\}_{k=1}^m$ be the list of external letters in ζ. Set

$$F(J_i^t) = \{i_k\} \cap J_i^t.$$

Assume first that $F(J_i^t) \neq \emptyset$ for all $t \in T_\zeta$ and all $i \in [1, q_t]$. Note that by item (vii) of the definition of the balanced evaluation tree this is the case once $T \neq \{\emptyset\}$. Set

$$U = \Big(\bigcup_{t \in T_\zeta} \bigcup_{i=1}^{q_t} [\min(J_i^t), \max(F(J_i^t))] \Big) \setminus \{i_k\}_{k=1}^m,$$

$$V = \Big(\bigcup_{t \in T_\zeta} \bigcup_{i=1}^{q_t} [\max(F(J_i^t)), \max(J_i^t)] \Big) \setminus \{i_k\}_{k=1}^m.$$

Now write $U = \{u_k\}_{k=1}^{p_u}$, $V = \{v_k\}_{k=1}^{p_v}$ as increasing sequences. Set (γ, ω) to be the right $\{u_k\}$-transfer of the pair (α, ζ) and (β, ξ) to be the left $\{v_k\}$-transfer of (γ, ω) (we view u_k's and v_k's as intervals that consist of a single point). We claim that the pair (β, ξ) satisfies all the assumptions of the lemma.

Item (i) follows from item (i) of Lemma 3.3.14. The latter lemma also implies that T_ζ is a balanced evaluation tree for ξ. Item (ii) follows from Lemma 3.5.3.

(iii). We show that T_ζ is a slim evaluation tree for ξ. Let $t \in T_\zeta$. Since T_ζ was slim for ζ, we have $\hat{\zeta}[I_t] = e$. Note that if $u_k \in U \cap R_t$, then $u_k + 1 \in R_t$ (by the construction of U). Similarly for $v_k \in V$, $v_k \in R_t$ implies $v_k - 1 \in R_t$. It now follows from item (iv) of Lemma 3.3.14 that $\hat{\xi}[I_t] = \hat{\zeta}[I_t] = e$ and therefore T_ζ is slim.

Finally, the simplicity of (β, ξ) is a consequence of items (iv) and (v) of Lemma 3.3.14.

So have we proved the lemma under the assumption that $F(J_i^t) \neq \emptyset$ for all $t \in T_\zeta$ and all $i \in [1, q_t]$. Suppose this assumption was false. By item (vii) of the definition of the balanced evaluation tree we get $T_\zeta = \{\emptyset\}$ and $F(I_\emptyset) = \emptyset$. Therefore $\zeta(i) \in A$ for all i. Set (β, ξ) to be the right $(i)_{i=1}^{n-1}$-transfer of (α, ζ). Then $\xi = e^\frown \ldots ^\frown e$ and obviously (β, ξ) is a simple f-pair of the same length and $T_\zeta = \{\emptyset\}$ is a simple balanced evaluation tree for ξ. □

Lemma 3.5.6. *Let (α, ζ) be a slim f-pair of length n with a slim evaluation tree T_ζ. Let $t \in T_\zeta$ be given and let $\{i_k\}_{k=1}^m \subseteq R_t$ be a symmetrization admissible list. If $\xi = \mathrm{Sym}(\alpha, \zeta; i', \{i_k\})$ for some $i' \in \{i_k\}_{k=1}^m$, then*

$$\rho(\alpha, \zeta) \geq \rho(\alpha, \xi).$$

Proof. Since ζ is slim, we have

$$\zeta(i_1) \cdot \zeta(i_2) \cdots \zeta(i_m) = e,$$

and by Proposition 3.4.3 we get

$$d(\alpha(i_1) \cdots \alpha(i_m), e) = d(\alpha(i_1) \cdots \alpha(i_m), \zeta(i_1) \cdots \zeta(i_m)) \leq \sum_{j=1}^m d(\alpha(i_j), \zeta(i_j)).$$

If $i' = i_k$, then

$$\rho(\alpha, \zeta) - \rho(\alpha, \xi) =$$
$$\sum_{j=1}^m d\big(\alpha(i_j), \zeta(i_j)\big) - d\big(\alpha(i_k), \alpha(i_{k-1})^{-1} \cdot \alpha(i_1)^{-1} \cdot \alpha(i_m)^{-1} \cdots \alpha(i_{k+1})^{-1}\big) =$$
$$\sum_{j=1}^m d\big(\alpha(i_j), \zeta(i_j)\big) - d\big(\alpha(i_1) \cdots \alpha(i_m), e\big) \geq 0.$$

This proves the lemma. □

Definition 3.5.7. A simple f-pair (α, ζ) is called *simple reduced* if α is a reduced form of f.

Lemma 3.5.8. *For any $f \in \coprod_A G_\lambda$*

$$\mathrm{N}(f) = \inf\{\rho(\alpha, \zeta) : (\alpha, \zeta) \textit{ is a simple reduced } f\textit{-pair}\}.$$

Proof. In view of Lemmas 3.5.2, 3.5.4, and 3.5.5, it is enough to show that for any simple f-pair (α, ζ) there is a simple reduced f-pair (β, ξ) such that $\rho(\alpha, \zeta) \geq \rho(\beta, \xi)$. Let (α, ζ) be a simple f-pair. Let (γ, ω) be a simple f-pair of the smallest length among all simple f-pairs (γ_0, ω_0) such that

$$\rho(\alpha, \zeta) \geq \rho(\gamma_0, \omega_0).$$

It is enough to show that γ is a reduced form of f. If $|\gamma| = 1$ this is obvious. Suppose $|\gamma| = n \geq 2$.

Claim 1. There is no $j \in [1, n]$ such that $\gamma(j) \in A$. Suppose this is false and there is such a $j \in [1, n]$.

Case 1. $\omega(j) \in A$. (In fact, since (γ, ω) is simple, $\omega(j) \in A$ implies $\omega(j) = e$, but this is not used here.) Suppose $j < n$. Since $\gamma(j) \in A$, $\omega(j) \in A$ and $\gamma(j+1) \cong \omega(j+1)$, we have $\gamma(j) \cdot \gamma(j+1) \cong \omega(j) \cdot \omega(j+1)$. Define (γ_1, ω_1) by

$$\gamma_1(i) = \begin{cases} \gamma(i) & \text{if } i < j; \\ \gamma(j) \cdot \gamma(j+1) & \text{if } i = j; \\ \gamma(i+1) & \text{if } i > j; \end{cases}$$

$$\omega_1(i) = \begin{cases} \omega(i) & \text{if } i < j; \\ \omega(j) \cdot \omega(j+1) & \text{if } i = j; \\ \omega(i+1) & \text{if } i > j. \end{cases}$$

It is easy to see that $|\gamma_1| = |\gamma| - 1$ and (γ_1, ω_1) is a congruent f-pair. Moreover, since by the two-sided invariance

$$d(\gamma(j)\gamma(j+1), \omega(j)\omega(j+1)) \leq d(\gamma(j), \omega(j)) + d(\gamma(j+1), \omega(j+1)),$$

we also have $\rho(\gamma, \omega) \geq \rho(\gamma_1, \omega_1)$. Since γ_1, ω_1 is a congruent f-pair, by Lemmas 3.5.4 and 3.5.5 there is a simple f-pair (γ_0, ω_0) such that $|\gamma_0| = |\gamma_1| = n - 1$ and $\rho(\gamma_0, \omega_0) = \rho(\gamma_1, \omega_1)$. This contradicts the choice of (γ, ω).

If $j = n$, define

$$\gamma_1(i) = \begin{cases} \gamma(i) & \text{if } i < j - 1; \\ \gamma(j-1) \cdot \gamma(j) & \text{if } i = j - 1; \\ \gamma(i+1) & \text{if } i > j - 1; \end{cases}$$

$$\omega_1(i) = \begin{cases} \omega(i) & \text{if } i < j - 1; \\ \omega(j-1) \cdot \omega(j) & \text{if } i = j - 1; \\ \omega(i+1) & \text{if } i > j - 1, \end{cases}$$

and proceed as before.

Case 2. $\omega(j) \notin A$. Let T_ω be a slim evaluation tree for ω. Let $t \in T_\omega$ be such that $j \in R_t$. Let $\{i_k\}_{k=1}^m$ be the list of external letters in R_t; this list is symmetrization admissible. Let $j_0 \in \{i_k\}_{k=1}^m$ be any such that $j_0 \neq j$, set $\omega_2 = \mathrm{Sym}(\gamma, \omega; j_0, \{i_k\})$. By Lemma 3.3.16 (γ, ω_2) is a slim f-pair and $\omega_2(j) = \gamma(j) \in A$. And we can decrease the length of the pair (γ, ω_2) as in the previous case. This proves the case and the claim.

Claim 2. There is no $j \in [1, n-1]$ such that $\gamma(j) \cong \gamma(j+1)$. Suppose this is false and there is such a $j \in [1, n-1]$. Note that by the previous claim $\gamma(j) \notin A$ and $\gamma(j+1) \notin A$. Hence there is $\lambda_0 \in \Lambda$ such that

$$\gamma(j),\ \gamma(j+1),\ \omega(j),\ \omega(j+1) \in G_{\lambda_0}.$$

Therefore $\gamma(j) \cdot \gamma(j+1) \cong \omega(j) \cdot \omega(j+1)$. The rest of the proof is similar to what we have done in the previous claim. Define (γ_3, ω_3) by

$$\gamma_3(i) = \begin{cases} \gamma(i) & \text{if } i < j \\ \gamma(j) \cdot \gamma(j+1) & \text{if } i = j \\ \gamma(i+1) & \text{if } i > j \end{cases}$$

$$\omega_3(i) = \begin{cases} \omega(i) & \text{if } i < j \\ \omega(j) \cdot \omega(j+1) & \text{if } i = j \\ \omega(i+1) & \text{if } i > j \end{cases}$$

Then $|\gamma_3| = |\gamma| - 1$, (γ_3, ω_3) is a congruent f-pair, and $\rho(\gamma, \omega) \geq \rho(\gamma_1, \omega_1)$. By Lemmas 3.5.4 and 3.5.5 there is a simple f-pair (γ_0, ω_0) such that $|\gamma_0| = |\gamma_3|$ and $\rho(\gamma_3, \omega_3) = \rho(\gamma_0, \omega_0)$, contradicting the choice of (γ, ω). The claim is proved.

From the second claim it follows that $\gamma(j) \ncong \gamma(j+1)$ for any $j \in [1, n-1]$ and therefore γ is reduced. ☐

Proposition 3.5.9. *Let $f \in \coprod_A G_\lambda$ be an element of length n. If α is a reduced form of f, then*

$$N(f) \geq \min\{d(\alpha(i), A) : i \in [1, n]\}.$$

Proof. Fix a reduced form α of f, the word α has length n. By Lemma 3.5.8 it remains to show that for any simple reduced f-pair (β, ξ) we have

$$\rho(\beta, \xi) \geq \min\{d(\alpha(i), A) : i \in [1, n]\}.$$

Let (β, ξ) be a simple reduced f-pair. Note that by Lemma 3.3.4 the length of β is n. Let T_ξ be a slim evaluation tree for ξ, and let $t \in T_\xi$ be a leaf (i.e., a node with no predecessors). Since I_t is ξ-congruent and (β, ξ) is a simple reduced pair, it follows that there is $i_0 \in I_t$ such that $\xi(i_0) = e$ (in fact, either $\xi(\min(I_t)) = e$ or $\xi(\min(I_t) + 1) = e$). By Lemma 3.3.4 there are $a_1, a_2 \in A$

such that $a_1\alpha(i_0)a_2 = \beta(i_0)$. By the two-sided invariance we get

$$\rho(\beta,\xi) \geq d(\beta(i_0), e) = d(a_1\alpha(i_0)a_2, e) = d(\alpha(i_0), a_1^{-1}a_2^{-1}) \geq d(\alpha(i_0), A). \quad \square$$

We are now ready to prove that the pseudo-metric $\underline{d}$ is, in fact, a metric.

Theorem 3.5.10. *If $\underline{d}$ is (as before) the pseudo-metric on $\coprod_A G_\lambda$ associated with the pseudo-norm N, $\underline{d}(f,e) = \mathrm{N}(f)$, then*

(i) $\underline{d}$ is a two-sided invariant metric on $\coprod_A G_\lambda$;

(ii) $\underline{d}$ extends d.

Proof. (i) By Proposition 3.5.1 we know that $\underline{d}$ is a tsi pseudo-metric. It only remains to show that $\underline{d}(f,e) = 0$ implies $f = e$. Let $f \in \coprod_A G_\lambda$ be such that $\underline{d}(f,e) = 0$, and let α be a reduced form of f. Suppose first that $|\alpha| \geq 2$ and therefore $\alpha(i) \notin A$ for all i by the definition of the reduced form. By Proposition 3.5.9 and since A is closed in G_λ for all λ, we have

$$\underline{d}(f,e) \geq \min\left\{d(\alpha(i), A) : i \in [1, |\alpha|]\right\} > 0.$$

Suppose now $|\alpha| = 1$ and therefore $\alpha = f$, $f \in G$, and the reduced form of f is unique. By Lemma 3.5.8 the distance $d(f,e)$ is given as the infimum over all simple reduced f-pairs, but there is only one such pair: (f,e), where f is viewed as a letter in G. Hence $d(f,e) = 0$ implies $f = e$.

