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OPTIMAL INTEGRABILITY THRESHOLD FOR GIBBS MEASURES ASSOCIATED WITH FOCUSING NLS ON THE TORUS

TADAHIRO OH, PHILIPPE SOSOE, AND LEONARDO TOLOMEO

Abstract. We study an optimal mass threshold for normalizability of the Gibbs measures associated with the focusing mass-critical nonlinear Schrödinger equation on the one-dimensional torus. In an influential paper, Lebowitz, Rose, and Speer (1988) proposed a critical mass threshold given by the mass of the ground state on the real line. We provide a proof for the optimality of this critical mass threshold. The proof also applies to the two-dimensional radial problem posed on the unit disc. In this case, we answer a question posed by Bourgain and Bulut (2014) on the optimal mass threshold.

Furthermore, in the one-dimensional case, we show that the Gibbs measure is indeed normalizable at the optimal mass threshold, thus answering an open question posed by Lebowitz, Rose, and Speer (1988). This normalizability at the optimal mass threshold is rather striking in view of the minimal mass blowup solution for the focusing quintic nonlinear Schrödinger equation on the one-dimensional torus.

Contents

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1. Introduction

1.1. **Focusing Gibbs measures.** In this paper, we continue the study of the focusing Gibbs measures for the nonlinear Schrödinger equations (NLS), initiated in the seminal papers by Lebowitz, Rose, and Speer [51] and Bourgain [9]. A focusing Gibbs measure ρ is a probability measure on functions / distributions with a formal density:

$$d\rho = Z^{-1}e^{-H(u)}du,$$

where Z denotes the partition function and the Hamiltonian functional $H(u)$ is given by

$$H(u) = \frac{1}{2}\int |\nabla u|^2 dx - \frac{1}{p}\int |u|^p dx. \tag{1.1}$$

The NLS equation:

$$i\partial_t u + \Delta u + |u|^{p-2}u = 0, \tag{1.2}$$

generated by the Hamiltonian functional $H(u)$, has been studied extensively as models for describing various physical phenomena ranging from Langmuir waves in plasmas to signal propagation in optical fibers [89, 40, 1]. Furthermore, the study of the equation (1.2) from the point of view of the (non-)equilibrium statistical mechanics has received wide attention; see for example [51, 9, 10, 11, 92, 93, 50, 12, 19, 26, 15]. See also [7] for a survey on the subject, more from the dynamical point of view. Our main goal in this paper is to study the construction of the focusing Gibbs measures on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the two-dimensional unit disc $\mathbb{D} \subset \mathbb{R}^2$ (under the radially symmetric assumption with the Dirichlet boundary condition) and determine optimal mass thresholds of their normalizability in the *critical* case. In particular, we resolve an issue in the Gibbs measure construction on $\mathbb{T}$ [51, Theorem 2.2] and also answer a question posed by Bourgain and Bulut [12, Remark 6.2] on the optimal mass threshold for the focusing Gibbs measure on the unit disc $\mathbb{D}$. Furthermore, in the case of the one-dimensional torus, we prove normalizability at the optimal mass threshold in spite of the existence of minimal mass blowup solution to NLS at this mass, thus answering an open question posed by Lebowitz, Rose, and Speer [51].

We first go over the case of the one-dimensional torus. Consider the mean-zero Brownian loop u on $\mathbb{T}$, defined by the Fourier-Wiener series:

$$u(x) = \sum_{n\neq 0} \frac{g_n(\omega)}{2\pi n} e^{2\pi i n x}, \tag{1.3}$$

where $\{g_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ denotes a sequence of independent standard complex-valued[1] Gaussian random variables. Then, the law μ_0 of the mean-zero Brownian loop u in (1.3) has the formal density given by

$$d\mu_0 = Z^{-1}e^{-\frac{1}{2}\int_{\mathbb{T}}|u'|^2dx}du.$$

The main difficulty in constructing the focusing Gibbs measures comes from the unboundedness-from-below of the Hamiltonian $H(u)$. This makes the problem very different from the defocusing case, which is a well studied subject in constructive Euclidean

[1]Namely, $\operatorname{Re} g_n$ and $\operatorname{Im} g_n$ are independent real-valued mean-zero Gaussian random variables with variance $\frac{1}{2}$.

quantum field theory. In [51], Lebowitz, Rose, and Speer proposed to consider the Gibbs measure with an L^2-cutoff:[2]

$$d\rho = Z_{p,K}^{-1}\, e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p\,dx}\,\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le K\}}\,d\mu_0$$

and claimed the following results.

Theorem 1.1. *Given $p > 2$ and $K > 0$, define the partition function $Z_{p,K}$ by*

$$Z_{p,K} = \mathbf{E}_{\mu_0}\Big[e^{\frac{1}{p}\int_{\mathbb{T}}|u|^p\,dx}\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le K\}}\Big], \tag{1.4}$$

where $\mathbf{E}_{\mu_0}$ denotes an expectation with respect to the law μ_0 of the mean-zero Brownian loop (1.3)*. Then, the following statements hold*:

(i) (subcritical case) *If $2 < p < 6$, then $Z_{p,K} < \infty$ for any $K > 0$.*

(ii) (critical case) *Let $p = 6$. Then, $Z_{6,K} < \infty$ if $K < \|Q\|_{L^2(\mathbb{R})}$, and $Z_{6,K} = \infty$ if $K > \|Q\|_{L^2(\mathbb{R})}$. Here, Q is the* (*unique*[3]) *optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ such that $\|Q\|_{L^6(\mathbb{R})}^6 = 3\|Q'\|_{L^2(\mathbb{R})}^2$.*

There remains a question of normalizability at the optimal threshold $K = \|Q\|_{L^2(\mathbb{R})}$ in the critical case ($p = 6$). We address this issue in Subsection 1.2.

Lebowitz, Rose, and Speer proved the non-normalizability result for $K > \|Q\|_{L^2(\mathbb{R})}$ in Theorem 1.1 (ii) by using a Cameron-Martin-type theorem and the following sharp Gagliardo-Nirenberg-Sobolev (GNS) inequality on $\mathbb{R}^d$:

$$\|u\|_{L^p(\mathbb{R}^d)}^p \le C_{\mathrm{GNS}}(d,p)\|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{d(p-2)}{2}}\|u\|_{L^2(\mathbb{R}^d)}^{2+\frac{p-2}{2}(2-d)} \tag{1.5}$$

with $d = 1$ and $p = 6$. See Section 3 for a further discussion on the sharp GNS inequality.

The threshold value $p = 6$ and the relevance of the GNS inequality can be understood at an intuitive level by formally rewriting (1.4) as a functional integral with respect to the (periodic) Gaussian free field (= the mean-zero Brownian loop in (1.3)):

$$Z_{p,K} \text{ “=” } \int_{\{\|u\|_{L^2(\mathbb{T})}\le K\}} e^{-\frac{1}{2}\int_{\mathbb{T}}|u'(x)|^2\,dx + \frac{1}{p}\int_{\mathbb{T}}|u(x)|^p\,dx}du. \tag{1.6}$$

Applying the GNS inequality (1.5), this quantity is bounded by

$$\int_{\{\|u\|_{L^2(\mathbb{T})}\le K\}} e^{-\frac{1}{2}\int_{\mathbb{T}}|u'(x)|^2\,dx + \frac{C_{\mathrm{GNS}}(1,p)}{p}K^{\frac{p+2}{2}}\left(\int_{\mathbb{T}}|u'(x)|^2\,dx\right)^{\frac{p-2}{4}}}\,du.$$

Thus, when $p < 6$ or when $p = 6$ and K is sufficiently small, we expect the Gaussian part of the measure to dominate, and hence the partition function to be finite.

Regarding the construction of the focusing Gibbs measure, a pleasing probabilistic proof of Theorem 1.1 based on this idea was given in [51], using the explicit joint density of the times that the Brownian path hits certain levels on a grid. Unfortunately, as pointed out by Carlen, Fröhlich, and Lebowitz [19, p. 315], there is a gap in the proof of Theorem 2.2 in [51]. More precisely, the proof in [51] seems to apply only to the case, where the expectation in the definition of $Z_{p,K}$ is taken with respect to a standard (“free”) Brownian motion started at 0, rather than the random periodic function (1.3).

[2] Recall that the L^2-norm is conserved under the NLS dynamics.

[3] Up to the symmetries.

Subsequently, a more analytic proof due to Bourgain appeared in [9], establishing normalizability of the focusing Gibbs measure (i.e. $Z_{p,K} < \infty$) for (i) $2 < p < 6$ and any $K > 0$ and for (ii) $p = 6$ and sufficiently small $K > 0$. His argument combines basic estimates for Gaussian vectors with the Sobolev embedding to identify the tail behavior of the random variable $\int_{\mathbb{T}} |u|^p \, dx$, subject to the condition $\|u\|_{L^2(\mathbb{T})} \le K$. It also applies to the case $p = 6$, but shows only that $Z_{6,K} < \infty$ for sufficiently small $K > 0$.

As the first main result in this paper, we obtain the optimal threshold when $p = 6$ claimed in Theorem 1.1 (ii) by proving $Z_{6,K} < \infty$ for any $K < \|Q\|_{L^2(\mathbb{R})}$. In particular, our argument resolves the issue in [51] mentioned above. Our proof is closer in spirit to Bourgain's, since it uses the series representation (1.3) of the Brownian loop, as opposed to the path space approach taken in [51]. In Section 2, we go over Bourgain's argument and point out that, in this approach, closing the gap between small K and the optimal threshold seems difficult. We then present our proof of the direct implication of Theorem 1.1 (ii) in Subsection 4.1. As in [51], the idea is to make rigorous the computation suggested by (1.6) by a finite dimensional approximation.

Remark 1.2. Theorem 1.1 also applies when we replace the mean-zero Brownian loop in (1.3) by the Ornstein-Uhlenbeck loop:

$$u(x) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{2\pi i n x}, \tag{1.7}$$

where $\langle n \rangle = (1 + 4\pi^2 |n|^2)^{\frac{1}{2}}$ and $\{g_n\}_{n \in \mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables. See Remark 4.1. The same comment applies to Theorem 1.4 below. The law μ of the Ornstein-Uhlenbeck loop has the formal density

$$d\mu = Z^{-1} e^{-\frac{1}{2}\|u\|^2_{H^1(\mathbb{T})}} du. \tag{1.8}$$

As seen in [9], μ is a more natural base Gaussian measure to consider for the nonlinear Schrödinger equations, due to the lack of the conservation of the spatial mean under the dynamics.

We also point out that Theorem 1.1 also holds in the real-valued setting. The same comment applies to Theorems 1.3 and 1.4. For example, this is relevant to the study of the generalized KdV equation (gKdV) on $\mathbb{T}$:

$$\partial_t u + \partial_x^3 u + \partial_x (u^{p-1}) = 0. \tag{1.9}$$

Our method also applies to the focusing Gibbs measures on the two-dimensional unit disc $\mathbb{D} \subset \mathbb{R}^2$, under the radially symmetric assumption with the Dirichlet boundary condition. In the subcritical case ($p < 4$), Tzvetkov [92] constructed the focusing Gibbs measures, along with the associated invariant dynamics. His analysis was complemented in [12] by a study of the critical case $p = 4$, under a small mass assumption. See also [92, 93, 12] for results in the defocusing case.

Our approach to Theorem 1.1 allows us to establish the optimal mass threshold in the critical case ($p = 4$), thus answering the question posed by Bourgain and Bulut in [12, Remark 6.2]. We first introduce some notations. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the unit disc. Let $J_0(r)$ be the Bessel function of order zero, defined by

$$J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x}{2}\right)^{2j},$$

and z_n, $n \geq 1$, be its successive, positive zeros. Then, it is known [92] that $\{e_n\}_{n\in\mathbb{N}}$ defined by

$$e_n(r) = \|J_0(z_n\cdot)\|_{L^2(\mathbb{D})}^{-1} J_0(z_n r), \quad 0 \leq r \leq 1,$$

forms an orthonormal basis of $L^2_{\text{rad}}(\mathbb{D})$, consisting of the radial eigenfunctions of the Dirichlet self-adjoint realization of $-\Delta$ on $\mathbb{D}$. Here, $L^2_{\text{rad}}(\mathbb{D})$ denotes the subspace of $L^2(\mathbb{D})$, consisting of radial functions. Now, consider the random series:

$$v(r) = \sum_{n=1}^{\infty} \frac{g_n(\omega)}{z_n} e_n(r), \quad r^2 = x^2 + y^2, \tag{1.10}$$

where $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of independent standard complex-valued Gaussian random variables.

Theorem 1.3. *Given $p > 2$ and $K > 0$, define the partition function $\widetilde{Z}_{p,K}$ by*

$$\widetilde{Z}_{p,K} = \mathbf{E}\Big[e^{\frac{1}{p}\int_{\mathbb{D}} |v|^p \,dx} \mathbf{1}_{\{\|v\|_{L^2(\mathbb{D})} \leq K\}}\Big], \tag{1.11}$$

where $\mathbf{E}$ denotes an expectation with respect to the law of the random series (1.10). *Then, the following statements hold*:

(i) (subcritical case) *If $p < 4$, then $\widetilde{Z}_{p,K} < \infty$ for any $K > 0$.*

(ii) (critical case) *Let $p = 4$. Then, $\widetilde{Z}_{4,K} < \infty$ if $K < \|Q\|_{L^2(\mathbb{R}^2)}$, and $\widetilde{Z}_{4,K} = \infty$ if $K > \|Q\|_{L^2(\mathbb{R}^2)}$, where Q is the optimizer in the Gagliardo-Nirenberg-Sobolev inequality* (1.5) *on $\mathbb{R}^2$ such that $\|Q\|^4_{L^4(\mathbb{R}^2)} = 2\|\nabla Q\|^2_{L^2(\mathbb{R}^2)}$.*

Part (i) of Theorem 1.3 is due to Tzvetkov [92]. In [12], Bourgain and Bulut considered the critical case ($p = 4$) and proved $\widetilde{Z}_{4,K} < \infty$ if $K \ll 1$, and $\widetilde{Z}_{4,K} = \infty$ if $K \gg 1$, leaving a gap.

Theorem 1.3 (ii) answers the question posed by Bourgain and Bulut in [12]. See Remark 6.2 in [12]. We present the proof of Theorem 1.3 (ii) in Subsection 4.2 and Section 5. In Subsection 4.2, we prove $\widetilde{Z}_{4,K} < \infty$ for the optimal range $K < \|Q\|_{L^2(\mathbb{R}^2)}$ by following our argument for Theorem 1.1 on $\mathbb{T}$. We point out that some care is needed here due to the growth of the L^4-norm of the eigenfunction e_n; see (4.15). In Section 5, we prove $\widetilde{Z}_{4,K} = \infty$ for $K > \|Q\|_{L^2(\mathbb{R}^2)}$. Our argument of the non-normalizability follows closely that on $\mathbb{T}$ by Lebowitz, Rose, and Speer [51].

1.2. **Integrability at the optimal mass threshold.** We now consider the normalizability issue of the focusing Gibbs measure on $\mathbb{T}$ in the critical case ($p = 6$) at the optimal threshold $K = \|Q\|_{L^2(\mathbb{R})}$, at which a phase transition takes place. Before doing so, let us first discuss the situation for the associated dynamical problem, namely, the focusing quintic NLS, (1.2) with $p = 6$. On the real line, the optimizer Q for the sharp Gagliardo-Nirenberg-Sobolev inequality (1.5) is the ground state for the associated elliptic problem (see (3.3) below). Then, by applying the pseudo-conformal transform to the solitary wave solution $Q(x)e^{2it}$, we obtain the minimal mass blowup solution to the focusing quintic NLS on $\mathbb{R}$. Here, the minimality refers to the fact that any solution to the focusing quintic NLS on $\mathbb{R}$ with $\|u\|_{L^2(\mathbb{R})} < \|Q\|_{L^2(\mathbb{R})}$ exists globally in time; see [94].

In [70], Ogawa and Tsutsumi constructed an analogous minimal mass blowup solution u_* with $\|u_*\|_{L^2(\mathbb{T})} = \|Q\|_{L^2(\mathbb{R})}$ to the focusing quintic NLS on the one-dimensional torus $\mathbb{T}$. It was also shown that, as time approaches a blowup time, the potential energy $\frac{1}{6}\int |u_*|^6 dx$ tends to ∞. In view of the structure of the partition function $Z_{6,K=\|Q\|_{L^2(\mathbb{R})}}$ in (1.4), this divergence of the potential energy seems to create a potential obstruction to the construction of the focusing Gibbs measure in the current setting. In [11], Bourgain wrote "One remarkable point concerning the normalizability problem for Gibbs-measures of NLS in the focusing case is its close relation to blowup phenomena in the classical theory. Roughly speaking, this may be understood as follows. After normalization, the measure would be forced to live essentially on "blowup data", which however is incompatible with the invariance properties under the flow."

In spite of the existence of the minimal mass blowup solution, we prove that the focusing critical Gibbs measure is normalizable at the optimal threshold $K = \|Q\|_{L^2(\mathbb{R})}$.

Theorem 1.4. *Let $K = \|Q\|_{L^2(\mathbb{R})}$. Then, the one-dimensional partition function $Z_{6,K}$ in* (1.4) *is finite.*

In view of the discussion above, Theorem 1.4 was unexpected and is rather surprising. Theorem 1.4 answers an open question posed by Lebowitz, Rose, and Speer in [51]. See Section 5 in [51]. Moreover, together with Theorem 1.1 (ii), Theorem 1.4 shows that the partition function $Z_{6,K}$ is not analytic in the cutoff parameter K, thus settling another question posed in [51, Remark 5.2]. Compare this with the subcritical case ($p < 6$), where the analyticity result of the partition function on the parameters (including the inverse temperature, which we do not consider here) was proved by Carlen, Fröhlich, and Lebowitz [19] (for slightly different Gibbs measures).

The proof of Theorem 1.4 is presented in Section 6 and constitutes the major part of this paper, involving ideas and techniques from various branches of mathematics: probability theory, functional inequalities, elliptic partial differential equations (PDEs), spectral analysis, etc. We break the proof into several steps:

(1) In the first step, we use a profile decomposition and establish a stability result for the GNS inequality (1.5); see Lemma 6.3. When combined with our proof of Theorem 1.1, this stability result shows that if the integration is restricted to the complement of $U_\varepsilon = \{u \in L^2(\mathbb{T}) : \|u - Q\|_{L^2(\mathbb{T})} < \varepsilon\}$ for suitable $\varepsilon \ll 1$, then the resulting partition function is finite. (In fact, we must exclude a neighborhood of the orbit of the ground state Q under translations, rescalings, and rotations, but we ignore this technicality here.) Thus, the question is reduced to the evaluation of the functional integral

$$\int_{U_\varepsilon} e^{-H(u)}\, \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})} \le K\}} du \tag{1.12}$$

in the neighborhood U_ε of (the orbit of) the ground state Q, where $H(u)$ is as in (1.1) with $p = 6$.

(2) In the second step, we show that when $\varepsilon > 0$ is sufficiently small, i.e. when U_ε lies in a sufficiently small neighborhood of the (approximate) soliton manifold $\mathcal{M} =$

$\{e^{i\theta}Q^{\rho}_{\delta,x_0} : 0 < \delta < \delta^*,\ x_0 \in \mathbb{T},\ \text{and } \theta \in \mathbb{R}\}$, where $Q^{\rho}_{\delta,x_0} = (\tau_{x_0}\rho)Q_{\delta,x_0}$ denotes the dilated and translated ground state (see (6.1.5) and (6.1.7)) and ρ is a suitable cutoff function for working on the torus $\mathbb{T} \cong [-\frac{1}{2}, \frac{1}{2})$ (see (6.1.6)), we can endow U_ε with an orthogonal coordinate system in terms of the (small) dilation parameter $0 < \delta < \delta^*$, the translation parameter $x_0 \in \mathbb{T}$, the rotation parameter $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, and the component $v \in L^2(\mathbb{T})$ orthogonal to the soliton manifold $\mathcal{M}$. See Propositions 6.4 and 6.9.

(3) We then introduce a change-of-variable formula and reduce the integral (1.12) to an integral in δ, x_0, θ, and v. See Lemma 6.10.

(4) In Subsection 6.6, we reduce the problem to estimating a certain Gaussian integral with the integrand given by

$$\exp\big(-(1-\eta^2)\langle Aw, w\rangle_{H^1(\mathbb{T})}\big)$$

for some small $\eta > 0$. Here, $A = A(\delta,\eta)$ denotes the operator on $H^1(\mathbb{T})$ defined in (6.7.1):

$$Aw = P^{H^1}_{V'}(1-\partial_x^2)^{-1}\bigg(\delta^{-2}P^{H^1}_{V'}w - (1+5\eta)(\rho Q_\delta)^4\Big(2\operatorname{Re}(P^{H^1}_{V'}w) + \frac{1}{2}P^{H^1}_{V'}w\Big)\bigg),$$

where $V' \subset H^1(\mathbb{T})$ is as in (6.6.23). See also (6.3.3). We point out that the operator A is closely related to the second variation $\delta^2 H$ of the Hamiltonian. See Lemma 6.16. In view of the compactness of the operator A, the issue is further reduced to estimating the eigenvalues of $\frac{1}{2}\operatorname{id} + (1-\eta^2)A$. Subsection 6.7 is devoted to the spectral analysis of the operator A.

For readers' convenience, we present the summary of the proof of Theorem 1.4 in Subsection 6.8.

Remark 1.5. There exists an extensive literature on the study of soliton-type behavior to dispersive PDEs on $\mathbb{R}^d$; see, for example, [66] and the references therein. In particular, there are existing works on $\mathbb{R}^d$ which are closely related to Steps (2) and (4) described above. In Remarks 6.5 and 6.17, we provide brief comparison of our work on $\mathbb{T}$ with those on $\mathbb{R}^d$, pointing out similarities and differences.

We now state a dynamical consequence of Theorems 1.1 and 1.4.

Corollary 1.6. *Let $p = 6$. Consider the Gibbs measure ρ with the formal density*

$$d\rho = Z^{-1}_{6,K}\, e^{\frac{1}{6}\int_{\mathbb{T}}|u|^6\,dx}\,\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le K\}}\,d\mu \tag{1.13}$$

where μ is the law of the Ornstein-Uhlenbeck loop in (1.7). *If $K \le \|Q\|_{L^2(\mathbb{R})}$, then the focusing quintic NLS,* (1.2) *with $p = 6$, on $\mathbb{T}$ is almost surely globally well-posed with respect to the Gibbs measure ρ. Moreover, the Gibbs measure ρ is invariant under the NLS dynamics.*

By imposing that the Ornstein-Uhlenbeck loop in (1.7) *is real-valued (i.e. $g_{-n} = g_n$, $n \in \mathbb{Z}$) or by replacing μ in* (1.13) *with the law μ_0 of the real-valued mean-zero Brownian loop in* (1.3) *(with $g_{-n} = g_n$, $n \in \mathbb{Z}\setminus\{0\}$), a similar result holds for the focusing quintic generalized KdV,* (1.9) *with $p = 6$.*

Strictly speaking, in the case of gKdV, the invariance is known only under the gauged dynamics (which is needed to prove local well-posedness in [20]) at this point. We, however, expect that the invariance of the Gibbs measure also holds for the (ungauged) gKdV; see [77].

When $K < \|Q\|_{L^2(\mathbb{R})}$, Corollary 1.6 follows from the deterministic local well-posedness results for the quintic NLS [8] and the quintic gKdV [20] in the spaces containing the support of the Gibbs measure, combined with Bourgain's invariant measure argument [9]. When $K = \|Q\|_{L^2(\mathbb{R})}$, the density $e^{\frac{1}{6}\int_{\mathbb{T}}|u|^6\,dx}\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le K\}}$ is only in $L^1(\mu)$ and thus Bourgain's invariant measure argument is not directly applicable. In this case, however, the desired claim follows from the corresponding result for $K = \|Q\|_{L^2(\mathbb{R})} - \varepsilon$, $\varepsilon > 0$, and the dominated convergence theorem by taking $\varepsilon \to 0$.

Remark 1.7. (i) A result analogous to Theorem 1.4 presumably holds for the two-dimensional radial Gibbs measure on $\mathbb{D}$ studied in Theorem 1.3. Moreover, we expect the analysis on $\mathbb{D}$ to be slightly simpler since the problem on $\mathbb{D}$ has fewer symmetries (in particular, no translation invariance). In order to limit the length of this paper, however, we do not pursue this issue here.

(ii) In [75], Quastel and the first author constructed the focusing Gibbs measure conditioned at a specified mass, provided that the mass is sufficiently small in the critical case ($p = 6$). This answered another question posed in [51]. See also [19, 13]. However, the argument in [75], based on Bourgain's approach, is not quantitative. Thus, it would be of interest to further investigate this problem to see if the focusing Gibbs measure conditioned at a specified mass in the critical case ($p = 6$) can be indeed constructed up to the optimal mass threshold as in Theorem 1.4. We point out that the key ingredients for the proof of Theorem 1.4 (see the steps (1) - (4) right after the statement of Theorem 1.4) hold true even in the case of the focusing critical Gibbs measure (with the critical power $p = 6$) restricted to the critical L^2-norm $\|u\|_{L^2(\mathbb{T})} = \|Q\|^2_{L^2(\mathbb{R})}$, and thus we expect that the focusing critical Gibbs measure at a specified mass is normalizable even at the optimal mass threshold. We, however, do not pursue this issue further in this paper.

Remark 1.8. (i) In view of the minimal mass blowup solution to NLS, the normalizability of the focusing critical Gibbs measure at the optimal threshold $K = \|Q\|_{L^2(\mathbb{R})}$ in Theorem 1.4 was somehow unexpected. As an afterthought, we may give some reasoning for this phenomenon, referring to certain properties of the minimal mass blowup solution on the real line (which are not known in the periodic setting).

The first result is the uniqueness / rigidity of the minimal mass blowup solution on $\mathbb{R}$ due to Merle [57], which states that if an H^1-solution u with $\|u\|_{L^2(\mathbb{R})} = \|Q\|_{L^2(\mathbb{R})}$ to the focusing quintic NLS on $\mathbb{R}$ blows up in a finite time, it must be the minimal mass blowup solution up to the symmetries of the equation.[4] See also an extension [52] of this result for rougher H^s-solution, $s > 0$, which holds only on $\mathbb{R}^d$ for $d \geq 4$ under the radial assumption. While an analogous result is not known in the periodic setting (and in low dimensions),

[4]In a recent preprint [29], Dodson extended this uniqueness / rigidity result of the minimal mass blowup solution to the $L^2(\mathbb{R})$-setting. We, however, point out that our understanding of the corresponding problem on the torus $\mathbb{T}$ is rather poor in this direction, in particular in a low regularity setting. For example, even local well-posedness in $L^2(\mathbb{T})$ of the focusing quintic NLS on $\mathbb{T}$ remains a challenging open problem after Bourgain's work [8]. See also [43].

these results may indicate non-existence of rough blowup solutions at the critical threshold $K = \|Q\|_{L^2(\mathbb{R})}$. Theorem 1.4 shows that this non-existence claim holds true probabilistically.

Another point is instability of the minimal mass blowup solution. The minimal mass blowup solution is intrinsically unstable because a mass subcritical perturbation leads to a globally defined solution. See also [63]. Such instability may be related to the fact that we do not see (rough perturbations of) the minimal mass blowup solution probabilistically.

Lastly, we mention the situation for the focusing quintic gKdV on the real line. While finite time blowup solutions to gKdV "near" the ground state are known to exist [58, 54], it is also known that there is *no* minimal mass blowup solution to the focusing quintic gKdV on the real line [55]. Thus, from the gKdV point of view, the normalizability in Theorem 1.4 is perhaps naturally expected (but we point out that analogues of the results in [58, 54, 55] are not known on $\mathbb{T}$).

(ii) In recent years, there has been a growing interest in studying nonlinear dispersive equations with random initial data; see, for example, [10, 17, 21, 53, 18, 4, 5, 83, 74, 71, 6, 27, 15]. In particular, there are recent works [71, 14, 34] on stability of finite time blowup solutions under rough and random perturbations. The so-called log-log blowup solutions to the mass-critical focusing NLS on $\mathbb{R}^d$ were constructed by Perelman [81] and Merle and Raphaël [59, 60, 61, 62]. When $d = 1, 2$, these log-log blowup solutions on $\mathbb{R}^d$ are known to be stable under H^s-perturbations for $s > 0$; see [84, 22]. When $d = 2$, Fan and Mendelson [34] proved stability of the log-log blowup solutions to the mass-critical focusing NLS under random but structured L^2-perturbations. (Some of) these results on the log-log blowup solutions (at least the deterministic ones) are expected to hold in the periodic setting due to the local-in-space nature of the blowup profile. See, for example, [82] for the construction of the log-log blowup solutions on a domain in $\mathbb{R}^2$. We point out that the log-log blowup solutions mentioned above have mass strictly greater than (but close to) the mass of the ground state, which is complementary to the regime we study in this paper with regard to the construction of the focusing critical Gibbs measure.

Remark 1.9. While the construction of the defocusing Gibbs measures has been extensively studied and well understood due to the strong interest in constructive Euclidean quantum field theory, the (non-)normalizability issue of the focusing Gibbs measures, going back to the work of Lebowitz, Rose, and Speer [51] and Brydges and Slade [16], is not fully explored. In [11], Bourgain wrote "It seems worthwhile to investigate this aspect [the (non-)normalizability issue of the focusing Gibbs measures] more as a continuation of $\big[$[51]$\big]$ and $\big[$[16]$\big]$." See related works [87, 12, 19, 72, 79, 73, 91] on the non-normalizability (and other issues) for focusing Gibbs measures.

In a recent series of works [72, 73], the first and third authors with Okamoto employed the variational approach due to Barashkov and Gubinelli [2] to study the following two *critical* focusing[5] models on the three-dimensional torus $\mathbb{T}^3$:

[5]By "focusing", we also mean the non-defocusing case, such as the cubic interaction appearing in (1.16), such that the interaction potential (for example, $\frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 \mathrm{d}x$ in (1.16)) is unbounded from above.

(i) the focusing Hartree Gibbs measure with a Hartree-type quartic interaction, formally written as

$$\mathrm{d}\rho(u) = Z^{-1} \exp\left(\frac{\sigma}{4}\int_{\mathbb{T}^3} (V * u^2) u^2 \mathrm{d}x\right) \mathrm{d}\mu_3(u), \tag{1.14}$$

where the coupling constant $\sigma > 0$ corresponds to the focusing interaction and V is (the kernel of) the Bessel potential of order $\beta > 0$ given by

$$V * f = \langle \nabla \rangle^{-\beta} f = (1-\Delta)^{-\frac{\beta}{2}} f.$$

Hereafter, μ_3 denotes the massive Gaussian free field on $\mathbb{T}^3$. When $\beta = 2$, the focusing Hartree model (1.14) turns out to be critical.

Recall that the Bessel potential of order β on $\mathbb{T}^3$ can be written (for some $c > 0$) as

$$V(x) = c|x|^{\beta-3} + K(x) \tag{1.15}$$

for $0 < \beta < 3$ and $x \in \mathbb{T}^3 \setminus \{0\}$, where K is a smooth function on $\mathbb{T}^3$. See Lemma 2.2 in [78]. Thus, when $\beta = 2$, the potential V essentially corresponds to the Coulomb potential $V(x) = |x|^{-1}$, which is of particular physical relevance.

(ii) the Φ^3_3-measure, formally written as

$$\mathrm{d}\rho(u) = Z^{-1} \exp\left(\frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 \mathrm{d}x\right) \mathrm{d}\mu_3(u), \tag{1.16}$$

where the coupling constant $\sigma \in \mathbb{R} \setminus \{0\}$ measures the strength of the cubic interaction. Since u^3 is not sign definite, the sign of σ does not play any role and, in particular, the problem is not defocusing even if $\sigma < 0$. We point out that the Φ^3_3-model makes sense only in the real-valued setting.

In the three-dimensional setting, the massive Gaussian free field μ_3 is supported on $H^s(\mathbb{T}^3) \setminus H^{-\frac{1}{2}}(\mathbb{T}^3)$ for any $s < -\frac{1}{2}$. Thus, the potentials in (1.14) and (1.16) do not make sense as they are given, and proper renormalizations need to be introduced. Furthermore, due to the focusing nature of the problems, one needs to endow the measures with certain taming. In [72, 73], the first and third authors with Okamoto studied the generalized grand-canonical Gibbs measure formulations of the focusing Hartree Gibbs measure in (1.14) and the Φ^3_3-measure in (1.16). For example, the generalized grand-canonical Gibbs measure formulations of the Φ^3_3-measure in (1.16) is given by

$$\mathrm{d}\rho(u) = Z^{-1} \exp\left(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3:\, \mathrm{d}x - A\left|\int_{\mathbb{T}^3} :u^2:\, dx\right|^3 - \infty\right) \mathrm{d}\mu_3(u), \tag{1.17}$$

where $:u^k:$ denotes the standard Wick renormalization and the term $-\infty$ denotes another (non-Wick) renormalization. See the work by Carlen, Fröhlich, and Lebowitz [19] for a discussion of the generalized grand-canonical Gibbs measure in the one-dimensional setting. See Remark 2.1 in [51].

In [72], the first and third authors with Okamoto established a phase transition in the following two respects: (i.a) the focusing Hartree Gibbs measure in (1.14) is constructible for $\beta > 2$, while it is not for $\beta < 2$ and (i.b) when $\beta = 2$, the focusing Hartree Gibbs measure is constructible in the weakly nonlinear regime $0 < \sigma \ll 1$, while it is not in the strongly nonlinear regime $\sigma \gg 1$. This shows that the focusing Hartree Gibbs measure is critical when $\beta = 2$.

In terms of scaling, the Φ^3_3-model corresponds to the focusing Hartree model in the critical case $\beta = 2$. Indeed, it was shown in [73] that the Φ^3_3-model is also critical, exhibiting the following phase transition; the Φ^3_3-measure is constructible in the weakly nonlinear regime $0 < |\sigma| \ll 1$, whereas it is not in the strongly nonlinear regime $|\sigma| \gg 1$. While the focusing Hartree Gibbs measure in (1.14) is absolutely continuous with respect the base massive Gaussian free field μ_3 even in the critical case ($\beta = 2$), it turned out that the Φ^3_3-measure in (1.16) is singular with respect to the base massive Gaussian free field μ_3. This singularity of the Φ^3_3-measure introduced additional difficulties (as compared to the focusing Hartree Gibbs measures studied in [72]) in both the measure (non-)construction part and the dynamical part in [73]. See [73] for a further discussion.

In view of the aforementioned results in [72, 73], it is of interest to investigate (existence of) a threshold value $\sigma_* > 0$ (depending on the models) such that the construction of the critical focusing Hartree Gibbs measure with $\beta = 2$ (and the Φ^3_3-measure, respectively) holds for $0 < \sigma < \sigma_*$ (and for $0 < |\sigma| < \sigma_*$, respectively), while the non-normalizability of the critical focusing Hartree Gibbs measure with $\beta = 2$ (and the Φ^3_3-measure, respectively) holds for $\sigma > \sigma_*$ (and for $|\sigma| > \sigma_*$, respectively). If such a threshold value σ_* could be determined, it would also be of interest to study normalizability at the threshold $\sigma = \sigma_*$ in the focusing Hartree case (and $|\sigma| = \sigma_*$ in the Φ^3_3-case), analogous to Theorem 1.4 in the one-dimensional case. Such a problem, however, requires optimizing all the estimates in the proofs in [72, 73] and is out of reach at this point.

Several comments are in order. As mentioned above, the Φ^3_3-model can be considered only in the real-valued setting. Furthermore, the critical focusing Hartree model with $\beta = 2$ and the Φ^3_3-model are mass-subcritical (whereas the critical cases studied in Theorems 1.1 (ii), 1.3 (ii), and 1.4 are all mass-critical). Hence, the critical nature of these models do not seem to have anything to do with finite-time blowup solutions (in particular to NLS) unlike Theorems 1.1, 1.3, and 1.4 studied in this paper. While we mentioned the results on the generalized grand-canonical Gibbs measure formulations (namely with a taming by the Wick-ordered L^2-norm) of the focusing Hartree measures and the Φ^3_3-measure, analogous results hold even when we consider the (non-)construction of these measures endowed with a Wick-ordered L^2-cutoff. See, for example, Remark 5.10 in [72]. We point out that even in this latter setting (namely with a Wick-ordered L^2-cutoff), what matters is the size of the coupling constant σ and the size of the Wick-ordered L^2-cutoff does not play any role (unlike Theorems 1.1, 1.3, and 1.4). See also [79].

Lastly, let us mention the dynamical aspects of these models. For both of the models (1.14) and (1.16), it is possible to study the standard (parabolic) stochastic quantization [80] (namely the associate stochastic nonlinear heat equation) and and the canonical stochastic quantization [88] (namely the associate stochastic damped nonlinear wave equation). In the parabolic case, well-posedness follows easily from the standard first order expansion as in [56, 10, 24]; see [72, Appendix A] and [32]. Due to a weaker smoothing property, the well-posedness issue in the hyperbolic setting becomes more challenging. By adapting the paracontrolled approach, originally introduced in the parabolic setting [37], to the wave setting [38], the first and third authors with Okamoto [72, 73] constructed global-in-time dynamics for the stochastic damped nonlinear wave equations associated with the focusing Hartree model (for $\beta \geq 2$) and the Φ^3_3-model. As for the focusing Hartree model endowed with a Wick-ordered L^2-cutoff, Bourgain [11] studied the associated Hartree NLS on $\mathbb{T}^3$ and constructed global-in-time dynamics when $\beta > 2$. This result was extended to the critical case ($\beta = 2$) in [72, 28], where, in [28], Deng, Nahmod, and Yue proved well-posedness[6] of the associated Hartree NLS on $\mathbb{T}^3$ by using the random averaging operators, originally introduced in [26].

1.3. **Notations.** We write $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some $C > 0$, Similarly, we write $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when we have $A \leq cB$ for some small $c > 0$. We may use subscripts to denote dependence on external parameters; for example, $A \lesssim_p B$ means $A \leq C(p)B$, where the constant $C(p)$ depends on a parameter p.

In the following, we deal with complex-valued functions viewed as elements in *real* Hilbert and Banach spaces. In particular, with $M = \mathbb{T}$, $\mathbb{D}$, or $\mathbb{R}$, the inner product on $H^s(M)$ is given by

$$\langle f, g\rangle_{H^s(M)} = \mathrm{Re} \int_M (1-\Delta)^s f(x)\overline{g(x)}\,\mathrm{d}x. \tag{1.18}$$

Note that with the inner product (1.18), the family $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ does not form an orthonormal basis of $L^2(\mathbb{T})$. Instead, we need to use $\{e^{2\pi inx}, ie^{2\pi inx}\}_{n\in\mathbb{Z}}$ as an orthonormal basis of $L^2(\mathbb{T})$. A similar comment applies to the case of the unit disc $\mathbb{D}$. We point out that the series representations such as (1.3) are not affected by whether we use the inner product (1.18) with the real part or that without the real part. For example, in (1.3), we have

$$g_n e^{2\pi inx} = (\mathrm{Re}\, g_n)e^{2\pi inx} + (\mathrm{Im}\, g_n)\, ie^{2\pi inx}.$$

Here, the right-hand side is more directly associated with the the inner product (1.18), while the left-hand side is associated with the inner product without the real part.

Given $N \in \mathbb{N}$, we denote by π_N the Dirichlet projection (for functions on $\mathbb{T}$) onto frequencies $\{|n| \leq N\}$:

$$\pi_N f(x) = \sum_{|n|\leq N} \widehat{f}(n)e^{2\pi inx} \tag{1.19}$$

and we set

$$E_N = \pi_N L^2(\mathbb{T}) = \mathrm{span}\big\{e^{2\pi inx} : |n| \leq N\big\}. \tag{1.20}$$

We also define $\pi_{\neq 0}$ to be the orthogonal projection onto the mean-zero part of a function:

$$\pi_{\neq 0} u = \sum_{n\in\mathbb{Z}\setminus\{0\}} \widehat{u}(n)e^{2\pi inx} \tag{1.21}$$

and set $\pi_0 = \mathrm{id} - \pi_{\neq 0}$.

[6]The local well-posedness argument in [28] applies to the range $\beta > \beta_*$ for some $\beta_* < 1$ sufficiently close to 1.

Given $k \in \mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$, let P_k be the Littlewood-Paley projection onto frequencies of order 2^k defined by

$$P_k u = \sum_{2^{k-1} < |n| \leq 2^k} \widehat{u}(n) e^{2\pi i n x}. \tag{1.22}$$

Similarly, set

$$P_{\leq k} u = \sum_{j=0}^{k} u_j = \sum_{|n| \leq 2^k} \widehat{u}(n) e^{2\pi i n x}, \tag{1.23}$$

$$P_{\geq k} u = \sum_{j=k}^{\infty} u_j = \sum_{|n| > 2^{k-1}} \widehat{u}(n) e^{2\pi i n x}. \tag{1.24}$$

Given measurable sets $A_1, \dots, A_k$, we use the following notation:

$$\mathbf{E}[f(u), A_1, \dots, A_k] = \mathbf{E}\bigg[f(u) \prod_{j=1}^{k} \mathbf{1}_{A_j} \bigg], \tag{1.25}$$

where $\mathbf{E}$ denotes an expectation with respect to a probability distribution for u under discussion.

This paper is organized as follows. In Section 2, we review Bourgain's argument from [9], which will be used in our proof of the direct implication of Theorem 1.1 (ii) in Subsection 4.1. In Section 3, we go over the Gagliardo-Nirenberg-Sobolev inequality (1.5) on $\mathbb{R}^d$ and discuss its variants on $\mathbb{T}$ and $\mathbb{D}$. In Section 4, we then establish the direct implications of Theorem 1.1 (ii) on the one-dimensional torus $\mathbb{T}$ (Subsection 4.1) and Theorem 1.3 (ii) on the unit disc $\mathbb{D}$ (Subsection 4.2). In Section 5, we prove the non-normalizability claim in Theorem 1.3 (ii). Finally, we prove normalizability of the focusing critical Gibbs measure at the optimal mass threshold (Theorem 1.4) in Section 6.

2. Review of Bourgain's argument

In this section, we reproduce Bourgain's argument in [9] for the proof of Theorem 1.1 (i). Part of the argument presented below will be used in Subsection 4.1. Let $2 < p \leq 6$ and u denote the mean-zero Brownian loop u in (1.3). Rewriting (1.4) as

$$Z_{p,K} \leq 1 + \int_0^\infty \lambda^{p-1} e^{\frac{1}{p}\lambda^p} \mathbf{P}\big(\|u\|_{L^p(\mathbb{T})} > \lambda, \|u\|_{L^2(\mathbb{T})} \leq K \big)\, d\lambda,$$

we see that it suffices to show that there exist $C > 0$ and $c > \frac{1}{p}$ such that

$$\mathbf{P}\big(\|u\|_{L^p(\mathbb{T})} > \lambda, \|u\|_{L^2(\mathbb{T})} \leq K \big) \leq C e^{-c\lambda^p} \tag{2.1}$$

for all sufficiently large $\lambda \gg 1$.

Given $k \in \mathbb{Z}_{\geq 0}$, we set

$$u_k = P_k u, \qquad u_{\leq k} = P_{\leq k} u, \qquad \text{and} \qquad u_{\geq k} = P_{\geq k} u,$$

where P_k, $P_{\le k}$, and $P_{\ge k}$ are as in (1.22), (1.23), and (1.24). By subadditivity, we have for any k:

$$\begin{aligned}\mathbf{P}\big(\|u_{\ge k}\|_{L^p(\mathbb{T})} > \lambda\big) &\le \mathbf{P}\bigg(\bigcup_{j=k}^{\infty}\{\|u_j\|_{L^p(\mathbb{T})} > \lambda_j\}\bigg)\\ &\le \sum_{j=k}^{\infty}\mathbf{P}(\|u_j\|_{L^p(\mathbb{T})} > \lambda_j),\end{aligned}\tag{2.2}$$

where $\{\lambda_j\}_{j=k}^{\infty}$ is a sequence of positive numbers such that

$$\sum_{j=k}^{\infty}\lambda_j = \lambda.\tag{2.3}$$

Then, by using Sobolev's inequality in the form of Bernstein's inequality, we have

$$\|u_j\|_{L^p(\mathbb{T})} \le C2^{j(\frac{1}{2}-\frac{1}{p})}\|u_j\|_{L^2(\mathbb{T})}.\tag{2.4}$$

Thus, with (1.3), the probability on the right-hand side of (2.2) is bounded by

$$\begin{aligned}\mathbf{P}\bigg(\|u_j\|_{L^2(\mathbb{T})} > \frac{\lambda_j}{C}2^{j(\frac{1}{p}-\frac{1}{2})}\bigg) &= \mathbf{P}\bigg(\sum_{2^{j-1}\le|n|<2^j}\frac{|g_n|^2}{n^2} > \frac{\lambda_j^2}{C_0^2}2^{2j(\frac{1}{p}-\frac{1}{2})}\bigg)\\ &\le \mathbf{P}\bigg(\sum_{2^{j-1}\le|n|<2^j}|g_n|^2 > \frac{\lambda_j^2}{4C_0^2}2^{(1+\frac{2}{p})j}\bigg),\end{aligned}\tag{2.5}$$

where $C_0 = (2\pi)^{-1}C$.

The next lemma follows from a simple calculation involving moment generating functions of Gaussian random variables. See for example [76] for a proof.

Lemma 2.1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be independent standard real-valued Gaussian random variables. Then, we have*

$$\mathbf{P}\bigg(\sum_{n=1}^{M}X_n^2 \ge R^2\bigg) \le e^{-\frac{R^2}{4}},$$

if $R \ge 3M^{\frac{1}{2}}$.

By applying Lemma 2.1, we can bound the probability (2.5) by

$$\exp\bigg(-\frac{\lambda_j^2}{16C_0^2}2^{(1+\frac{2}{p})j}\bigg),\tag{2.6}$$

provided

$$\lambda_j \ge 6C_0 2^{-\frac{1}{p}j}.\tag{2.7}$$

By choosing

$$\lambda_j = \lambda(1-2^{-r})2^{kr}2^{-jr}$$

for $0 < r < \frac{1}{p}$, both conditions (2.7) and (2.3) are satisfied for all large k (and $j \ge k$). For such k, the probability in (2.6) is then bounded by

$$\exp\bigg(-\frac{\lambda^2(1-2^{-r})^2}{16C_0^2}2^{2kr}2^{(1+\frac{2}{p}-2r)j}\bigg).$$

Summing over $j \geq k$ in (2.2), we find that

$$\mathbf{P}\big(\|u_{\geq k}\|_{L^p(\mathbb{T})} > \lambda\big) \leq C_r \exp\bigg(- \frac{\lambda^2(1-2^{-r})^2}{16C_0^2} 2^{(1+\frac{2}{p})k}\bigg). \tag{2.8}$$

By applying Bernstein's inequality again with the restriction $\|u\|_{L^2(\mathbb{T})} \leq K$, we have

$$\|u_{\leq k-1}\|_{L^p(\mathbb{T})} \leq C2^{k(\frac{1}{2}-\frac{1}{p})}\|u_{\leq k-1}\|_{L^2(\mathbb{T})} \leq C2^{k(\frac{1}{2}-\frac{1}{p})}K. \tag{2.9}$$

Hence, by setting

$$k = \log_2 \bigg(\frac{\lambda}{2CK}\bigg)^{\frac{2p}{p-2}},$$

it follows from (2.9) that

$$\|u_{\leq k-1}\|_{L^p(\mathbb{T})} \leq \frac{\lambda}{2}. \tag{2.10}$$

Therefore, from (2.8) and (2.10), we obtain

$$\begin{aligned}
&\mathbf{P}\Big(\|u\|_{L^p(\mathbb{T})} > \lambda, \|u\|_{L^2(\mathbb{T})} \leq K\Big)\\
&\quad \leq \mathbf{P}\bigg(\|u_{\leq k-1}\|_{L^p(\mathbb{T})} > \frac{\lambda}{2}, \|u\|_{L^2(\mathbb{T})} \leq K\bigg) + \mathbf{P}\bigg(\|u_{\geq k}\|_{L^p(\mathbb{T})} > \frac{\lambda}{2}\bigg)\\
&\quad \leq C_r \exp\bigg(- \frac{(1-2^{-r})^2}{64C_0^2 \cdot (2CK)^{\frac{2(p+2)}{p-2}}} \lambda^{\frac{4p}{p-2}}\bigg)
\end{aligned}$$

for all sufficiently large $\lambda \gg 1$. Note that the exponent $\frac{4p}{p-2}$ beats the exponent p in (2.1) if (i) $p < 6$ or (ii) $p = 6$ and K is sufficiently small. Determining the optimal threshold for K would presumably require a delicate optimization of λ_j in (2.2), an exact Gaussian tail bound to replace the appraisal (2.6), and an optimal inequality to replace the applications of Bernstein's inequality in (2.4) and (2.9) to determine the precise tail behavior of $\|u\|_{L^p(\mathbb{T})}$ given $\|u\|_{L^2(\mathbb{T})} \leq K$. We did not attempt this calculation. Even if it is possible to carry out, such an approach would likely lead to a less transparent argument than the one we propose in Section 4. Moreover, our argument is easily adapted to the case of the two-dimensional unit disc $\mathbb{D}$.

3. Sharp Gagliardo-Nirenberg-Sobolev inequality

The optimizers for the Gagliardo-Nirenberg-Sobolev interpolation inequality with the optimal constant:

$$\|u\|_{L^p(\mathbb{R}^d)}^p \leq C_{\text{GNS}}(d,p)\|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{(p-2)d}{2}}\|u\|_{L^2(\mathbb{R}^d)}^{2+\frac{p-2}{2}(2-d)} \tag{3.1}$$

play an important role in the study of the focusing Gibbs measures. The following result on the optimal constant $C_{\text{GNS}}(d,p)$ and optimizers is due to Nagy [64] for $d = 1$ and Weinstein [94] for $d \geq 2$. See also Appendix B in [90].

Proposition 3.1. *Let $d \geq 1$ and let* (i) *$p > 2$ if $d = 1, 2$ and* (ii) *$2 < p < \frac{2d}{d-2}$ if $d \geq 3$. Consider the functional*

$$J^{d,p}(u) = \frac{\|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{(p-2)d}{2}}\|u\|_{L^2(\mathbb{R}^d)}^{2+\frac{p-2}{2}(2-d)}}{\|u\|_{L^p(\mathbb{R}^d)}^p} \tag{3.2}$$

on $H^1(\mathbb{R}^d)$. Then, the minimum

$$C_{\text{GNS}}^{-1}(d,p) := \inf_{\substack{u\in H^1(\mathbb{R}^d)\\ u\neq 0}} J^{d,p}(u)$$

is attained at a function $Q \in H^1(\mathbb{R}^d)$ which is a positive, radial, and exponentially decaying solution to the following semilinear elliptic equation on $\mathbb{R}^d$:

$$(p-2)d\Delta Q + 4Q^{p-1} - (4+(p-2)(2-d))Q = 0 \tag{3.3}$$

with the minimal L^2-norm (namely, the ground state). Moreover, we have

$$C_{\text{GNS}}(d,p) = \frac{p}{2}\|Q\|_{L^2(\mathbb{R}^d)}^{2-p}. \tag{3.4}$$

See [36] for a pleasant exposition, including a proof of the uniqueness of positive solutions to (3.3), following [49].

Remark 3.2. Recall from Theorem 1.1 in [36] that any optimizer u for (3.1) is of the form $u(x) = cQ(b(x-a))$ for some $a \in \mathbb{R}^d$, $b > 0$, and $c \in \mathbb{C}\setminus\{0\}$. In particular, u is positive (up to a multiplicative constant).

The scale invariance of the minimization problem implies that these inequalities also hold on the finite domains $\mathbb{T}$ and $\mathbb{D}$ (essentially) with the same optimal constants.

Lemma 3.3. (i) *Let $p > 2$. Then, given any $m > 0$, there is a constant $C = C(m) > 0$ such that*

$$\|u\|_{L^p(\mathbb{T})}^p \le \big(C_{\text{GNS}}(1,p) + m\big)\|u'\|_{L^2(\mathbb{T})}^{\frac{p-2}{2}}\|u\|_{L^2(\mathbb{T})}^{\frac{p+2}{2}} + C(m)\|u\|_{L^2(\mathbb{T})}^p \tag{3.5}$$

for any $u \in H^1(\mathbb{T})$. We point out that there exists no $C_0 > 0$ such that the Gagliardo-Nirenberg-Sobolev inequality:

$$\|u\|_{L^p(\mathbb{T})}^p \le C_0\|u'\|_{L^2(\mathbb{T})}^{\frac{p-2}{2}}\|u\|_{L^2(\mathbb{T})}^{\frac{p+2}{2}} \tag{3.6}$$

holds for all functions in $H^1(\mathbb{T})$.

By restricting our attention to mean-zero functions belonging to

$$H_0^1(\mathbb{T}) := \big\{u \in H^1(\mathbb{T}) : \textstyle\int_{\mathbb{T}} u\,\mathrm{d}x = 0\big\},$$

the Gagliardo-Nirenberg-Sobolev inequality (3.1) *holds true on $\mathbb{T}$. Namely, we have*

$$\|u\|_{L^p(\mathbb{T})}^p \le C_{\text{GNS}}(1,p)\|u'\|_{L^2(\mathbb{T})}^{\frac{p-2}{2}}\|u\|_{L^2(\mathbb{T})}^{\frac{p+2}{2}} \tag{3.7}$$

for any function $u \in H_0^1(\mathbb{T})$. In fact, (3.7) *holds for any u belonging to*

$$H_{00}^1(\mathbb{T}) := \big\{u \in H^1(\mathbb{T}) : \operatorname{Re} u(x_1) = \operatorname{Im} u(x_2) = 0 \text{ for some } x_1, x_2 \in \mathbb{T}\big\}. \tag{3.8}$$

(ii) *Let $p > 2$. Then, we have*

$$\|u\|_{L^p(\mathbb{D})}^p \le C_{\text{GNS}}(2,p)\|\nabla u\|_{L^2(\mathbb{D})}^{p-2}\|u\|_{L^2(\mathbb{D})}^2 \tag{3.9}$$

for any $u \in H^1(\mathbb{D})$ vanishing on $\partial\mathbb{D}$.

Proof. (i) For the proof of (3.5), see Lemma 4.1 in [51]. As for the failure of the GNS inequality (3.6) for general $u \in H^1(\mathbb{T})$, we first note that (3.6) does not hold for constant functions. Moreover, given a function $u \in H^1(\mathbb{T})$, by simply considering $u + C$ for large $C \gg 1$ and observing that the left-hand side of (3.6) on $\mathbb{T}$ grows faster than the right-hand side as $C \to \infty$, we see that the inequality (3.6) on $\mathbb{T}$ does not hold for (non-constant) functions in general, unless they have mean zero. Hence, we restrict our attention to mean-zero functions.

For mean-zero functions on $\mathbb{T}$, Sobolev's inequality on $\mathbb{T}$ (see [3]) and an interpolation yield the GNS inequality (3.6) with some constant $C_0 > 0$ for any $u \in H^1_0(\mathbb{T})$. In fact, the GNS inequality (3.6) on $\mathbb{T}$ for mean-zero functions holds with $C_0 = C_{\mathrm{GNS}}(1,p)$ coming from the GNS inequality (3.1) on $\mathbb{R}$. Suppose that $u \in H^1_0(\mathbb{T})$ is a real-valued mean-zero function on $\mathbb{T}$. Then, by the continuity of u, there exists a point $x_0 \in \mathbb{T}$ such that $u(x_0) = 0$. By setting

$$v(x) = \begin{cases} u(x + x_0 - \frac{1}{2}), & \text{for } x \in [-\frac{1}{2}, \frac{1}{2}], \\ 0, & \text{for } |x| > \frac{1}{2}, \end{cases}$$

we can apply the GNS inequality (3.1) on $\mathbb{R}$ to v and conclude that the GNS inequality (3.6) on $\mathbb{T}$ holds for u with $C_0 = C_{\mathrm{GNS}}(1,p)$. Now, given complex-valued $u \in H^1_0(\mathbb{T})$, write $u = u_1 + iu_2$, where $u_1 = \operatorname{Re} u$ and $u_2 = \operatorname{Im} u$. Since u has mean zero on $\mathbb{T}$, its real and imaginary parts also have mean zero. In particular, there exists $x_j \in \mathbb{T}$ such that $u_j(x_j) = 0$, $j = 1, 2$. Hence, by the argument above, we see that for u_1 and u_2, the GNS inequality (3.6) holds with $C_0 = C_{\mathrm{GNS}}(1,p)$. We now proceed as in Step 2 of the proof of Theorem A.1 in [36]. By Hölder's inequality (in j), we have

$$\sum_{j=1}^{2} \Big(\|u_j'\|_{L^2(\mathbb{T})}^{\frac{p-2}{2}} \|u_j\|_{L^2(\mathbb{T})}^{\frac{p+2}{2}} \Big)^{\frac{2}{p}} \le \Big(\|u'\|_{L^2(\mathbb{T})}^{\frac{p-2}{2}} \|u\|_{L^2(\mathbb{T})}^{\frac{p+2}{2}} \Big)^{\frac{2}{p}}. \tag{3.10}$$

Then, by the triangle inequality, the GNS inequality (3.6) with $C_0 = C_{\mathrm{GNS}}(1,p)$ for u_j, $j = 1, 2$, and (3.10), we obtain

$$\begin{aligned} \|u\|_{L^p(\mathbb{T})}^p = \|u_1^2 + u_2^2\|_{L^{\frac{p}{2}}(\mathbb{T})}^{\frac{p}{2}} &\le \Big(\sum_{j=1}^{2} \|u_j\|_{L^p(\mathbb{T})}^2 \Big)^{\frac{p}{2}} \\ &\le C_{\mathrm{GNS}}(1,p) \|u'\|_{L^2(\mathbb{T})}^{\frac{p-2}{2}} \|u\|_{L^2(\mathbb{T})}^{\frac{p+2}{2}}. \end{aligned} \tag{3.11}$$

This proves the GNS inequality (3.7) for $u \in H^1_0(\mathbb{T})$. Note that the argument above shows that the GNS inequality (3.7) on $\mathbb{T}$ indeed holds for any u belonging to a larger class $H^1_{00}(\mathbb{T})$ defined in (3.8).

(ii) Given a function $u \in H^1(\mathbb{D})$ vanishing on $\partial\mathbb{D}$, we can extend u on $\mathbb{D}$ to $\bar{u} \in H^1(\mathbb{R}^2)$ by setting $\bar{u} \equiv 0$ on $\mathbb{R}^2 \setminus \mathbb{D}$. Then, applying (3.1) on $\mathbb{R}^2$, we obtain the sharp GNS inequality (3.9) on $\mathbb{D}$ □

We conclude this section by stating non-existence of optimizers for the Gagliardo-Nirenberg-Sobolev inequality (3.7) on $\mathbb{T}$ among mean-zero functions and for the Gagliardo-Nirenberg-Sobolev inequality (3.9) on $\mathbb{D}$ among functions in $H^1(\mathbb{D})$, vanishing on $\partial\mathbb{D}$.

Lemma 3.4. (i) *There exists no optimizer in $H^1_0(\mathbb{T})$ for the Gagliardo-Nirenberg-Sobolev inequality* (3.7) *on* $\mathbb{T}$.

(ii) *There exists no optimizer in* $H^1(\mathbb{D})$*, vanishing on* $\partial\mathbb{D}$*, for the Gagliardo-Nirenberg-Sobolev inequality* (3.9) *on* $\mathbb{D}$.