(ii) Fix $g_1, g_2 \in G$ and suppose first that $g_1 \ncong g_2$. Let (α,ζ) be a simple reduced $g_1g_2^{-1}$-pair. We claim that there is $a \in A$ such that $g_1a = \alpha(1)$, and $a^{-1}g_2^{-1} = \alpha(2)$. Indeed,

$$\begin{aligned}\alpha(1)\alpha(2) = g_1g_2^{-1} \implies g_2g_1^{-1}\alpha(1)\alpha(2) = e \implies g_1^{-1}\alpha(1) \in A \implies \\ \exists a \in A \text{ such that } \alpha(1) = g_1a, \text{ and } \alpha(2) = a^{-1}g_2^{-1}.\end{aligned}$$

Moreover, since $g_1 \ncong g_2$ and since (α,ζ) is congruent, we get $\zeta = e^\frown e$ and thus

$$\begin{aligned}\underline{d}(g_1,g_2) = \underline{d}(g_1g_2^{-1}, e) = \inf\{\rho(g_1a^\frown a^{-1}g_2^{-1}, e^\frown e) : a \in A\} = \\ \inf\{d(g_1, a^{-1}) + d(a^{-1}, g_2) : a \in A\} = d(g_1,g_2).\end{aligned}$$

If $g_1 \cong g_2$, then there is only one simple reduced $g_1g_2^{-1}$-pair, namely (g_1g^{-1}, e) and the item follows. $\square$

3.6 Properties of Graev metrics

Theorem 3.5.10 allows us to make the following definition: the metric $\underline{d}$ constructed in the previous section is called the *Graev metric* on the free product of groups (G_λ, d_λ) with amalgamation over A.

Theorem 3.2.3 implies that the Graev metric on a free group is, in some sense, computable, that is if one can compute the metric on the base, then to find the norm of an element f in the free group one has to calculate the function ρ for only *finitely many* trivial words, moreover those words are constructable from the letters of f. For the case of free products without amalgamation, i.e., when $A = \{e\}$, we have a similar result (see Corollary 3.6.4 below).

Definition 3.6.1. Let (α, ζ) be a slim f-pair with a slim evaluation tree T_ζ. The pair (α, ζ) is called *symmetric with respect to the tree* T_ζ if for each $t \in T_\zeta$ there are a symmetrization admissible list $\{i_{t,k}\}_{k=1}^{m_t}$ and $j_t \in \{i_{t,k}\}_{k=1}^{m_t}$ such that

$$\zeta = \mathrm{Sym}(\alpha, \zeta; j_t, \{i_{t,k}\}_{k=1}^{m_t}).$$

An f-pair (α, ζ) is called *symmetric* if there is a slim evaluation tree T_ζ such that (α, ζ) is a symmetric f-pair with respect to T_ζ.

Remark 3.6.2. Note that for any word α there are only finitely many words ζ such that (α, ζ) is symmetric.

Proposition 3.6.3. *If* $f \in \coprod_A G_\lambda$, *then*

$$N(f) = \inf\{\rho(\alpha, \xi) : (\alpha, \xi) \textit{ is a symmetric reduced } f\textit{-pair}\}.$$

Proof. By Lemma 3.5.8 it is enough to show that for any simple reduced f-pair (α, ζ) there is a symmetric reduced f-pair (α, ξ) such that

$$\rho(\alpha, \zeta) \geq \rho(\alpha, \xi).$$

Let (α, ζ) be a simple reduced f-pair, and let T_ζ be a slim evaluation tree for ζ. We construct a new slim evaluation tree T_ζ^* for ζ with the following property: for any $t \in T_\zeta^*$ and any $i \in R_\zeta^*$ if $\zeta(i) = e$, then t is a leaf and, moreover, $R_t^* = I_t^* = \{i\}$.

Let $\{j_k\}_{k=1}^m$ be such that $\zeta(j_k) = e$ for all k and $\zeta(j) = e$ implies $j = j_k$ for some $k \in [1, m]$. We construct a sequence of slim evaluation trees $T_\zeta^{(k)}$ for ζ and claim that $T_\zeta^{(m)}$ is as desired. Set $T_\zeta^{(0)} = T_\zeta$. Suppose $T_\zeta^{(k)}$ has been constructed. Let $t_0 \in T_\zeta^{(k)}$ be such that $j_{k+1} \in R_{t_0}^{(k)}$. If $|R_{t_0}^{(k)}| = 1$, that is if $R_{t_0}^{(k)} = I^{(k)} = \{j_{k+1}\}$, then do nothing: set $T_\zeta^{(k+1)} = T_\zeta^{(k)}$.

Suppose $|R_{t_0}^{(k)}| > 1$. Let s be a symbol for a new node. For all $t \in T_\zeta^{(k)} \setminus \{t_0\}$ set

$$T_\zeta^{(k+1)} = T_\zeta^{(k)} \cup \{s\}, \; I_t^{(k+1)} = I_t^{(k)}, \; I_s^{(k+1)} = [j_{k+1}, j_{k+1}] = \{j_{k+1}\}.$$

We need to turn the set $T_\zeta^{(k+1)}$ into a tree. For that let the ordering of the nodes in $T_\zeta^{(k+1)}$ extend the ordering of the nodes of $T_\zeta^{(k)}$. To finish the construction it remains to define the place for the node s inside $T_\zeta^{(k+1)}$ and an interval $I_{t_0}^{(k+1)}$.

- If j_{k+1} is the minimal element of $R_{t_0}^{(k)}$, i.e., if $j_{k+1} = \min(R_{t_0}^{(k)})$, then set $I_{t_0}^{(k+1)} = [\min(I_{t_0}^{(k)}) + 1, \max(I_{t_0}^{(k)})]$. Let $t_1 \in T_\zeta^{(k)}$ be such that $(t_0, t_1) \in E(T_\zeta^{(k)})$. Set $(s, t_1) \in E(T_\zeta^{(k+1)})$, or in other words, $s \prec t_1$ in $T_\zeta^{(k+1)}$.
- If j_{k+1} is the maximal element of $R_{t_0}^{(k)}$, i.e., if $j_{k+1} = \max(R_{t_0}^{(k)})$, then set $I_{t_0}^{(k+1)} = [\min(I_{t_0}^{(k)}), \max(I_{t_0}^{(k)}) - 1]$. Let $t_1 \in T_\zeta^{(k)}$ be such that $(t_0, t_1) \in E(T_\zeta^{(k)})$. Set $(s, t_1) \in E(T_\zeta^{(k+1)})$, or in other words, $s \prec t_1$ in $T_\zeta^{(k+1)}$.
- If j_{k+1} is neither maximal nor minimal element of $R_{t_0}^{(k)}$, then set $I_{t_0}^{(k+1)} = I_{t_0}^{(k)}$ and $(s, t_0) \in E(T_\zeta^{(k+1)})$.

It is straightforward to check that $T_\zeta^{(k+1)}$ is a slim evaluation tree for ζ.

Finally, we define $T_\zeta^* = T_\zeta^m$. Then T_ζ^* is a slim evaluation tree for ζ and, by construction, if j is such that $\zeta(j) = e$, then $I_{t_0}^* = \{j\}$ for some $t_0 \in T_\zeta^*$.

Let $\{i_k\}_{k=1}^p$ be the list of external letters of ζ. Set

$$F_t^* = \begin{cases} R_t^* \cap \{i_k\}_{k=1}^p & \text{if } R_t^* \cap \{i_k\}_{k=1}^p \neq \emptyset; \\ I_t^* & \text{otherwise.} \end{cases}$$

Note that F_t^* is symmetrization admissible for all t. Let $\{t_j\}_{j=1}^N$ be the list of nodes of T_ζ^*. For any $j \in [1, N]$ pick some l_j such that $l_j \in F_{t_j}$. Set $\xi_0 = \zeta$ and construct inductively

$$\xi_{k+1} = \mathrm{Sym}(\alpha, \xi_k; l_{k+1}, F_{t_{k+1}}).$$

Finally, set $\xi = \xi_N$. It follows from Lemma 3.3.16 that (α, ξ) is a slim f-pair and is symmetric with respect to T_ζ^* by construction. Lemma 3.5.6 implies

$$\rho(\alpha, \zeta) \geq \rho(\alpha, \xi)$$

as desired. □

If $A = \{e\}$, that is we have a free product without amalgamation, then for any $f \in \coprod_A G_\lambda$ there is exactly one reduced word $\alpha \in \mathrm{Words}(G)$ such that $\hat{\alpha} = f$. This observation together with Remark 3.6.2 gives us the following

Corollary 3.6.4. *If $A = \{e\}$, then for any $f \in \coprod_A G_\lambda$*

$$N(f) = \min\{\rho(\alpha, \xi) : (\alpha, \xi) \textit{ is a symmetric reduced } f\textit{-pair}\}.$$

We can now prove an analog of Proposition 3.2.4 for the Graev metrics on the free products with amalgamation.

Proposition 3.6.5. *The Graev metric $\underline{d}$ has the following properties:*

(i) if (T, d_T) is a tsi group, $\phi_\lambda : G_\lambda \to T$ are K-Lipschitz homomorphisms (K does not depend on λ) such that for all $a \in A$ and all $\lambda_1, \lambda_2 \in \Lambda$

$$\phi_{\lambda_1}(a) = \phi_{\lambda_2}(a),$$

then there exist a unique K-Lipschitz homomorphism $\phi : \coprod_A G_\lambda \to T$ that extends ϕ_λ;

(ii) let $H_\lambda < G_\lambda$ be subgroups such that $A < H_\lambda$ for all λ and think of $\coprod_A H_\lambda$ as being a subgroup of $\coprod_A G_\lambda$. Endow H_λ with the metric induced from G_λ. The Graev metric on $\coprod_A H_\lambda$ is the same as the induced Graev metric from $\coprod_A G_\lambda$. Moreover, if H_λ are closed subgroups, then $\coprod_A H_\lambda$ is a closed subgroup $\coprod_A G_\lambda$;

(iii) let δ be any other tsi metric on the amalgam $\coprod_A G_\lambda$. If δ extends d, then $\delta(f_1, f_2) \le \underline{d}(f_1, f_2)$ for all $f_1, f_2 \in \coprod_A G_\lambda$, i.e., $\underline{d}$ is maximal among all the tsi metrics that extend d;

(iv) if $\Lambda' = \{\lambda \in \Lambda : G_\lambda \neq A\}$ and $|\Lambda'| \ge 2$, then

$$\chi(\coprod\nolimits_A G_\lambda) = \max\big\{\aleph_0, \sup\{\chi(G_\lambda) : \lambda \in \Lambda\}, |\Lambda'|\big\}.$$

In particular, if Λ is at most countable and G_λ are all separable, then the amalgam is also separable.

Proof. (i) By the universal property for the free products with amalgamation there is a unique extension of the homomorphisms ϕ_λ to a homomorphism $\phi : \coprod_A G_\lambda \to T$, it remains to check that ϕ is K-Lipschitz. Let (α, ζ) be a congruent f-pair of length n. Then

$$\begin{aligned} K\rho(\alpha,\zeta) = &\sum_{i=1}^n K d(\alpha(i), \zeta(i)) \ge \sum_{i=1}^n d_T\big(\phi(\alpha(i)), \phi(\zeta(i))\big) \ge \\ &d_T(\phi(\hat{\alpha}), \phi(\hat{\zeta})) = d_T(\phi(f), e). \end{aligned}$$

And therefore

$$K\underline{d}(f, e) = \inf\{K\rho(\alpha,\zeta) : (\alpha,\zeta) \text{ is a congruent } f\text{-pair}\} \ge d_T(\phi(f), e).$$

Hence ϕ is K-Lipschitz.

(ii) Let $\underline{d}_H$ be the Graev metric on $\coprod_A H_\lambda$ and $\underline{d}$ be the Graev metric on $\coprod_A G_\lambda$. From Proposition 3.6.3 it follows that $\underline{d}_H = \underline{d}|_{\coprod_A H_\lambda}$.