Proof. (i) Define the functional $J^{1,p}_{\mathbb{T}}(u)$ by

$$J^{1,p}_{\mathbb{T}}(u) = \frac{\|u'\|_{L^2(\mathbb{T})}^{\frac{p-2}{2}}\|u\|_{L^2(\mathbb{T})}^{\frac{p+2}{2}}}{\|u\|_{L^p(\mathbb{T})}^p}$$

as in (3.2) and consider the minimization problem over $H^1_{00}(\mathbb{T})$:

$$\inf_{\substack{u\in H^1_{00}(\mathbb{T})\\ u\neq 0}} J^{1,p}_{\mathbb{T}}(u). \tag{3.12}$$

It follows from (3.11) that this infimum is bounded below by $C_{\text{GNS}}(1,p)^{-1} > 0$. Suppose that there exists a mean-zero optimizer $u_* \in H^1_0(\mathbb{T}) \subset H^1_{00}(\mathbb{T})$ for (3.12). Then, by adapting the argument in Step 2 of the proof of Theorem A.1 in [36] (see (3.10) and (3.11) above) to the case of the one-dimensional torus $\mathbb{T}$, we see that either (i) one of $\operatorname{Re} u_*$ or $\operatorname{Im} u_*$ is identically equal to 0, or (ii) both $\operatorname{Re} u_*$ and $\operatorname{Im} u_*$ are optimizers for (3.12) and $|\operatorname{Re} u_*| = \lambda|\operatorname{Im} u_*|$ for some $\lambda > 0$. In either case, one of $\operatorname{Re} u_*$ or $\operatorname{Im} u_*$ is an optimizer for (3.12). Without loss of generality, suppose that $\operatorname{Re} u_*$ is a (non-zero) optimizer for (3.12). Note that $\operatorname{Re} u_* \in H^1_{00}(\mathbb{T})$ and that its positive and negative parts also belong to $H^1_{00}(\mathbb{T})$ (but not to $H^1_0(\mathbb{T})$). Then, by the argument in Step 2 of the proof of Theorem A.1 in [36] once again, we conclude that $\operatorname{Re} u_*$ is either non-negative or non-positive. This, however, is a contradiction to the fact that u (and hence $\operatorname{Re} u_*$) has mean zero on $\mathbb{T}$. This argument shows that there is no mean-zero optimizer for the GNS inequality (3.7) on $\mathbb{T}$.

(ii) Suppose that there exists an optimizer u in $H^1(\mathbb{D})$, vanishing on $\partial\mathbb{D}$. We can extend u on $\mathbb{D}$ to a function in $H^1(\mathbb{R}^2)$ by setting $u \equiv 0$ on $\mathbb{R}^2 \setminus \mathbb{D}$, which would be a (non-zero) non-negative optimizer for (3.1) on $\mathbb{R}^2$ with compact support, which is a contradiction (see Theorem 1.1 in [36]). □

4. Integrability below the threshold

4.1. **On the one-dimensional torus.** We first present the proof of the direct implication in Theorem 1.1 (ii). Namely, we show that the partition function $Z_{6,K}$ in (1.4) is finite, provided that $K < \|Q\|_{L^2(\mathbb{R})}$. See [51, Theorem 2.2 (b)] for the converse. All the norms are taken over the one-dimensional torus $\mathbb{T}$, unless otherwise stated.

In the following, $\mathbf{E}$ denotes an expectation with respect to the mean-zero Brownian loop u in (1.3) (in particular u has mean-zero on $\mathbb{T}$). Given $\lambda > 0$, we use the notation (1.25) and write

$$\begin{aligned} Z_{p,K} &= \mathbf{E}\Big[e^{\frac{1}{p}\int_{\mathbb{T}}|u(x)|^p\,dx}, \|u\|_{L^2} \le K\Big] \\ &= \mathbf{E}\Big[e^{\frac{1}{p}\int_{\mathbb{T}}|u(x)|^p\,dx}, \|u_{\ge 0}\|_{L^p} \le \lambda, \|u\|_{L^2} \le K\Big] \\ &\quad + \mathbf{E}\Big[e^{\frac{1}{p}\int_{\mathbb{T}}|u(x)|^p\,dx}, \|u_{\ge 0}\|_{L^p} > \lambda, \|u\|_{L^2} \le K\Big]. \end{aligned}$$

Here, we used the fact that $P_{\ge 0} = \text{Id}$ on mean-zero functions.

Fir $k \geq 1$, define E_k by

$$\begin{aligned} E_k &= \big\{ \|u_{\geq 0}\|_{L^p} > \lambda, \dots, \|u_{\geq k-1}\|_{L^p} > \lambda, \|u_{\geq k}\|_{L^p} \leq \lambda \big\} \\ &\subset \big\{ \|u_{\geq k-1}\|_{L^p} > \lambda, \|u_{\geq k}\|_{L^p} \leq \lambda \big\}. \end{aligned} \tag{4.1}$$

Note that the sets E_k's are disjoint and that $\sum_{k=1}^N \mathbf{1}_{E_k}$ increases to $\mathbf{1}_{\{\|u\|_{L^p}>\lambda\}}$ almost surely as $N \to \infty$ since u in (1.3) belongs almost surely to $L^p(\mathbb{T})$ for any finite p. Hence, by the monotone convergence theorem, we obtain

$$\begin{aligned} Z_{p,K} &= \mathbf{E}\Big[e^{\frac{1}{p}\int_\mathbb{T} |u(x)|^p \, \mathrm{d}x}, \|u\|_{L^p} \leq \lambda, \, \|u\|_{L^2} \leq K \Big] \\ &\quad + \sum_{k=1}^\infty \mathbf{E}\Big[e^{\frac{1}{p}\int_\mathbb{T} |u(x)|^p \, \mathrm{d}x}, E_k, \, \|u\|_{L^2} \leq K \Big]. \end{aligned} \tag{4.2}$$

The first term on the right-hand side of (4.2) is clearly finite for any finite $\lambda, K > 0$. Hence, in view of (4.1), it suffices to show that

$$\mathbf{E}\Big[e^{\frac{1}{p}\int_\mathbb{T} |u(x)|^p \, \mathrm{d}x}, \|u_{\geq k-1}\|_{L^p} > \lambda, \|u_{\geq k}\|_{L^p} \leq \lambda, \, \|u\|_{L^2} \leq K \Big] \tag{4.3}$$

is summable in $k \in \mathbb{N}$.

Given an integer p, we have

$$|u|^p = |u_{\leq k-1} + u_{\geq k}|^p \leq \sum_{\ell=0}^p \binom{p}{\ell} |u_{\leq k-1}|^{p-\ell} |u_{\geq k}|^\ell.$$

Integrating and applying Hölder's inequality followed by Young's inequality, we have, for any u satisfying $\|u_{\geq k}\|_{L^p} \leq \lambda$,

$$\begin{aligned} \int_\mathbb{T} |u|^p \, \mathrm{d}x &\leq \int_\mathbb{T} |u_{\leq k-1}|^p \, \mathrm{d}x + \sum_{\ell=1}^p \binom{p}{\ell} \|u_{\leq k-1}\|_{L^p}^{p-\ell} \lambda^\ell \\ &\leq \int_\mathbb{T} |u_{\leq k-1}|^p \, \mathrm{d}x + \sum_{\ell=1}^p \binom{p}{\ell} \Big(\frac{p-\ell}{p} \varepsilon \|u_{\leq k-1}\|_{L^p}^p + \frac{\ell}{p} \varepsilon^{-\frac{p-\ell}{\ell}} \lambda^p \Big) \\ &\leq \big(1 + (2^p - 1)\varepsilon\big) \int_\mathbb{T} |u_{\leq k-1}|^p \, \mathrm{d}x + C_p(\varepsilon) \lambda^p \end{aligned} \tag{4.4}$$

for some small $\varepsilon > 0$ (to be chosen later), where in the last step we used

$$\sum_{\ell=1}^p \frac{p-\ell}{p} \binom{p}{\ell} \leq \sum_{\ell=1}^p \binom{p}{\ell} \leq 2^p - 1.$$

Hence, by letting

$$\delta = \delta(p, \varepsilon) = (2^p - 1)\varepsilon, \tag{4.5}$$

the quantity in (4.3) is bounded by

$$e^{C_p(\varepsilon)\lambda^p} \mathbf{E}\Big[e^{\frac{(1+\delta)}{p}\int_\mathbb{T} |u_{\leq k-1}(x)|^p \, \mathrm{d}x}, \|u_{\geq k-1}\|_{L^p} > \lambda, \|u\|_{L^2} \leq K \Big]. \tag{4.6}$$

Now, let $\lambda = 1$ and $p = 6$. By Lemma 3.3 (i) for some small $m > 0$ (to be chosen later) with $\|u\|_{L^2} \leq K$, there is a constant $C(m) > 0$ such that (4.6) is now bounded by

$$e^{C_6(\varepsilon) + C(m)K^6} \mathbf{E}\Big[e^{\frac{(C_{\mathrm{GNS}}(1,6)+m)K^4(1+\delta)}{6}\int_\mathbb{T} |u'_{\leq k-1}(x)|^2 \, \mathrm{d}x}, \|u_{\geq k-1}\|_{L^6} > 1 \Big]$$

By Hölder's inequality,

$$\begin{aligned}&\le e^{C_6(\varepsilon)+C(m)K^6}\bigg\{\mathbf{E}\Big[e^{\frac{(C_{\mathrm{GNS}}(1,6)+m)K^4(1+\eta)(1+\delta)}{6}\int_{\mathbb{T}}|u'_{\le k-1}(x)|^2\,\mathrm{d}x}\Big]\bigg\}^{\frac{1}{1+\eta}}\\ &\quad\times\Big\{\mathbf{P}\big(\|u_{\ge k-1}\|_{L^6}>1\big)\Big\}^{\frac{\eta}{1+\eta}}.\end{aligned}\tag{4.7}$$

By (2.8), we have

$$\Big\{\mathbf{P}\big(\|u_{\ge k-1}\|_{L^6}>1\big)\Big\}^{\frac{\eta}{1+\eta}}\le C^{\frac{\eta}{1+\eta}}\exp\bigg(-C\frac{\eta}{1+\eta}2^{\frac{4}{3}k}\bigg).\tag{4.8}$$

Recall that

$$\mathbf{E}[e^{tX^2}]=(1-2t)^{-\frac{1}{2}}\tag{4.9}$$

for $X\sim\mathcal{N}_{\mathbb{R}}(0,1)$ and $t<\frac{1}{2}$. Then, using (4.9) and (3.4) in Proposition 3.1, the expectation in (4.7) is bounded by

$$\begin{aligned}&\mathbf{E}\Big[e^{\frac{(C_{\mathrm{GNS}}(1,6)+m)K^4(1+\eta)(1+\delta)}{6}\int_{\mathbb{T}}|u'_{\le k-1}(x)|^2\,\mathrm{d}x}\Big]\\ &\quad=\prod_{1\le|n|\le 2^{k-1}}\mathbf{E}\Big[e^{\frac{(C_{\mathrm{GNS}}(1,6)+m)K^4(1+\eta)(1+\delta)}{6}|g_n|^2}\Big]\\ &\quad\le\bigg(1-2\frac{(C_{\mathrm{GNS}}(1,6)+m)K^4(1+\eta)(1+\delta)}{6}\bigg)^{-2^k}\\ &\quad=\bigg(1-\frac{K^4}{\|Q\|^4_{L^2(\mathbb{R})}}\Big(1+\frac{m}{3}\|Q\|^4_{L^2(\mathbb{R})}\Big)(1+\eta)(1+\delta)\bigg)^{-2^k}.\end{aligned}\tag{4.10}$$

Note that, under $K<\|Q\|_{L^2(\mathbb{R})}$ and (4.5), we can choose $m,\varepsilon,\eta>0$ sufficiently small such that

$$\frac{K^4}{\|Q\|^4_{L^2(\mathbb{R})}}\Big(1+\frac{m}{3}\|Q\|^4_{L^2(\mathbb{R})}\Big)(1+\eta)(1+\delta)<c<1,$$

guaranteeing the application of (4.9) in the computation above.

Finally, summing (4.3) over $k\in\mathbb{N}$ with (4.6), (4.7), (4.8), and (4.10), we have

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathbf{E}\Big[e^{\frac{1}{6}\int_{\mathbb{T}}|u(x)|^6\,\mathrm{d}x},\|u_{\ge k-1}\|_{L^6}>\lambda,\|u_{\ge k}\|_{L^6}\le\lambda,\ \|u\|_{L^2}\le K\Big]\\ &\quad\le e^{C(\varepsilon,m,K)}C^{\frac{\eta}{1+\eta}}\sum_{k=1}^{\infty}\exp\bigg(-C\frac{\eta}{1+\eta}2^{\frac{4}{3}k}\bigg)\exp\bigg(\frac{2^k}{1+\eta}\log\frac{1}{1-c}\bigg)\\ &\quad<\infty.\end{aligned}\tag{4.11}$$

Therefore, we conclude that the partition function $Z_{6,K}$ is finite for any $K<\|Q\|_{L^2(\mathbb{R})}$. This completes the proof of Theorem 1.1.

Remark 4.1. As mentioned in Remark 1.2, Theorem 1.1 (ii) holds for the Ornstein-Uhlenbeck loop in (1.7). We first note that (2.8) also holds for the Ornstein-Uhlenbeck loop u since (2.5) holds even if we replace n^2 by $(2\pi)^{-2} + n^2$. Hence, from (1.7) and (4.8), we have

$$
\begin{aligned}
(4.7) &\le e^{C_6(\varepsilon)+C(m)K^6} \bigg\{ \mathbf{E}_\mu \Big[e^{\frac{(C_{\text{GNS}}(1,6)+m)K^4(1+\eta)(1+\delta)}{6} \int_{\mathbb{T}} |\langle \partial_x \rangle u_{\le k-1}(x)|^2 \, \mathrm{d}x} \Big] \bigg\}^{\frac{1}{1+\eta}} \\
&\quad \times \Big\{ \mathbf{P}\big(\|u_{\ge k-1}\|_{L^6} > 1 \big) \Big\}^{\frac{\eta}{1+\eta}} \\
&\le C^{\frac{\eta}{1+\eta}} \exp\bigg(-C \frac{\eta}{1+\eta} 2^{\frac{4}{3}k} \bigg) \prod_{0 \le |n| \le 2^{k-1}} \mathbf{E}\Big[e^{\frac{(C_{\text{GNS}}(1,6)+m)K^4(1+\eta)(1+\delta)}{6} |g_n|^2} \Big],
\end{aligned}
$$

where $\mathbf{E}_\mu$ denotes an expectation with respect to the law of the Ornstein-Uhlenbeck loop u and $\langle \partial_x \rangle = \sqrt{1 - \partial_x^2}$. The rest of the argument follows as above, thus establishing Theorem 1.1 (ii) for the Ornstein-Uhlenbeck loop u in (1.7).

4.2. **On the two-dimensional disc.** Next, we prove the normalizability of the Gibbs measure on $\mathbb{D}$ stated in Theorem 1.3 (ii). Namely, we show that the partition function $\widetilde{Z}_{4,K}$ in (1.11) is finite, provided that $K < \|Q\|_{L^2(\mathbb{R}^2)}$. The proof is based on a computation analogous to that in Subsection 4.1. As we see below, however, we need to proceed with more care, partially due to the eigenfunction estimate, which makes the computation barely work on $\mathbb{D}$; compare (4.11) and (4.23).

We first recall the following simple corollary of Fernique's theorem [35]. See also Theorem 2.7 in [25].[7]

Lemma 4.2. *There exists a constant $c > 0$ such that if X is a mean-zero Gaussian process with values in a separable Banach space B with $\mathbf{E}\big[\|X\|_B\big] < \infty$, then*

$$
\int e^{c \frac{\|X\|_B^2}{(\mathbf{E}[\|X\|_B])^2}} \, \mathrm{d}\mathbf{P} < \infty. \tag{4.12}
$$

In particular, we have

$$
\mathbf{P}\Big(\|X\|_B \ge t \mathbf{E}\big[\|X\|_B\big] \Big) \le e^{-ct^2} \tag{4.13}
$$

for any $t > 1$.

Recall from [92, Lemmas 2.1 and 2.2] the asymptotic formula for the eigenvalue z_n:

$$
z_n = \pi \bigg(n - \frac{1}{4} \bigg) + O\bigg(\frac{1}{n} \bigg) \tag{4.14}
$$

and the eigenfunction estimate:

$$
\|e_n\|_{L^4(\mathbb{D})} \lesssim \big[\log(2+n) \big]^{\frac{1}{4}}. \tag{4.15}
$$

As in Section 2, we define the spectral projection of v by

$$
v_k := \sum_{2^{k-1} < n \le 2^k} \widehat{v}(n) e_n
$$

[7] In the context of Theorem 2.7 on [25], we set $x = \frac{X}{\mathbf{E}[\|X\|_B]}$. Then, by Markov's inequality and choosing $r \gg 1$, we have

$$
\log\bigg(\frac{\mu(\|x\|_B > r)}{\mu(\|x\|_B \le r)} \bigg) = \log\bigg(\frac{\mu\big(\|X\|_B > r\mathbf{E}[\|X\|_B]\big)}{1 - \mu\big(\|X\|_B > r\mathbf{E}[\|X\|_B]\big)} \bigg) \le \log \frac{1}{r-1} \le -2
$$

without using any fine property of X. Then, (4.12) and (4.13) follow in view of Remark 2.8 in [25].

for $k \in \mathbb{Z}_{\geq 0}$, where $\widehat{v}(n) = \int_{\mathbb{D}} v e_n \mathrm{d}x$. We also define $v_{\leq k}$ and $v_{\geq k}$ in an analogous manner.

In the following, $\mathbf{E}$ denotes an expectation with respect to the random Fourier series v in (1.10) and all the norms are taken over the unit disc $\mathbb{D} \subset \mathbb{R}^2$ unless otherwise stated. By Minkowski's integral inequality with (4.14) and (4.15), we have

$$\begin{aligned} \mathbf{E}\big[\|v_j\|_{L^4}\big] &\lesssim \bigg\| \bigg(\sum_{2^{j-1} < n \leq 2^j} \frac{1}{{z_n}^2} e_n^2 \bigg)^{\frac{1}{2}} \bigg\|_{L^4} \\ &\leq \bigg(\sum_{2^{j-1} < n \leq 2^j} \frac{1}{{z_n}^2} \|e_n\|_{L^4}^2 \bigg)^{\frac{1}{2}} \lesssim \langle j \rangle^{\frac{1}{4}} 2^{-\frac{1}{2}j} \end{aligned} \tag{4.16}$$

for any $j \in \mathbb{Z}_{\geq 0}$. Then, applying Lemma 4.2 and (4.16) with suitable $\varepsilon_j \sim \langle j \rangle^{-2}$ such that $\sum_{j \geq k} \varepsilon_j \leq 1$, we have

$$\begin{aligned} \mathbf{P}\big(\|v_{\geq k}\|_{L^4} \geq \lambda\big) &\leq \sum_{j \geq k} \mathbf{P}\big(\|v_j\|_{L^4} \geq \varepsilon_j \lambda\big) \\ &= \sum_{j \geq k} \mathbf{P}\bigg(\|v_j\|_{L^4} \geq \frac{\varepsilon_j \lambda}{\mathbf{E}\big[\|v_j\|_{L^4}\big]} \mathbf{E}\big[\|v_j\|_{L^4}\big] \bigg) \\ &\leq \sum_{j \geq k} e^{-c \varepsilon_j^2 \lambda^2 \langle j \rangle^{-\frac{1}{2}} 2^j} \\ &\leq C e^{-c' \lambda^2 \langle k \rangle^{-\frac{9}{2}} 2^k} \end{aligned} \tag{4.17}$$

for some constant $C > 0$, uniformly in $k \in \mathbb{Z}_{\geq 0}$ and $\lambda \geq 1$. Then, it follows from the Borel-Cantelli lemma that

$$\limsup_{k \to \infty} \frac{\|v_{\geq k}\|_{L^4}}{\langle k \rangle^3} = 0 \tag{4.18}$$

with probability 1.

Define a set F_k by

$$F_k = \big\{ \|v_{\geq j}\|_{L^4} > \langle j \rangle^3, \ j = 0, 1, \dots, k-1, \quad \text{and} \quad \|v_{\geq k}\|_{L^p} \leq \langle k \rangle^3 \big\}.$$

By definition, F_k's are disjoint and, from (4.18), we have

$$\mathbf{P}\bigg(\bigcup_{k=1}^{\infty} F_k \bigg) = 1.$$

This implies that $\sum_{k=1}^N \mathbf{1}_{F_k}$ increases to $\mathbf{1}_{\{\|v\|_{L^4} > 1\}}$ almost surely as $N \to \infty$.

Starting from $\widetilde{Z}_{4,K}$ in (1.11), we reproduce the computations in (4.2), (4.4), and (4.6) with $p = 4$ and $\lambda = 1$ by replacing u in (1.3), the sets E_k, and the integrals over $[0, 1]$ with

v in (1.10), the sets F_k, and integrals over $\mathbb{D}$, respectively. We find

$$\begin{aligned}\widetilde{Z}_{4,K} &\le \mathbf{E}\Big[e^{\frac{1}{4}\int_{\mathbb{D}}|v(x)|^4\,dx}, \|v\|_{L^4}\le 1,\ \|v\|_{L^2}\le K\Big]\\&\quad+\sum_{k=1}^\infty \mathbf{E}\Big[e^{\frac{1}{4}\int_{\mathbb{D}}|v(x)|^4\,dx}, \|v_{\ge k-1}\|_{L^4}>\langle k-1\rangle^3,\\&\qquad\qquad \|v_{\ge k}\|_{L^4}\le \langle k\rangle^3, \|v\|_{L^2}\le K\Big]\\&\le C+\sum_{k=1}^\infty e^{C_4(\varepsilon)\langle k\rangle^{12}}\mathbf{E}\Big[e^{\frac{(1+\delta)}{4}\int_{\mathbb{D}}|v_{\le k-1}(x)|^4\,dx},\\&\qquad\qquad \|v_{\ge k-1}\|_{L^4}>\langle k-1\rangle^3, \|v\|_{L^2}\le K\Big],\end{aligned}\tag{4.19}$$

where $\delta=\delta(4)=48\varepsilon$ is as in (4.5) with $p=4$. As before, it remains to show that the series in (4.19) is convergent.

From Lemma 3.3 (ii), we have

$$\|v\|_{L^4(\mathbb{D})}^4 \le C_{\text{GNS}}(2,4)\|v\|_{L^2(\mathbb{D})}^2\|\nabla v\|_{L^2(\mathbb{D})}^2$$

for any $v\in H_0^1(\mathbb{D})$ with

$$C_{\text{GNS}}(2,4)=2\|Q\|_{L^2(\mathbb{R}^2)}^{-2},$$

where Q is the optimizer for the Gagliardo-Nirenberg-Sobolev inequality (3.1) on $\mathbb{R}^2$. Then, by recalling

$$\begin{aligned}\int_{\mathbb{D}}|\nabla v_{\le k-1}(x)|^2\,dx &= -\int_{\mathbb{D}} v_{\le k-1}(x)\overline{\Delta v_{\le k-1}(x)}\,dx\\&= -\sum_{n\le 2^{k-1}}\frac{|g_n|^2}{z_n^2}\int_0^1 e_n(r)\Delta_r e_n(r) r\,dr\\&= \sum_{n\le 2^{k-1}}|g_n|^2.\end{aligned}$$

and applying Hölder's inequality, the expectation in the summands on the right-hand side of (4.19) is bounded by

$$\begin{aligned}&\mathbf{E}\Big[e^{\frac{C_{\text{GNS}}(2,4)K^2(1+\delta)}{4}\int_{\mathbb{D}}|\nabla v_{\le k-1}(x)|^2\,dx}, \|v_{\ge k-1}\|_{L^4}>\langle k-1\rangle^3\Big]\\&\quad= \mathbf{E}\Big[e^{\frac{C_{\text{GNS}}(2,4)K^2(1+\delta)}{4}\sum_{n\le 2^{k-1}}|g_n|^2}, \|v_{\ge k-1}\|_{L^4}>\langle k-1\rangle^3\Big]\\&\quad\le \Big\{\mathbf{E}\Big[e^{\frac{C_{\text{GNS}}(2,4)K^2(1+\eta)(1+\delta)}{4}\sum_{n\le 2^{k-1}}|g_n|^2}\Big]\Big\}^{\frac{1}{1+\eta}}\Big\{\mathbf{P}\big(\|v_{\ge k-1}\|_{L^4}>\langle k-1\rangle^3\big)\Big\}^{\frac{\eta}{1+\eta}}.\end{aligned}\tag{4.20}$$

The first factor on the right-hand side of (4.20) can be computed exactly as in (4.10), using (4.9). Namely, provided that

$$\begin{aligned}C(K,\delta,\eta) &:= C_{\text{GNS}}(2,4)\frac{K^2}{2}(1+\eta)(1+\delta)\\&= \frac{K^2}{\|Q\|_{L^2(\mathbb{R}^2)}^2}(1+\eta)(1+\delta)<1,\end{aligned}\tag{4.21}$$

it is finite and equals

$$\left(1-\frac{K^2}{\|Q\|_{L^2(\mathbb{R}^2)}^2}(1+\eta)(1+\delta)\right)^{-2^{k-1}}. \tag{4.22}$$

Given $K < \|Q\|_{L^2(\mathbb{R}^2)}$, we choose δ and η sufficiently small such that (4.21) holds. Then, from (4.20) with (4.17) and (4.22), we conclude that $\widetilde{Z}_{4,K}$ in (4.19) is bounded by

$$\begin{aligned}\widetilde{Z}_{4,K} &\le C + \sum_{k=1}^{\infty} e^{C_4(\varepsilon)\langle k\rangle^{12}} \\ &\qquad \times \exp\left(\frac{2^{k-1}}{1+\eta}\log\frac{1}{1-C(K,\delta,\eta)}\right)\exp\left(-\frac{c''\eta}{1+\eta}\langle k\rangle^{\frac{3}{2}}2^k\right) \\ &< \infty.\end{aligned} \tag{4.23}$$

This proves the normalizability of the Gibbs measure on $\mathbb{D}$ claimed in Theorem 1.3 (ii).

5. Non-integrability above the threshold on the disc

In this section, we discuss the non-normalizability of the Gibbs measure on $\mathbb{D}$ stated in Theorem 1.3 (ii). Namely, when $K > \|Q\|_{L^2(\mathbb{R}^2)}$, we prove

$$\widetilde{Z}_{4,K} = \mathbf{E}\Big[e^{\frac{1}{4}\int_{\mathbb{D}}|v|^4\,\mathrm{d}x}\mathbf{1}_{\{\|v\|_{L^2(\mathbb{D})}\le K\}}\Big] = \infty. \tag{5.1}$$

Here, Q is the ground state on $\mathbb{R}^2$ as in Proposition 3.1. Since the proof of (5.1) is essentially identical to that for the non-normalizability of the Gibbs measure on the torus $\mathbb{T}$ (see [51, Theorem 2.2 (b)]), we keep our presentation brief.

Let $K_0 = \|Q\|_{L^2(\mathbb{R}^2)}$ and fix $K > K_0$. Choose $\alpha > 1$ such that

$$\|\alpha Q\|_{L^2(\mathbb{R}^2)} < K. \tag{5.2}$$

Then, by setting

$$H_{\mathbb{R}^2}(v) := \frac{1}{2}\int_{\mathbb{R}^2}|\nabla v|^2\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^2}|v|^4\mathrm{d}x, \tag{5.3}$$

it follows from Proposition 3.1 (in particular (3.4)) that $H_{\mathbb{R}^2}(Q) = 0$. As a result, we have

$$H_{\mathbb{R}^2}(\alpha Q) < 0.$$

Given $\rho > 0$, define the L^2-invariant scaling operator D_λ on $\mathbb{R}^2$ by setting

$$D_\lambda(f) = \lambda^{-1}f(\lambda^{-1}x).$$

Then, we have

$$\begin{aligned}\|D_{\rho^{-1}}(f)\|_{L^2(\mathbb{R}^2)}^2 &= \|f\|_{L^2(\mathbb{R}^2)}^2, \\ \|D_{\rho^{-1}}(f)\|_{L^4(\mathbb{R}^2)}^4 &= \rho^2\|f\|_{L^4(\mathbb{R}^2)}^4, \\ \|\nabla D_{\rho^{-1}}(f)\|_{L^2(\mathbb{R}^2)}^2 &= \rho^2\|\nabla f\|_{L^2(\mathbb{R}^2)}^2, \\ \|\partial_r^2 D_{\rho^{-1}}(f)\|_{L^1(\mathbb{R}^2)} &= \rho\|\partial_r^2 f\|_{L^1(\mathbb{R}^2)},\end{aligned} \tag{5.4}$$

where we assume f to be radial for the last identity and ∂_r denotes the directional derivative in the radial direction.

For each $\rho \gg 1$, we define $Q_{\alpha,\rho} \in C^\infty_{\text{rad}}(\mathbb{D})$ by setting

$$Q_{\alpha,\rho}(x) = \alpha D_{\rho^{-1}}(Q)(x) = \alpha\rho Q(\rho x)$$

for $|x| \le 1 - \frac{1}{\rho}$ and $Q_{\alpha,\rho} \equiv 0$ on $\partial\mathbb{D}$. Then, thanks to the exponential decay of the ground state Q on $\mathbb{R}^2$, it follows from (5.2), (5.3), and (5.4) that

$$\begin{aligned} H_{\mathbb{D}}(Q_{\alpha,\rho}) := \frac{1}{2}\int_{\mathbb{D}} |\nabla Q_{\alpha,\rho}|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{D}} |Q_{\alpha,\rho}|^4 \mathrm{d}x \le -A_1\rho^2 < 0,\\ \|Q_{\alpha,\rho}\|^2_{L^2(\mathbb{D})} < K^2 - \eta_0,\\ \|Q_{\alpha,\rho}\|^4_{L^4(\mathbb{D})} \le A_2\rho^2,\\ \|\partial_r^2 Q_{\alpha,\rho}\|_{L^1(\mathbb{D})} \le A_3\rho \end{aligned} \tag{5.5}$$

for some $A_1, A_2, A_3 > 0$ and sufficiently small $\eta_0 > 0$, uniformly in large $\rho \gg 1$.

We need the following simple calculus lemma which follows from a straightforward computation. See Lemma 4.2 in [51].

Lemma 5.1. *Given $\eta > 0$, there exists small $\varepsilon > 0$ such that for any $v_1, v_2 \in C_0(\mathbb{D})$ satisfying $\|v_1 - v_2\|_{L^\infty(\mathbb{D})} < \varepsilon$, we have*

$$\Big| \|v_1\|^p_{L^p(\mathbb{D})} - \|v_2\|^p_{L^p(\mathbb{D})} \Big| < \eta\big(\|v_1\|^p_{L^p(\mathbb{D})} + 1\big)$$

for $p = 2, 4$. Here, $C_0(\mathbb{D})$ denotes the collection of continuous complex-valued functions on $\mathbb{D}$, vanishing on the boundary $\partial\mathbb{D}$.

Given $\varepsilon > 0$, let $B_\varepsilon(Q_{\alpha,\rho}) \subset C_0(\mathbb{D})$ be the ball of radius ε centered at $Q_{\alpha,\rho}$:

$$B_\varepsilon(Q_{\alpha,\rho}) = \big\{ v \in C_0(\mathbb{D}) : \|v - Q_{\alpha,\rho}\|_{L^\infty(\mathbb{D})} < \varepsilon \big\}.$$

Let $\eta \le \frac{\eta_0}{K^2 + 1 - \eta_0}$. Then, from Lemma 5.1 and (5.5), there exists small $\varepsilon > 0$ such that

$$\begin{aligned} \widetilde{Z}_{4,K} &= \int_{C_{\text{rad},0}(\mathbb{D})} e^{\frac{1}{4}\int_{\mathbb{D}} |v(x)|^4 \,\mathrm{d}x} \mathbf{1}_{\{\|v\|_{L^2(\mathbb{D})} \le K\}} \mathbf{P}(\mathrm{d}v)\\ &\ge \int_{B_\varepsilon(Q_{\alpha,\rho})} e^{\frac{1}{4}\int_{\mathbb{D}} |v(x)|^4 \,\mathrm{d}x}\, \mathbf{P}(\mathrm{d}v)\\ &\ge \exp\bigg(\frac{1-\eta}{4} \int_{\mathbb{D}} |Q_{\alpha,\rho}(x)|^4 \,\mathrm{d}x - \frac{\eta}{4} \bigg) \mathbf{P}\big(B_\varepsilon(Q_{\alpha,\rho})\big), \end{aligned} \tag{5.6}$$

where $C_{\text{rad},0}(\mathbb{D}) = \big\{ v \in C_0(\mathbb{D}) : v, \text{ radial} \big\}$ and $\mathbf{P}$ denotes the Gaussian probability measure with respect to the random series (1.10).

Since $Q_{\alpha,\rho} \in C^\infty_{\text{rad},0}(\mathbb{D}) \subset H^1_{\text{rad, }0}(\mathbb{D}) = \big\{ v \in H^1_0(\mathbb{D}) : v, \text{ radial} \big\}$, we can apply the Cameron-Martin theorem (Theorem 2.8 in [23]) and obtain

$$\begin{aligned} \mathbf{P}\big(B_\varepsilon(Q_{\alpha,\rho})\big) &= e^{-\frac{1}{2}\int_{\mathbb{D}} |\nabla Q_{\alpha,\rho}(x)|^2 \mathrm{d}x} \int_{B_\varepsilon(0)} e^{\operatorname{Re}\int_{\mathbb{D}} v\Delta Q_{\alpha,\rho} \mathrm{d}x}\, \mathbf{P}(\mathrm{d}v)\\ &\ge e^{-\frac{1}{2}\int_{\mathbb{D}} |\nabla Q_{\alpha,\rho}(x)|^2 \mathrm{d}x} e^{-\varepsilon \|\partial_r^2 Q_{\alpha,\rho}\|_{L^1(\mathbb{D})}} \mathbf{P}(B_\varepsilon(0))\\ &\ge e^{-\frac{1}{2}\int_{\mathbb{D}} |\nabla Q_{\alpha,\rho}(x)|^2 \mathrm{d}x} e^{-\varepsilon A_3 \rho} \mathbf{P}\big(B_\varepsilon(0)\big), \end{aligned} \tag{5.7}$$

where we used (5.5) in the last step.

Hence, by further imposing $\eta \le \frac{2A_1}{A_2}$, we conclude from (5.6) and (5.7) with $\mathbf{P}\big(B_\varepsilon(0)\big) = C_\varepsilon > 0$ and (5.5) that

$$\begin{aligned}\widetilde{Z}_{4,K} &\ge C_\varepsilon \exp\bigg(- H_{\mathbb{D}}(Q_{\alpha,\rho}) - \frac{\eta}{4}\|Q_{\alpha,\rho}\|^4_{L^4(\mathbb{D})} - \varepsilon A_3\rho - \frac{\eta}{4}\bigg)\\ &\ge C_\varepsilon \exp\bigg(A_1\rho^2 - \frac{\eta}{4}A_2\rho^2 - \varepsilon A_3\rho - \frac{\eta}{4}\bigg)\\ &\longrightarrow \infty,\end{aligned}$$

as $\rho \to \infty$. This proves the non-normalizability of the Gibbs measure on $\mathbb{D} \subset \mathbb{R}^2$, when $K > \|Q\|_{L^2(\mathbb{R}^2)}$.

6. Integrability at the optimal mass threshold

In this section, we present the proof of Theorem 1.4. Namely, we show that the $Z_{6,K} < \infty$ when

$$K = \|Q\|_{L^2(\mathbb{R})}.$$

We fix $d = 1$, $p = 6$, and $C_{\text{GNS}} = C_{\text{GNS}}(1, 6)$. In this section, we prove the normalizability when u is the Ornstein-Uhlenbeck loop in (1.7):

$$u(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle} e^{2\pi i n x}, \tag{6.0.1}$$

where $\langle n\rangle = (1+4\pi^2|n|^2)^{\frac{1}{2}}$. Namely, expectations are taken with respect to the law μ of the Ornstein-Uhlenbeck loop in (6.0.1). In this non-homogeneous setting, the problem is not scaling invariant and some extra care is needed. See for example the proofs of Lemma 6.3 and Proposition 6.4. In Remark 6.20, we indicate the necessary modifications for handling the case of the mean-zero Brownian loop in (1.3).

6.1. Rescaled and translated ground state.

In the following, we compare a function u on the circle $\mathbb{T}$ to translations and rescalings of the ground state Q. For this purpose, we introduce the L^2-invariant scaling operator D_λ on $\mathbb{R}$ by

$$D_\lambda f(x) = \lambda^{-\frac{1}{2}} f(\lambda^{-1}x). \tag{6.1.1}$$

Then, given $\delta > 0$, we set

$$Q_\delta(x) = D_\delta Q(x) = \delta^{-\frac{1}{2}} Q(\delta^{-1}x).$$

While the ground state Q is defined on $\mathbb{R}$, we now interpret it as a function of $x \in [-\frac{1}{2}, \frac{1}{2})$ which we identify with the torus $\mathbb{T} \cong [-\frac{1}{2}, \frac{1}{2})$. Then, it follows from (3.3) that

$$\partial_x^2 Q_\delta + Q_\delta^5 - 2\delta^{-2}Q_\delta = 0 \tag{6.1.2}$$

for $x \in (-\frac{1}{2}, \frac{1}{2})$. As a consequence, we have

$$\langle \partial_x^2 Q_\delta, v\rangle_{L^2(\mathbb{T})} = -\langle Q_\delta^5, v\rangle_{L^2(\mathbb{T})} + 2\delta^{-2}\langle Q_\delta, v\rangle_{L^2(\mathbb{T})}. \tag{6.1.3}$$

Given $x_0 \in \mathbb{T}$, we also introduce the translation operator τ_{x_0} defined by

$$\tau_{x_0} f(x) = f(x - x_0), \tag{6.1.4}$$

where $x - x_0$ is interpreted mod 1, taking values in $[-\frac{1}{2}, \frac{1}{2})$. The rescaled and translated version of the ground state:

$$Q_{\delta,x_0}(x) := \tau_{x_0} D_\delta Q(x) = \delta^{-\frac{1}{2}} Q(\delta^{-1}(x - x_0)) \tag{6.1.5}$$

plays an important role in our analysis. By this definition, we have $Q_\delta = Q_{\delta,0}$. In the following, we use ∂_δ and ∂_{x_0} to denote differentiations with respect to the scaling and translation parameters, respectively. When there is no confusion, we also denote by D_λ and τ_{x_0} the scaling and translation operators for functions on the real line.

When restricted to the torus, the rescaled and translated version of the ground state belongs to $H^1(\mathbb{T})$, but not to $H^2(\mathbb{T})$ (nor to $H^k(\mathbb{T})$ for any higher k). This is due to the fact that Q'_δ on $[-\frac{1}{2}, \frac{1}{2})$, truncated at $x = \pm\frac{1}{2}$, does not respect the periodic boundary condition. In order to overcome this problem, we introduce an even cutoff function $\rho \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\operatorname{supp}(\rho) \subset \big[-\tfrac{1}{4}, \tfrac{1}{4}\big] \qquad \text{and} \qquad \rho \equiv 1 \text{ on } \big[-\tfrac{1}{8}, \tfrac{1}{8}\big], \tag{6.1.6}$$

and consider ρQ_δ, which is smooth at $x = \pm\frac{1}{2}$. In the following, we will view

$$Q_\delta^\rho := \rho Q_\delta = \rho D_\delta Q$$

as a function on the torus. By setting

$$Q_{\delta,x_0}^\rho := \tau_{x_0}(\rho Q_\delta) = \rho(x - x_0)\delta^{-\frac{1}{2}} Q(\delta^{-1}(x - x_0)), \tag{6.1.7}$$

we will also view Q_{δ,x_0}^ρ as a function on the torus.

Note that since Q decays exponentially as $|x| \to \infty$ on $\mathbb{R}$, we have

$$\begin{aligned} \|Q_{\delta,x_0}^\rho\|_{L^2(\mathbb{T})}^2 &= \delta^{-1} \int_{\mathbb{R}} \rho^2(x) Q^2(\delta^{-1}x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}} \rho^2(\delta x) Q^2(x)\, \mathrm{d}x \\ &= \|Q\|_{L^2(\mathbb{R})}^2 + O(\exp(-c\delta^{-1})) \end{aligned} \tag{6.1.8}$$

as $\delta \to 0$.

Remark 6.1. From (6.1.5), we have

$$\partial_\delta Q_\delta(x) = -\tfrac{1}{2}\delta^{-\frac{3}{2}} Q(\delta^{-1}x) - \delta^{-\frac{3}{2}}(\delta^{-1}x) Q'(\delta^{-1}x), \tag{6.1.9}$$

which is an even function. Similarly, $\partial_\delta Q_\delta^\rho = \partial_\delta(\rho Q_\delta)$ is an even function. On the other hand, we have

$$\partial_{x_0} Q_\delta(x) = \partial_{x_0} Q_{\delta,x_0}(x)|_{x_0=0} = -\delta^{-\frac{3}{2}} Q'(\delta^{-1}x), \tag{6.1.10}$$

which is an odd function. Similarly,

$$\partial_{x_0} Q_\delta^\rho = \partial_{x_0} Q_{\delta,x_0}^\rho|_{x_0=0} = \partial_{x_0}(\tau_{x_0}(\rho Q_\delta))|_{x_0=0} \tag{6.1.11}$$

is also an odd function. By writing

$$\langle \partial_\delta Q_\delta^\rho, \partial_{x_0} Q_\delta^\rho \rangle_{H^k(\mathbb{T})} = \langle \partial_\delta Q_\delta^\rho, (1 - \partial_x^2)^k \partial_{x_0} Q_\delta^\rho \rangle_{L^2(\mathbb{T})},$$

we see that $\partial_\delta Q_\delta^\rho$ and $\partial_{x_0} Q_\delta^\rho$ are orthogonal in $H^k(\mathbb{T})$, $k \in \mathbb{Z}_{\geq 0}$, since $(1 - \partial_x^2)\partial_{x_0} Q_\delta^\rho$ is an odd function.

Similarly, for given $x_0 \in \mathbb{T}$, we have

$$\begin{aligned} \partial_\delta Q_{\delta,x_0}(x) &= -\tfrac{1}{2}\delta^{-\frac{3}{2}}Q(\delta^{-1}(x-x_0)) - \delta^{-\frac{3}{2}}(\delta^{-1}(x-x_0))Q'(\delta^{-1}(x-x_0)),\\ \partial_{x_0} Q_{\delta,x_0}(x) &= -\delta^{-\frac{3}{2}}Q'(\delta^{-1}(x-x_0)). \end{aligned} \tag{6.1.12}$$

By parity considerations of these functions centered at $x = x_0$, we also conclude that $\partial_\delta Q^\rho_{\delta,x_0}$ and $\partial_{x_0} Q^\rho_{\delta,x_0}$ are orthogonal on $H^k(\mathbb{T})$, $k \in \mathbb{Z}_{\geq 0}$.

Given $\delta > 0$ and $x_0 \in \mathbb{R}$, let $Q_{\delta,x_0} = \tau_{x_0} D_\delta Q$ as a function on the real line. Then, a similar consideration shows orthogonality of $\partial_\delta Q_{\delta,x_0}$ and $\partial_{x_0} Q_{\delta,x_0}$ in $\dot{H}^k(\mathbb{R})$, $k \in \mathbb{Z}_{\geq 0}$.

Remark 6.2. The set

$$\mathcal{M} = \big\{e^{i\theta}Q^\rho_{\delta,x_0} = e^{i\theta}\tau_{x_0}(\rho Q_\delta) : \delta \in \mathbb{R}_+,\, x_0 \in \mathbb{T}, \text{ and } \theta \in \mathbb{R}\big\} \subset H^1(\mathbb{T})$$

is a smooth manifold of dimension 3, embedded in $H^1(\mathbb{T})$. The presence of the cutoff function ρ is fundamental for this to be true. Indeed, if we instead consider the set

$$\mathcal{M}' = \{e^{i\theta}Q_{\delta,x_0} : \delta \in \mathbb{R}_+,\, x_0 \in \mathbb{T}, \text{ and } \theta \in \mathbb{R}\} \subset H^1(\mathbb{T}),$$

then it turns out that $\mathcal{M}'$ is *not* a Lipschitz submanifold of $H^1(\mathbb{T})$ (at least with the parametrization induced by (θ, x_0, δ)). Indeed, if $\theta = 0, \delta = 1$, $0 < x_0 \ll 1$, recalling that Q' is odd, we have

$$\begin{aligned} \|Q_{1,0} - Q_{1,x_0}\|^2_{\dot{H}^1(\mathbb{T})} &= \int_{-\frac{1}{2}+x_0}^{\frac{1}{2}} |Q'(x) - Q'(x-x_0)|^2 dx\\ &\quad + \int_{-\frac{1}{2}}^{-\frac{1}{2}+x_0} |Q'(x) - Q'(x-x_0+1)|^2 dx\\ &\gtrsim \int_{-\frac{1}{2}}^{-\frac{1}{2}+x_0} \big|Q'\big(-\tfrac{1}{2}\big)\big|^2 dx\\ &\gtrsim |x_0|, \end{aligned}$$

and thus $\|Q_{1,0} - Q_{1,x_0}\|_{\dot{H}^1(\mathbb{T})} \gtrsim |x_0|^{\frac{1}{2}}$, which shows that the dependence of Q_{δ,x_0} in x_0 is not Lipschitz. Similar considerations hold for $e^{i\theta}Q_{\delta,x_0}$ for every value of θ, δ, x_0. In view of Lemmas 6.10 and 6.11 below, it is crucial that the set $\mathcal{M}$ is a C^2-manifold.

6.2. **Stability of the optimizers of the GNS inequality.** We begin by establishing stability of the optimizers of the Gagliardo-Nirenberg-Sobolev inequality (3.1); if u is "far" from all rescalings and translations of the ground state Q in the L^2-sense, then the GNS inequality (3.1) is "far" from being sharp in the sense of (6.2.1) below.

Given $\gamma > 0$, define the set S_γ by setting

$$\begin{aligned} S_\gamma = \big\{&u \in L^2(\mathbb{T}) : \|u\|_{L^2(\mathbb{T})} \leq \|Q\|_{L^2(\mathbb{R})},\\ &\|P_{\leq k}\pi_{\neq 0}u\|^6_{L^6(\mathbb{T})} \leq (C_{\text{GNS}} - \gamma)\|P_{\leq k}u\|^2_{\dot{H}^1(\mathbb{T})}\|Q\|^4_{L^2(\mathbb{R})} \text{ for all } k \geq 1\big\}. \end{aligned} \tag{6.2.1}$$

Here, $P_{\leq k}u$ denotes the Dirichlet projector onto the frequencies $\{|n| \leq 2^k\}$ defined in (1.23) and $\pi_{\neq 0}$ denotes the projection onto the mean-zero part defined in (1.21).

Fix $\gamma > 0$. Young's inequality and (6.2.1) yield

$$\begin{aligned}\|P_{\le k}u\|_{L^6(\mathbb{T})}^6 &\le \frac{C_{\mathrm{GNS}} - \frac{\gamma}{2}}{C_{\mathrm{GNS}} - \gamma}\|P_{\le k}\pi_{\ne 0}u\|_{L^6(\mathbb{T})}^6 + C_\gamma|\pi_0 u|^6 \\ &\le \big(C_{\mathrm{GNS}} - \tfrac{\gamma}{2}\big)\|P_{\le k}u\|_{\dot H^1(\mathbb{T})}^2\|Q\|_{L^2(\mathbb{R})}^4 + C_\gamma\|Q\|_{L^2(\mathbb{R})}^6\end{aligned} \tag{6.2.2}$$

for any $u \in S_\gamma$, where $\pi_0 = \mathrm{id} - \pi_{\ne 0}$. Then, by repeating the argument in Subsection 4.1 with (6.2.2), we can show that

$$\mathbf{E}\Big[e^{\frac{1}{6}\int_{\mathbb{T}}|u(x)|^6\,dx}, S_\gamma\Big] \le C(\gamma) < \infty. \tag{6.2.3}$$

The main goal of this subsection is to prove the following "stability" result. In view of (6.2.3), this lemma allows us to restrict our attention to a small neighborhood of the orbit of the ground state Q in the subsequent subsections.

Lemma 6.3. *Let ρ be as in (6.1.6). Given any $\varepsilon > 0$ and $\delta^* > 0$, there exists $\gamma(\varepsilon, \delta^*) > 0$ such that the following holds; suppose that a function $u \in L^2(\mathbb{T})$ with $\|u\|_{L^2(\mathbb{T})} \le \|Q\|_{L^2(\mathbb{R})}$ satisfies*

$$\|u - e^{i\theta}Q^\rho_{\delta,x_0}\|_{L^2(\mathbb{T})} \ge \varepsilon \tag{6.2.4}$$

for all $0 < \delta < \delta^*$, $x_0 \in \mathbb{T}$, and $\theta \in \mathbb{R}$, where $Q^\rho_{\delta,x_0} = \tau_{x_0}(\rho Q_\delta)$ is as in (6.1.7). Then, we have $u \in S_{\gamma(\varepsilon,\delta^*)}$.

Recall that there exists no optimizer for the GNS inequality (3.7) on the torus $\mathbb{T}$; see Lemmas 3.3 and 3.4. Lemma 6.3 states that "almost optimizers" of the GNS inequality (3.7) on $\mathbb{T}$ exist only in a small $L^2(\mathbb{T})$-neighborhood of $e^{i\theta}Q^\rho_{\delta,x_0}$ for $0 < \delta < \delta^*$, where $\delta^* > 0$ is any given small number. This is not surprising since (i) the ground state Q (up to symmetries) is the unique optimizer for the GNS inequality (3.1) on the real line $\mathbb{R}$ (see Remark 3.2) and (ii) we see from the definition (6.1.5) that, as $\delta \to 0$, $Q^\rho_\delta = \rho Q_\delta$ on $\mathbb{T} \cong [-\frac{1}{2}, \frac{1}{2})$ becomes a more and more accurate approximation of (a dilated copy of) the ground state Q on $\mathbb{R}$ (and thus becomes a more and more accurate almost optimizer for the GNS inequality (3.7) on the torus $\mathbb{T} \cong [-\frac{1}{2}, \frac{1}{2})$).

Proof. We first make preliminary computations which allow us to reduce the problem to the mean-zero case. Suppose that $u \in H^1(\mathbb{T})$ satisfies

$$\|u\|_{L^2(\mathbb{T})} \le \|Q\|_{L^2(\mathbb{R})} \tag{6.2.5}$$

but $u \notin S_\gamma$ for some $\gamma > 0$. Then, from the GNS inequality (3.7) on $\mathbb{T}$ for mean-zero functions (see Lemma 3.3) and the definition (6.2.1) of S_γ, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned}(C_{\mathrm{GNS}} - \gamma)\|P_{\le k}u\|_{\dot H^1(\mathbb{T})}^2\|Q\|_{L^2(\mathbb{R})}^4 &< \|P_{\le k}\pi_{\ne 0}u\|_{L^6(\mathbb{T})}^6 \\ &\le C_{\mathrm{GNS}}\|P_{\le k}u\|_{\dot H^1(\mathbb{T})}^2\|\pi_{\ne 0}u\|_{L^2(\mathbb{T})}^4.\end{aligned} \tag{6.2.6}$$

Thus, from (6.2.5) and (6.2.6), we obtain

$$\begin{aligned}|\pi_0 u|^2 = \|\pi_0 u\|_{L^2(\mathbb{T})}^2 &= \|u\|_{L^2(\mathbb{T})}^2 - \|\pi_{\ne 0}u\|_{L^2(\mathbb{T})}^2 \\ &< \bigg(1 - \sqrt{\frac{C_{\mathrm{GNS}} - \gamma}{C_{\mathrm{GNS}}}}\bigg)\|Q\|_{L^2(\mathbb{R})}^2 = O(\gamma)\|Q\|_{L^2(\mathbb{R})}^2\end{aligned}$$

as $\gamma \to 0$. Hence, if we have (6.2.4) for some $\varepsilon > 0$, then there exists $\gamma_0 = \gamma_0(\varepsilon) > 0$ such that

$$\|\pi_{\neq 0} u - e^{i\theta} Q^{\rho}_{\delta, x_0}\|_{L^2(\mathbb{T})} \geq \frac{\varepsilon}{2}. \tag{6.2.7}$$

for any $u \notin S_\gamma$ with $0 < \gamma < \gamma_0$.

We prove the lemma by contradiction. Suppose that there is no such $\gamma(\varepsilon, \delta^*)$. Then, there exist $\varepsilon > 0$, $\delta^* > 0$, $\{u_n\}_{n\in\mathbb{N}} \subset L^2(\mathbb{T})$ with

$$\|u_n\|_{L^2(\mathbb{T})} \leq \|Q\|_{L^2(\mathbb{R})}, \tag{6.2.8}$$

and $\gamma_n \to 0$ such that

$$\|u_n - e^{i\theta} Q^{\rho}_{\delta, x_0}\|_{L^2(\mathbb{T})} \geq \varepsilon \tag{6.2.9}$$

for any $x_0 \in \mathbb{T}$, $0 < \delta < \delta^*$, and $\theta \in \mathbb{R}$ but $u_n \notin S_{\gamma_n}$. By the discussion above, in particular from (6.2.7), there exists $N_0(\varepsilon) \in \mathbb{N}$ such that

$$\|\pi_{\neq 0} u_n - e^{i\theta} Q^{\rho}_{\delta, x_0}\|_{L^2(\mathbb{T})} \geq \frac{\varepsilon}{2}$$

for any $x_0 \in \mathbb{T}$, $0 < \delta < \delta^*$, $\theta \in \mathbb{R}$, and $n \geq N_0$ but $\pi_{\neq 0} u_n \notin S_{\gamma_n}$. Therefore, without loss of generality, we may assume that u_n, satisfying (6.2.8) and (6.2.9), has mean zero (i.e. $u_n = \pi_{\neq 0} u_n$) and derive a contradiction.

By definition (6.2.1) of S_γ, for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$\|P_{\leq k_n} u_n\|^6_{L^6(\mathbb{T})} > (C_{\text{GNS}} - \gamma_n)\|P_{\leq k_n} u_n\|^2_{\dot{H}^1(\mathbb{R})}\|Q\|^4_{L^2(\mathbb{R})}, \tag{6.2.10}$$

since $u_n \notin S_{\gamma_n}$. Then, from (3.1) and (6.2.10) with (6.2.8) and $\gamma_n \to 0$, we see that

$$\|P_{\leq k_n} u_n\|_{L^2(\mathbb{T})} \longrightarrow \|Q\|_{L^2(\mathbb{R})} \tag{6.2.11}$$

as $n \to \infty$. In view of the upper bound (6.2.8), we also have $\|u_n\|_{L^2(\mathbb{T})} \to \|Q\|_{L^2(\mathbb{R})}$. Hence, by the Pythagorean theorem, we obtain

$$\|u_n - P_{\leq k_n} u_n\|_{L^2(\mathbb{T})} \longrightarrow 0. \tag{6.2.12}$$

Furthermore, we claim that

$$\|P_{\leq k_n} u_n\|_{\dot{H}^1(\mathbb{T})} \longrightarrow \infty. \tag{6.2.13}$$

Otherwise, we would have $\|P_{\leq k_n} u_n\|_{\dot{H}^1(\mathbb{T})} \leq C < \infty$ for all $n \in \mathbb{N}$ and thus there exists a subsequence, still denoted by $\{P_{\leq k_n} u_n\}_{n\in\mathbb{N}}$, converging weakly to some u in $\dot{H}^1(\mathbb{T})$. Then, from the compact embedding[8] of $\dot{H}^1(\mathbb{T})$ into $L^2(\mathbb{T}) \cap L^6(\mathbb{T})$, we see that $P_{\leq k_n} u_n$ converges strongly to u in $L^2(\mathbb{T}) \cap L^6(\mathbb{T})$. Hence, from (6.2.8) and then (6.2.10), we obtain

$$\begin{aligned} C_{\text{GNS}}\|u\|^2_{\dot{H}^1(\mathbb{T})}\|u\|^4_{L^2(\mathbb{T})} &\leq \liminf_{n\to\infty}(C_{\text{GNS}} - \gamma_n)\|P_{\leq k_n} u_n\|^2_{\dot{H}^1(\mathbb{T})}\|Q\|^4_{L^2(\mathbb{R})} \\ &\leq \liminf_{n\to\infty} \|P_{\leq k_n} u_n\|^6_{L^6(\mathbb{T})} = \|u\|^6_{L^6(\mathbb{T})}. \end{aligned}$$

This would imply that u is a mean-zero optimizer of the Gagliardo-Nirenberg-Sobolev inequality (3.1) on the torus $\mathbb{T}$, which is a contradiction to Lemma 3.4. Therefore, (6.2.13) must hold.

By continuity, there exists a point $x_n \in \mathbb{T}$ such that

$$|P_{\leq k_n} u_n(x_n)| \leq \|P_{\leq k_n} u_n\|_{L^2(\mathbb{T})}. \tag{6.2.14}$$

[8]Recall that we work with mean-zero functions on $\mathbb{T}$.

With $\beta_n = \|P_{\le k_n} u_n\|_{\dot{H}^1(\mathbb{T})}^{-1} \to 0$, define $v_n : \mathbb{R} \to \mathbb{C}$ by

$$v_n(x) = \begin{cases} P_{\le k_n} u_n(x + x_n - \frac{1}{2}), & \text{for } x \in [-\frac{1}{2}, \frac{1}{2}], \\ 0, & \text{for } |x| > \frac{1}{2} + \beta_n, \end{cases} \tag{6.2.15}$$

and by linear interpolation for $\frac{1}{2} < |x| \le \frac{1}{2} + \beta_n$, where the addition here is understood mod 1. Then, from (6.2.14), we have $|v_n(\pm\frac{1}{2})| \le \|P_{\le k_n} u_n\|_{L^2(\mathbb{T})}$. Moreover, from (6.2.15) with (6.2.8) and (6.2.13), we have $v_n \in H^1(\mathbb{R})$ with

$$\|v_n\|_{L^2(\mathbb{R})} \le \sqrt{1 + 2\beta_n}\, \|P_{\le k_n} u_n\|_{L^2(\mathbb{T})} \le \sqrt{1 + 2\beta_n}\, \|Q\|_{L^2(\mathbb{R})} \tag{6.2.16}$$

for any $n \in \mathbb{N}$, and

$$\|v_n\|_{\dot{H}^1(\mathbb{R})}^2 = \|P_{\le k_n} u_n\|_{\dot{H}^1(\mathbb{T})}^2 + 2\beta_n^{-1} |P_{\le k_n} u_n(x_n)|^2 \longrightarrow \infty, \tag{6.2.17}$$

as $n \to \infty$. Hence, from (6.2.10), (6.2.16), and (6.2.17) with (6.2.14) and (6.2.8), we have

$$\|v_n\|_{L^6(\mathbb{R})}^6 > (C_{\mathrm{GNS}} - \gamma_n)\alpha_n \|v_n\|_{\dot{H}^1(\mathbb{T})}^2 \|v_n\|_{L^2(\mathbb{R})}^4, \tag{6.2.18}$$

where

$$\alpha_n = (1 + 2\beta_n)^{-2}(1 - 2\beta_n \|Q\|_{L^2(\mathbb{R})}^2).$$

Since $\beta_n = \|P_{\le k_n} u_n\|_{\dot{H}^1(\mathbb{T})}^{-1} \to 0$, we have $\alpha_n \to 1$ as $n \to \infty$. Hence, we conclude from (6.2.18) that there exists $\widetilde{\gamma}_n \to 0$ such that

$$\|v_n\|_{L^6(\mathbb{R})}^6 > (C_{\mathrm{GNS}} - \widetilde{\gamma}_n) \|v_n\|_{\dot{H}^1(\mathbb{T})}^2 \|v_n\|_{L^2(\mathbb{R})}^4. \tag{6.2.19}$$

With the scaling operator D_λ as in (6.1.1), let

$$w_n = D_{\|v_n\|_{\dot{H}^1(\mathbb{R})}} v_n. \tag{6.2.20}$$

Then, from (6.2.11), (6.2.16), and (6.2.19), we have

$$\|w_n\|_{\dot{H}^1(\mathbb{R})} = 1, \tag{6.2.21}$$

$$\|w_n\|_{L^2(\mathbb{R})} = \|v_n\|_{L^2(\mathbb{R})} \le \sqrt{1 + 2\beta_n}\, \|Q\|_{L^2(\mathbb{R})} \text{ and } \|w_n\|_{L^2(\mathbb{R})} \to \|Q\|_{L^2(\mathbb{R})}, \tag{6.2.22}$$

$$\|w_n\|_{L^6(\mathbb{R})}^6 > (C_{\mathrm{GNS}} - \widetilde{\gamma}_n) \|w_n\|_{L^2(\mathbb{R})}^4. \tag{6.2.23}$$

Since $\{w_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R})$, we can invoke the profile decomposition [39, Proposition 3.1] for the (subcritical) Sobolev embedding: $H^1(\mathbb{R}) \hookrightarrow L^6(\mathbb{R})$. See also Theorem 4.6 in [42]. There exist $J^* \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$, a sequence $\{\phi^j\}_{j=1}^{J^*}$ of non-trivial $H^1(\mathbb{R})$-functions, and a sequence $\{x_n^j\}_{j=1}^{J^*}$ for each $n \in \mathbb{N}$ such that up to a subsequence, still denoted by $\{w_n\}_{n \in \mathbb{N}}$, we have

$$w_n(x) = \sum_{j=1}^{J} \phi^j(x - x_n^j) + r_n^J(x) \tag{6.2.24}$$

for each finite $0 \le J \le J^*$, where the remainder term r_n^J satisfies

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|r_n^J\|_{L^6(\mathbb{R})} = 0. \tag{6.2.25}$$

Here, $\lim_{J\to\infty} f(J) := f(J^*)$ if $J^* < \infty$. Moreover, for any finite $0 \le J \le J^*$, we have

$$\|w_n\|_{L^2(\mathbb{R})}^2 = \sum_{j=1}^{J} \|\phi^j\|_{L^2(\mathbb{R})}^2 + \|r_n^J\|_{L^2(\mathbb{R})}^2 + o(1), \tag{6.2.26}$$

$$\|w_n\|_{\dot H^1(\mathbb{R})}^2 = \sum_{j=1}^{J} \|\phi^j\|_{\dot H^1(\mathbb{R})}^2 + \|r_n^J\|_{\dot H^1(\mathbb{R})}^2 + o(1), \tag{6.2.27}$$

as $n \to \infty$, and

$$\limsup_{n\to\infty} \|w_n\|_{L^6(\mathbb{R})}^6 = \sum_{j=1}^{J^*} \|\phi^j\|_{L^6(\mathbb{R})}^6. \tag{6.2.28}$$

From (6.2.21) and (6.2.27) and taking $n \to \infty$, we obtain

$$\sup_{j=1,\dots,J^*} \|\phi^j\|_{\dot H^1(\mathbb{R})}^2 \le \sum_{j=1}^{J^*} \|\phi^j\|_{\dot H^1(\mathbb{R})}^2 \le 1, \tag{6.2.29}$$

where an equality holds at the first inequality if and only if $J^* = 0$ or 1.