For the moreover part suppose that H_λ are closed in G_λ for all $\lambda \in \Lambda$. Set $H = \bigcup_{\lambda\in\Lambda} H_\lambda$. Note that H is a closed subset of G. Suppose towards a contradiction that there exists $f \in \coprod_A G_\lambda$ such that $f \notin \coprod_A H_\lambda$, but $f \in \overline{\coprod_A H_\lambda}$. Let $\alpha \in \mathrm{Words}(G)$ be a reduced form of f, and let $n = |\alpha|$. Set

$$\begin{aligned} \epsilon_1 &= \min\big\{d(\alpha(i), A) : i \in [1, n]\big\}, \\ \epsilon_2 &= \min\big\{d(\alpha(i), H) : i \in [1, n],\ \alpha(i) \notin H\big\}. \end{aligned}$$

Note that $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Let $i_0 \in [1, n]$ be the largest such that $\alpha(i_0) \notin H$. By Lemma 3.3.4 the numbers ϵ_i and i_0 are independent of the choice of the reduced form α. Set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Let $h \in \coprod_A H_\lambda$ be such that $\underline{d}(f, h) < \epsilon$. By Lemma 3.5.8 there is a simple reduced fh^{-1}-pair (β, ξ) such that $\rho(\beta, \xi) < \epsilon$. Let T_ξ be a slim evaluation tree for ξ, and let $t_0 \in T_\xi$ be such that $i_0 \in R_{t_0}$. It is easy to see that there is a word α' such that α' is a reduced form of f, $\alpha'(i) = \beta(i)$ for all $i \in [1, i_0 - 1]$, and $\alpha'(i_0) = \beta(i_0) \cdot h_0$ for some $h_0 \in H$. Without loss of generality assume that $\alpha' = \alpha$. Note that $\beta(i) \in H$ for all $i > i_0$.

We claim that $i_0 = \min(R_{t_0})$. Suppose not. Let $j_0 \in R_{t_0}$ be such that $j_0 < i_0$ and $[j_0 + 1, i_0 - 1] \cap R_{t_0} = \emptyset$ (i.e., j_0 is the predecessor of i_0 in R_{t_0}). Let $I = [j_0 + 1, i_0 - 1]$. Because T_ξ is slim, $\hat{\xi}[I] = e$. Since β is reduced and (β, ξ) is congruent, there is $i_1 \in I$ such that $\xi(i_1) \in A$ (in fact, $\xi(i_1) = e$). But then

$$\rho(\beta, \xi) \geq d(\beta(i_1), \xi(i_1)) \geq d(\alpha(i_1), A) \geq \epsilon,$$

contradicting the assumption $\rho(\beta, \xi) < \epsilon$. The claim is proved.

Therefore $i_0 = \min(R_{t_0})$. Let $\{j_k\}_{k=1}^m$ be the list of external letters of ξ, and let $F_{t_0} = R_{t_0} \cap \{j_k\}_{k=1}^m$. We know that $\xi(i_0) \notin A$, since otherwise $\rho(\beta, \xi) \geq \epsilon$. Thus $i_0 \in F_{t_0}$. Let $\xi' = \mathrm{Sym}(\beta, \xi; i_0, F_{t_0})$. By Lemma 3.5.6 $\rho(\beta, \xi) \geq \rho(\beta, \xi')$. Since $\beta(i) \in H$ for all $i > i_0$, we get $\xi'(i) \in H$ for all $i \in R_{t_0} \setminus \{i_0\}$. Let λ_0 be such that $\xi'(i) \in H_{\lambda_0}$ for all $i \in R_{t_0} \setminus \{i_0\}$. Since $\hat{\xi}'[R_{t_0}] = e$, it follows that $\xi'(i_0) \in H_{\lambda_0}$ as well. Finally, we get

$$\rho(\beta, \xi) \geq \rho(\beta, \xi') \geq d(\beta(i_0), \xi'(i_0)) \geq d(\alpha(i_0), H_{\lambda_0}) \geq \epsilon,$$

contradiction the choice of (β, ξ). Therefore there is no $f \in \overline{\coprod_A H_\lambda}$ such that $f \notin \coprod_A H_\lambda$.

(iii) Let $f \in \coprod_A G_\lambda$ be given, let (α, ζ) be a congruent f-pair of length n. Since δ extends d, we get

$$\delta(f, e) \leq \sum_{i=1}^{n} \delta(\alpha(i), \zeta(i)) = \sum_{i=1}^{n} d(\alpha(i), \zeta(i)).$$

By taking the infimum over all such pairs (α, ζ) we get $\delta(f, e) \leq \underline{d}(f, e)$. By the left-invariance $\delta(f_1, f_2) \leq \underline{d}(f_1, f_2)$ for all $f_1, f_2 \in \coprod_A G_\lambda$.

(iv) If $|\Lambda'| \geq 2$, then $\coprod_A G_\lambda$ is an infinite metric space, therefore $\chi(\coprod_A G_\lambda) \geq \aleph_0$. Since $G_\lambda < \coprod_A G_\lambda$, it follows that $\chi(\coprod_A G_\lambda) \geq \chi(G_\lambda)$. We now show that $\chi(\coprod_A G_\lambda) \geq |\Lambda'|$. It is enough to consider the case $|\Lambda'| \geq \aleph_0$. There is an $\epsilon_0 > 0$ such that

$$\left|\left\{\lambda \in \Lambda : \sup\{d(g, A) : g \in G_\lambda\} > \epsilon_0\right\}\right| = |\Lambda'|.$$

For any such λ choose a $g_\lambda \in G_\lambda$ such that $d(g_\lambda, A) > \epsilon_0$. The family $\{g_\lambda\}_{\lambda \in \Lambda}$ is $2\epsilon_0$-separated and hence $\chi(\coprod_A G_\lambda) \geq |\Lambda'|$.

Finally, for the reverse inequality, let $F_\lambda \subseteq G_\lambda$ be dense sets such that

$|F_\lambda| = \chi(G_\lambda)$. The set

$$\Big\{\hat{\alpha} : \alpha \in \mathrm{Words}(\bigcup_{\lambda\in\Lambda} F_\lambda)\Big\}$$

is dense in $\coprod_A G_\lambda$ and

$$\Big|\mathrm{Words}(\bigcup_{\lambda\in\Lambda} F_\lambda)\Big| = \max\Big\{\aleph_0, \sup\{\chi(G_\lambda) : \lambda \in \Lambda\}, |\Lambda'|\Big\}.$$

□

3.6.1 Factors of Graev metrics.

Note that one can naturally view G as a pointed metric space (G, e, d), and the identity map $G \mapsto \coprod_A G_\lambda$ is 1-Lipschitz (in fact, we have shown in Theorem 3.5.10 that it is an isometric embedding). We can construct the Graev metric on the free group $(F(G), d_F)$, and by item (i) of Proposition 3.2.4 there is a 1-Lipschitz homomorphism

$$\phi : F(G) \to \coprod\nolimits_A G_\lambda$$

such that $\phi(g) = g$ for all $g \in G$. Since G generates $\coprod_A G_\lambda$, the map ϕ is onto. Let $\mathfrak{N} = \ker(\phi)$ be the kernel of this homomorphism. If d_0 is the factor metric on $F(G)/\mathfrak{N}$ (see the remark after Proposition 3.4.4), then $(F(G)/\mathfrak{N}, d_0)$ is a tsi group and $F(G)/\mathfrak{N}$ is isomorphic to $\coprod_A G_\lambda$ as an abstract group.

Proposition 3.6.6. *In the above setting* $(F(G)/\mathfrak{N}, d_0)$ *is* isometrically *isomorphic to* $(\coprod_A G_\lambda, \underline{d})$.

Proof. We recall the definition of the factor metric: for $f_1\mathfrak{N}, f_2\mathfrak{N} \in F(G)/\mathfrak{N}$

$$d_0(f_1\mathfrak{N}, f_2\mathfrak{N}) = \inf\{d_F(f_1h_1, f_2h_2) : h_1, h_2 \in \mathfrak{N}\}.$$

Of course, by construction $F(G)/\mathfrak{N}$ is isomorphic to $\coprod_A G_\lambda$ and we check that the natural isomorphism is an isometry.

Let $f' \in \coprod_A G_\lambda$, and let $w \in \mathrm{Words}(G)$ be reduced form of f'. We can naturally view w as a reduced form of the element in $F(G)$, call it f. It is enough to show that for any such f and f' we have

$$d_0(f'\mathfrak{N}, \mathfrak{N}) = \underline{d}(f, e).$$

Note that if $h \in \mathfrak{N}$, then $h^\sharp \in \mathfrak{N}$ (for the definition of $h^\sharp$ see Subsection 3.2.1). Therefore by Proposition 3.2.5

$$d_0(f'\mathfrak{N}, \mathfrak{N}) = \inf\{d_F(f'h, e) : h \in \mathfrak{N}\} = \inf\{d_F(f'h^\sharp, e) : h \in \mathfrak{N}\}.$$

If $h \in \mathfrak{N}$ and $\gamma \in \mathrm{Words}(G)$ is the reduced form of $h^\sharp \in F(G)$, then

$$d_F(f'h^\sharp, e) = \inf\Big\{\rho\big(w^\frown\gamma, (w^\frown\gamma)^\theta\big) : \theta \text{ is a match on } [1, |w^\frown\gamma|]\Big\}.$$

Since $w, \gamma \in \text{Words}(G)$ and since $\hat{\gamma} = e$, we get $\underline{d}(f, e) \leq d_0(w\mathfrak{N}, \mathfrak{N})$. Since f was arbitrary and because of left-invariance of the metrics $\underline{d}$ and d_0, we get $\underline{d} \leq d_0$.

For the reverse inequality note that d_0 is a two-sided invariant metric on $\coprod_A G_\lambda$ and it extends the metric d on G, therefore by item (iii) of Proposition 3.6.5 we have $d_0 \leq \underline{d}$ and hence $d_0 = \underline{d}$. □

3.6.2 Graev metrics for products of Polish groups

We would like to note that the construction of metrics on the free products with amalgamation works well with respect to group completions. Let us be more precise. Suppose we start with tsi groups (G_λ, d_λ) and a common closed subgroup $A < G_\lambda$, assume additionally that all the groups G_λ are complete as metrics spaces. The group $(\coprod_A G_\lambda, \underline{d})$, in general, is not complete, so let's take its group completion (for tsi groups this is the same as the metric completion), which we denote by $(\overline{\coprod_A G_\lambda}, \underline{d})$. We have an analog of item (i) of Proposition 3.6.5 for complete tsi groups. But first we need a simple lemma.

Lemma 3.6.7. *Let (H_1, d_1) and (H_2, d_2) be complete tsi groups, $\Lambda < H_1$ be a dense subgroup and $\phi : \Lambda \to H_2$ be a K-Lipschitz homomorphism. Then ϕ extends uniquely to a K-Lipschitz homomorphism*

$$\psi : H_1 \to H_2.$$

Proof. Let $h \in H_1$ and let $\{b_n\}_{n=1}^{\infty} \subseteq \Lambda$ be such that $b_n \to h$. Since ψ is K-Lipschitz, we have

$$d_2(\psi(b_n), \psi(b_m)) \leq K d_1(b_n, b_m).$$

Hence $\{\psi(b_n)\}_{n=1}^{\infty}$ is a d_2-Cauchy sequence, and thus there is $f \in H_2$ such that $\psi(b_n) \to f$. Set $\psi(h) = f$. This extends ψ to a map $\psi : H_1 \to H_2$ and it is easy to see that is extension is still K-Lipschitz. □

Combining the above result with item (i) of Proposition 3.6.5 we get

Proposition 3.6.8. *Let (T, d_T) be a complete tsi group, let $\phi_\lambda : G_\lambda \to T$ be K-Lipschitz homomorphisms such that for all $a \in A$ and all $\lambda_1, \lambda_2 \in \Lambda$*

$$\phi_{\lambda_1}(a) = \phi_{\lambda_2}(a).$$

There exist a unique K-Lipschitz homomorphism $\phi : \overline{\coprod_A G_\lambda} \to T$ such that ϕ extends ϕ_λ for all λ.

This proposition together with item (iv) of Proposition 3.6.5 shows that there are countable coproducts in the category of tsi Polish metric groups and 1-Lipschitz homomorphisms.

3.6.3 Tsi groups with no Lie sums and Lie brackets

In [26] L. van den Dries and S. Gao gave an example of a group, which they denote by F, and a two-sided invariant metric d on F such that the completion $(\overline{F}, d)$ of this group has neither Lie sums nor Lie brackets. More precisely, they constructed two one-parameter subgroups

$$A_i = \left(f_t^{(i)}\right)_{t \in \mathbb{R}} < \overline{F} \quad i = 1, 2,$$

such that neither Lie sum nor Lie bracket of A_1 and A_2 exist.

Their group can be nicely explained in out setting. It turns out that the group F that they have constructed is isometrically isomorphic to the group $\mathbb{Q} * \mathbb{Q}$ with the Graev metric (and the metrics on the copies of the rationals are the usual absolute-value metrics). The group completion of $\mathbb{Q} * \mathbb{Q}$ is then the same as the group completion of the group $\mathbb{R} * \mathbb{R}$ with the Graev metric. And moreover, A_1 and A_2 are just the one-parameter subgroups given by the $\mathbb{R}$ factors.

3.7 Metrics on SIN groups

Recall that topological group is SIN if for every open neighborhood of the identity there is a smaller open neighborhood $V \subseteq G$ such that $gVg^{-1} = V$ for all $g \in G$. SIN stands for Small Invariant Neighborhoods. It is well-knows that a metrizable topological group admits a compatible two-sided invariant metric if and only if it is a SIN group.

Suppose G_λ are metrizable topological groups that admit compatible two-sided invariant metrics and $A < G_\lambda$ is a common closed subgroup. It is natural to ask whether one can find compatible tsi metrics d_λ that agree on A.