From (3.1) and (6.2.23) with $\widetilde\gamma_n \to 0$, we have

$$C_{\text{GNS}} \lim_{n\to\infty} \|w_n\|_{L^2(\mathbb{R})}^4 = \limsup_{n\to\infty} \|w_n\|_{L^6(\mathbb{R})}^6.$$

By (6.2.28) followed by (3.1), (6.2.29), and (6.2.26) with $\ell^2 \subset \ell^4$,

$$\begin{aligned} C_{\text{GNS}} \lim_{n\to\infty} \|w_n\|_{L^2(\mathbb{R})}^4 &= \sum_{j=1}^{J^*} \|\phi^j\|_{L^6(\mathbb{R})}^6 \\ &\le C_{\text{GNS}} \sum_{j=1}^{J^*} \|\phi^j\|_{\dot H^1(\mathbb{R})}^2 \|\phi^j\|_{L^2(\mathbb{R})}^4 \\ &\le C_{\text{GNS}} \sum_{j=1}^{J^*} \|\phi^j\|_{L^2(\mathbb{R})}^4 \\ &\le C_{\text{GNS}} \lim_{n\to\infty} \|w_n\|_{L^2(\mathbb{R})}^4. \end{aligned} \tag{6.2.30}$$

Here, an equality holds if and only if $J^* = 0$ or 1. If $J^* = 0$, then it follows from (6.2.28) that w_n tends to 0 in $L^6(\mathbb{R})$ as $n \to \infty$. Then, from (6.2.23), we see that w_n tends to 0 in $L^2(\mathbb{R})$. This is a contradiction to (6.2.22). Hence, we must have $J^* = 1$. In this case, (6.2.30) (with $J^* = 1$) holds with equalities and thus we see that ϕ^1 is an optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$.

Hence, we conclude from Remark 3.2 that there exist $\sigma \ne 0$, $\lambda > 0$, and $x_0 \in \mathbb{R}$ such that

$$\phi^1 = \sigma \tau_{x_0} D_\lambda Q.$$

From the last step in (6.2.30) (with $J^* = 1$ and an equality) with (6.2.22), we have

$$\|\phi^1\|_{L^2(\mathbb{R})} = \lim_{n\to\infty} \|w_n\|_{L^2(\mathbb{R})} = \|Q\|_{L^2(\mathbb{R})},$$

which implies that $\sigma = e^{i\theta}$ for some $\theta \in \mathbb{R}$. From (6.2.24) and (6.2.25) with $J = J^* = 1$, we see that $\tau_{-x_n^1} w_n$ converges weakly to ϕ^1 in $L^2(\mathbb{R})$ (which follows from the (weak) convergence of $\tau_{-x_n^1} w_n$ to ϕ^1 in $L^6(\mathbb{R})$), while (6.2.22) implies convergence of the L^2-norms. Hence, we obtain strong convergence in $L^2(\mathbb{R})$:

$$\|w_n - e^{i\theta}\tau_{x_0+x_n^1} D_\lambda Q\|_{L^2(\mathbb{R})} \longrightarrow 0.$$

Hence, from (6.2.20), we have

$$\|v_n - e^{i\theta}\tau_{y_n} D_{\lambda_n} Q\|_{L^2(\mathbb{R})} \longrightarrow 0, \tag{6.2.31}$$

where $y_n = \|v_n\|^{-1}_{\dot{H}^1(\mathbb{R})}(x_0 + x_n^1)$ and $\lambda_n = \|v_n\|^{-1}_{\dot{H}^1(\mathbb{R})}\lambda \to 0$ in view of (6.2.17).

Recall that v_n is supported on $[-\frac{1}{2} - \beta_n, \frac{1}{2} + \beta_n]$ with the L^2-norm on $[-\frac{1}{2}, \frac{1}{2}]^c$ bounded by $\sqrt{2\beta_n}\,\|Q\|_{L^2(\mathbb{R})}$ (which tends to 0 as $n \to \infty$). Thus, it follows from (6.2.31) that

$$\begin{aligned} J_n &:= \int_{|y| \ge \frac{1}{2\lambda_n}} |Q(y - \lambda_n^{-1} y_n)|^2 dy \\ &= \|\tau_{y_n} D_{\lambda_n} Q\|_{L^2([-\frac{1}{2},\frac{1}{2}]^c)} \longrightarrow 0, \end{aligned} \tag{6.2.32}$$

as $n \to \infty$. In the following, we denote by Q_{λ_n, y_n} the dilated and translated ground state Q, viewed as a periodic function on $\mathbb{T}$, and by $\tau_{y_n} D_{\lambda_n} Q$ the dilated and translated ground state viewed as a function on the real line. By possibly choosing a subsequence, we assume that $y_n \ge 0$ without loss of generality. Write

$$\mathbb{T} \cong [-\tfrac{1}{2}, \tfrac{1}{2}) = [-\tfrac{1}{2} + y_n, \tfrac{1}{2}) \cup [-\tfrac{1}{2}, -\tfrac{1}{2} + y_n) =: I_{1,n} \cup I_{2,n}.$$

Note that while Q_{λ_n, y_n} and $\tau_{y_n} D_{\lambda_n} Q$ coincide on $I_{1,n}$, they do not coincide on $I_{2,n}$. Thanks to the exponential decay of the ground state Q, we have

$$\|\tau_{y_n} D_{\lambda_n} Q\|_{L^2(I_{2,n})} \le \|D_{\lambda_n} Q\|_{L^2((-\infty, -\frac{1}{2}])} = O(\exp(-c\lambda_n^{-1})) \longrightarrow 0. \tag{6.2.33}$$

On the other hand, on $I_{2,n}$, we have $Q_{\lambda_n, y_n}(x) = \lambda_n^{-\frac{1}{2}} Q(\lambda_n^{-\frac{1}{2}}(x + 1 - y_n))$. Thus, from a change of variables and (6.2.32), we have

$$\|Q_{\lambda_n, y_n}\|_{L^2(I_{2,n})} \le \|\tau_{y_n} D_{\lambda_n} Q\|_{L^2([\frac{1}{2}, \infty))} \longrightarrow 0. \tag{6.2.34}$$

Therefore, from (6.2.31), (6.2.33) and (6.2.34), we obtain

$$\begin{aligned} &\|v_n - e^{i\theta} Q_{\lambda_n, y_n}\|_{L^2(\mathbb{T})} \\ &\quad \le \|v_n - e^{i\theta}\tau_{y_n} D_{\lambda_n} Q\|_{L^2([-\frac{1}{2},\frac{1}{2}])} + \|Q_{\lambda_n, y_n}\|_{L^2(I_{2,n})} + \|\tau_{y_n} D_{\lambda_n} Q\|_{L^2(I_{2,n})} \\ &\quad \longrightarrow 0, \end{aligned} \tag{6.2.35}$$

as $n \to \infty$. Moreover, since $\lambda_n \to 0$, it follows from (6.1.6) that

$$\begin{aligned} \|e^{i\theta} Q^\rho_{\lambda_n, y_n} - e^{i\theta} Q_{\lambda_n, y_n}\|_{L^2(\mathbb{T})} &= \|\rho Q_{\lambda_n, 0} - Q_{\lambda_n, 0}\|_{L^2(\mathbb{T})} \\ &\le \|Q_{\lambda_n, 0}\|_{L^2([-\frac{1}{8}, \frac{1}{8}]^c)} \\ &= \|Q\|_{L^2([-\frac{\lambda_n}{8}, \frac{\lambda_n}{8}]^c)} \to 0. \end{aligned} \tag{6.2.36}$$

Finally, by combining (6.2.12), (6.2.15), (6.2.35), and (6.2.36), we obtain a contradiction to (6.2.9). This completes the proof of Lemma 6.3. □

6.3. **Orthogonal coordinate system in a neighborhood of the soliton manifold.** In view of Lemma 6.3 and (6.2.3), in order to prove $Z_{6,K=\|Q\|_{L^2(\mathbb{R})}} < \infty$, it suffices to show that

$$\mathbf{E}\Big[e^{\frac{1}{6}\int_{\mathbb{T}}|u(x)|^6\,dx}\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le\|Q\|_{L^2(\mathbb{R})}\}}, U_\varepsilon(\delta^*)\Big] < \infty \tag{6.3.1}$$

for some small $\varepsilon, \delta^* > 0$, where $U_\varepsilon(\delta^*)$ is defined by

$$\begin{aligned} U_\varepsilon(\delta^*) = \big\{u \in L^2(\mathbb{T}) : \|u - e^{i\theta}Q^\rho_{\delta,x_0}\|_{L^2(\mathbb{T})} < \varepsilon \\ \text{for some } 0 < \delta < \delta^*,\ x_0 \in \mathbb{T},\ \text{and } \theta \in \mathbb{R}\big\} \end{aligned} \tag{6.3.2}$$

with $Q^\rho_{\delta,x_0} = \tau_{x_0}(\rho Q_\delta)$ as in (6.1.7). Namely, the domain of integration $U_\varepsilon(\delta^*)$ is an ε-neighborhood of the (approximate) soliton manifold $\mathcal{M} = \mathcal{M}(\delta^*)$:

$$\mathcal{M} = \big\{e^{i\theta}Q^\rho_{\delta,x_0} = e^{i\theta}\tau_{x_0}(\rho Q_\delta) : 0 < \delta < \delta^*,\ x_0 \in \mathbb{T},\ \text{and } \theta \in \mathbb{R}\big\}.$$

Here, we say "approximate" due to the presence of the cutoff function ρ. See Remark 6.2. Recall that the expectation in (6.3.1) is taken with respect to the Ornstein-Uhlenbeck loop in (6.0.1) whose law is given by the Gaussian measure μ in (1.8) with the Cameron-Martin space $H^1(\mathbb{T})$. Our main goal in this subsection is to endow $U_\varepsilon(\delta^*)$ with an "orthogonal" coordinate system, where the orthogonality is measured in terms of $H^1(\mathbb{T})$. This then allows us to introduce a change of variables for the integration in (6.3.1); see Subsection 6.5.

Given $0 < \delta \ll 1$, $x_0 \in \mathbb{T}$, and $0 \le \theta < 2\pi$, we define $V_{\delta,x_0,\theta} = V_{\delta,x_0,\theta}(\mathbb{T})$ by

$$\begin{aligned} V_{\delta,x_0,\theta} = \big\{u \in L^2(\mathbb{T}) : &\langle u, (1-\partial_x^2)e^{i\theta}\partial_\delta Q^\rho_{\delta,x_0}\rangle_{L^2(\mathbb{T})} = 0, \\ &\langle u, (1-\partial_x^2)e^{i\theta}\partial_{x_0} Q^\rho_{\delta,x_0}\rangle_{L^2(\mathbb{T})} = 0, \\ &\langle u, (1-\partial_x^2)ie^{i\theta} Q^\rho_{\delta,x_0}\rangle_{L^2(\mathbb{T})} = 0\big\}. \end{aligned} \tag{6.3.3}$$

We point out that, due to the insufficient regularity of $u \in L^2(\mathbb{T})$, instead of the H^1-inner product, we measure the orthogonality in (6.3.3) with respect to the L^2-inner product with a weight $(1-\partial_x^2)$ so that the inner products in (6.3.3) are well defined for $u \in L^2(\mathbb{T})$. Here, we emphasize that the inner product on $L^2(\mathbb{T})$ defined in (1.18) is real-valued. It is easy to check that $(\tau_{x_0}\rho)\partial_\delta Q_{\delta,x_0}$ and $\partial_{x_0}(\tau_{x_0}\rho)Q_{\delta,x_0}$ are orthogonal in $L^2(\mathbb{T})$ and $H^1(\mathbb{T})$ (see Remark 6.1), and similarly that they are orthogonal to $i(\tau_{x_0}\rho)Q_{\delta,x_0}$ in $H^1(\mathbb{T})$ (viewed as a real vector space with the inner product in (1.18)). Hence, the space $V_{\delta,x_0,\theta}$ denotes a *real* subspace of codimension 3 in $L^2(\mathbb{T})$, orthogonal (with the weight $(1-\partial_x^2)$) to the tangent vectors $e^{i\theta}\partial_\delta Q^\rho_{\delta,x_0}$, $e^{i\theta}\partial_{x_0} Q^\rho_{\delta,x_0} = e^{i\theta}\partial_{x_0}\big(\tau_{x_0}(\rho Q_\delta)\big)$, and $\partial_\theta(e^{i\theta}Q^\rho_{\delta,x_0}) = ie^{i\theta}Q^\rho_{\delta,x_0}$ of the soliton manifold $\mathcal{M}$ (with $0 < \delta \ll 1$). The following proposition shows that a small neighborhood of $\mathcal{M}$ can be endowed with an orthogonal coordinate system in terms of $e^{i\theta}\partial_\delta Q^\rho_{\delta,x_0}$, $e^{i\theta}\partial_{x_0} Q^\rho_{\delta,x_0}$, $ie^{i\theta}Q^\rho_{\delta,x_0}$, and $V_{\delta,x_0,\theta}$.

Proposition 6.4. *Given small $\varepsilon_1 > 0$, there exist $\varepsilon = \varepsilon(\varepsilon_1) > 0$ and $\delta^* = \delta^*(\varepsilon_1) > 0$ such that*

$$\begin{aligned} U_\varepsilon(\delta^*) \subset \big\{u \in L^2(\mathbb{T}) : \|u - e^{i\theta}Q^\rho_{\delta,x_0}\|_{L^2(\mathbb{T})} < \varepsilon_1,\ u - e^{i\theta}Q^\rho_{\delta,x_0} \in V_{\delta,x_0,\theta}(\mathbb{T}) \\ \textit{for some } 0 < \delta < \delta^*,\ x_0 \in \mathbb{T},\ \textit{and } \theta \in \mathbb{R}\big\}. \end{aligned}$$

Remark 6.5. The series of works by Nakanishi and Schlag [65, 67, 68, 69] and Krieger, Nakanishi, and Schlag [44, 45, 46, 47, 48] use a coordinate system similar to the one given by Proposition 6.4,[9] but centered around a *single* soliton. See, for example, Section 2.5 in [45]. Thanks to the symmetries of the problem on $\mathbb{R}^d$, it is easy to extend the coordinate system to a tubular neighborhood of the soliton manifold in these works. In the setting of Proposition 6.4 on the torus $\mathbb{T}$, however, we lack dilation symmetry, which makes it impossible to use such a soft argument to conclude the desired result (namely, endow $U_\varepsilon(\delta^*)$ with an orthogonal coordinate system).

Proof. We first show that the claimed result holds true in the case of the real line (without the extra factor $\tau_{x_0}\rho$). Given $\gamma_0 \in \mathbb{R}$, $0 < \delta \ll 1$, $x_0 \in \mathbb{T}$, and $0 \le \theta < 2\pi$, we define $V^{\gamma_0}_{\delta,x_0,\theta} = V^{\gamma_0}_{\delta,x_0,\theta}(\mathbb{R})$ by

$$\begin{aligned} V^{\gamma_0}_{\delta,x_0,\theta} = \big\{u \in L^2(\mathbb{R}) : &\langle u, (\gamma_0^2 - \partial_x^2)e^{i\theta}\partial_\delta Q_{\delta,x_0}\rangle_{L^2(\mathbb{R})} = 0,\\ &\langle u, (\gamma_0^2 - \partial_x^2)e^{i\theta}\partial_{x_0} Q_{\delta,x_0}\rangle_{L^2(\mathbb{R})} = 0,\\ &\langle u, (\gamma_0^2 - \partial_x^2)ie^{i\theta} Q_{\delta,x_0}\rangle_{L^2(\mathbb{R})} = 0\big\}, \end{aligned} \tag{6.3.4}$$

where the inner product on $L^2(\mathbb{R})$ is real-valued as defined in (1.18).

Consider the map $H : \big(\mathbb{R} \times L^2(\mathbb{R})\big) \times \big(\mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \times V^0_{1,0,0}\big) \to L^2(\mathbb{R})$ defined by

$$H(\gamma_0, u, \delta, x_0, \theta, w) = u - e^{i\theta}Q_{\delta,x_0} - P_{V^{\gamma_0}_{\delta,x_0,\theta}} w, \tag{6.3.5}$$

where $P_{V^{\gamma_0}_{\delta,x_0,\theta}}$ is the projection onto $V^{\gamma_0}_{\delta,x_0,\theta}(\mathbb{R})$ in $L^2(\mathbb{R})$. It is easy to see that H is Fréchet differentiable and $H(0, Q_{1,0}, 1, 0, 0, 0) = 0$. Moreover, the Fréchet derivative of H in the $\big(\mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \times V^0_{1,0,0}\big)$-variable at $(0, Q_{1,0}, 1, 0, 0, 0)$ is

$$(\mathrm{d}_2 H)(0, Q_{1,0}, 1, 0, 0, 0) = -\mathrm{d}(e^{i\theta}Q_{\delta,x_0})|_{(\delta,x_0,\theta)=(1,0,0)} - \mathrm{id}_{V^0_{1,0,0}},$$

where

$$\big(\mathrm{d}(e^{i\theta}Q_{\delta,x_0})\big)(\alpha,\beta,\gamma) = \alpha e^{i\theta}\partial_\delta Q_{\delta,x_0} + \beta e^{i\theta}\partial_{x_0}Q_{\delta,x_0} + \gamma i e^{i\theta}Q_{\delta,x_0}$$

for $(\alpha,\beta,\gamma) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

The image of $Z = \mathrm{d}(e^{i\theta}Q_{\delta,x_0})|_{(\delta,x_0,\theta)=(1,0,0)}$ has dimension 3, while the image of $\mathrm{id}_{V^0_{1,0,0}}$, namely the subspace $V^0_{1,0,0}$, has codimension 3. Moreover, if u lies in the intersection of the image of Z and the image of $\mathrm{id}_{V^0_{1,0,0}}$, then the definition (6.3.4) of $V^0_{1,0,0}$ and the orthogonality of $e^{i\theta}\partial_\delta Q_{\delta,x_0}$, $e^{i\theta}\partial_{x_0}Q_{\delta,x_0}$, and $ie^{i\theta}Q_{\delta,x_0}$ in $\dot H^1(\mathbb{R})$ (see Remark 6.1) allow us to conclude that $u = 0$. Hence, $\mathrm{d}_2 H$ is invertible at $(0, Q_{1,0}, 1, 0, 0, 0)$. By the implicit function theorem, ([31, Theorem 26.27]), there exists a neighborhood in $\mathbb{R}\times L^2(\mathbb{R})$ of $(0, Q_{1,0})$ of the form:

$$W = \big\{|\gamma| < \gamma_*\big\} \times \big\{\|u - Q_{1,0}\|_{L^2(\mathbb{R})} < \varepsilon\big\}, \tag{6.3.6}$$

and a C^1-function $b = (\delta, x_0, \theta, w) : W \to \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \times V^0_{1,0,0}$ such that

$$b(0, Q_{1,0}) = (1, 0, 0, 0) \qquad \text{and} \qquad H\big(\gamma_0, u, b(\gamma_0, u)\big) = 0. \tag{6.3.7}$$

Namely, from (6.3.5) and (6.3.7), we have

$$u = e^{i\theta}Q_{\delta,x_0} + v \tag{6.3.8}$$

[9]Without the multiplication by (a translate) of the cutoff function ρ.

for some $(\delta, x_0, \theta, v) \in \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \times V^{\gamma_0}_{\delta,x_0,\theta}$. Moreover, by continuity b with (6.3.7), if we choose $\gamma_* = \gamma_*(\varepsilon_0)$ and $\varepsilon = \varepsilon(\varepsilon_0)$ sufficiently small, then we can guarantee

$$\|v\|_{L^2(\mathbb{R})} < \varepsilon_0. \tag{6.3.9}$$

Now, suppose that

$$\|u - e^{i\theta_0} Q_{\delta_0,x_0}\|_{L^2(\mathbb{R})} < \varepsilon \tag{6.3.10}$$

for some $(\delta_0, x_0, \theta_0) \in \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ with $0 < \delta_0 < \gamma_*$. Note that we have

$$Q_{\delta_0,x_0} = \tau_{x_0} D_{\delta_0} Q_{1,0} \tag{6.3.11}$$

where D_{δ_0} is the scaling operator on the real line defined in (6.1.1) and τ_{x_0} denotes the translation operator for functions on the real line. By setting

$$T = e^{i\theta_0} \tau_{x_0} D_{\delta_0}, \tag{6.3.12}$$

we can rewrite (6.3.10) as

$$\|T^{-1}u - Q_{1,0}\|_{L^2(\mathbb{R})} < \varepsilon$$

since T is an isometry on $L^2(\mathbb{R})$. Then, it follows from (6.3.6) that $(\delta_0, T^{-1}u) \in W$. Hence, from (6.3.8), we have

$$T^{-1}u = e^{i\theta} Q_{\delta,x} + v \tag{6.3.13}$$

for some $(\delta, x, \theta, v) \in \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \times V^{\delta_0}_{\delta,x,\theta}$ near $(1, 0, 0, 0)$. Therefore, we obtain

$$u = e^{i\theta_1} Q_{\delta_1,x_1} + Tv \tag{6.3.14}$$

for some $(\delta_1, x_1, \theta_1) \in \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ such that

$$e^{i\theta_1} Q_{\delta_1,x_1} = T e^{i\theta} Q_{\delta,x}. \tag{6.3.15}$$

From (6.3.12) and (6.3.15) with (6.3.9), we can easily check that $Tv \in V^1_{\delta_1,x_1,\theta_1}$ and $\|Tv\|_{L^2(\mathbb{R})} < \varepsilon_0$ for $v \in V^{\delta_0}_{\delta,x,\theta}$ in (6.3.13). Hence, we conclude that u in (6.3.14) has the desired form in this real line case.

We now prove the claim in the case of the torus $\mathbb{T}$. In this case, a scaling argument such as (6.3.11) no longer works and we need to proceed with care. By a translation and a rotation, we may assume that $u \in L^2(\mathbb{T})$ satisfies

$$\|u - \rho Q_{\delta_0}\|_{L^2(\mathbb{T})} < \varepsilon, \tag{6.3.16}$$

where $Q_{\delta_0} = Q_{\delta_0,0}$. Extending u by 0 outside $[-\frac{1}{2}, \frac{1}{2}]$, we obtain a function in $L^2(\mathbb{R})$, which we compare to the rescaled soliton on the real line. From (6.1.6) and (6.3.16), we have

$$\begin{aligned}
\|u - Q_{\delta_0}\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} |u(x) - \rho(x)Q_{\delta_0}(x) - (1-\rho(x))Q_\delta(x)|^2 \,\mathrm{d}x \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} |u(x) - \rho(x)Q_{\delta_0}(x)|^2 \,\mathrm{d}x \\
&\quad + 2\,\mathrm{Re}\Big(\int_{\mathbb{R}} (u(x) - \rho(x)Q_{\delta_0}(x))(1-\rho(x))Q_\delta(x)\,\mathrm{d}x \Big) \\
&\quad + \int_{\mathbb{R}} |(1-\rho(x))Q_\delta(x)|^2 \,\mathrm{d}x \\
&\le \int_{-\frac{1}{2}}^{\frac{1}{2}} |u(x) - \rho(x)Q_{\delta_0}(x)|^2 \,\mathrm{d}x \\
&\quad + 2\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} |u(x) - \rho(x)Q_{\delta_0}(x)|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} |(1-\rho(x))Q_\delta(x)|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} \\
&\quad + \int_{\mathbb{R}} |(1-\rho(x))Q_\delta(x)|^2 \,\mathrm{d}x \\
&\le \varepsilon + O(\exp(-c\delta_0^{-1}))
\end{aligned}$$

and thus, for sufficiently small $\delta_0 = \delta_0(\varepsilon) > 0$, we have

$$\|u - Q_{\delta_0}\|_{L^2(\mathbb{R})} < 2\varepsilon.$$

Hence, from the discussion above on the real line case, we have

$$u = e^{i\theta_1} Q_{\delta_1,x_1} + v \tag{6.3.17}$$

for some $(\delta_1, x_1, \theta_1)$ near $(\delta_0, 0, 0)$ and $v \in V^1_{\delta_1,x_1,\theta_1}(\mathbb{R})$ with $\|v\|_{L^2(\mathbb{R})} < \varepsilon_0 \ll 1$. Since $u = 0$ and $Q_{\delta_1,x_1} = O_{L^2(\mathbb{R})}(\exp(-c\delta_1^{-1}))$ outside $[-\frac{1}{2}, \frac{1}{2}]$, we have

$$\|v\|_{L^2([-\frac{1}{2},\frac{1}{2}]^c)} = O(\exp(-c\delta_1^{-1})). \tag{6.3.18}$$

Moreover, from (6.1.6) and (6.1.7), we have

$$\|e^{i\theta_1} Q_{\delta_1,x_1} - e^{i\theta_1} Q^\rho_{\delta_1,x_1}\|_{L^2(\mathbb{R})} = O(\exp(-c\delta_1^{-1})) \tag{6.3.19}$$

Define the map $F = F_{\delta_1,x_1,\theta_1} : \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \times V_{\delta_1,x_1,\theta_1}(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$F(\delta, x, \theta, v) = e^{i\theta} Q^\rho_{\delta,x} + P_{V_{\delta,x,\theta}} v,$$

where $Q^\rho_{\delta,x} = (\tau_x\rho)Q_{\delta,x}$ and $P_{V_{\delta,x,\theta}}$ is the projection onto $V_{\delta,x,\theta}(\mathbb{T})$ in $L^2(\mathbb{T})$. Then, from (6.3.17), we claim that

$$\|u - F(\delta_1, x_1, \theta_1, v_1)\|_{L^2(\mathbb{T})} = O(\exp(-c\delta_1^{-1})), \tag{6.3.20}$$

where $v_1 = P_{V_{\delta_1,x_1,\theta_1}}(v|_{\mathbb{T}})$. Note that we have

$$\|v_1\|_{L^2(\mathbb{T})} \lesssim \varepsilon_0 \ll 1. \tag{6.3.21}$$

Let us verify (6.3.20). For $v \in V^1_{\delta_1,x_1,\theta_1}(\mathbb{R})$ in (6.3.17), we have

$$\begin{aligned} &\langle v, (1-\partial_x^2)e^{i\theta_1}\partial_\delta Q_{\delta_1,x_1}\rangle_{L^2(\mathbb{R})} = \langle v, (1-\partial_x^2)e^{i\theta_1}\partial_{x_0} Q_{\delta_1,x_1}\rangle_{L^2(\mathbb{R})} \\ &\quad = \langle v, (1-\partial_x^2)ie^{i\theta_1} Q_{\delta_1,x_1}\rangle_{L^2(\mathbb{R})} = 0. \end{aligned} \tag{6.3.22}$$

Recalling from Remark 6.1 the orthogonality of $e^{i\theta_1}\partial_\delta Q^\rho_{\delta_1,x_1}$, $e^{i\theta_1}\partial_{x_0} Q^\rho_{\delta_1,x_1}$, and $ie^{i\theta_1}Q^\rho_{\delta_1,x_1}$ in $H^2(\mathbb{T})$, we have

$$\begin{aligned} v_1 &= P_{V_{\delta_1,x_1,\theta_1}}(v|_\mathbb{T}) \\ &= v - \frac{\langle v, (1-\partial_x^2)e^{i\theta_1}\partial_\delta Q^\rho_{\delta_1,x_1}\rangle_{L^2(\mathbb{T})}}{\|(1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)e^{i\theta_1}\partial_\delta Q^\rho_{\delta_1,x_1} \\ &\quad - \frac{\langle v, (1-\partial_x^2)e^{i\theta_1}\partial_{x_0} Q^\rho_{\delta_1,x_1}\rangle_{L^2(\mathbb{T})}}{\|(1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)e^{i\theta_1}\partial_{x_0} Q^\rho_{\delta_1,x_1} \\ &\quad - \frac{\langle v, (1-\partial_x^2)ie^{i\theta_1} Q^\rho_{\delta_1,x_1}\rangle_{L^2(\mathbb{T})}}{\|(1-\partial_x^2) Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)ie^{i\theta_1} Q^\rho_{\delta_1,x_1}. \end{aligned} \tag{6.3.23}$$

Then, from the exponential decay of the ground state, (6.3.18), (6.3.19), and (6.3.22), we obtain

$$\|v - v_1\|_{L^2(\mathbb{T})} = O(\exp(-c\delta_1^{-1})) \tag{6.3.24}$$

for $0 < \delta_1 \ll 1$. Here, we used the polynomial bounds (in δ_1^{-1}) on $\|(1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}$, $\|(1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}$, and $\|(1-\partial_x^2)Q^\rho_{\delta_1,x_1}\|^2_{L^2}$. See (6.3.43), (6.3.46), and (6.3.49) below. Hence, from (6.3.17), (6.3.24), and $F(\delta_1, x_1, \theta_1, v_1) = e^{i\theta_1}Q_{\delta_1,x_1} + v_1$, we obtain (6.3.20).

By another translation and rotation, we may assume that $x_1 = 0$ and $\theta_1 = 0$ in (6.3.20). Namely, we have

$$\|u - F(\delta_1, 0, 0, v_1)\|_{L^2(\mathbb{T})} = O(\exp(-c\delta_1^{-1})). \tag{6.3.25}$$

Hence, to finish the proof, we show that (6.3.25) guarantees that u lies in the image of $F = F_{\delta_1,0,0}$. For this purpose, we recall the following version of the inverse function theorem; see [31, Theorem 26.29]. See also [31, Lemma 26.28]. In the following, $\|\cdot\|$ denotes the operator norm.

Lemma 6.6. *Given Banach spaces X and Y, let $f: U \to Y$ be a C^1-map from an open subset $U \subset X$ to Y. Suppose $x_0 \in U$ such that $\mathrm{d}f(x_0)$ is invertible. If there exists $R > 0$ such that $\overline{B^X(x_0,R)} \subset U$ and*

$$\sup_{x \in B^X(x_0,R)} \|(\mathrm{d}f(x_0))^{-1}\mathrm{d}f(x) - \mathrm{id}_X\| = \kappa < 1, \tag{6.3.26}$$

then f is invertible (with a C^1-inverse) on $B^X(x_0,R)$. Moreover, letting $y_0 = f(x_0) \in Y$, we have

$$B^Y(y_0, r) \subset f(B^X(x_0, (1-\kappa)^{-1}\|\mathrm{d}f(x_0)^{-1}\|r))$$

for any $r < (1-\kappa)R/\|\mathrm{d}f(x_0)^{-1}\|$. Here, $B^Z(z_0,R)$ denotes the ball of radius R in $Z = X$ or Y centered at z_0.

Our goal is thus to estimate the quantity κ in (6.3.26), with f replaced by $F = F_{\delta_1,0,0}$, to conclude from (6.3.25) that u is in the image of F. We begin by computing $\big(\mathrm{d}F(\delta_1,0,0,v_1)\big)^{-1}$ and its norm.

Lemma 6.7. *There is a constant $C > 0$ such that*

$$\big\| \big(\mathrm{d}F_{\delta_1,0,0}(\delta_1, 0, 0, v_1)\big)^{-1} \big\| \le C$$

for all sufficiently small $\delta_1 > 0$ and $\|v_1\|_{L^2(\mathbb{T})} \ll 1$.

We assume Lemma 6.7 for now and proceed with the proof of Proposition 6.4. The proofs of this lemma and Lemma 6.8 below will be presented at the end of this subsection.

Let $(\delta, x_0, \theta_0, w) \in \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \times V_{\delta_1,0,0}(\mathbb{T})$ such that $|\delta - \delta_1| \ll 1$, $|x_0| \ll 1$, $|\theta_0| \ll 1$, and $\|w\|_{L^2(\mathbb{T})} \ll 1$. In the following, we compute

$$\begin{aligned}\big(\mathrm{d}F(\delta_1, 0, 0, v_1)\big)^{-1}\mathrm{d}F(\delta, x_0, \theta_0, w) : &\,\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times V_{\delta_1,0,0}(\mathbb{T}) \\ &\to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times V_{\delta_1,0,0}(\mathbb{T})\end{aligned}$$

and compare it with the identity operator. Given a vector $\mathbf{v} = (\alpha, \beta, \gamma, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times V_{\delta_1,0,0}(\mathbb{T})$, we have

$$\begin{aligned}\widetilde{u} &:= \big(\mathrm{d}F(\delta, x_0, \theta_0, w)\big)\mathbf{v} \\ &= \alpha e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0} + \beta e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0} + \gamma i e^{i\theta_0} Q^\rho_{\delta,x_0} \\ &\quad + \big(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha, \beta, \gamma)\big)w + P_{V_{\delta,x_0,\theta_0}} v,\end{aligned} \tag{6.3.27}$$

where $\big(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha, \beta, \gamma)\big)w$ is given by

$$\big(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha, \beta, \gamma)\big)w = \alpha\partial_\delta P_{V_{\delta,x_0,\theta_0}} w + \beta\partial_{x_0} P_{V_{\delta,x_0,\theta_0}} w + \gamma\partial_{\theta_0} P_{V_{\delta,x_0,\theta_0}} w. \tag{6.3.28}$$

Suppose that for $\widetilde{\mathbf{v}} = (\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{v})$, we have

$$\big(\mathrm{d}F(\delta, x_0, \theta_0, w)\big)\mathbf{v} = \widetilde{u} = \big(\mathrm{d}F(\delta_1, 0, 0, v_1)\big)\widetilde{\mathbf{v}}. \tag{6.3.29}$$

Namely, we have

$$\begin{aligned}\widetilde{\mathbf{v}} &= (\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{v}) \\ &= \big(\mathrm{d}F(\delta_1, 0, 0, v_1)\big)^{-1}\widetilde{u} = \big(\mathrm{d}F(\delta_1, 0, 0, v_1)\big)^{-1}\big(\mathrm{d}F(\delta, x_0, \theta_0, w)\big)\mathbf{v}.\end{aligned} \tag{6.3.30}$$

Then, in view of the hypothesis in Lemma 6.6, our goal is to estimate $\mathbf{v} - \widetilde{\mathbf{v}}$.

Lemma 6.8. *There exist $0 < \delta_1, \varepsilon_0 \ll 1$ such that given any $v_1 \in V_{\delta_1,0,0}(\mathbb{T})$ with $\|v_1\|_{L^2(\mathbb{T})} \lesssim \varepsilon_0 \ll 1$ and given any $(\delta, x_0, \theta_0, w) \in \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \times V_{\delta_1,0,0}(\mathbb{T})$ with $|\delta - \delta_1| \ll \delta_1$, $|x_0| \ll 1$, $|\theta_0| \ll 1$, and $\|w - v_1\|_{L^2(\mathbb{T})} \ll 1$, we have*

$$\widetilde{\alpha} = \alpha + O\Big(A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha, \beta, \gamma, v)\Big), \tag{6.3.31}$$

$$\widetilde{\beta} = \beta + O\Big(A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha, \beta, \gamma, v)\Big), \tag{6.3.32}$$

$$\widetilde{\gamma} = \gamma + O\Big(\delta_1^{-1} A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha, \beta, \gamma, v)\Big), \tag{6.3.33}$$

$$\widetilde{v} = v + O_{L^2(\mathbb{T})}\Big(\delta_1^{-1} A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha, \beta, \gamma, v)\Big) \tag{6.3.34}$$

for any $(\alpha, \beta, \gamma, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times V_{\delta_1,0,0}(\mathbb{T})$, where $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{v})$ is defined by (6.3.30) and $A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha, \beta, \gamma, v)$ is defined by

$$\begin{aligned}A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha, \beta, \gamma, v) = &\Big(\delta_1^{-1}(|\delta - \delta_1| + |x_0|) + |\theta_0| + \exp(-c\delta_1^{-1}) + \|w - v_1\|_{L^2(\mathbb{T})}\Big) \\ &\times \Big(|\alpha| + |\beta| + \delta_1|\gamma| + \delta_1\|v\|_{L^2(\mathbb{T})} + \|v_1\|_{L^2(\mathbb{T})}\Big).\end{aligned}$$

Now, given small $\delta_1 > 0$, let us choose $|\delta - \delta_1| + |x_0| + |\theta_0| + \|w - v_1\|_{L^2(\mathbb{T})} \lesssim \delta_1^3$. Then, Lemmas 6.7 and 6.8 allow us to apply Lemma 6.6 with $R \sim \delta_1^3$ and $\kappa \sim \delta_1$ and conclude that the image of $F = F_{\delta_1,0,0}$ contains a ball of radius $r \sim \delta_1^3$ around $F(\delta_1, 0, 0, v_1)$. Recalling (6.3.25), we see that u indeed lies in the image of F. Lastly, we need to choose $\delta_1 = \delta_1(\varepsilon_1) > 0$ sufficiently small such that $R = c\delta_1^3 \le \varepsilon_1$. This concludes the proof of Proposition 6.4. $\square$

We conclude this subsection by presenting the proofs of Lemmas 6.7 and 6.8. In the following, $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{T})$.

Proof of Lemma 6.7. Let $(\alpha, \beta, \gamma, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times V_{\delta_1,0,0}(\mathbb{T})$. Then, with $F = F_{\delta_1,0,0}$, we have

$$\begin{aligned} \widetilde{v} &:= \big(\mathrm{d}F(\delta_1, 0, 0, v_1)\big)(\alpha, \beta, \gamma, v) \\ &= \alpha \partial_\delta Q_{\delta_1}^\rho + \beta \partial_{x_0} Q_{\delta_1}^\rho + \gamma i Q_{\delta_1}^\rho \\ &\quad + \big(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha, \beta, \gamma)\big) v_1|_{(\delta,x_0,\theta_0)=(\delta_1,0,0)} + v \in L^2(\mathbb{T}), \end{aligned} \tag{6.3.35}$$

where $\partial_{x_0} Q_{\delta_1}^\rho = \partial_{x_0}\big((\tau_{x_0}\rho) Q_{\delta_1,x_0})\big)|_{x_0=0}$ is as in (6.1.11) and the fourth term on the right-hand side is as in (6.3.28).

• **Case 1:** We first consider the case when $v_1 = 0$.

Then, by the orthogonality of $\partial_\delta Q_{\delta_1}^\rho$, $\partial_{x_0} Q_{\delta_1}^\rho$, and $iQ_{\delta_1}^\rho$ in $H^1(\mathbb{T})$ (see Remark 6.1) and the definition (6.3.3) of $V_{\delta_1,0,0}$, we have

$$\alpha = \frac{\langle \widetilde{v}, (1 - \partial_x^2)\partial_\delta Q_{\delta_1}^\rho \rangle}{\langle \partial_\delta Q_{\delta_1}^\rho, (1 - \partial_x^2)\partial_\delta Q_{\delta_1}^\rho \rangle}, \tag{6.3.36}$$

$$\beta = \frac{\langle \widetilde{v}, (1 - \partial_x^2)\partial_{x_0} Q_{\delta_1}^\rho \rangle}{\langle \partial_{x_0} Q_{\delta_1}^\rho, (1 - \partial_x^2)\partial_{x_0} Q_{\delta_1}^\rho \rangle}, \tag{6.3.37}$$

$$\gamma = \frac{\langle \widetilde{v}, (1 - \partial_x^2) i Q_{\delta_1}^\rho \rangle}{\langle Q_{\delta_1}^\rho, (1 - \partial_x^2) Q_{\delta_1}^\rho \rangle}, \tag{6.3.38}$$

$$v = \widetilde{v} - \alpha \partial_\delta Q_{\delta_1}^\rho - \beta \partial_{x_0} Q_{\delta_1}^\rho - \gamma i Q_{\delta_1}^\rho. \tag{6.3.39}$$

From (6.3.36), we obtain

$$|\alpha| \le \|\widetilde{v}\|_{L^2(\mathbb{T})} \frac{\|(1 - \partial_x^2)\partial_\delta Q_{\delta_1}^\rho\|_{L^2(\mathbb{T})}}{|\langle \partial_\delta Q_{\delta_1}^\rho, (1 - \partial_x^2)\partial_\delta Q_{\delta_1}^\rho \rangle|}. \tag{6.3.40}$$

By a direct computation with (6.1.8) and (6.1.9), we have

$$\begin{aligned} &\langle \partial_\delta Q_{\delta_1}^\rho, (1 - \partial_x^2)\partial_\delta Q_{\delta_1}^\rho \rangle \\ &\quad = \langle \rho \partial_\delta Q_{\delta_1}, (1 - \partial_x^2)(\rho \partial_\delta Q_{\delta_1}) \rangle \\ &\quad = \delta_1^{-4} \int_{-\frac{1}{2\delta_1}}^{\frac{1}{2\delta_1}} \rho(\delta_1 x) A_1(x) \\ &\qquad \times \Big(\delta_1^2\big(\rho(\delta_1 x) - \rho''(\delta_1 x)\big)A_1(x) - 2\delta_1 \rho'(\delta_1 x) A_2(x) - A_3(x)\Big)\mathrm{d}x \\ &\quad = \delta_1^{-4} \int_{\mathbb{R}} A_1(x)\big(\delta_1^2 A_1(x) - A_3(x)\big)\mathrm{d}x + O(\exp(-c\delta_1^{-1})) \\ &\quad \sim \delta_1^{-4} \end{aligned} \tag{6.3.41}$$

for $0 < \delta_1 \ll 1$, where

$$\begin{aligned} A_1(x) &:= -\frac{1}{2}Q(x) - xQ'(x), \qquad A_2(x) := -\frac{3}{2}Q'(x) - xQ''(x), \\ \text{and} \qquad A_3(x) &:= -\frac{5}{2}Q''(x) - xQ'''(x). \end{aligned} \tag{6.3.42}$$

By a similar computation, we have

$$\begin{aligned} \|(1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\|^2_{L^2(\mathbb{T})} &= \delta_1^{-6}\int_{\mathbb{R}} \big(\delta_1^2 A_1(x) - A_3(x)\big)^2 \mathrm{d}x + O\big(\exp(-c\delta_1^{-1})\big) \\ &\sim \delta_1^{-6} \end{aligned} \tag{6.3.43}$$

for $0 < \delta_1 \ll 1$. Hence, from (6.3.40), (6.3.41), and (6.3.43) with (6.3.35), we obtain

$$|\alpha| \lesssim \delta_1 \big\| \big(\mathrm{d}F(\delta_1, 0,0,0)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})} \tag{6.3.44}$$

for $0 < \delta_1 \ll 1$.

Proceeding analogously, we have

$$\langle \partial_{x_0} Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta_1}\rangle \sim \delta_1^{-4}, \tag{6.3.45}$$

$$\|(1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta_1}\|^2_{L^2(\mathbb{T})} \sim \delta_1^{-6} \tag{6.3.46}$$

for $0 < \delta_1 \ll 1$. Hence, from (6.3.37), (6.3.45), and (6.3.46), we obtain

$$|\beta| \lesssim \delta_1 \big\| \big(\mathrm{d}F(\delta_1, 0,0,0)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})} \tag{6.3.47}$$

for $0 < \delta_1 \ll 1$. Similarly, we have

$$\langle Q^\rho_{\delta_1}, (1-\partial_x^2) Q^\rho_{\delta_1}\rangle \sim \delta_1^{-2}, \tag{6.3.48}$$

$$\|(1-\partial_x^2) Q^\rho_{\delta_1}\|^2_{L^2(\mathbb{T})} \sim \delta_1^{-4} \tag{6.3.49}$$

for $0 < \delta_1 \ll 1$. Hence, from (6.3.38), (6.3.48), and (6.3.49), we obtain

$$|\gamma| \lesssim \big\| \big(\mathrm{d}F(\delta_1, 0,0,0)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})}. \tag{6.3.50}$$

Lastly, from (6.3.39), (6.3.44), (6.3.47), and (6.3.50) with $\|\partial_\delta Q^\rho_{\delta_1}\|_{L^2(\mathbb{T})} \sim \|\partial_{x_0} Q^\rho_{\delta_1}\|_{L^2(\mathbb{T})} \sim \delta_1^{-1}$ and $\|Q^\rho_{\delta_1}\|_{L^2(\mathbb{T})} \sim 1$, we obtain

$$\|v\|_{L^2(\mathbb{T})} \lesssim \big\| \big(\mathrm{d}F(\delta_1, 0,0,0)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})} \tag{6.3.51}$$

for $0 < \delta_1 \ll 1$. Therefore, we conclude from (6.3.44), (6.3.47), (6.3.50), and (6.3.51) that the inverse $\big(\mathrm{d}F(\delta_1,0,0,0)\big)^{-1}$ of $\mathrm{d}F$ at $(\delta_1,0,0,0)$ has a norm bounded by a constant, uniformly in sufficiently small $\delta_1 > 0$.

• **Case 2:** Next, we consider the case when $v_1 \neq 0$ with $\|v_1\|_{L^2(\mathbb{T})} \ll 1$.
In this case, from (6.3.35), we have

$$\alpha = \frac{\langle \widetilde{v}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle} - \frac{\langle V_1, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}, \tag{6.3.52}$$

$$\beta = \frac{\langle \widetilde{v}, (1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta_1}\rangle}{\langle \partial_{x_0} Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta_1}\rangle} - \frac{\langle V_1, (1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta_1}\rangle}{\langle \partial_{x_0} Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta_1}\rangle}, \tag{6.3.53}$$

$$\gamma = \frac{\langle \widetilde{v}, (1-\partial_x^2) i Q^\rho_{\delta_1}\rangle}{\langle Q^\rho_{\delta_1}, (1-\partial_x^2) Q^\rho_{\delta_1}\rangle} - \frac{\langle V_1 (1-\partial_x^2) i Q^\rho_{\delta_1}\rangle}{\langle Q^\rho_{\delta_1}, (1-\partial_x^2) Q^\rho_{\delta_1}\rangle}, \tag{6.3.54}$$

$$v = \widetilde{v} - \alpha\partial_\delta Q^\rho_{\delta_1} - \beta\partial_{x_0} Q^\rho_{\delta_1} - \gamma i Q^\rho_{\delta_1} - V_1. \tag{6.3.55}$$

where $V_1 = V_1(\alpha,\beta,\gamma)$ is given by

$$V_1 := \mathrm{d}P_{V_{\delta_1,0,0}}(\alpha,\beta,\gamma)\big)v_1 = \big(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha,\beta,\gamma)\big)v_1|_{(\delta,x_0,\theta_0)=(\delta_1,0,0)}. \tag{6.3.56}$$

From (6.3.28) and (6.3.23), we have

$$\begin{aligned}
&(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha,\beta,\gamma))v_1 \\
&\quad = -\alpha\partial_\delta\bigg(\frac{\langle v_1, (1-\partial_x^2)e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2)\partial_\delta Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}\bigg) \\
&\quad\quad - \beta\partial_{x_0}\bigg(\frac{\langle v_1, (1-\partial_x^2)e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2)\partial_\delta Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}\bigg) \\
&\quad\quad - \gamma\partial_{\theta_0}\bigg(\frac{\langle v_1, (1-\partial_x^2)e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2)\partial_\delta Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}\bigg) \\
&\quad\quad - \alpha\partial_\delta\bigg(\frac{\langle v_1, (1-\partial_x^2)e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0}\bigg) \\
&\quad\quad - \beta\partial_{x_0}\bigg(\frac{\langle v_1, (1-\partial_x^2)e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0}\bigg) \\
&\quad\quad - \gamma\partial_{\theta_0}\bigg(\frac{\langle v_1, (1-\partial_x^2)e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0}\bigg) \\
&\quad\quad - \alpha\partial_\delta\bigg(\frac{\langle v_1, (1-\partial_x^2)ie^{i\theta_0} Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2) Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)ie^{i\theta_0} Q^\rho_{\delta,x_0}\bigg) \\
&\quad\quad - \beta\partial_{x_0}\bigg(\frac{\langle v_1, (1-\partial_x^2)ie^{i\theta_0} Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2) Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)ie^{i\theta_0} Q^\rho_{\delta,x_0}\bigg) \\
&\quad\quad - \gamma\partial_{\theta_0}\bigg(\frac{\langle v_1, (1-\partial_x^2)ie^{i\theta_0} Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2) Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)ie^{i\theta_0} Q^\rho_{\delta,x_0}\bigg).
\end{aligned} \tag{6.3.57}$$

The derivatives appearing above are given by

$$
\begin{aligned}
&\partial_{\kappa_1}\left(\frac{\langle v_1,(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\rangle}{\|(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\right)\\
&\qquad=\frac{\langle v_1,(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\rangle}{\|(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)\partial^2_{\kappa_1,\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\\
&\qquad\quad+\frac{\langle v_1,(1-\partial_x^2)\partial^2_{\kappa_1,\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\rangle}{\|(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\|^2_{L^2(\mathbb{T})}}(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\\
&\qquad\quad-2\langle(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0}),(1-\partial_x^2)\partial^2_{\kappa_1,\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\rangle\\
&\qquad\quad\times\frac{\langle v_1,(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\rangle}{\|(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})\|^4_{L^2(\mathbb{T})}}(1-\partial_x^2)\partial_{\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})
\end{aligned}
\tag{6.3.58}
$$

for $\kappa_1,\kappa_2\in\{\delta,x_0,\theta_0\}$. By estimating (6.3.58) with (6.3.41), (6.3.43), (6.3.45), (6.3.46), (6.3.48), (6.3.49), and similar computations for $(1-\partial_x^2)\partial^2_{\kappa_1,\kappa_2}(e^{i\theta_0}Q^{\rho}_{\delta,x_0})$, it is a simple task to show

$$\big\|\big(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha,\beta,\gamma)\big)v_1\big\|_{L^2(\mathbb{T})}\lesssim\big(\delta^{-1}(|\alpha|+|\beta|)+|\gamma|\big)\|v_1\|_{L^2(\mathbb{T})}, \tag{6.3.59}$$

uniformly in $x_0\in\mathbb{T}$ and $\theta_0\in\mathbb{R}/2\pi\mathbb{Z}$. In particular, with (6.3.56), we have

$$\|V_1\|_{L^2(\mathbb{T})}\lesssim\big(\delta_1^{-1}(|\alpha|+|\beta|)+|\gamma|\big)\|v_1\|_{L^2(\mathbb{T})}, \tag{6.3.60}$$

From (6.3.52), (6.3.41), and (6.3.43), we obtain

$$|\alpha|\lesssim\delta_1\big\|\big(\mathrm{d}F(\delta_1,0,0,v_1)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})}+\delta_1\|V_1\|_{L^2(\mathbb{T})}. \tag{6.3.61}$$

Similarly, from (6.3.53), we have

$$|\beta|\lesssim\delta_1\big\|\big(\mathrm{d}F(\delta_1,0,0,v_1)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})}+\delta_1\|V_1\|_{L^2(\mathbb{T})}. \tag{6.3.62}$$

Compare these with (6.3.44) and (6.3.47). From (6.3.61) and (6.3.62) with (6.3.60), we then obtain

$$|\alpha|+|\beta|\lesssim\delta_1\big\|\big(\mathrm{d}F(\delta_1,0,0,v_1)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})}+\delta_1|\gamma|\|v_1\|_{L^2(\mathbb{T})}, \tag{6.3.63}$$

provided that $\|v_1\|_{L^2(\mathbb{T})}\ll 1$. By a similar computation with (6.3.54), we have

$$|\gamma|\lesssim\big\|\big(\mathrm{d}F(\delta_1,0,0,v_1)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})}+\|V_1\|_{L^2(\mathbb{T})}. \tag{6.3.64}$$

Then, from (6.3.64), (6.3.60), and (6.3.63), we obtain

$$|\gamma|\lesssim\big\|\big(\mathrm{d}F(\delta_1,0,0,v_1)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})}. \tag{6.3.65}$$

Hence, from (6.3.63) and (6.3.65), we obtain

$$|\alpha|+|\beta|\lesssim\delta_1\big\|\big(\mathrm{d}F(\delta_1,0,0,v_1)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})}. \tag{6.3.66}$$

Proceeding as in Case 1 with (6.3.55), (6.3.65), and (6.3.66), we obtain

$$\|v\|_{L^2(\mathbb{T})}\lesssim\big\|\big(\mathrm{d}F(\delta_1,0,0,v_1)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})}+\|V_1\|_{L^2(\mathbb{T})}. \tag{6.3.67}$$

Then, using (6.3.60), (6.3.65), and (6.3.66), we conclude from (6.3.67) that

$$\|v\|_{L^2(\mathbb{T})}\lesssim\big\|\big(\mathrm{d}F(\delta_1,0,0,v_1)\big)(\alpha,\beta,\gamma,v)\big\|_{L^2(\mathbb{T})}.$$

This completes the proof of Lemma 6.7. □

Next, we present the proof of Lemma 6.8.

Proof of Lemma 6.8. From (6.3.29), we have

$$\begin{aligned}\widetilde{u} &= \big(\mathrm{d}F(\delta_1, 0, 0, v_1)\big)(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{v}) \\ &= \widetilde{\alpha}\partial_\delta Q^\rho_{\delta_1} + \widetilde{\beta}\partial_{x_0} Q^\rho_{\delta_1} + \widetilde{\gamma} i Q^\rho_{\delta_1} + V_1 + \widetilde{v},\end{aligned} \tag{6.3.68}$$

where $V_1 = V_1(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}) = \big(\mathrm{d}P_{V_{\delta_1,0,0}}(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})\big)v_1$ is as in (6.3.56) with α, β, γ replaced by $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$. Then, using (6.3.27), we can write the first component $\widetilde{\alpha}$ of $\widetilde{\mathbf{v}}$ as

$$\begin{aligned}\widetilde{\alpha} &= \frac{\langle \widetilde{u}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle} - \frac{\langle V_1, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle} \\ &= \frac{1}{\|\partial_\delta Q^\rho_{\delta_1}\|^2_{H^1(\mathbb{T})}}\langle \alpha e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0} + \beta e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0} + \gamma i e^{i\theta_0} Q^\rho_{\delta,x_0} \\ &\qquad\qquad + P_{V_{\delta,x_0,\theta_0}} v, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle \\ &\quad + \frac{\langle W - V_1, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle},\end{aligned} \tag{6.3.69}$$

where $W = W(\alpha, \beta, \gamma) = \big(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha, \beta, \gamma)\big)w$.

The main term of the numerator in the last expression is $\alpha\langle e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle$. Since ρ and Q are real-valued, we see that

$$\begin{aligned}\langle e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle &= \cos\theta_0 \cdot \langle \partial_\delta Q^\rho_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle \\ &= \big(1 + O(\theta_0^2)\big)\langle \partial_\delta Q^\rho_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle\end{aligned}$$

for $|\theta_0| \ll 1$. Recall from (6.1.12) and (6.3.42) that

$$\partial_\delta Q_{\delta,x_0} = \delta^{-\frac{3}{2}} A_1(\delta^{-1}(x - x_0)) \qquad \text{and} \qquad \partial_\delta Q_{\delta_1} = \delta_1^{-\frac{3}{2}} A_1(\delta_1^{-1}x). \tag{6.3.70}$$

Without loss of generality, assume that $0 < x_0 \ll 1$. By the mean value theorem, we have

$$\delta^{-\frac{3}{2}}\big(A_1(\delta^{-1}(x - x_0)) - A_1(\delta^{-1}x)\big) \sim \delta^{-\frac{5}{2}} A_1'(\delta^{-1}x)|x_0| \tag{6.3.71}$$

for $x \in [-\frac{3}{8}, \frac{3}{8}]$. Then, by (6.3.70), (6.3.71), and repeating a computation analogous to (6.3.41) with (6.1.7) and the mean value theorem applied to $\tau_{x_0}\rho - \rho$, we have

$$\begin{aligned}&\langle \partial_\delta Q^\rho_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle \\ &= \langle \partial_\delta Q^\rho_{\delta}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle + \langle \rho\delta^{-\frac{3}{2}}(A_1(\delta^{-1}(\cdot - x_0)) - A_1(\delta^{-1}\cdot)), (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle \\ &\quad + \langle (\tau_{x_0}\rho - \rho)\partial_\delta Q_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle \\ &= \langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle + O\big(\delta_1^{-5}(|x_0| + |\delta - \delta_1|)\big) + O(\exp(-c\delta_1^{-1})).\end{aligned} \tag{6.3.72}$$

Hence, from (6.3.41) and (6.3.72), we obtain

$$\frac{\langle e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle} = 1 + O\big(\theta_0^2 + \delta_1^{-1}(|x_0| + |\delta - \delta_1|) + \exp(-c\delta_1^{-1})\big). \tag{6.3.73}$$

A similar computation with the orthogonality of $\partial_\delta Q^\rho_{\delta_1}$ and $\partial_{x_0} Q^\rho_{\delta_1}$ in $H^1(\mathbb{T})$ (where $\partial_{x_0} Q^\rho_{\delta_1}$ is as in (6.1.11)) gives

$$\langle \partial_{x_0} Q^\rho_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle = O\big(\delta_1^{-5}(|x_0|+|\delta-\delta_1|)+\exp(-c\delta_1^{-1})\big),$$

and together with (6.3.41), we obtain

$$\frac{\langle e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle} = O\big(\delta_1^{-1}(|x_0|+|\delta-\delta_1|)+\exp(-c\delta_1^{-1})\big). \tag{6.3.74}$$

By Cauchy-Schwarz inequality with (6.3.41), (6.3.43), and $\|Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}\sim 1$, we have

$$\left|\frac{\langle ie^{i\theta_0} Q^\rho_{\delta,x_0}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}\right| \lesssim \delta_1|\sin(\theta_0)| = O(\delta_1|\theta_0|). \tag{6.3.75}$$

From (6.1.12) and (6.3.42), we have

$$\partial_x^2\partial_\delta Q_{\delta,x_0} = \delta^{-\frac{7}{2}}A_3(\delta^{-1}(x-x_0)) \qquad \text{and} \qquad \partial_x^2\partial_\delta Q_{\delta_1} = \delta_1^{-\frac{7}{2}}A_3(\delta_1^{-1}x).$$

Then, by (6.3.3) and repeating a similar computation as above with the mean value theorem, we have

$$\begin{aligned}
&\left|\frac{\langle P_{V_{\delta,x_0,\theta_0}} v, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}\right| \\
&\quad = \left|\frac{\langle P_{V_{\delta,x_0,\theta_0}} v, (1-\partial_x^2)(\partial_\delta Q^\rho_{\delta_1} - e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}\right| \\
&\quad \lesssim \big(\delta_1|\theta_0| + |x_0| + |\delta-\delta_1| + \exp(-c\delta_1^{-1})\big)\|v\|_{L^2(\mathbb{T})}.
\end{aligned} \tag{6.3.76}$$

We now consider the last term in (6.3.69). We write $W - V_1$ as

$$\begin{aligned}
W - V_1 &= \big(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha,\beta,\gamma)\big)(w-v_1) \\
&\qquad + \big(\mathrm{d}P_{V_{\delta,x_0,\theta_0}}(\alpha,\beta,\gamma) - \mathrm{d}P_{V_{\delta_1,0,0}}(\alpha,\beta,\gamma)\big)v_1 \\
&\qquad + \big(\mathrm{d}P_{V_{\delta_1,0,0}}(\alpha,\beta,\gamma) - \mathrm{d}P_{V_{\delta_1,0,0}}(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma})\big)v_1 \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{6.3.77}$$

By Cauchy-Schwarz inequality with (6.3.59), (6.3.41), and (6.3.43), we have

$$\begin{aligned}
\left|\frac{\langle \mathrm{I}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}\right| &\lesssim \frac{\delta_1}{\delta}\big(|\alpha|+|\beta|+\delta|\gamma|\big)\|w-v_1\|_{L^2(\mathbb{T})} \\
&\lesssim \big(|\alpha|+|\beta|+\delta|\gamma|\big)\|w-v_1\|_{L^2(\mathbb{T})}.
\end{aligned} \tag{6.3.78}$$

Proceeding as above with (6.3.57), (6.3.58), and the mean value theorem, we have

$$\begin{aligned}
\|\mathrm{II}\|_{L^2(\mathbb{T})} &\lesssim \big(|\alpha|+|\beta|+\delta|\gamma|\big) \\
&\qquad \times \big(\delta_1^{-1}|\theta_0| + \delta_1^{-2}(|x_0|+|\delta-\delta_1|) + \exp(-c\delta_1^{-1})\big)\|v_1\|_{L^2(\mathbb{T})}.
\end{aligned} \tag{6.3.79}$$

Thus, we obtain

$$\begin{aligned}
&\left|\frac{\langle \mathrm{II}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}\right| \\
&\quad \lesssim \big(|\alpha|+|\beta|+\delta|\gamma|\big)\big(|\theta_0| + \delta_1^{-1}(|x_0|+|\delta-\delta_1|) + \exp(-c\delta_1^{-1})\big)\|v_1\|_{L^2(\mathbb{T})}.
\end{aligned} \tag{6.3.80}$$

Lastly, we consider Ⅲ in (6.3.77). Recalling that $v_1 \in V_{\delta_1,0,0}$, it follows from (6.3.57) with the orthogonality of $\partial_\delta Q^\rho_{\delta_1}$, $\partial_{x_0} Q^\rho_{\delta_1}$, and $iQ^\rho_{\delta_1}$ in $H^2(\mathbb{T})$ that

$$\begin{aligned}\langle \text{Ⅲ}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle &= \big\langle \big(\mathrm{d}P_{V_{\delta_1,0,0}}(\alpha-\widetilde\alpha, \beta-\widetilde\beta, \gamma-\widetilde\gamma)\big)v_1, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\big\rangle\\ &= -(\alpha-\widetilde\alpha)\langle v_1, (1-\partial_x^2)\partial_\delta^2 Q^\rho_{\delta_1}\rangle - (\beta-\widetilde\beta)\langle v_1, (1-\partial_x^2)\partial_{x_0}\partial_\delta Q^\rho_{\delta_1}\rangle\\ &\quad - (\gamma-\widetilde\gamma)\langle v_1, (1-\partial_x^2) i\partial_\delta Q^\rho_{\delta_1}\rangle.\end{aligned}$$

Then, from $\|v_1\|_{L^2(\mathbb{T})} \lesssim \varepsilon_0 \ll 1$, we have

$$\frac{\langle \text{Ⅲ}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle}{\langle \partial_\delta Q^\rho_{\delta_1}, (1-\partial_x^2)\partial_\delta Q^\rho_{\delta_1}\rangle} = c_1(\alpha-\widetilde\alpha) + c_2(\beta-\widetilde\beta) + c_3\delta_1(\gamma-\widetilde\gamma), \tag{6.3.81}$$

where $c_j = c_j(\varepsilon_0) = O(\varepsilon_0)$, $j = 1,2,3$.