Question 3.7.1. Let G_1 and G_2 be metrizable SIN topological groups, and let $A < G_i$ be a common closed subgroup. Are there compatible tsi metrics d_i on G_i such that

$$d_1(a_1, a_2) = d_2(a_1, a_2)$$

for all $a_1, a_2 \in A$?

We do not know the answer to this question. Before discussing some partial results let us recall the notion of a Birkhoff-Kakutani family of neighborhoods.

Definition 3.7.2. Let G be a topological group. A family $\{U_i\}_{i=0}^{\infty}$ of open neighborhoods of the identity $e \in G$ is called *Birkhoff-Kakutani* if the following conditions are met:

(i) $U_0 = G$;

(ii) $\bigcap_i U_i = e$;

(iii) $U_i^{-1} = U_i$;

(iv) $U_{i+1}^3 \subseteq U_i$.

If additionally

(v) $gU_ig^{-1} = U_i$ for all $g \in G$,

then the sequence is called *conjugacy invariant*.

It is well known (see, for example, [4]) that a topological group G admits a Birkhoff-Kakutani family if and only if it is metrizable. Moreover, let $\{U_i\}_{i=0}^{\infty}$ be a Birkhoff-Kakutani family in a group G, for $g_1, g_2 \in G$ set

$$\eta(g_1, g_2) = \inf\{2^{-n} : g_2^{-1}g_1 \in U_n\},$$

$$d(g_1, g_2) = \inf\Big\{\sum_{i=1}^{n-1} \eta(f_i, f_{i+1}) : \{f_i\}_{i=1}^n \subseteq G,\ f_1 = g_1, f_n = g_2\Big\}.$$

Then the function d is a compatible left-invariant metric on G and for all $g_1, g_2 \in G$

$$\frac{1}{2}\eta(g_1, g_2) \leq d(g_1, g_2) \leq \eta(g_1, g_2).$$

We call this metric d a *Birkhoff-Kakutani metric* associated with the family $\{U_i\}$.

A metrizable topological group admits a compatible tsi metric if and only if there is a conjugacy invariant Birkhoff-Kakutani family, and moreover, if $\{U_i\}$ is conjugacy invariant, then the metric d constructed above is two-sided invariant.

Proposition 3.7.3. *Let G_1 and G_2 be metrizable SIN groups, let $A < G_i$ be a common subgroup. There are compatible tsi metrics d_i on G_i such that $d_1|_A$ is bi-Lipschitz equivalent to $d_2|_A$, i.e, there is $K > 0$ such that*

$$\frac{1}{K}d_1(a_1, a_2) \leq d_2(a_1, a_2) \leq Kd_1(a_1, a_2)$$

for all $a_1, a_2 \in A$.

Proof. Since G_1 and G_2 are metrizable, we can fix two compatible metrics μ_1 and μ_2 on G_1 and G_2 respectively such that μ_i-diam$(G_i) < 1$. We construct conjugacy invariant Birkhoff-Kakutani families $\{U_i^{(j)}\}_{i=0}^{\infty}$ for G_j, $j = 1, 2$, such that

(i) $U_{2i+1}^{(1)} \cap A \subseteq U_{2i}^{(2)} \cap A$;

(ii) $U_{2i+2}^{(2)} \cap A \subseteq U_{2i+1}^{(1)} \cap A$.

For the base of construction let $U_0^j = G_j$. Suppose we have constructed $\{U_i^{(j)}\}_{i=1}^N$ and suppose N is even (if N is odd, switch the roles of G_1 and G_2). If $V = U_N^{(2)} \cap A$, then V is an open neighborhood of the identity in A and therefore there is an open set $U \subseteq G_1$ such that $U \cap A = V$. Let $U_{N+1}^{(1)} \subseteq G_1$ be any open neighborhood of the identity such that $(U^{(1)}_{N+1})^{-1} = U^{(1)}_{N+1}$, $gU^{(1)}_{N+1}g^{-1} = U^{(1)}_{N+1}$ for all $g \in G_1$, μ_1-diam$(U^{(1)}_{N+1}) < 1/N$ and

$$(U^{(1)}_{N+1})^3 \subseteq U \cap U^{(1)}_N.$$

Such a $U^{(1)}_{N+1}$ exists because G_1 is SIN. Set $U^{(2)}_{N+1}$ to be any open symmetric neighborhood of $e \in G_2$ such that $(U^{(2)}_{N+1})^3 \subseteq U^{(2)}_N$.

It is straightforward to check that such sequences $\{U^{(j)}_i\}_{i=1}^{\infty}$ indeed satisfy all the requirements. If d_j are the Birkhoff-Kakutani metrics that correspond to the families $\{U^{(j)}_i\}$, then for all $a_1, a_2 \in A$

$$\frac{1}{2}\eta_1(a_1, a_2) \leq \eta_2(a_1, a_2) \leq 2\eta_1(a_1, a_2),$$

whence

$$\frac{1}{4}d_1(a_1, a_2) \leq d_2(a_1, a_2) \leq 4d_1(a_1, a_2),$$

and therefore $d_1|_A$ and $d_2|_A$ are bi-Lipschitz equivalent with a constant $K = 4$. □

Remark 3.7.4. It is, of course, straightforward to generalize the above construction to the case of finitely many groups G_j, but we do not know if the result is true for infinitely many groups G_j.

Remark 3.7.5. Note that one can always multiply the metric d_2 by a suitable constant (which is 4 in the above construction) to assure that $d_1|_A \leq d_2|_A$. We use this observation later in Remark 3.7.7.

Proposition 3.7.6. *Let G be a topological group, $A < G$ be a closed subgroup of G, N_G be a tsi norm on G, N_A be a tsi norm on A and suppose that for all $a \in A$*

$$N_A(a) \leq N_G(a).$$

There exists a compatible norm N on G such that

(i) N extends N_A, that is $N_A(a) = N(a)$ for all $a \in A$;

(ii) $N \leq N_G$.

If, moreover, A is a normal subgroup of G, then N is two-sided invariant.

Proof. For $g \in G$ set

$$N(g) = \inf\{N_A(a) + N_G(a^{-1}g) : a \in A\}.$$

We claim that N is a pseudo-norm on G.

- $N(e) = 0$ is obvious.
- For any $g \in G$ and any $a \in A$ by the two-sided invariance of N_G

$$N_A(a) + N_G(a^{-1}g) = N_A(a^{-1}) + N_G(g^{-1}a) = N_A(a^{-1}) + N_G(ag^{-1})$$

and therefore $N(g) = N(g^{-1})$.

- If $g_1, g_2 \in G$, then

$$\begin{aligned} N(g_1 g_2) = &\inf\{N_A(a) + N_G(a^{-1}g_1g_2) : a \in A\} = \\ &\inf\{N_A(a_1a_2) + N_G(a_2^{-1}a_1^{-1}g_1g_2) : a_1, a_2 \in A\} \leq \\ &\inf\{N_A(a_1) + N_A(a_2) + N_G(a_1^{-1}g_1) + N_G(g_2a_2^{-1}) : a_1, a_2 \in A\} = \\ &\inf\{N_A(a_1) + N_G(a_1^{-1}g_1) : a_1 \in A\} + \\ &\inf\{N_A(a_2) + N_G(a_2^{-1}g_2) : a_2 \in A\} = \\ &N(g_1) + N(g_2). \end{aligned}$$

Next we show that N is a compatible pseudo-norm. For a sequence $\{g_n\}_{n=1}^{\infty} \subseteq G$ we have

$$\begin{aligned} N(g_n) \to 0 \iff &\exists \{a_n\}_{n=1}^{\infty} \subseteq A \quad N_A(a_n) + N_G(a_n^{-1}g_n) \to 0 \iff \\ &a_n \to e \text{ and } a_n^{-1}g_n \to e \iff \\ &g_n \to e. \end{aligned}$$

In particular, N is a norm.

(i) Now we claim that N extends N_A. Let $b \in A$. Using $N_G \geq N_A$ we get

$$\begin{aligned} N(b) = &\inf\{N_A(a) + N_G(a^{-1}b) : a \in A\} \geq \\ &\inf\{N_A(a) + N_A(a^{-1}b) : a \in A\} \geq N_A(b). \end{aligned}$$

On the other hand

$$N(b) \leq N_A(b) + N_G(b^{-1}b) = N_A(b),$$

and therefore $N(b) = N_A(b)$.

(ii) Finally, for any $g \in G$ we have

$$\begin{aligned} N(g) = &\inf\{N_A(a) + N_G(a^{-1}g) : a \in A\} \leq \\ &\inf\{N_G(a) + N_G(a^{-1}g) : a \in A\} \leq \\ &N_G(e) + N_G(g) = N_G(g), \end{aligned}$$

and therefore $N \leq N_G$.

For the moreover part suppose that A is a normal subgroup. If $g_1 \in G$, then

$$\begin{aligned} N(g_1 g g_1^{-1}) = &\inf\{N_A(a) + N_G(a^{-1}g_1gg_1^{-1}) : a \in A\} = \\ &\inf\{N_A(g_1^{-1}ag_1) + N_G(g_1^{-1}a^{-1}g_1g) : a \in A\} = N(g), \end{aligned}$$

and so N is two-sided invariant. □

Remark 3.7.7. Proposition 3.7.3 (with Remark 3.7.5) and Proposition 3.7.6 together yield a positive answer to Question 3.7.1 when A is a normal subgroup of one of G_j.

It is natural to ask whether it is really necessarily to assume in Proposition 3.7.6 the existence of a norm N_G such that $N_A \leq N_G$. The following example shows that this assumption cannot be dropped.

Example. Let G be the discrete Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},$$

and let A be the center of G

$$A = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

The subgroup A is, of course, isomorphic to the group of integers $\mathbb{Z}$. Let d be a metric on A given by the absolute value: $d(b_1, b_2) = |b_1 - b_2|$. We claim that this tsi metric can not be extended to a tsi (in fact, even to a left-invariant) metric on G. Indeed, suppose there is such an extension $\underline{d}$. The group G is generated by the three matrices:

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } z = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that $z^{(n^2)} = [x^n, y^n] = x^n y^n x^{-n} y^{-n}$. Therefore

$$n^2 = d(z^{(n^2)}, e) = \underline{d}(z^{(n^2)}, e) = \underline{d}(x^n y^n x^{-n} y^{-n}, e) \leq 2n\big(\underline{d}(x, e) + \underline{d}(y, e)\big),$$

for all n, which is absurd.

3.8 Induced metrics

In this section (G, d) denotes a tsi group, and $A < G$ is a closed subgroup. This section is a preparation for the HNN construction, which is given in the next section. Let $\langle t \rangle$ denote a copy of the free group on one element t, i.e., a copy of the integers, with the usual metric $d(t^m, t^n) = |m - n|$. The Graev metric on the free product $G * \langle t \rangle$ is denoted again by the letter d. Consider the subgroup of the free product generated by G and tAt^{-1}; it not hard to check that, in fact, as an abstract group it is isomorphic to the free product $G * tAt^{-1}$. Thus we have two metrics on the group $G * tAt^{-1}$: one is just the metric d, the other one is the Graev metric on this free product; denote the latter by $\underline{d}$. When are these two metrics the same? It turns out that they are the same if and only if the diameter of A is at most 1. The proof of this fact is the core of this section.

We can naturally view $\mathrm{Words}(G \cup tAt^{-1})$ as a subset of $\mathrm{Words}(G \cup \langle t \rangle)$ by treating a letter $tat^{-1} \in tAt^{-1}$ as a word $t^\frown a^\frown t^{-1} \in \mathrm{Words}(G \cup \langle t \rangle)$. In what follows we identify $\mathrm{Words}(G \cup tAt^{-1})$ with a subset of $\mathrm{Words}(G \cup \langle t \rangle)$.

Let $f \in G * tAt^{-1}$ be given and let $\alpha \in \mathrm{Words}(G \cup tAt^{-1})$ be the reduced form of f. Note that since we have a free product (no amalgamation), reduced form is unique. The word $\alpha \in \mathrm{Words}(G \cup \langle t \rangle)$ can be written as

$$\alpha = g_1{}^\frown t^\frown a_1{}^\frown t^{-1} g_2{}^\frown t^\frown a_2{}^\frown t^{-1\frown} \cdots {}^\frown t^\frown a_n{}^\frown t^{-1\frown} g_{n+1},$$

where $g_i \in G$, $a_i \in A$, and also g_1 or g_{n+1} may be absent.

Lemma 3.5.8 implies

$$d(f, e) = \inf\{\rho(\alpha, \zeta) : (\alpha, \zeta) \text{ is a congruent } f\text{-pair}\},$$

and notice that the infimum is taken over all pairs with the same first coordinate α — the reduced form of f. We can also impose some restrictions on ζ and change the infimum to a minimum, but we do not need this for a moment.

In the rest of the section ζ, ξ, δ denote words in the alphabet $G \cup \langle t \rangle$.