Combining (6.3.69), (6.3.73) - (6.3.78), (6.3.80), and (6.3.81), we have

$$(1+c_1)(\widetilde\alpha-\alpha) + c_2(\widetilde\beta-\beta) + c_3\delta_1(\widetilde\gamma-\gamma) = O\big(A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha,\beta,\gamma,v)\big). \tag{6.3.82}$$

By a similar computation, we also obtain

$$c_4(\widetilde\alpha-\alpha) + (1+c_5)(\widetilde\beta-\beta) + c_6\delta_1(\widetilde\gamma-\gamma) = O\big(A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha,\beta,\gamma,v)\big), \tag{6.3.83}$$

where $c_j = c_j(\varepsilon_0) = O(\varepsilon_0)$, $j = 4,5,6$.

As for the estimate (6.3.33), it follows from (6.3.68) with (6.3.27) that

$$\begin{aligned}\widetilde\gamma &= \frac{\langle \widetilde u, (1-\partial_x^2) iQ^\rho_{\delta_1}\rangle}{\langle Q^\rho_{\delta_1}, (1-\partial_x^2)Q^\rho_{\delta_1}\rangle} - \frac{\langle V_1, (1-\partial_x^2) iQ^\rho_{\delta_1}\rangle}{\langle Q^\rho_{\delta_1}, (1-\partial_x^2)Q^\rho_{\delta_1}\rangle}\\ &= \frac{1}{\|Q^\rho_{\delta_1}\|^2_{H^1(\mathbb{T})}}\langle \alpha e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0} + \beta e^{i\theta_0}\partial_{x_0}Q^\rho_{\delta,x_0} + \gamma i e^{i\theta_0} Q^\rho_{\delta,x_0}\\ &\qquad\qquad + P_{V_{\delta,x_0,\theta_0}}v, (1-\partial_x^2) iQ^\rho_{\delta_1}\rangle\\ &\quad + \frac{\langle W - V_1, (1-\partial_x^2) iQ^\rho_{\delta_1}\rangle}{\langle Q^\rho_{\delta_1}, (1-\partial_x^2)Q^\rho_{\delta_1}\rangle}.\end{aligned} \tag{6.3.84}$$

Then, proceeding as before with the mean value theorem, (6.3.48), and (6.3.49). the main contribution to (6.3.84) is given by

$$\begin{aligned}&\frac{\langle \gamma i e^{i\theta_0} Q^\rho_{\delta,x_0}, (1-\partial_x^2) iQ^\rho_{\delta_1}\rangle}{\langle Q^\rho_{\delta_1}, (1-\partial_x^2)Q^\rho_{\delta_1}\rangle}\\ &\quad = \gamma\Big(1 + O\big(\theta_0^2 + \delta_1^{-1}(|x_0| + |\delta-\delta_1|) + \exp(-c\delta_1^{-1})\big)\Big),\end{aligned} \tag{6.3.85}$$

while the contribution to $\widetilde\gamma$ from the terms involving α and β can be bounded by

$$\big(|\alpha| + |\beta|\big)\big(\delta_1^{-1}|\theta_0| + \delta_1^{-2}(|x_0| + |\delta-\delta_1|) + \exp(-c\delta_1^{-1})\big). \tag{6.3.86}$$

Proceeding as in (6.3.76), we have

$$\begin{aligned}\bigg|\frac{\langle P_{V_{\delta,x_0,\theta_0}}v, (1-\partial_x^2) iQ^\rho_{\delta_1}\rangle}{\langle Q^\rho_{\delta_1}, (1-\partial_x^2)Q^\rho_{\delta_1}\rangle}\bigg| &= \bigg|\frac{\langle P_{V_{\delta,x_0,\theta_0}}v, i(1-\partial_x^2)(Q^\rho_{\delta_1} - e^{i\theta_0}Q^\rho_{\delta,x_0}\rangle}{\langle Q^\rho_{\delta_1}, (1-\partial_x^2)Q^\rho_{\delta_1}\rangle}\bigg|\\ &\lesssim \big(|\theta_0| + \delta^{-1}(|x_0| + |\delta-\delta_1|) + \exp(-c\delta_1^{-1})\big)\|v\|_{L^2(\mathbb{T})}.\end{aligned} \tag{6.3.87}$$

By Cauchy-Schwarz inequality with (6.3.77) and (6.3.59), we have

$$\bigg|\frac{\langle \mathrm{I}, (1-\partial_x^2) i Q_{\delta_1}^\rho\rangle}{\langle Q_{\delta_1}^\rho, (1-\partial_x^2) Q_{\delta_1}^\rho\rangle}\bigg| \lesssim \big(\delta^{-1}(|\alpha|+|\beta|)+|\gamma|\big)\|w-v_1\|_{L^2(\mathbb{T})}. \tag{6.3.88}$$

From (6.3.79), we have

$$\begin{aligned}\bigg|\frac{\langle \mathrm{II}, (1-\partial_x^2) i Q_{\delta_1}^\rho\rangle}{\langle Q_{\delta_1}^\rho, (1-\partial_x^2) Q_{\delta_1}^\rho\rangle}\bigg| &\lesssim \big(|\alpha|+|\beta|+\delta|\gamma|\big) \\ &\quad\times \big(\delta_1^{-1}|\theta_0| + \delta_1^{-2}(|x_0|+|\delta-\delta_1|) + \exp(-c\delta_1^{-1})\big)\|v_1\|_{L^2(\mathbb{T})}.\end{aligned} \tag{6.3.89}$$

Next, we consider III in (6.3.77). Recalling that $v_1 \in V_{\delta_1,0,0}$, it follows from (6.3.57) with the orthogonality of $\partial_\delta Q_{\delta_1}^\rho$, $\partial_{x_0} Q_{\delta_1}^\rho$, and $iQ_{\delta_1}^\rho$ in $H^2(\mathbb{T})$ that

$$\begin{aligned}\langle \mathrm{III}, (1-\partial_x^2) i Q_{\delta_1}^\rho\rangle &= -(\alpha-\widetilde{\alpha})\langle v_1, (1-\partial_x^2) i\partial_\delta Q_{\delta_1}^\rho\rangle - (\beta-\widetilde{\beta})\langle v_1, (1-\partial_x^2) i\partial_{x_0} Q_{\delta_1}^\rho\rangle \\ &\quad + (\gamma-\widetilde{\gamma})\langle v_1, (1-\partial_x^2) Q_{\delta_1}^\rho\rangle.\end{aligned}$$

Then, from $\|v_1\|_{L^2(\mathbb{T})} \lesssim \varepsilon_0 \ll 1$, we have

$$\frac{\langle \mathrm{III}, (1-\partial_x^2) i Q_{\delta_1}^\rho\rangle}{\langle Q_{\delta_1}^\rho, (1-\partial_x^2) Q_{\delta_1}^\rho\rangle} = c_7\delta_1^{-1}(\alpha-\widetilde{\alpha}) + c_8\delta_1^{-1}(\beta-\widetilde{\beta}) + c_9(\gamma-\widetilde{\gamma}), \tag{6.3.90}$$

where $c_j = c_j(\varepsilon_0) = O(\varepsilon_0)$, $j = 7, 8, 9$.

Hence, from (6.3.84) - (6.3.90), we obtain

$$\begin{aligned}c_7\delta_1^{-1}(\widetilde{\alpha}-\alpha) + c_8\delta_1^{-1}(\widetilde{\beta}-\beta) + (1+c_9)(\widetilde{\gamma}-\gamma) \\ = O\big(\delta_1^{-1} A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha,\beta,\gamma,v)\big).\end{aligned} \tag{6.3.91}$$

By solving the system of linear equations (6.3.82), (6.3.83), and (6.3.91) for $\widetilde{\alpha}-\alpha$, $\widetilde{\beta}-\beta$, and $\widetilde{\gamma}-\gamma$, we obtain (6.3.31), (6.3.32), and (6.3.33).

Finally, we turn to (6.3.34). By (6.3.68) and (6.3.27), $\widetilde{v}$ is given by

$$\begin{aligned}\widetilde{v} &= \widetilde{u} - \widetilde{\alpha}\partial_\delta Q_{\delta_1}^\rho - \widetilde{\beta}\partial_{x_0} Q_{\delta_1}^\rho - \widetilde{\gamma} i Q_{\delta_1}^\rho - V_1 \\ &= \alpha e^{i\theta_0}\partial_\delta Q_{\delta,x_0}^\rho + \beta e^{i\theta_0}\partial_{x_0} Q_{\delta,x_0}^\rho + \gamma i e^{i\theta_0} Q_{\delta,x_0}^\rho \\ &\quad + P_{V_{\delta,x_0,\theta_0}} v - \widetilde{\alpha}\partial_\delta Q_{\delta_1}^\rho - \widetilde{\beta}\partial_{x_0} Q_{\delta_1}^\rho - \widetilde{\gamma} i Q_{\delta_1}^\rho \\ &\quad + W - V_1.\end{aligned} \tag{6.3.92}$$

For $v \in V_{\delta_1,0,0}(\mathbb{T})$, from (6.3.3), (6.3.43), (6.3.46), (6.3.49), and computations similar to (6.3.73), we have

$$\begin{aligned}
P_{V_{\delta,x_0,\theta_0}} v &= v - \frac{\langle v, (1-\partial_x^2) e^{i\theta_0} \partial_\delta Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2)\partial_\delta Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2) e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0} \\
&\quad - \frac{\langle v, (1-\partial_x^2) e^{i\theta_0} \partial_{x_0} Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2) e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0} \\
&\quad - \frac{\langle v, (1-\partial_x^2) i e^{i\theta_0} Q^\rho_{\delta,x_0}\rangle}{\|(1-\partial_x^2) Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2) i e^{i\theta_0} Q^\rho_{\delta,x_0} \\
&= v - \frac{\langle v, (1-\partial_x^2) (e^{i\theta_0} \partial_\delta Q^\rho_{\delta,x_0} - \partial_\delta Q^\rho_{\delta_1})\rangle}{\|(1-\partial_x^2)\partial_\delta Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2) e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0} \\
&\quad - \frac{\langle v, (1-\partial_x^2) (e^{i\theta_0} \partial_{x_0} Q^\rho_{\delta,x_0} - \partial_{x_0} Q^\rho_{\delta_1})\rangle}{\|(1-\partial_x^2)\partial_{x_0} Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2) e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0} \\
&\quad - \frac{\langle v, (1-\partial_x^2) i (e^{i\theta_0} Q^\rho_{\delta,x_0} - Q^\rho_{\delta_1})\rangle}{\|(1-\partial_x^2) Q^\rho_{\delta,x_0}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2) i e^{i\theta_0} Q^\rho_{\delta,x_0} \\
&= v + O_{L^2(\mathbb{T})}\Big(\big(\delta_1^{-1}(|x_0| + |\delta - \delta_1|) + |\theta_0| + \exp(-c\delta_1^{-1})\big) \|v\|_{L^2(\mathbb{T})}\Big).
\end{aligned} \tag{6.3.93}$$

(Note that we could have obtained the same estimate integrating (6.3.59) over the curve $(\alpha_s, \beta_s, \gamma_s) = (s(\delta - \delta_1), sx_0, s\theta_0), 0 \le s \le 1$).

From (6.1.7), (6.3.70), and (6.3.31), we have

$$\begin{aligned}
&\alpha e^{i\theta_0}\partial_\delta Q^\rho_{\delta,x_0} - \widetilde{\alpha}\partial_\delta Q^\rho_{\delta_1} \\
&\quad = \alpha(\tau_{x_0}\rho)(e^{i\theta_0}\partial_\delta Q_{\delta,x_0} - \partial_\delta Q_{\delta_1}) \\
&\quad\quad + \alpha((\tau_{x_0}\rho)\partial_\delta Q_{\delta_1} - \rho\partial_\delta Q_{\delta_1}) + (\alpha - \widetilde{\alpha})\partial_\delta Q^\rho_{\delta_1} \\
&\quad = O_{L^2(\mathbb{T})}\Big(\big(\delta_1^{-2}(|x_0| + |\delta - \delta_1|) + \delta_1^{-1}|\theta_0| + \exp(-c\delta_1^{-1})\big)|\alpha| + \delta_1^{-1}|\alpha - \widetilde{\alpha}|\Big) \\
&\quad = O_{L^2(\mathbb{T})}\Big(\delta_1^{-1} A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha,\beta,\gamma,v)\Big).
\end{aligned} \tag{6.3.94}$$

Similarly, from (6.1.11) and (6.3.32), we have

$$\begin{aligned}
&\beta e^{i\theta_0}\partial_{x_0} Q^\rho_{\delta,x_0} - \widetilde{\beta}\partial_{x_0} Q^\rho_{\delta_1} \\
&\quad = \beta\big(e^{i\theta_0}\partial_{x_0}\big((\tau_{x_0}\rho)Q_{\delta,x_0}\big) - \partial_{x_0}((\tau_{x_0}\rho)Q_{\delta_1,0})\big) \\
&\quad\quad + \beta\big(\partial_{x_0}((\tau_{x_0}\rho)Q_{\delta_1,0}) - \partial_{x_0}(\tau_{x_0}(\rho Q_\delta))|_{x_0=0}\big) + (\beta - \widetilde{\beta})\partial_{x_0} Q^\rho_{\delta_1} \\
&\quad = O_{L^2(\mathbb{T})}\Big(\big(\delta_1^{-2}(|x_0| + |\delta - \delta_1|) + \delta_1^{-1}|\theta_0| + \exp(-c\delta_1^{-1})\big)|\beta| + \delta_1^{-1}|\beta - \widetilde{\beta}|\Big) \\
&\quad = O_{L^2(\mathbb{T})}\Big(\delta_1^{-1} A_{\delta_1,v_1,\delta,x_0,\theta_0,w}(\alpha,\beta,\gamma,v)\Big).
\end{aligned} \tag{6.3.95}$$

From (6.3.33), we have

$$\begin{aligned}\gamma i e^{i\theta_0} Q^\rho_{\delta,x_0} - \widetilde{\gamma} i Q^\rho_{\delta_1} &= \gamma i(\tau_{x_0}\rho)(e^{i\theta_0}Q_{\delta,x_0} - Q_{\delta_1}) + \gamma i((\tau_{x_0}\rho)Q_{\delta_1} - \rho Q_{\delta_1}) + (\gamma - \widetilde{\gamma}) i Q^\rho_{\delta_1} \\ &= O_{L^2(\mathbb{T})}\Big(\big(\delta_1^{-1}(|x_0| + |\delta - \delta_1|) + |\theta_0| + \exp(-c\delta_1^{-1})\big)|\gamma| + |\gamma - \widetilde{\gamma}|\Big) \\ &= O_{L^2(\mathbb{T})}\Big(\delta_1^{-1} A_{\delta_1, v_1, \delta, x_0, \theta_0, w}(\alpha, \beta, \gamma, v)\Big).\end{aligned} \tag{6.3.96}$$

Finally, from (6.3.77), (6.3.59), (6.3.79), (6.3.31), (6.3.32), and (6.3.33) with $\|v_1\|_{L^2(\mathbb{T})} \lesssim \varepsilon_0 \ll 1$, we have

$$\begin{aligned}\|W - V_1\|_{L^2(\mathbb{T})} &\lesssim \big(\delta^{-1}(|\alpha| + |\beta|) + |\gamma|\big)\|w - v_1\|_{L^2(\mathbb{T})}, \\ &\quad + \big(|\alpha| + |\beta| + \delta|\gamma|\big) \\ &\qquad \times \big(\delta_1^{-1}|\theta_0| + \delta_1^{-2}(|x_0| + |\delta - \delta_1|) + \exp(-c\delta_1^{-1})\big)\|v_1\|_{L^2(\mathbb{T})} \\ &\quad + \big(\delta_1^{-1}(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|) + |\gamma - \widetilde{\gamma}|\big)\|v_1\|_{L^2(\mathbb{T})} \\ &\lesssim \delta_1^{-1} A_{\delta_1, v_1, \delta, x_0, \theta_0, w}(\alpha, \beta, \gamma, v).\end{aligned} \tag{6.3.97}$$

Hence, we obtain (6.3.34) from (6.3.92) together with (6.3.93) - (6.3.97). This concludes the proof of Lemma 6.8. □

6.4. **Orthogonal coordinate system in the finite-dimensional setting.** Given $N \in \mathbb{N}$, let π_N and E_N be as in (1.19) and (1.20), respectively. Given ε, δ_*, $\delta^* > 0$, we define $U_\varepsilon(\delta_*, \delta^*)$ by

$$\begin{aligned}U_\varepsilon(\delta_*, \delta^*) = \big\{u \in L^2(\mathbb{T}) : \|u - e^{i\theta}Q^\rho_{\delta,x_0}\|_{L^2(\mathbb{T})} < \varepsilon \\ \text{for some } \delta_* < \delta < \delta^*,\ x_0 \in \mathbb{T}, \text{ and } \theta \in \mathbb{R}\big\}.\end{aligned} \tag{6.4.1}$$

By modifying the proof of Proposition 6.4, we obtain the following proposition on an orthogonal coordinate system in the finite-dimensional setting.

Proposition 6.9. *Let $N \in \mathbb{N}$. Given small $\varepsilon_1 > 0$, there exist $N_0 = N_0(\varepsilon_1) \in \mathbb{N}$, $\varepsilon = \varepsilon(\varepsilon_1) > 0$, $\delta^* = \delta^*(\varepsilon_1) > 0$, and $\delta_* = \delta_*(\varepsilon_1, N) > 0$ with*

$$\lim_{N\to\infty} \delta_*(\varepsilon_1, N) = 0$$

(for each fixed $\varepsilon_1 > 0$) such that

$$\begin{aligned}&U_\varepsilon(\delta_*, \delta^*) \cap E_N \\ &\quad \subset \big\{u \in E_N : \|u - e^{i\theta}\pi_N Q^\rho_{\delta,x_0}\|_{L^2(\mathbb{T})} < \varepsilon_1,\ u - e^{i\theta}\pi_N Q^\rho_{\delta,x_0} \in V_{\delta,x_0,\theta}(\mathbb{T}) \\ &\qquad \text{for some } 0 < \delta < \delta^*,\ x_0 \in \mathbb{T}, \text{ and } \theta \in \mathbb{R}\big\}.\end{aligned}$$

Before proceeding to the proof of Proposition 6.9, let us first discuss properties of truncated solitons. Note that the frequency truncation operator π_N is parity-preserving; $\pi_N Q^\rho_\delta = \pi_N(\rho Q_\delta)$ is an even function for any $\delta > 0$. It also follows from (6.1.9) and (6.1.10) that $\pi_N \partial_\delta Q^\rho_\delta$ is an even function, while $\pi_N \partial_{x_0} Q^\rho_\delta$ is an odd function. Hence, they are orthogonal in $H^k(\mathbb{T})$, $k \in \mathbb{Z}_{\geq 0}$. Moreover, the operator π_N also commutes with the pointwise conjugation, so $\pi_N Q^\rho_\delta$, $\pi_N \partial_\delta Q^\rho_\delta$, and $\pi_N \partial_{x_0} Q^\rho_\delta$ are all real functions. Therefore, $\pi_N i Q^\rho_\delta$ is orthogonal[10] to both $\pi_N \partial_\delta Q^\rho_\delta$ and $\pi_N \partial_{x_0} Q^\rho_\delta$ in $H^k(\mathbb{T})$, $k \in \mathbb{Z}_{\geq 0}$. By a similar consideration

[10] Recall that we view $H^k(\mathbb{T})$ as a Hilbert space over reals.

centred at $x = x_0$, we also conclude that $\pi_N e^{i\theta}\partial_\delta Q^\rho_{\delta,x_0}$, $\pi_N e^{i\theta}\partial_{x_0} Q^\rho_{\delta,x_0}$, and $\pi_N ie^{i\theta} Q^\rho_{\delta,x_0}$ are pairwise orthogonal in $H^k(\mathbb{T})$, $k \in \mathbb{Z}_{\geq 0}$.

In the following, we use $\mathcal{F}_\mathbb{R}$ to denote the Fourier transform of a function on the real line. By (6.1.6), a change of variables, and the exponential decay of Q, we have

$$\begin{aligned}\widehat{Q^\rho_\delta}(n) &= \delta^{-\frac12}\int_{-\frac12}^{\frac12}\rho(x)Q(\delta^{-1}x)e^{-2\pi inx}dx = \delta^{\frac12}\int_\mathbb{R}\rho(\delta x)Q(x)e^{-2\pi i(\delta n)x}dx\\ &= \delta^{\frac12}\mathcal{F}_\mathbb{R}(Q)(\delta n) + O(\exp(-c\delta^{-1}))\\ &\sim \delta^{\frac12}\mathcal{F}_\mathbb{R}(Q)(\delta n)\end{aligned}$$

for $0 < \delta \ll 1$, provided that $\delta|n| \lesssim 1$. With A_1 as in (6.3.42), an analogous computation yields

$$\begin{aligned}\widehat{\partial_\delta Q^\rho_\delta}(n) &= \delta^{-\frac12}\mathcal{F}_\mathbb{R}(A_1)(\delta n) + O(\exp(-c\delta^{-1}))\\ &\sim \delta^{-\frac12}\mathcal{F}_\mathbb{R}(A_1)(\delta n)\end{aligned} \tag{6.4.2}$$

for $0 < \delta \ll 1$, provided that $\delta|n| \lesssim 1$.

Fix small $\gamma > 0$ and set $\delta_* = \delta_*(N) > 0$ such that

$$(\delta_*)^{1+\gamma} \gtrsim N^{-1} \tag{6.4.3}$$

Then, by a Riemann sum approximation with (6.4.2), we then have

$$\begin{aligned}\delta^{2k+2}\|\pi_N\partial_\delta Q^\rho_\delta\|^2_{\dot H^k(\mathbb{T})} &\gtrsim \sum_{\substack{|n|\leq N\\ |n|\lesssim\delta^{-1}}}(\delta n)^{2k}\delta|\mathcal{F}_\mathbb{R}(A_1)(\delta n)|^2 \sim \|\pi^\mathbb{R}_{\min(\delta N,1)}A_1\|^2_{\dot H^k(\mathbb{R})}\\ &\geq c_k > 0\end{aligned}$$

for $\delta_* < \delta \ll 1$, where $\pi^\mathbb{R}_N$ denotes the Dirichlet projection onto frequencies $\{|\xi| \leq N\}$ for functions on the real line. Since the estimate above holds independently of the base point $x_0 \in \mathbb{T}$, we have

$$\|\pi_N\partial_\delta Q^\rho_{\delta,x_0}\|^2_{H^k(\mathbb{T})} \gtrsim \delta^{-2k-2} \tag{6.4.4}$$

uniformly for $\delta_* < \delta \ll 1$ and $x_0 \in \mathbb{T}$. On the other hand, by integration by parts $2K$ times together with the exponential decay of the ground state and (6.4.3), we have

$$\begin{aligned}\widehat{\partial_\delta Q^\rho_\delta}(n) &= \delta^{-\frac12}\int_\mathbb{R}\rho(\delta x)A_1(x)e^{-2\pi i(\delta n)x}dx\\ &= O\big(\delta^{-2K-\frac12}|n|^{-2K}\big).\end{aligned} \tag{6.4.5}$$

Then, with $\pi_N^\perp = \mathrm{id} - \pi_N$, it follows from (6.4.3) and (6.4.5) that

$$\|\pi_N^\perp\rho\partial_\delta Q_{\delta,x_0}\|^2_{H^k(\mathbb{T})} \lesssim \delta^K \tag{6.4.6}$$

for any $K \in \mathbb{N}$, uniformly in $\delta_* < \delta \ll 1$ and $x_0 \in \mathbb{T}$. Similarly, we have

$$\begin{aligned}\|\pi_N\partial_{x_0}Q^\rho_{\delta,x_0}\|^2_{H^k(\mathbb{T})} &\gtrsim \delta^{-2k-2},\\ \|\pi_N^\perp\partial_{x_0}Q^\rho_{\delta,x_0}\|^2_{H^k(\mathbb{T})} &\lesssim \delta^K,\\ \|\pi_N Q^\rho_{\delta,x_0}\|^2_{H^k(\mathbb{T})} &\gtrsim \delta^{-2k},\\ \|\pi_N^\perp Q^\rho_{\delta,x_0}\|^2_{H^k(\mathbb{T})} &\lesssim \delta^K\end{aligned} \tag{6.4.7}$$

for any $k, K \in \mathbb{N}$, uniformly in $\delta_* < \delta \ll 1$ and $x_0 \in \mathbb{T}$.

Proof of Proposition 6.9. The proof of this proposition is based on a small modification of the proof of Proposition 6.4. We only go over the main steps, indicating required modifications. By a translation and a rotation, we may assume that $u \in E_N$ satisfies

$$\|u - \rho Q_{\delta_0}\|_{L^2(\mathbb{T})} < \varepsilon.$$

From the real line case discussed in the proof of Proposition 6.4 (see (6.3.17) above), we have

$$u = e^{i\theta_1} Q_{\delta_1, x_1} + v, \tag{6.4.8}$$

for some $(\delta_1, x_1, \theta_1)$ near $(\delta_0, 0, 0)$ and

$$v \in V^1_{\delta_1, x_1, \theta_1}(\mathbb{R}) \quad \text{with} \quad \|v\|_{L^2(\mathbb{R})} < \varepsilon_0 \ll 1, \tag{6.4.9}$$

provided that $\delta_0 = \delta_0(\varepsilon_0) > 0$ is sufficiently small.