3.8.1 Hereditary words

Definition 3.8.1. A trivial word $\zeta \in \mathrm{Words}(G \cup \langle t \rangle)$ is called *hereditary* if $\zeta(i) \in \langle t \rangle \setminus \{e\}$ implies $\zeta(i) = t^{\pm 1}$ for all $i \in [1, n]$. A congruent f-pair (α, ζ), where $f \in G * tAt^{-1}$, is called *hereditary* if α is the reduced form of f, ζ is hereditary, and moreover,

$$\zeta(i) = t^{\pm 1} \implies \zeta(i) = \alpha(i).$$

Lemma 3.8.2. *Let $f \in G * tAt^{-1}$, and let $\alpha \in \mathrm{Words}(G \cup \langle t \rangle)$ be the reduced form of f. If (α, ζ) is a congruent f-pair, then there exists a trivial word $\xi \in \mathrm{Words}(G \cup \langle t \rangle)$ such that (α, ξ) is a hereditary f-pair and $\rho(\alpha, \xi) \leq \rho(\alpha, \zeta)$.*

Proof. Let T_ζ be an evaluation tree for ζ. Fix $t \in T_\zeta$. Suppose there exists $j \in R_t$ such that $\alpha(j) = t^{\pm 1}$ and neither $\zeta(j) = \alpha(j)$ nor $\zeta(j) = e$. Since $\zeta(j) \neq e$ and because the pair (α, ζ) is congruent, it must be the case that $\zeta(j) = t^M$ for some $M \neq 0$. Let $\{i_k\}_{k=1}^m \subseteq R_t$ be the complete list of external letters of ζ in R_t, note that $j \in \{i_k\}_{k=1}^m$. Since R_t is ζ-congruent, we have $\zeta(i_k) \cong t$ for all $k \in [1, m]$. Note that since we have a free product, any evaluation tree is, in fact, slim, and any congruent f-pair is, in fact, a simple f-pair. So we can perform a symmetrization. Set

$$\delta = \mathrm{Sym}(\alpha, \zeta; i_1, \{i_k\}).$$

By Lemma 3.5.6 $\rho(\alpha, \delta) \le \rho(\alpha, \zeta)$ and also for all $i \in R_t$ we have

$$\alpha(i) = t^{\pm 1} \implies (\alpha(i) = \delta(i)) \text{ or } (\delta(i) = e) \text{ or } (i = i_1).$$

Let $\epsilon_k \in \{-1, +1\}$ be such that $\alpha(i_k) = t^{\epsilon_k}$. For all $k \in [2, m]$

$$\delta(i_k) = \alpha(i_k) = t^{\epsilon_k}.$$

Let N be such that $\delta(i_1) = t^N$. Note that since $\hat{\delta}[I_t] = e$,

$$N + \epsilon_2 + \ldots + \epsilon_m = 0.$$

We now construct a word $\bar{\xi}$ as follows.

Case 0. If $N = 0$ or $N = \epsilon_1$, then set $\bar{\xi} = \delta$.

In cases below we assume $N \notin \{0, \epsilon_1\}$.

Case 1. Suppose $\text{sign}(N) = \text{sign}(\epsilon_1)$. Find different indices $k_1, \ldots, k_{|N|-1}$ such that $\text{sign}(N) = -\text{sign}(\epsilon_{k_p})$ for all $p \in [1, |N| - 1]$. Set

$$\bar{\xi}(i) = \begin{cases} \delta(i) & \text{if } i \notin \{i_{k_p}\}_{p=1}^{|N|-1} \text{ and } i \neq i_1; \\ \alpha(i_1) & \text{if } i = i_1; \\ e & \text{if } i \in \{i_{k_p}\}_{p=1}^{|N|-1}. \end{cases}$$

Case 2. Suppose $\text{sign}(N) = -\text{sign}(\epsilon_1)$. Find different indices $k_1, \ldots, k_{|N|}$ such that $\text{sign}(N) = -\text{sign}(\epsilon_{k_p})$ for all $p \in [1, |N|]$. Set

$$\bar{\xi}(i) = \begin{cases} \delta(i) & \text{if } i \notin \{i_{k_p}\}_{p=1}^{|N|} \text{ and } i \neq i_1; \\ e & \text{if } i \in \{i_{k_p}\}_{p=1}^{|N|} \text{ or } i = i_1. \end{cases}$$

It is easy to check that $\rho(\alpha, \delta) = \rho(\alpha, \bar{\xi})$ and $\hat{\bar{\xi}} = e$. Moreover, for all $i \in R_t$ if $\alpha(i) = t^{\pm 1}$, then either $\bar{\xi}(i) = \alpha(i)$ or $\bar{\xi}(i) = e$.

Now apply the same procedure for all $t \in T_\zeta$ and denote the result by ξ. The word ξ is as desired. □

To analyze the structure of hereditary words we introduce the following notion of a structure tree.

Definition 3.8.3. Let ζ be a hereditary word of length n. A tree T_ζ together with a function that assigns to a node $t \in T_\zeta$ an interval $I_t \subseteq [1, n]$ is called a *structure tree for* ζ if for all $s, t \in T_\zeta$ the following conditions are met:

(i) $I_\emptyset = [1, n]$;

(ii) $\hat{\zeta}[I_t] = e$;

(iii) if $t \neq \emptyset$, then $\zeta(\min(I_t)) = t^{\pm 1}$ and $\zeta(\max(I_t)) = t^{\mp 1}$ (in particular $\zeta(\min(I_t)) = \zeta(\max(I_t))^{-1}$).

Set $R_t = I_t \setminus \bigcup_{t' \prec t} I_{t'}$; then also

(v) for all $i \in R_t$ if $i \notin \{\min(I_t), \max(I_t)\}$, then $\zeta(i) \in G$ (in particular $R_t \setminus \{\min(I_t), \max(I_t)\}$ is ζ-congruent);

(vi) $\zeta(i) \in G$ for all $i \in R_\emptyset$ (in general $R_\emptyset$ may be empty);

(vii) if $H(t) \leq H(s)$ and $I_s \cap I_t \neq \emptyset$, then $s \prec t$ or $s = t$;

(viii) if $s \prec t$ and $t \neq \emptyset$, then

$$\min(I_t) < \min(I_s) < \max(I_s) < \max(I_t).$$

Lemma 3.8.4. *If ζ is a hereditary word of length n, then*

$$|\{i \in [1, n] : \zeta(i) = t\}| = |\{i \in [1, n] : \zeta(i) = t^{-1}\}|.$$

Proof. Let $\{i_k\}_{k=1}^m$ be the list of letters such that

(i) $\zeta(i_k) = t^{\epsilon_k}$ for some $\epsilon_k \in \{-1, 1\}$;

(ii) $\zeta(i) = t^\epsilon$, $\epsilon \in \{-1, 1\}$, implies $i = i_k$ for some k.

Since $\hat{\zeta} = e$, we get

$$\epsilon_1 + \ldots + \epsilon_m = 0,$$

and therefore

$$|\{i \in [1, n] : \zeta(i) = t\}| = |\{i \in [1, n] : \zeta(i) = t^{-1}\}|. \qquad \square$$

Lemma 3.8.5. *Let ζ be a hereditary word of length n. If there is $i \in [1, n]$ such that $\zeta(i) = t$, then there is an interval $I \subseteq [1, n]$ such that*

(i) $\zeta(\min(I)) = t^{\pm 1}$ *and* $\zeta(\max(I)) = t^{\mp 1}$;

(ii) $\zeta(i) \in G$ *for all* $i \in I \setminus \{\min(I), \max(I)\}$;

(iii) $\hat{\zeta}[I] = e$.

Proof. Let $I_1, \ldots, I_m$ be the list of intervals such that

(i) $\zeta(\min(I_k)) = t^{\pm 1}$, $\zeta(\max(I_k)) = t^{\mp 1}$;

(ii) $\zeta(i) \in G$ for all $i \in I_k \setminus \{\min(I_k), \max(I_k)\}$;

(iii) $\max(I_k) \leq \min(I_{k+1})$;

(iv) if I is an interval that satisfies (i) and (ii) above, then $I = I_k$ for some $k \in [1, m]$.

It follows from Lemma 3.8.4 that the list of such intervals is nonempty. Let $J_0, \dots, J_m$ be the complementary intervals:

$$J_0 = [1, \min(J_1) - 1], \quad J_m = [\max(J_m) + 1, n],$$

$$J_k = [\max(I_k) + 1, \min(I_{k+1}) + 1] \quad \text{for } k \in [2, m-1].$$

Some (and even all) of the intervals J_k may be empty. If for some $j_1, j_2 \in J_k$ we have $\zeta(j_1) = t^{\epsilon_1}$, $\zeta(j_2) = t^{\epsilon_2}$, then $\epsilon_1 = \epsilon_2$, and moreover, $\max(I_k) = \zeta(j_1) = \min(I_{k+1})$. It is now easy to see that $\hat{\zeta}[I_k] \neq e$ for all $k \in [1, m]$ implies $\hat{\zeta} \neq e$, contradicting the assumption that ζ is trivial. □

Lemma 3.8.6. *If ζ is a hereditary word of length n, then there is a structure tree T_ζ for ζ.*

Proof. We prove the lemma by induction on $|\{i \in [1, n] : \zeta(i) = t\}|$. For the base of induction suppose that $\zeta(i) \neq t$ for all i. By the definition of a hereditary word and by Lemma 3.8.4 we have $\zeta(i) \in G$ for all $i \in [1, n]$. Set $T_\zeta = \{\emptyset\}$ and $I_\emptyset = [1, n]$. It is easy to see that this gives a structure tree.

Suppose now there is $i \in [1, n]$ such that $\zeta(i) = t$. Apply Lemma 3.8.5 and let I denote an interval granted by this lemma. Let m be the length of I. If $m = n$, that is if $I = [1, n]$, then set $T_\zeta = \{\emptyset, t\}$ with $t \prec \emptyset$ and $I_t = I_\emptyset = [1, n]$. One checks that this is a structure tree. Assume now that $m < n$. Define a word δ of length $n - m$ by

$$\delta(i) = \begin{cases} \zeta(i) & \text{if } i < \min(I) \\ \zeta(i + m) & \text{if } i \geq \min(I). \end{cases}$$

The word δ is a hereditary word and

$$|\{i \in [1, |\delta|] : \delta(i) = t\}| < |\{i \in [1, n] : \zeta(i) = t\}|.$$

Therefore, by induction hypothesis, there is a structure tree T_δ and intervals J_t, $t \in T_\delta$, for the word δ. Let s be a symbol for a new node. Set $T_\zeta = T_\delta \cup \{s\}$. If $\min(I) = 1$ or $\max(I) = n$, set $(s, \emptyset) \in E(T_\delta)$. Otherwise let $t \in T_\delta$ be the minimal node such that $\min(J_t) < \min(I) \leq \max(J_t)$ (t may still be the root $\emptyset$) and set $(s, t) \in E(T_\delta)$. Finally, define for $t \in T_\delta$

$$I_t = \begin{cases} J_t & \text{if } \max(J_t) < \min(I); \\ [\min(J_t), \max(J_t) + m] & \text{if } \min(J_t) \leq \min(I) \leq \max(J_t); \\ [\max(J_t) + m, \max(J_t) + m] & \text{if } \min(I) < \min(J_t). \end{cases}$$

and set $I_s = I$.

It is now straightforward to check that T_ζ is a structure tree for ζ. □

3.8.2 From hereditary to rigid words

From now on A will denote a closed subgroup of G of diameter $\mathrm{diam}(A) \leq 1$, unless stated otherwise.

Lemma 3.8.7. *If (G, d) is a tsi group, then for all $g_1, \ldots, g_{n-1} \in G$, for all $a_1, \ldots, a_n \in A$ such that $d(a_i, e) \leq 1$*

$$d(g_1 \cdots g_{n-1}, a_1 g_1 a_2 \cdots a_{n-1} g_{n-1} a_n) \leq n$$

Proof. By induction. For $n = 2$ we have

$$d(g_1, a_1 g_1 a_2) \leq d(g_1, a_1 g_1) + d(a_1 g_1, a_1 g_1 a_2) = d(e, a_1) + d(e, a_2) \leq 2.$$

For the step of induction

$$\begin{aligned}
&d(g_1 \cdots g_{n-1}, a_1 g_1 a_2 \cdots a_{n-1} g_{n-1} a_n) \leq \\
&d(g_1 \cdots g_{n-1}, g_1 \cdots g_{n-1} a_n) + d(g_1 \cdots g_{n-1} a_n, a_1 g_1 a_2 \cdots a_{n-1} g_{n-1} a_n) = \\
&d(e, a_n) + d(g_1 \cdots g_{n-2}, a_1 g_1 a_2 \cdots g_{n-2} a_{n-1}) \leq 1 + (n-1) = n.
\end{aligned}$$

And the lemma follows. □

Let β be a word of the form

$$\beta = g_0{}^\frown t^\frown a_1{}^\frown t^{-1}{}^\frown g_1{}^\frown t^\frown a_2{}^\frown t^{-1}{}^\frown \cdots {}^\frown g_{n-1}{}^\frown t^\frown a_n{}^\frown t^{-1}{}^\frown g_n,$$

where $g_i \in G$ and $a_i \in A$.

Define a word δ by setting for $i \in [1, |\beta|]$

$$\delta(i) = \begin{cases} e & \text{if } i = 1 \mod 4; \\ t & \text{if } i = 2 \mod 4; \\ e & \text{if } i = 3 \mod 4; \\ t^{-1} & \text{if } i = 0 \mod 4. \end{cases}$$

Or, equivalently,

$$\delta = e^\frown t^\frown e^\frown t^{-1} e^\frown \cdots {}^\frown e^\frown t^\frown e^\frown t^{-1}{}^\frown e.$$

Set $\xi = \mathrm{Sym}(\beta, \delta; 1, \{4k+1\}_{k=0}^{n})$.

Lemma 3.8.8. *Let β, ξ be as above. If ζ is a trivial word of length $|\beta|$ that is congruent to β and such that $\zeta(i) \in G$ for all i, in other words if*

$$\zeta = h_0{}^\frown e^\frown h_1{}^\frown e^\frown h_2{}^\frown e^\frown h_3{}^\frown e^\frown \cdots {}^\frown h_{2n-2}{}^\frown e^\frown h_{2n-1}{}^\frown e^\frown h_{2n},$$

where $h_i \in G$, then $\rho(\beta, \xi) \leq \rho(\beta, \zeta)$.

Proof. By the two-sided invariance

$$\rho(\beta,\zeta) \geq d(g_0a_1g_1a_2\cdots g_{n-1}a_ng_n,e)+2n.$$

On the other hand

$$\begin{aligned}\rho(\beta,\xi) = &\sum_{i=1}^{n} d(a_i,e)+d(g_0g_1\cdots g_n,e) \leq \\ &n+d(g_0g_1\cdots g_n,e) \leq \\ &n+d(g_0g_1\cdots g_n, g_0a_1g_1a_2\cdots g_{n-1}a_ng_n)+d(g_0a_1g_1a_2\cdots g_{n-1}a_ng_n,e) = \\ &n+d(g_1\cdots g_{n-1}, a_1g_1\cdots g_{n-1}a_n)+d(g_0a_1g_1a_2\cdots g_{n-1}a_ng_n,e) \leq \\ &[\text{by Lemma 3.8.7}]\ 2n+d(g_0a_1g_1a_2\cdots g_{n-1}a_ng_n,e).\end{aligned}$$

Hence $\rho(\beta,\xi) \leq \rho(\beta,\zeta)$. ☐

Suppose we have words

$$\begin{aligned}\nu_k &= g_{(k,1)}{}^\frown\cdots{}^\frown g_{(k,q_k)}, \quad \text{where } g_{(k,j)}\in G \text{ and } k\in[0,n],\\ \mu_k &= a_{(k,1)}{}^\frown\cdots{}^\frown a_{(k,p_k)}, \quad \text{where } a_{(k,j)}\in A \text{ and } k\in[1,n].\end{aligned}$$

And let $\bar\beta$ be the word

$$\bar\beta = \nu_0{}^\frown t^\frown\mu_1{}^\frown t^{-1}\nu_1{}^\frown\cdots{}^\frown\nu_{n-1}{}^\frown t^\frown\mu_n{}^\frown t^{-1\frown}\nu_n.$$

Let $\{i_k\}_{k=1}^n$, $\{i'_k\}_{k=1}^n$ be indices such that

(i) $i_k < i_{k+1}$, $i'_k < i'_{k+1}$;

(ii) $\beta(i_k)=t$, $\beta(i'_k)=t^{-1}$;

(iii) if $\beta(i)=t$, then $i=i_k$ for some $k\in[1,n]$; if $\beta(i)=t^{-1}$, then $i=i'_k$ for some $k\in[1,n]$.

In other words

$$i_k = \sum_{l=0}^{k-1} q_l + \sum_{l=1}^{k-1} p_k + 2(k-1)+1, \quad i'_k = i_k+p_k+1.$$

Define the word δ of length $|\bar\beta|$ by

$$\delta(i) = \begin{cases} e & \text{if } \bar\beta(i)\in G;\\ \bar\beta(i) & \text{if } \bar\beta(i)=t^{\pm1}.\end{cases}$$

Let $\{j_k\}_{k=1}^m$ be the enumeration of the set

$$[1,|\bar\beta|]\setminus\bigcup_{k=1}^n [i_k,i'_k].$$

Set inductively

$$\xi_0 = \mathrm{Sym}(\bar{\beta}, \delta; j_1, \{j_k\}),$$
$$\xi_{l+1} = \mathrm{Sym}(\bar{\beta}, \xi_l; j_1^{(l+1)}, \{j_k^{(l+1)}\}),$$

where $j_k^{(l)} = i_l + k$, $l \in [1, n]$, $k \in [1, p_k]$. Finally set $\bar{\xi} = \xi_n$.

Example. For example, if

$$\bar{\beta} = g_1{}^\frown g_2{}^\frown t^\frown a_1{}^\frown a_2{}^\frown a_3{}^\frown t^{-1\frown} g_3,$$

then

$$\delta = e^\frown e^\frown t^\frown e^\frown e^\frown e^\frown t^{-1\frown} e,$$
$$\xi_0 = x^\frown g_2{}^\frown t^\frown e^\frown e^\frown e^\frown t^{-1\frown} g_3, \quad x = g_3^{-1} g_2^{-1},$$
$$\xi_1 = x^\frown g_2{}^\frown t^\frown y^\frown a_2{}^\frown a_3 t^{-1\frown} g_3, \quad y = a_3^{-1} a_2^{-1}.$$

Lemma 3.8.9. *Let $\bar{\beta}, \bar{\xi}$ be as above. If ζ is a trivial word of length $|\bar{\beta}|$ that is congruent to $\bar{\beta}$ and such that $\zeta(i) \in G$ for all i, then $\rho(\bar{\beta}, \bar{\xi}) \le \rho(\bar{\beta}, \zeta)$.*

Proof. Set

$$\beta = \hat{\nu}_0{}^\frown t^\frown \hat{\mu}_1{}^\frown t^{-1\frown} \ldots {}^\frown \hat{\mu}_n{}^\frown t^{-1\frown} \hat{\nu}_n,$$
$$\xi' = \hat{\bar{\xi}}[1, i_1 - 1]^\frown t^\frown \hat{\bar{\xi}}[i_1 + 1, i_1' - 1]^\frown t^{-1\frown} \ldots {}^\frown \hat{\bar{\xi}}[i_n + 1, i_n' - 1]^\frown t^{-1\frown} \hat{\bar{\xi}}[i_n' + 1, n],$$
$$\zeta' = \hat{\zeta}[1, i_1 - 1]^\frown t^\frown \hat{\zeta}[i_1 + 1, i_1' - 1]^\frown t^{-1\frown} \ldots {}^\frown \hat{\zeta}[i_n + 1, i_n' - 1]^\frown t^{-1\frown} \hat{\zeta}[i_n' + 1, n].$$

If ξ is as in Lemma 3.8.8, then $\xi' = \xi$ and

$$\rho(\bar{\beta}, \zeta) \ge [\text{by tsi}]\ \rho(\beta, \zeta') \ge [\text{by Lemma 3.8.8}]\ \rho(\beta, \xi) = \rho(\beta, \xi') = \rho(\bar{\beta}, \bar{\xi}). \quad \square$$

Let γ be a word of the form

$$\gamma = a_0{}^\frown t^{-1\frown} g_0{}^\frown t^\frown a_1{}^\frown t^{-1\frown} g_1{}^\frown t^\frown \cdots {}^\frown a_{n-1}{}^\frown t^{-1\frown} g_{n-1}{}^\frown t^\frown a_n,$$

where $g_i \in G$ and $a_i \in A$. Let ζ be a trivial word of the same length that is congruent to γ and such that $\zeta(i) \in G$ for all i. In other words

$$\zeta = h_0{}^\frown e^\frown h_1{}^\frown e^\frown h_2{}^\frown e^\frown h_3{}^\frown e^\frown \cdots {}^\frown h_{2n-2}{}^\frown e^\frown h_{2n-1}{}^\frown e^\frown h_{2n},$$

where $h_i \in G$. Define a word δ by

$$\delta(i) = \begin{cases} a_0 & \text{if } i = 1; \\ e & \text{if } i = 1 \mod 4 \text{ and } 1 < i < 4n+1; \\ t & \text{if } i = 2 \mod 4; \\ e & \text{if } i = 3 \mod 4; \\ t^{-1} & \text{if } i = 0 \mod 4; \\ a_0^{-1} & \text{if } i = 4n+1. \end{cases}$$

Or, equivalently,

$$\delta = a_0{}^\frown t^\frown e^\frown t^{-1\frown} e^\frown \cdots {}^\frown e^\frown t^\frown e^\frown t^{-1\frown} a_0^{-1}.$$

Set $\xi = \mathrm{Sym}(\gamma, \delta; 3, \{4k-1\}_{k=1}^n)$.

Example. For example, if

$$\gamma = a_0{}^\frown t^{-1\frown} g_0{}^\frown t^\frown a_1{}^\frown t^{-1\frown} g_1{}^\frown t^\frown a_2,$$

then

$$\delta = a_0{}^\frown t^{-1\frown} e^\frown t^\frown e^\frown t^{-1\frown} e^\frown t^\frown a_0^{-1},$$
$$\xi = a_0{}^\frown t^{-1\frown} g_1^{-1\frown} t^\frown e^\frown t^{-1\frown} g_1{}^\frown t^\frown a_0^{-1}.$$

Lemma 3.8.10. *If γ, ζ, ξ are as above, then $\rho(\gamma, \xi) \le \rho(\gamma, \zeta)$.*

Proof. By the two-sided invariance

$$\rho(\gamma, \zeta) \ge d(a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n, e) + 2n.$$

On the other hand

$$\begin{aligned}
\rho(\gamma, \xi) = & d(a_0 a_n, e) + n - 1 + d(g_0 g_1 \cdots g_n, e) \le \\
& n + d(g_0 g_1 \ldots g_{n-1}, a_0^{-1} a_n^{-1}) + d(a_0^{-1} a_n^{-1}, e) \\
& n + 1 + d(g_0 g_1 \cdots g_{n-1}, a_0^{-1} a_n^{-1}) \le \\
& n + 1 + d(a_0 g_0 g_1 \cdots g_{n-2} g_{n-1} a_n, a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n) + \\
& \quad d(a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n, e) = \\
& n + 1 + d(g_1 \cdots g_{n-2}, a_1 g_1 \cdots g_{n-2} a_{n-1}) + \\
& \quad d(a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n, e) \le [\text{by Lemma 3.8.7}] \\
& n + 1 + n - 1 + d(a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n, e) \le \rho(\gamma, \zeta).
\end{aligned}$$

And the lemma follows. □

Suppose we have words

$$\mu_k = a_{(k,1)}{}^\frown \cdots {}^\frown a_{(k,p_k)}, \quad \text{where } a_{(k,j)} \in A \text{ and } k \in [0,n],$$
$$\nu_k = g_{(k,1)}{}^\frown \cdots {}^\frown g_{(k,q_k)}, \quad \text{where } g_{(k,j)} \in G \text{ and } k \in [1,n],$$

and let $\bar{\gamma}$ be the word

$$\bar{\gamma} = \mu_0{}^\frown t^{-1}{}^\frown \nu_0{}^\frown t^\frown \mu_1{}^\frown \cdots {}^\frown \mu_{n-1}{}^\frown t^{-1}{}^\frown \nu_{n-1}{}^\frown t^\frown \mu_n.$$

Let $\{i_k\}_{k=1}^n$, $\{i'_k\}_{k=1}^n$ be indices such that

(i) $i_k < i_{k+1}$, $i'_k < i'_{k+1}$;

(ii) $\gamma(i_k) = t^{-1}$, $\gamma(i'_k) = t$;

(iii) if $\gamma(i) = t^{-1}$, then $i = i_k$ for some $k \in [1,n]$; if $\gamma(i) = t$, then $i = i'_k$ for some $k \in [1,n]$.

Define the word δ of length $|\bar{\gamma}|$ by

$$\delta(i) = \begin{cases} e & \text{if } \bar{\gamma}(i) \in G; \\ \bar{\gamma}(i) & \text{if } \bar{\gamma}(i) = t^{\pm 1}. \end{cases}$$

Let $\{j_k\}_{k=1}^m$ be the enumeration of the set

$$\bigcup_{k=1}^n [i_k + 1, i'_k - 1].$$

Set inductively

$$\xi_0 = \mathrm{Sym}(\bar{\gamma}, \delta; j_1, \{j_k\}),$$
$$\xi_{l+1} = \mathrm{Sym}(\bar{\gamma}, \xi_l; j_1^{(l+1)}, \{j_k^{(l+1)}\}),$$

where $j_k^{(l)} = i'_l + k$ and $l \in [1,n]$, $k \in [1, p_k]$. Finally set

$$\bar{\xi} = \mathrm{Sym}(\bar{\gamma}, \xi_n; 1, [1, i_1 - 1] \cup [i'_n + 1, n]).$$

Example. For example, if

$$\bar{\gamma} = a_1{}^\frown a_2{}^\frown t^{-1}{}^\frown g_1{}^\frown t^\frown a_3{}^\frown a_4{}^\frown t^{-1}{}^\frown g_2{}^\frown g_3{}^\frown t^\frown a_5,$$

then

$$\delta = e^\frown e^\frown t^{-1}{}^\frown e^\frown t^\frown e^\frown e^\frown t^{-1}{}^\frown e^\frown e^\frown t^\frown e,$$
$$\xi_0 = e^\frown e^\frown t^{-1} x^\frown t^\frown e^\frown e^\frown t^{-1}{}^\frown g_2{}^\frown g_3{}^\frown t^\frown e, \quad x = g_3^{-1} g_2^{-1}$$
$$\xi_1 = e^\frown e^\frown t^{-1}{}^\frown x^\frown t^\frown a_4^{-1}{}^\frown a_4{}^\frown t^{-1}{}^\frown g_2{}^\frown g_3{}^\frown t^\frown e,$$
$$\xi = y^\frown a_2{}^\frown t^{-1}{}^\frown x^\frown t^\frown a_4^{-1}{}^\frown a_4{}^\frown t^{-1}{}^\frown g_2{}^\frown g_3{}^\frown t^\frown a_5, \quad y = a_5^{-1} a_2^{-1}.$$

Lemma 3.8.11. *Let $\bar{\gamma}, \bar{\xi}$ be as above. If ζ is a trivial word of length $|\bar{\gamma}|$ that is congruent to $\bar{\gamma}$ and such that $\zeta(i) \in G$ for all i, then $\rho(\bar{\gamma}, \bar{\xi}) \leq \rho(\bar{\gamma}, \zeta)$.*

Proof. Proof is similar to the proof of Lemma 3.8.9. □

Definition 3.8.12. Let (α, ζ) be a hereditary f-pair of length n. It is called *rigid* if for all $i \in [1, n]$

$$\alpha(i) = t^{\pm 1} \implies \zeta(i) = \alpha(i).$$

Here is an example of a rigid pair:

$$\alpha = g_0 {}^\frown t {}^\frown a_1 {}^\frown t^{-1} {}^\frown g_1 {}^\frown t {}^\frown a_2 {}^\frown t^{-1} {}^\frown g_2,$$
$$\zeta = g_2^{-1} g_1^{-1} {}^\frown t {}^\frown e {}^\frown t^{-1} {}^\frown g_1 {}^\frown t {}^\frown e {}^\frown t^{-1} {}^\frown g_2.$$

Lemma 3.8.13. *Let $f \in G * tAt^{-1}$, and let $\alpha \in \mathrm{Words}(G \cup \langle t \rangle)$ be the reduced form of f. If (α, ζ) is a hereditary f-pair, then there exists a rigid f-pair (α, ξ) such that $\rho(\alpha, \zeta) \geq \rho(\alpha, \xi)$. Moreover, if for some i one has $\alpha(i) = t$, then $\xi(i + 1) \in A$.*

Proof. Let (α, ζ) be hereditary and let T_ζ be a structure tree for ζ. If $t \in T_\zeta$ and $Q_t = R_t \setminus \{\min(I_t), \max(I_t)\}$, then, depending on whether $\zeta(\min(I_t)) = t^{-1}$ or $\zeta(\min(I_t)) = t$, we have

$$\alpha[Q_t] = \bar{\beta} = g_{(0,1)} {}^\frown \cdots {}^\frown g_{(0,q_0)} {}^\frown t {}^\frown a_{(1,1)} {}^\frown \cdots {}^\frown a_{(1,p_1)} {}^\frown t^{-1} {}^\frown \cdots$$
$$\cdots {}^\frown t {}^\frown a_{(n,1)} {}^\frown \cdots {}^\frown a_{(n,p_n)} {}^\frown t^{-1} {}^\frown g_{(n,1)} \cdots g_{(n,q_n)},$$

or

$$\alpha[Q_t] = \bar{\gamma} = a_{(0,1)} {}^\frown \cdots {}^\frown a_{(0,p_0)} {}^\frown t^{-1} {}^\frown g_{(0,1)} {}^\frown \cdots {}^\frown g_{(0,q_1)} {}^\frown t {}^\frown \cdots$$
$$\cdots {}^\frown t^{-1} {}^\frown g_{(n-1,1)} {}^\frown \cdots {}^\frown g_{(n-1,q_n)} {}^\frown t {}^\frown a_{(n,1)} \cdots a_{(n,p_n)},$$

where $a_{(i,j)} \in A$ and $g_{(i,j)} \in G$.

Let $\bar{\xi}_t$ be as in Lemma 3.8.9 or in Lemma 3.8.11 depending on whether $\alpha[Q_t] = \bar{\beta}$ or $\alpha[Q_t] = \bar{\gamma}$ and set

$$\xi[Q_t] := \bar{\xi}_t, \quad \xi(\min(I_t)) = \alpha(\min(I_t)), \quad \xi(\max(I_t)) = \alpha(\max(I_t)) \quad \text{if } t \neq \emptyset,$$

$$\xi[R_\emptyset] := \bar{\xi}_\emptyset, \quad \text{if } t = \emptyset.$$

Do this for all $t \in T_\zeta$. Then (α, ξ) is a rigid f-pair and

$$\rho(\alpha, \zeta) \geq \rho(\alpha, \xi) \text{ [by Lemma 3.8.9 and Lemma 3.8.11]}.$$

The moreover part follows immediately from the construction of ξ. □

Theorem 3.8.14. *Let (G, d) be a tsi group, $A < G$ be a closed subgroup, not necessarily of diameter at most one. If d and $\underline{d}$ are as before (see the beginning of Section 3.8), then $d = \underline{d}$ if and only if* $\mathrm{diam}(A) \leq 1$.

Proof. First we show that the condition $\mathrm{diam}(A) \leq 1$ is necessary. Suppose $\mathrm{diam}(A) > 1$ and let $a \in A$ be such that $d(a, e) > 1$. Then

$$\underline{d}(ata^{-1}t^{-1}, e) = d(a, e) + d(ta^{-1}t^{-1}, e) = d(a, e) + d(a^{-1}, e) = 2d(a, e) > 2,$$
$$d(ata^{-1}t^{-1}, e) = d(ata^{-1}t^{-1}, aea^{-1}e) \leq$$
$$d(a, a) + d(t, e) + d(a^{-1}, a^{-1}) + d(t^{-1}, e) = 2.$$

And so $\underline{d} \neq d$.

Suppose now $\mathrm{diam}(A) \leq 1$. Let $f \in G * tAt^{-1}$ be given and let α be the reduced form of f. If (α, ζ) is a congruent f-pair, then by Lemma 3.8.2 and Lemma 3.8.13 there is a rigid f-pair (α, ξ) such that $\rho(\alpha, \xi) \leq \rho(\alpha, \zeta)$ and $\alpha(i) = t$ implies $\xi(i+1) \in A$. Hence we can view ξ as an element in $\mathrm{Words}(G \cup tAt^{-1})$. Since ζ was arbitrary, it follows that $\underline{d}(f, e) \leq d(f, e)$. The inverse inequality $d(f, e) \leq \underline{d}(f, e)$ follows from item (iii) of Proposition 3.6.5. Thus $\underline{d}(f, e) = d(f, e)$, and, by the left invariance, $\underline{d}(f_1, f_2) = d(f_1, f_2)$ for all $f_1, f_2 \in G * tAt^{-1}$. □

Proposition 3.8.15. *Let (G, d) be a tsi group, $A < G$ be a subgroup and $\underline{d}$ be the Graev metric on the free product $G * \langle t \rangle$. We can naturally view $G * tAt^{-1}$ as a subgroup of $G * \langle t \rangle$. If A is closed in G, then $G * tAt^{-1}$ is closed in $G * \langle t \rangle$*

Proof. The proof is similar in spirit to the proof of item (ii) of Proposition 3.6.5, but requires some additional work. Suppose the statement is false and there is $f \in G * \langle t \rangle$ such that $f \notin G * tAt^{-1}$, but $f \in \overline{G * tAt^{-1}}$. Let $\alpha \in \mathrm{Words}(G \cup \langle t \rangle)$ be the reduced form of f, $n = |\alpha|$. We show that this is impossible and $f \in G * tAt^{-1}$. The proof goes by induction on n.

Base of induction. For the base of induction we consider cases $n \in \{1, 2\}$. If $n = 1$, then either $f \in G$ or $f = t^k$ for some $k \neq 0$. Since $G < G * tAt^{-1}$, it must be the case that $f = t^k$. Let $h \in G * tAt^{-1}$ be such that $\underline{d}(f, h) < 1$, where $\underline{d}$ is the Graev metric on $G * \langle t \rangle$. Let $\phi_1 : G \to \mathbb{Z}$ be the trivial homomorphism: $\phi_1(g) = 0$ for all $g \in G$; and let $\phi_2 : \langle t \rangle \to \mathbb{Z}$ be the natural isomorphism: $\phi_2(t^k) = k$. By item (i) of Proposition 3.6.5 ϕ_1 and ϕ_2 extend to a 1-Lipschitz homomorphism $\phi : G * \langle t \rangle \to \mathbb{Z}$. But $d(\phi(f), \phi(h)) = |k| \geq 1$. We get a contradiction with the assumption $\underline{d}(f, h) < 1$.

Note that for any $h \in G * tAt^{-1}$

$$f \in \left(\overline{G * tAt^{-1}}\right) \setminus G * tAt^{-1} \implies gh, hg \in \left(\overline{G * tAt^{-1}}\right) \setminus G * tAt^{-1}.$$

Using this observation the case $n = 2$ follows from the case $n = 1$. Indeed, $n = 2$ implies $\alpha = g^\frown t^k$ or $\alpha = t^{k\frown} g$ for some $g \in G$, $k \neq 0$. Multiplying f by g from either left or right brings us to the case $n = 1$.

Step of induction. Without loss of generality we may assume that $\alpha(n) = t^k$ for some $k \neq 0$. Indeed, if $\alpha(n) = g$ for some $g \in G$, then we can substitute fg^{-1} for f. Assume that $\alpha = \alpha_0 {}^\frown t^{k_1} {}^\frown g {}^\frown t^{k_2}$, where $k_1, k_2 \neq 0$ and $g \in G$. We claim that $k_1 = 1$, $k_2 = -1$, and $g \in A$. Set

$$\epsilon_1 = \min\{d(\alpha(i), e) : i \in [1, n]\},$$
$$\epsilon_2 = \begin{cases} 1 & \text{if } \forall i \ \alpha(i) \in G \implies \alpha(i) \in A, \\ \min\{d(\alpha(i), A) : \alpha(i) \in G \setminus A\} & \text{otherwise.} \end{cases}$$

And let $\epsilon = \min\{1, \epsilon_1, \epsilon_2\}$. Note that $\epsilon > 0$.

Since $f \in \overline{G * tAt^{-1}}$, there is $h \in G * tAt^{-1}$ such that $\underline{d}(f, h) < \epsilon$. Therefore there is a reduced simple fh^{-1}-pair (β, ξ) such that $\rho(\beta, \xi) < \epsilon$. Let γ be the reduced form of h^{-1}. Suppose first that $k_2 \neq -1$. Assume for simplicity that $\beta = \alpha {}^\frown \gamma$ (in general the first letter of γ may get canceled; the proof for the general case is the same, it is just notationally simpler to assume that $\beta = \alpha {}^\frown \gamma$). Let T_ξ be the slim evaluation tree for ξ, and let $s_0 \in T_\xi$ be such that $n \in R_{s_0}$.

We claim that $n = \min(R_{s_0})$. If this is not the case, then there is $i_0 \in R_{s_0}$ such that $i_0 < n$ and $[i_0 + 1, n - 1] \cap R_{s_0} = \emptyset$. Since α is reduced, $i_0 < n - 1$. If $I = [i_0 + 1, n - 1]$, then $\hat{\xi}[I] = e$ and so there is $j_0 \in I$ such that $\xi(j_0) = e$. Therefore

$$\rho(\beta, \xi) \geq d(\beta(j_0), \xi(j_0)) = d(\alpha(j_0), e) \geq \epsilon_1 \geq \epsilon.$$

Contradicting the choice of the pair (β, ξ).

Thus $n = \min(R_{s_0})$. Let $j_1, \ldots, j_p$ be such that

(i) $j_k \in R_{s_0}$ for all $k \in [1, p]$;

(ii) $j_k < j_{k+1}$;

(iii) $\xi(j_k) \neq e$;

(iv) $\xi(j) \neq e$ and $j \in R_{s_0}$ implies $j = j_k$ for some k.

In fact, we can always modify the tree to assure that $\xi(j) \neq e$ for all $j \in R_{s_0}$, but this is not used here. In this notation $j_1 = n$. Since $\rho(\beta, \xi) < 1$, we get $\beta(j_k) = \xi(j_k) = t^{\pm 1}$ for all $k \in [2, p]$. If $I_k = [j_k + 1, j_{k+1} - 1]$ for $k \in [1, p - 1]$, then $\hat{\xi}[I_k] = e$ for all k, whence for any $k \in [1, p - 1]$

$$|\{i \in I_k : \xi(i) = t\}| = |\{i \in I_k : \xi(i) = t^{-1}\}|.$$

This implies

$$\xi(j_2) = t, \xi(j_3) = t^{-1}, \xi(j_4) = t, \ldots, \xi(j_p) = t^{((-1)^p)}.$$

Finally, since $\hat{\xi}[R_{s_0}] = e$, we get $\xi(j_1) = t^{-1}$ or $\xi(j_1) = e$, depending on whether p is even or odd. But since by assumption $k_2 \neq 0$ we get $k_2 = -1$.

We have proved that $k_2 = -1$. The next step is to show that $g \in A$. We have two cases.

Case 1. $\gamma(1) \in G$. In this case we have $\beta = \alpha^\frown\gamma$. Let $s_1 \in T_\xi$ be such that $n-1 \in R_{s_1}$. Similarly to the previous step one shows that $n-1 = \min(R_{s_1})$. Let $R_{s_1} = \{j_k\}_{k=1}^p$, where $j_k < j_{k+1}$. In particular, $n-1 = j_1$. Set $I_k = [j_k+1, j_{k+1}-1]$. From $\hat{\xi}[I_k] = e$ it follows

$$|\{i \in I_k : \xi(i) = t\}| = |\{i \in I_k : \xi(i) = t^{-1}\}|.$$

Therefore $\xi(j_k) \in A$ for all $k \in [2, p]$. And so $\xi(j_1) \in A$ as well. Finally, if $g \notin A$, then

$$\rho(\beta, \xi) \geq d(\beta(n-1), \xi(n-1)) \geq d(g, A) \geq \epsilon_2 \geq \epsilon.$$

And again we have a contradiction with the choice of (β, ξ).

Case 2. $\gamma(1) = t$. In this case $\alpha = \alpha_0{}^\frown t^{k_1}{}^\frown g^\frown t^{-1}$ and $\gamma = t^\frown a^\frown t^{-1}{}^\frown\gamma_0$, for some $a \in A$ and a word γ_0. If $g \notin A$ then $\beta = \alpha_0{}^\frown t^{k_1}{}^\frown ga^\frown t^{-1}{}^\frown\gamma_0$. And we are essentially in Case 1. Therefore by the proof of Case 1 we get $ga \in A$, but then $g \in A$.

Thus $g \in A$. The proof of $k_1 = 1$ is similar to the proof of $k_2 = -1$ given earlier, and we omit the details.

We have shown that $\alpha = \alpha_0{}^\frown t^\frown a^\frown t^{-1}$. If $f' = fta^{-1}t^{-1}$, then α_0 is the reduced form of f' and $f' \in \overline{G * tAt^{-1}} \setminus G * tAt^{-1}$. We proceed by induction on the length of α. □

3.9 HNN extensions of groups with tsi metrics

We now turn to the HNN construction itself. There are several ways to build an HNN extension. We will follow the original construction of G. Higman, B. H. Neumann and H. Neumann from [8], because their approach hides a lot of complications into the amalgamation of groups, and we have already constructed Graev metrics on amalgams in the previous sections.

Let us briefly remind what an HNN extension is. Let G be an abstract group, $A, B < G$ be isomorphic subgroups and $\phi : A \to B$ be an isomorphism between them. An HNN extension of (G, ϕ) is a pair (H, t), where t is a new symbol and $H = \langle G, t | tat^{-1} = \phi(a), a \in A \rangle$. The element t is called a *stable letter* of the HNN extension.

3.9.1 Metrics on HNN extensions

Theorem 3.9.1. *Let (G, d) be a tsi group, $\phi : A \to B$ be a d-isometric isomorphism between the closed subgroups A, B. Let H be the HNN extension of (G, ϕ) in the abstract sense, and let t be the stable letter of the HNN extension.*

If $\operatorname{diam}(A) \leq K$*, then there is a tsi metric* $\underline{d}$ *on* H *such that* $\underline{d}|_G = d$ *and* $\underline{d}(t,e) = K$*.*

Proof. First assume that $K = 1$. Let $\langle u \rangle$ and $\langle v \rangle$ be two copies of the group $\mathbb{Z}$ of the integers with the usual metric. Form the free products $(G * \langle u \rangle, d_u)$ and $(G * \langle v \rangle, d_v)$, where d_u, d_v are the Graev metrics. Since $\operatorname{diam}(A) = \operatorname{diam}(B) \leq 1$, by Theorem 3.8.14 the Graev metric on $G * uAu^{-1}$ is the restriction of d_u onto $G * uAu^{-1}$, and, similarly, the Graev metric on $G * vBv^{-1}$ is just the restriction of d_v. Let $\psi : G * uAu^{-1} \to G * vBv^{-1}$ be an isomorphism that is uniquely defined by

$$\psi(g) = g, \quad \psi(uau^{-1}) = v\phi(a)v^{-1}, \quad a \in A, \ g \in G.$$

By Theorem 3.8.14 ψ is an isometry. Also, by Proposition 3.8.15 $G * uAu^{-1}$ and $G * vBv^{-1}$ are closed subgroups of $G * \langle u \rangle$ and $G * \langle v \rangle$ respectively. Hence by the results of Section 3.5 we can amalgamate $G * \langle u \rangle$ and $G * \langle v \rangle$ over $G * uAu^{-1} = G * vBv^{-1}$. Denote the result of this amalgamation by $(\widetilde{H}, \underline{d})$. Then

$$uau^{-1} = v\phi(a)v^{-1} \quad \text{for all } a \in A,$$

and therefore $v^{-1}uau^{-1}v = \phi(a)$. If $H = \langle G, v^{-1}u \rangle$, then $(H, v^{-1}u)$ is an HNN extension of (G, ϕ) and $\underline{d}|_{H_\phi}$ is a two-sided invariant metric on H, which extends d.

This was done under the assumption that $K = 1$. The general case can be reduced to this one. If $d' = (1/K)d$, then d' is a tsi metric on G, ϕ is a d'-isometric isomorphism and d'-$\operatorname{diam}(A) \leq 1$. By the above construction there is a tsi metric $\underline{d}'$ on H such that $\underline{d}'|_G = d'$. Now set $\underline{d} = K\underline{d}'$. □

It is, of course, natural to ask if the condition of having a bounded diameter is crucial. The answer to this question is not known, but here is a necessary condition.

Proposition 3.9.2. *Let* (G, d) *be a tsi group,* $\phi : A \to B$ *be a* d*-isometric isomorphism, and* H *be the HNN extension of* (G, ϕ) *with the stable letter* t*. If* d *is extended to a tsi metric* d' *on* H*, then*

$$\sup\{d'(a, \phi(a)) : a \in A\} < \infty.$$

Proof. If $K = d'(t, e)$, then for any $a \in A$

$$\begin{aligned} d'(a, \phi(a)) =& d'(a, tat^{-1}) = d'(a^{-1}tat^{-1}, e) = \\ & d'(a^{-1}tat^{-1}, a^{-1}eae) \leq d'(t, e) + d'(t^{-1}, e) = 2K. \end{aligned}$$

Therefore $\sup\{d'(a, \phi(a)) : a \in A\} \leq 2K$. □

Question 3.9.3. Is this condition also sufficient? To be precise, suppose (G, d) is a tsi group, $\phi : A \to B$ is a d-isometric isomorphism between closed subgroups A, B, and suppose that

$$\sup \left\{ d(a, \phi(a)) : a \in A \right\} < \infty.$$

Does there exist a tsi metric $\underline{d}$ on the HNN extension H of (G, ϕ) such that $\underline{d}|_G = d$?

3.9.2 Induced conjugation and HNN extension

Recall that a topological group G is called SIN if for every open $U \subseteq G$ such that $e \in U$ there is an open subset $V \subseteq U$ such that $gVg^{-1} = V$ for all $g \in G$. A metrizable group admits a compatible two-sided invariant metric if and only if it is SIN.

Theorem 3.9.4. *Let G be a SIN metrizable group. Let $\phi : A \to B$ be a topological isomorphism between two closed subgroups. There exist a SIN metrizable group H and an element $t \in H$ such that $G < H$ is a topological subgroup and $tat^{-1} = \phi(a)$ for all $a \in A$ if and only if there is a compatible tsi metric d on G such that ϕ becomes a d-isometric isomorphisms.*

Proof. Necessity of the condition is obvious: if d is a compatible tsi metric on H, then ϕ is $d|_G$-isometric. We show sufficiency. Let d be a compatible tsi metric on G such that ϕ is a d-isometric isomorphism. If $d'(g, e) = \min\{d(g, e), 1\}$, then d' is also a compatible tsi metric on G, ϕ is a d'-isometric isomorphism, and d'-diam$(A) \leq 1$ (because d'-diam$(G) \leq 1$). Apply Theorem 3.9.1 to get an extension of d' to a tsi metric on H, where (H, t) is the HNN extension of (G, ϕ). Then (H, t) satisfies the conclusions of the theorem. □

Corollary 3.9.5. *Let G be a SIN metrizable group. Let $\phi : A \to B$ be a topological group isomorphism. If A and B are discrete, then there is a topology on the HNN extension of (G, ϕ) such that G is a closed subgroup of H and H is SIN and metrizable.*

Proof. Let d be a compatible tsi metric on G. Since A and B are discrete, there exists constant $c > 0$ such that

$$\inf\{d(a_1, a_2) : a_1, a_2 \in A\} \geq c, \quad \inf\{d(b_1, b_2) : b_1, b_2 \in B, b_1 \neq b_2\} \geq c.$$

If $d'(g_1, g_2) = \min\{d(g_1, g_2), c\}$, then d' is a compatible tsi metric on G and ϕ is a d'-isometric isomorphism. Theorem 3.9.4 finishes the proof. □

Corollary 3.9.6. *Let $(G, +)$ be an abelian SIN metrizable group. If $\phi : G \to G$ is given by $\phi(x) = -x$, then there is a SIN metrizable topology on the HNN extension H of (G, ϕ) that extends the topology of G.*

Proof. If d is a compatible tsi metric on G such that d-diam$(G) \leq 1$, then ϕ is a d-isometric isomorphism and we apply Theorem 3.9.4. □

Definition 3.9.7. Let G be a topological group. Elements $g_1, g_2 \in G$ are said to be *induced conjugated* if there exist a topological group H and an element $t \in H$ such that $G < H$ is a topological subgroup and $tg_1t^{-1} = g_2$.

Example 3.9.8. Let $(\mathbb{T}, +)$ be a circle viewed as a compact abelian group, and let $g_1, g_2 \in \mathbb{T}$. The elements g_1 and g_2 are induced conjugated if and only if one of the two conditions is satisfied:

(i) g_1 and g_2 are periodic elements of the same period;

(ii) $g_1 = \pm g_2$.

Proof. The sufficiency of any of these conditions follows from Corollary 3.9.5 and Corollary 3.9.6. We need to show the necessity. If g_1 and g_2 are induced conjugated, then they have the same order. If the order of g_i is finite, we are done. Suppose the order is infinite. The groups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are naturally isomorphic (as topological groups) via the map $\phi(kg_1) = kg_2$. This map extends to a continuous isomorphism $\phi : \mathbb{T} \to \mathbb{T}$, because $\mathbb{T}$ is compact and $\langle g_i \rangle$ is dense in $\mathbb{T}$. But there are only two continuous isomorphisms of the circle: $\phi = id$ and $\phi = -id$. Thus $g_1 = \pm g_2$. □

Example 3.9.9. Let $G = \mathbb{T}^{\mathbb{Z}}$ be a product of circles, and let $S : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ be the shift map $S(x)(n) = x(n+1)$ for all $x \in \mathbb{T}^{\mathbb{Z}}$ and all $n \in \mathbb{Z}$. The group $\mathbb{T}^{\mathbb{Z}}$ is monothetic and abelian. If $x = \{a_n\}_{n \in \mathbb{Z}}$, where a_n's and 1 are linearly independent over $\mathbb{Q}$, then $\langle x \rangle$ is dense in $\mathbb{T}^{\mathbb{Z}}$. Since S is an automorphism, x and $S(x)$ are topologically similar. We claim that x and $S(x)$ are not induced conjugated in any SIN metrizable group H.

Proof. Suppose H is a SIN metrizable group, G is a topological subgroup of H and $t \in H$ is such that $txt^{-1} = S(x)$. If $\phi_t : H \to H$ is given by $\phi_t(y) = tyt^{-1}$, then $\phi_t(mx) = S(mx)$ for all $m \in \mathbb{Z}$ and hence, by continuity and density of $\langle x \rangle$, $\phi_t(y) = S(y)$ for all $y \in \mathbb{T}^{\mathbb{Z}}$. If d is a compatible tsi metric on H, then ϕ_t is a d-isometric isomorphism. Therefore for $x_0 \in \mathbb{T}^{\mathbb{Z}}$,

$$x_0(n) = \begin{cases} 1/2 & \text{if } n = 0; \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$d(\phi_t^m(x_0), e) = d(\phi_t^m(x_0), \phi_t^m(e)) = d(x_0, e) = \text{const} > 0,$$

but $S^m(x_0) \to 0$, when $m \to \infty$. This contradicts $\phi_t(y) = S(y)$ for all $y \in \mathbb{T}^{\mathbb{Z}}$. □

Chapter 4

References

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