Given $N \in \mathbb{N}$, define the map $F^N = F^N_{\delta_1, x_1, \theta_1} : \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \times V_{\delta_1, x_1, \theta_1}(\mathbb{T}) \cap E_N \to L^2(\mathbb{T})$ by

$$F^N(\delta, x, \theta, v) = \pi_N e^{i\theta} Q^\rho_{\delta, x} + P_{V_{\delta,x,\theta} \cap E_N} v, \tag{6.4.10}$$

where $P_{V_{\delta,x,\theta} \cap E_N}$ is the projection onto $V_{\delta,x,\theta} \cap E_N$ in $L^2(\mathbb{T})$. From (6.4.8) and (6.4.10) with $v_1 = P_{V_{\delta_1,x_1,\theta_1} \cap E_N}(v|_{\mathbb{T}})$, we have

$$u - F^N(\delta_1, x_1, \theta_1, v_1) = e^{i\theta_1}(Q_{\delta_1, x_1} - \pi_N Q^\rho_{\delta_1, x_1}) + (v - v_1). \tag{6.4.11}$$

From the orthogonality of $\pi_N e^{i\theta_1} \partial_\delta Q^\rho_{\delta_1, x_1}$, $\pi_N e^{i\theta_1} \partial_{x_0} Q^\rho_{\delta_1, x_1}$, and $\pi_N i e^{i\theta_1} Q^\rho_{\delta_1, x_1}$ in $H^2(\mathbb{T})$, we have

$$\begin{aligned} v_1 &= P_{V_{\delta_1,x_1,\theta_1} \cap E_N}(v|_{\mathbb{T}}) \\ &= \pi_N v - \frac{\langle \pi_N v, (1-\partial_x^2) e^{i\theta_1} \partial_\delta Q^\rho_{\delta_1,x_1} \rangle_{L^2(\mathbb{T})}}{\|(1-\partial_x^2)\pi_N \partial_\delta Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2)\pi_N e^{i\theta_1} \partial_\delta Q^\rho_{\delta_1,x_1} \\ &\quad - \frac{\langle \pi_N v, (1-\partial_x^2) e^{i\theta_1} \partial_{x_0} Q^\rho_{\delta_1,x_1} \rangle_{L^2(\mathbb{T})}}{\|(1-\partial_x^2)\pi_N \partial_{x_0} Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2)\pi_N e^{i\theta_1} \partial_{x_0} Q^\rho_{\delta_1,x_1} \\ &\quad - \frac{\langle \pi_N v, (1-\partial_x^2) i e^{i\theta_1} Q^\rho_{\delta_1,x_1} \rangle_{L^2(\mathbb{T})}}{\|(1-\partial_x^2)\pi_N Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2)\pi_N i e^{i\theta_1} Q^\rho_{\delta_1,x_1}. \end{aligned} \tag{6.4.12}$$

Then, we can proceed as in the proof of Proposition 6.4 with (6.4.4), (6.4.6), (6.4.7), and (6.4.9) to estimate $\pi_N v - v_1$. For example, the second term on the right-hand side of (6.4.12) can be written as

$$\begin{aligned} &\frac{\langle v, (1-\partial_x^2) e^{i\theta_1} \partial_\delta Q^\rho_{\delta_1,x_1} \rangle_{L^2(\mathbb{T})}}{\|(1-\partial_x^2)\pi_N \partial_\delta Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2)\pi_N e^{i\theta_1} \partial_\delta Q^\rho_{\delta_1,x_1} \\ &\quad - \frac{\langle v, (1-\partial_x^2) \pi_N^\perp e^{i\theta_1} \partial_\delta Q^\rho_{\delta_1,x_1} \rangle_{L^2(\mathbb{T})}}{\|(1-\partial_x^2)\pi_N \partial_\delta Q^\rho_{\delta_1,x_1}\|^2_{L^2(\mathbb{T})}} (1-\partial_x^2)\pi_N e^{i\theta_1} \partial_\delta Q^\rho_{\delta_1,x_1}. \end{aligned} \tag{6.4.13}$$

Thanks to (6.4.4), (6.4.9), and the exponential decay of the ground state, the first term in (6.4.13) is bounded as $O_{L^2(\mathbb{T})}\big(\exp(-c\delta_1^{-1})\big)$. On the other hand, from (6.4.4) and (6.4.6), we can bound the second term in (6.4.13) as $O_{L^2(\mathbb{T})}(\delta_1^K)$ for any $K \in \mathbb{N}$. By estimating the third and fourth terms on the right-hand side of (6.4.12) in an analogous manner, we obtain

$$\|\pi_N v - v_1\|_{L^2(\mathbb{T})} = O(\delta_1^K) \tag{6.4.14}$$

for any $K \in \mathbb{N}$.

From (6.4.8), $u \in E_N$, and (6.4.7), we have

$$\|\pi_N^\perp v\|_{L^2(\mathbb{T})} = O(\delta_1^K) \tag{6.4.15}$$

for any $K \in \mathbb{N}$. Putting (6.4.14) and (6.4.15) together, we obtain

$$\|v - v_1\|_{L^2(\mathbb{T})} = O(\delta_1^K) \tag{6.4.16}$$

for $\delta_* < \delta_1 \ll 1$, where $\delta_* = \delta_*(N)$ satisfies (6.4.3). Finally, we conclude from (6.4.11), (6.3.19), (6.4.7), and (6.4.16) that

$$\|u - F^N(\delta_1, x_1, \theta_1, v_1)\|_{L^2(\mathbb{T})} = O(\delta_1^K) \tag{6.4.17}$$

for $\delta_* < \delta_1 \ll 1$, where $\delta_* = \delta_*(N)$ satisfies (6.4.3).

By our choice of $\delta_* = \delta_*(N)$ such that (6.4.4) and (6.4.7) hold, we see that Lemmas 6.7 and 6.8 applied to $F^N = F^N_{\delta_1,0,0}$ hold uniformly for $\delta_* < \delta_1 \ll 1$, since (6.4.4) and (6.4.7) provide lower bounds on the denominators of the various terms appearing in the proofs of these lemmas. This allows us to apply the inverse function theorem (Lemma 6.6) with $R \sim \delta_1^3$, $\kappa \sim \delta_1$, and $r \sim \delta_1^3$. By taking $K > 3$ in (6.4.17), we conclude that u lies in the image of $F^N = F^N_{\delta_1,x_1,\theta_1}$. Lastly, we need to choose $\delta_* = \delta_*(\varepsilon_1, N) > 0$ and $\delta^* = \delta^*(\varepsilon_1) > 0$ sufficiently small such that $\delta_* < \delta^* \lesssim \varepsilon_1^{\frac{1}{3}}$. This concludes the proof of Proposition 6.9. $\square$

6.5. **A change-of-variable formula.** Our main goal is to prove the bound (6.3.1). In the remaining part of the paper, we fix small $\delta^* > 0$ and set

$$U_\varepsilon = U_\varepsilon(\delta^*),$$

where $U_\varepsilon(\delta^*)$ is defined in (6.3.2) for given small $\varepsilon > 0$. Recalling the low regularity of the Ornstein-Uhlenbeck loop in (1.7), we only work with functions $u \in H^s(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T})$, $s < \frac{1}{2}$, for the μ-integration. Thus, with a slight abuse of notations, we redefine $U_\varepsilon = U_\varepsilon(\delta^*)$ in (6.3.2) to mean $U_\varepsilon \setminus H^{\frac{1}{2}}(\mathbb{T})$. Namely, when we write $U_\varepsilon = U_\varepsilon(\delta^*)$ in the following, it is understood that we take an intersection with $\big(H^{\frac{1}{2}}(\mathbb{T})\big)^c$.

Given small $\delta > 0$ and $x_0 \in \mathbb{T}$, let $V_{\delta,x_0,0} = V_{\delta,x_0,0}(\mathbb{T})$ be as in (6.3.3) with $\theta = 0$. In view of Proposition 6.4, the convention above: $U_\varepsilon = U_\varepsilon \cap \big(H^{\frac{1}{2}}(\mathbb{T})\big)^c$, and the fact that $Q^\rho_{\delta,x_0} \in H^1(\mathbb{T})$, we redefine $V_{\delta,x_0,0}$ to mean $V_{\delta,x_0,0} \cap H^1(\mathbb{T})$. Namely, when we write $V_{\delta,x_0,0}$ in the following, it is understood that we take an intersection with $H^1(\mathbb{T})$. Now, let $\mu^\perp_{\delta,x_0}$ denote the Gaussian measure with $V_{\delta,x_0,0} \subset H^1(\mathbb{T})$ as its Cameron-Martin space. Then, we have the following lemma on a change of variables for the μ-integration over U_ε.

Lemma 6.10. *Fix $K > 0$ and sufficiently small $\delta^* > 0$. Let $F(u) \geq 0$ be a functional of u on $\mathbb{T}$ which is continuous in some topology $H^s(\mathbb{T})$, $s < \frac{1}{2}$, such that $F \leq C$ for some C and $F(u) = 0$ if $\|u\|_{L^2(\mathbb{T})} > K$. Then, there is a locally finite measure $\mathrm{d}\sigma(\delta)$ on $(0, \delta^*)$ such* *that*

$$\int_{U_\varepsilon} F(u)\,\mu(\mathrm{d}u) \le \iiiint_{U_\varepsilon} F\Big(e^{i\theta}(Q^\rho_{\delta,x_0}+v)\Big) \\ \times e^{-\frac{1}{2}\|Q^\rho_{\delta,x_0}\|^2_{H^1(\mathbb{T})} - \langle (1-\partial_x^2)Q^\rho_{\delta,x_0}, v\rangle_{L^2(\mathbb{T})}} \mu^\perp_{\delta,x_0}(\mathrm{d}v)\mathrm{d}\sigma(\delta)\mathrm{d}x_0\mathrm{d}\theta, \tag{6.5.1}$$

where the domain of the integration on the right-hand side is to be interpreted as

$$\Big\{(\delta,x_0,\theta,v)\in(0,\delta^*)\times\mathbb{T}\times(\mathbb{R}/2\pi\mathbb{Z})\times V_{\delta,x_0,0}(\mathbb{T}) : e^{i\theta}(Q^\rho_{\delta,x_0}+v)\in U_\varepsilon\Big\}.$$

Moreover, σ is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}\delta$ *on* $(0,\delta^*)$, *satisfying* $|\frac{\mathrm{d}\sigma}{\mathrm{d}\delta}| \lesssim \delta^{-20}$.

The proof of Lemma 6.10 is based on a finite dimensional approximation and an application of the following lemma.

Lemma 6.11. *Let $M\subset\mathbb{R}^n$ be a closed submanifold of dimension d and $\mathcal{N}$ be its normal bundle. Then, there is a neighborhood U of M such that U is diffeomorphic to a subset $\mathcal{M}$ of $\mathcal{N}$ via the map $\phi:\mathcal{N}\to\mathbb{R}^n$ given by*

$$\phi(x,v) = x+v \tag{6.5.2}$$

for $x\in M$ and $v\in T_xM^\perp$. Furthermore, the following estimate holds for any non-negative measurable function $f:\mathbb{R}^n\to\mathbb{R}$ and any open set $V\subset\{(x,v):|v|\le 1\}\subset\mathcal{N}$:

$$\int_{\phi(V)} f(z)\mathrm{d}z \le C_d \int_M \bigg(\int_{T_xM^\perp} f(x+v)\mathbf{1}_V(x,v)\mathrm{d}v\bigg)\mathrm{d}\sigma(x). \tag{6.5.3}$$

Here, the measure $\mathrm{d}\sigma$ *is defined by*

$$\mathrm{d}\sigma(x) = \Big(1+\sup_{k=1,\dots,d}\|\nabla t_k(x)\|^d\Big)\mathrm{d}\omega(x),$$

where $\mathrm{d}\omega(x)$ is the surface measure on M and $\{t_k(x)\}_{k=1}^d = \{(t_k^1(x),\dots,t_k^n(x))\}_{k=1}^d$ is an orthonormal basis for the tangent space T_xM with the expression $\|\nabla t_k(x)\|$ defined by

$$\|\nabla t_k(x)\| = \bigg(\sum_{j=1}^n\sum_{i=1}^d \Big|\frac{\partial}{\partial y_i}(t_k^j\circ\varphi^{-1})(y)\Big|^2\bigg)^{\frac{1}{2}}$$

for any coordinate chart φ on a neighborhood of $x\in M$ such that $y=(y_1,\dots,y_d)=\varphi(x)$ and $d(\varphi^{-1})(y)$ is an isometry with its image. Note that the constant in (6.5.3) *is independent of the dimension n of the ambient space $\mathbb{R}^n$.*

Proof. Given $x\in M$, let $w_1(x),\dots,w_{n-d}(x)$ be an orthonormal basis of $T_xM^\perp$. Consider the map

$$\psi: M_x\times\mathbb{R}^{n-d}_\alpha\to\mathbb{R}^n, \qquad \psi(x,\alpha) = x+\sum_{j=1}^{n-d}\alpha_j w_j(x), \tag{6.5.4}$$

where $\alpha=(\alpha_1,\dots,\alpha_{n-d})$. Recalling that $\{w_j(x)\}_{j=1}^{n-d}$ is an orthonormal basis of $T_xM^\perp$, it follows from the area formula (see [33, Theorem 2 on p. 99]) with (6.5.2), (6.5.4), and

$V \subset \{(x,v) : |v| \le 1\}$ and then applying a change of variables that

$$\begin{aligned}
&\int_{\phi(V)} f(z)\mathrm{d}z \\
&\quad \le \int_{\phi(V)} \#\bigg\{(x,\alpha) \in M \times \mathbb{R}^{n-d} : \Big(x, \sum_{j=1}^{n-d} \alpha_j w_j(x)\Big) \in V,\ \psi(x,\alpha) = z\bigg\} f(z)\mathrm{d}z \\
&\quad = \iint_{M_x \times \{\alpha \in \mathbb{R}^{n-d} : |\alpha| \le 1\}} f(\psi(x,\alpha)) \mathbf{1}_V\bigg(\Big(x, \sum_{j=1}^{n-d} \alpha_j w_j(x)\Big)\bigg) |J_\psi(x,\alpha)| \mathrm{d}\alpha \mathrm{d}\omega(x) \\
&\quad \le \int_M \bigg(\int_{T_x M^\perp} f(x+v) \mathbf{1}_V(x,v)\mathrm{d}v\bigg) \sup_{|\alpha|\le 1} |J_\psi(x,\alpha)| \mathrm{d}\omega(x),
\end{aligned}$$

where J_ψ is the determinant of the differential of the map ψ. Hence, the bound (6.5.3) follows once we prove

$$|J_\psi(x,\alpha)| \le C_d \Big(1 + \sup_{k=1,\dots,d} \|\nabla t_k(x)\|\Big)^d \tag{6.5.5}$$

for every $\alpha \in \mathbb{R}^{n-d}$ with $|\alpha| \le 1$, where $\{t_k(x)\}_{k=1}^d$ is an orthonormal basis for the tangent space $T_x M$.

Recall that the tangent space of $M \times \mathbb{R}^{n-d}$ at the point (x,α) is isomorphic to $T_x M \times \mathbb{R}^{n-d}$. Then, denoting by $\{e_j\}_{j=1}^{n-d}$ the standard basis of $\mathbb{R}^{n-d}$, it follows from (6.5.4) that[11]

$$\begin{aligned}
\mathrm{d}\psi[t_k] &= t_k + \sum_{j=1}^{n-d} \alpha_j \mathrm{d}w_j[t_k], \qquad k = 1, \dots, d, \\
\mathrm{d}\psi[e_j] &= w_j, \qquad j = 1, \dots, n-d.
\end{aligned}$$

Hence, by taking $(t_1, \dots, t_d, e_1, \dots, e_{n-d})$ (and $(t_1, \dots, t_d, w_1, \dots, w_{n-d})$, respectively) as the (orthonormal) basis of the domain $T_x M \times \mathbb{R}^{n-d}$ (and the codomain $\mathbb{R}^n$, respectively), the matrix representation of $\mathrm{d}\psi$ is given by

$$A(x,\alpha) = \begin{pmatrix} \mathrm{id}_{d\times d} + B & 0 \\ D & \mathrm{id}_{(n-d)\times(n-d)} \end{pmatrix},$$

where $B = B(\alpha) = \{B(\alpha)_{h,k}\}_{1\le h,k\le d}$ is given by

$$B(\alpha)_{h,k} = \sum_{j=1}^{n-d} \alpha_j \langle \mathrm{d}w_j[t_k], t_h\rangle_{\mathbb{R}^n}. \tag{6.5.6}$$

Thus, we have

$$|J_\psi(x,\alpha)| = |\det A(x,\alpha)| = |\det(\mathrm{id}_{d\times d} + B(\alpha))| \lesssim_d 1 + \sup_{1\le h,k\le d} |B(\alpha)_{h,k}|^d$$

for any $\alpha \in \mathbb{R}^{n-d}$ with $|\alpha| \le 1$. Therefore, the bound (6.5.5) (and hence (6.5.3)) follows once we prove

$$\sup_{1\le h,k\le d} |B(\alpha)_{h,k}| \le \sup_{k=1,\dots,d} \|\nabla t_k(x)\|, \tag{6.5.7}$$

uniformly in $\alpha \in \mathbb{R}^{n-d}$ with $|\alpha| \le 1$.

[11]Hereafter, we suppress the x-dependence of $w_j = w_j(x)$ and $t_k = t_k(x)$ when there is no confusion.

By differentiating the orthogonality relation $\langle w_j(x), t_k(x)\rangle_{\mathbb{R}^n} = 0$, we obtain

$$\langle \mathrm{d}w_j[\tau], t_k\rangle_{\mathbb{R}^n} = -\langle w_j, \mathrm{d}t_k[\tau]\rangle_{\mathbb{R}^n} \tag{6.5.8}$$

for any $\tau \in T_x M$. Thus, from (6.5.6), (6.5.8), Cauchy-Schwarz inequality with $|\alpha| \le 1$, and the orthonormality of $\{w_j\}_{j=1}^{n-d}$, we have

$$\begin{aligned}
|B(\alpha)_{h,k}| &= \bigg| \sum_{j=1}^{n-d} \alpha_j \langle \mathrm{d}w_j[t_k], t_h\rangle_{\mathbb{R}^n} \bigg| \\
&= \bigg| \sum_{j=1}^{n-d} \alpha_j \langle w_j, \mathrm{d}t_h[t_k]\rangle_{\mathbb{R}^n} \bigg| \\
&\le \bigg(\sum_{j=1}^{n-d} |\alpha_j|^2 \bigg)^{\frac{1}{2}} \bigg(\sum_{j=1}^{n-d} \langle w_j, \mathrm{d}t_h[t_k]\rangle_{\mathbb{R}^n}^2 \bigg)^{\frac{1}{2}} \\
&\le \big| \mathrm{d}t_h[t_k] \big| \\
&\le \sup_{h=1,\dots,d} \|\nabla t_h(x)\|,
\end{aligned}$$

where we used Cauchy-Schwarz inequality once again in the last step. This proves (6.5.7) and hence concludes the proof of Lemma 6.11. □

We now present the proof of Lemma 6.10.

Proof of Lemma 6.10. Let E_N and $U_\varepsilon(\delta_*, \delta^*)$ as in (1.20) and (6.4.1). By Proposition 6.9, given any $u \in U_\varepsilon(\delta_*, \delta^*) \cap E_N$, there exist coordinates $(\delta, x_0, \theta) \in (0, \delta^*) \times \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ and $v \in V_{\delta, x_0, \theta} \cap E_N$ such that

$$u = e^{i\theta} \pi_N Q^\rho_{\delta, x_0} + v.$$

Given (δ, x_0, θ), we let $v_j = v_j(\delta, x_0, \theta)$, $j = 1, \dots, 4N-1$, denote an $H^1(\mathbb{T})$-orthonormal basis[12] of $V_{\delta, x_0, \theta} \cap E_N$. Then, from (6.0.1) with $g = \{g_n\}_{|n| \le N}$, we have

$$\begin{aligned}
&\int_{U_\varepsilon(\delta_*, \delta^*)} F(\pi_N u) \mu(\mathrm{d}u) \\
&\quad = \frac{1}{(2\pi)^{2N+1}} \int_{\mathbb{C}^{2N}} \mathbf{1}_{U_\varepsilon(\delta_*, \delta^*)} \bigg(\sum_{|n| \le N} \frac{g_n}{\langle n \rangle} e^{2\pi i n x} \bigg) F\bigg(\sum_{|n| \le N} \frac{g_n}{\langle n \rangle} e^{2\pi i n x} \bigg) e^{-\frac{|g|^2}{2}} \mathrm{d}g
\end{aligned}$$

[12]Once again, recall that we view $H^k(\mathbb{T})$ as a Hilbert space over reals.

From Lemma 6.11 with $y = \{y_j\}_{j=1}^{4N-1} \in \mathbb{R}^{4N-1}$,

$$
\begin{aligned}
&\lesssim \frac{1}{(2\pi)^{(4N-1)/2}} \iint \mathbf{1}_{U_\varepsilon(\delta_*,\delta^*)}\bigg(e^{i\theta}\pi_N Q^\rho_{\delta,x_0} + \sum_{j=1}^{4N-1} y_j v_j\bigg) \\
&\qquad \times F\bigg(e^{i\theta}\pi_N Q^\rho_{\delta,x_0} + \sum_{j=1}^{4N-1} y_j v_j\bigg) \\
&\qquad \times e^{-\frac{1}{2}\|e^{i\theta}\pi_N Q^\rho_{\delta,x_0} + \sum_{j=1}^{4N-1} y_j v_j\|^2_{H^1}} \,\mathrm{d}y\mathrm{d}\sigma_N(\delta, x_0, \theta) \\
&= \iint \mathbf{1}_{U_\varepsilon(\delta_*,\delta^*)}\Big(e^{i\theta}\pi_N (Q^\rho_{\delta,x_0} + v)\Big) F\Big(e^{i\theta}\pi_N (Q^\rho_{\delta,x_0} + v)\Big) \\
&\qquad \times e^{-\frac{1}{2}\|\pi_N Q^\rho_{\delta,x_0}\|^2_{H^1(\mathbb{T})} - \langle (1-\partial_x^2)\pi_N Q^\rho_{\delta,x_0}, v\rangle_{L^2(\mathbb{T})}} \mu^\perp_{\delta,x_0}(\mathrm{d}v)\mathrm{d}\sigma_N(\delta, x_0, \theta),
\end{aligned}
\tag{6.5.9}
$$

where $\mu^\perp_{\delta,x_0}$ denotes the Gaussian measure with $V_{\delta,x_0,0} \subset H^1(\mathbb{T})$ as its Cameron-Martin space.[13] Here, the measure σ_N is given by

$$
d\sigma_N(\delta, x_0, \theta) = \Big(1 + \sup_{k=1,2,3} \|\nabla t_k(\delta, x_0, \theta)\|^3\Big)\mathrm{d}\omega_N(\delta, x_0, \theta),
$$

where $t_k = t_k(N, \delta, x_0, \theta)$, $k = 1, 2, 3$, are the orthonormal vectors obtained by applying the Gram-Schmidt orthonormalization procedure in $H^1(\mathbb{T})$ to $\big\{\pi_N\partial_\delta(e^{i\theta}Q^\rho_{\delta,x_0})$, $\pi_N\partial_{x_0}(e^{i\theta}Q^\rho_{\delta,x_0})$, $\pi_N\partial_\theta(e^{i\theta}Q^\rho_{\delta,x_0})\big\}$ and the surface measure ω_N is given by

$$
\mathrm{d}\omega_N(\delta, x_0, \theta) = |\gamma_N(\delta, x_0, \theta)|\mathrm{d}\delta\mathrm{d}x_0\mathrm{d}\theta_0
$$

with $\gamma_N(\delta, x_0, \theta)$ given by

$$
\begin{aligned}
&\gamma_N(\delta, x_0, \theta) \\
&= \det\begin{pmatrix}
\langle \pi_N\partial_\delta(e^{i\theta}Q^\rho_{\delta,x_0}), t_1\rangle_{H^1(\mathbb{T})} & \langle \pi_N\partial_{x_0}(e^{i\theta}Q^\rho_{\delta,x_0}), t_1\rangle_{H^1(\mathbb{T})} & \langle \pi_N\partial_\theta(e^{i\theta}Q^\rho_{\delta,x_0}), t_1\rangle_{H^1(\mathbb{T})} \\
\langle \pi_N\partial_\delta(e^{i\theta}Q^\rho_{\delta,x_0}), t_2\rangle_{H^1(\mathbb{T})} & \langle \pi_N\partial_{x_0}(e^{i\theta}Q^\rho_{\delta,x_0}), t_2\rangle_{H^1(\mathbb{T})} & \langle \pi_N\partial_\theta(e^{i\theta}Q^\rho_{\delta,x_0}), t_2\rangle_{H^1(\mathbb{T})} \\
\langle \pi_N\partial_\delta(e^{i\theta}Q^\rho_{\delta,x_0}), t_3\rangle_{H^1(\mathbb{T})} & \langle \pi_N\partial_{x_0}(e^{i\theta}Q^\rho_{\delta,x_0}), t_3\rangle_{H^1(\mathbb{T})} & \langle \pi_N\partial_\theta(e^{i\theta}Q^\rho_{\delta,x_0}), t_3\rangle_{H^1(\mathbb{T})}
\end{pmatrix} \\
&= \det\begin{pmatrix}
\langle \partial_\delta(e^{i\theta}Q^\rho_{\delta,x_0}), t_1\rangle_{H^1(\mathbb{T})} & \langle \partial_{x_0}(e^{i\theta}Q^\rho_{\delta,x_0}), t_1\rangle_{H^1(\mathbb{T})} & \partial_\theta e^{i\theta}\langle Q^\rho_{\delta,x_0}, t_1\rangle_{H^1(\mathbb{T})} \\
\langle \partial_\delta(e^{i\theta}Q^\rho_{\delta,x_0}), t_2\rangle_{H^1(\mathbb{T})} & \langle \partial_{x_0}(e^{i\theta}Q^\rho_{\delta,x_0}), t_2\rangle_{H^1(\mathbb{T})} & \partial_\theta e^{i\theta}\langle Q^\rho_{\delta,x_0}, t_2\rangle_{H^1(\mathbb{T})} \\
\langle \partial_\delta(e^{i\theta}Q^\rho_{\delta,x_0}), t_3\rangle_{H^1(\mathbb{T})} & \langle \partial_{x_0}(e^{i\theta}Q^\rho_{\delta,x_0}), t_3\rangle_{H^1(\mathbb{T})} & \partial_\theta e^{i\theta}\langle Q^\rho_{\delta,x_0}, t_3\rangle_{H^1(\mathbb{T})}
\end{pmatrix}.
\end{aligned}
\tag{6.5.10}
$$

Note that $\{t_k\}_{k=1}^3$ are chosen so that $t_k(\delta, x_0, \theta) = \tau_{x_0} t_k(\delta, 0, \theta)$, where τ_{x_0} is the translation defined in (6.1.4). Together with invariance under multiplication by a unitary complex number, we obtain

$$
\gamma_N(\delta, x_0, \theta) = \gamma_N(\delta, 0, 0) \tag{6.5.11}
$$

for any $x_0 \in \mathbb{T}$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$. From (6.5.11) and (6.5.10) with (6.3.41), (6.3.45), and (6.3.48), we have

$$
\begin{aligned}
|\gamma_N(\delta, x_0, \theta)| = |\gamma_N(\delta, 0, 0)| &\lesssim \|\partial_\delta Q^\rho_\delta\|^3_{H^1(\mathbb{T})} + \|\partial_{x_0} Q^\rho_\delta\|^3_{H^1(\mathbb{T})} + \|Q^\rho_\delta\|^3_{H^1(\mathbb{T})} \\
&\lesssim \delta^{-6}.
\end{aligned}
$$

[13] In the last step of (6.5.9), we used the decomposition $\mu^\perp_{\delta,x_0} = \mu^\perp_{\delta,x_0,\le N} \otimes \mu^\perp_{\delta,x_0,>N}$, where $\mu^\perp_{\delta,x_0,\le N}$ (and $\mu^\perp_{\delta,x_0,>N}$, respectively) denotes the Gaussian measure with $V_{\delta,x_0,0} \cap E_N$ (and $V_{\delta,x_0,0} \cap \pi_N^\perp H^1(\mathbb{T})$, respectively) as its Cameron-Martin space.

A computation analogous to (6.3.41), (6.3.45), and (6.3.48) together with (6.4.4) and (6.4.7) shows

$$\sup_{k=1,2,3} \|\nabla t_k(\delta,x_0,\theta)\| \lesssim \sup_{\kappa_1,\kappa_2\in\{\delta,x_0,\theta\}} \|\partial^2_{\kappa_1,\kappa_2}(\pi_N e^{i\theta}Q^\rho_{\delta,x_0})\|_{H^1} \lesssim \delta^{-3},$$

uniformly in $N\in\mathbb{N}$.

Therefore, by taking the limit $N\to\infty$ in (6.5.9), the dominated convergence theorem yields

$$\begin{aligned}&\int_{U_\varepsilon(\delta_*,\delta^*)} F(u)\mathrm{d}\mu(u)\\ &\le \iiiint_{U_\varepsilon(\delta_*,\delta^*)} F\Big(e^{i\theta}(Q^\rho_{\delta,x_0}+v)\Big)\\ &\qquad\times e^{-\frac12\|Q^\rho_{\delta,x_0}\|^2_{H^1(\mathbb{T})}-\langle(1-\partial_x^2)Q^\rho_{\delta,x_0},v\rangle_{L^2(\mathbb{T})}}\mu^\perp_{\delta,x_0}(\mathrm{d}v)\mathrm{d}\sigma(\delta)\mathrm{d}x_0\mathrm{d}\theta.\end{aligned}$$

Finally, in view of Proposition 6.9, by taking $\delta_*\to 0$, we obtain (6.5.1). □

6.6. **Further reductions.** The forthcoming calculations aim to prove (6.3.1). We first apply the change of variables (Lemma 6.10) to the integral in (6.3.1). In the remaining part of this section, we set $K=\|Q\|_{L^2(\mathbb{R})}$ and use $\langle\cdot,\cdot\rangle$ to denote the inner product in $L^2(\mathbb{T})$, unless otherwise specified. Note that the integrand

$$\exp\bigg(\frac16\int_{\mathbb{T}}|u|^6\mathrm{d}x\bigg)\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le K\}}$$

is neither bounded nor continuous. The lack of continuity is due to the sharp cutoff $\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le K\}}$. Since the set of discontinuity has μ-measure zero, we may start with a smooth cutoff and then pass to the limit. We can also replace this integrand by a bounded one, as long as the bounds we obtain are uniform, at the cost of a simple approximation argument which we omit.

By applying Lemma 6.10 to the integral in (6.3.1) and using also the translation invariance of the surface measure (namely, the independence of all the quantities from x_0), we have

$$\begin{aligned}&\int_{U_\varepsilon}\exp\bigg(\frac16\int_{\mathbb{T}}|u|^6\,\mathrm{d}x\bigg)\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le K\}}\,\mu(\mathrm{d}u)\\ &\le\iiiint_{U_\varepsilon} e^{F(v)}\mathbf{1}_{\{\|Q^\rho_{\delta,x_0}+v\|_{L^2(\mathbb{T})}\le K\}}\,\mu^\perp_{\delta,x_0}(\mathrm{d}v)d\sigma(\delta)\mathrm{d}x_0\mathrm{d}\theta\\ &=2\pi\iint_{U_\varepsilon} e^{F(v)}\mathbf{1}_{\{\|Q^\rho_\delta+v\|_{L^2(\mathbb{T})}\le K\}}\,\mu^\perp_\delta(\mathrm{d}v)\mathrm{d}\sigma(\delta),\end{aligned}\tag{6.6.1}$$

where $\mu^\perp_\delta=\mu^\perp_{\delta,0}$ and

$$F(v)=\frac16\int_{\mathbb{T}}|Q^\rho_\delta+v|^6\,\mathrm{d}x-\frac12\|Q^\rho_\delta\|^2_{H^1(\mathbb{T})}-\langle(1-\partial_x^2)Q^\rho_\delta,v\rangle.\tag{6.6.2}$$

A direct computation shows[14]

$$\begin{aligned}
&\frac{1}{6}\int_{\mathbb{T}} |Q_\delta^\rho + v|^6 \,\mathrm{d}x \\
&\quad = \frac{1}{6}\int (Q_\delta^\rho)^6 \,\mathrm{d}x + \langle (Q_\delta^\rho)^5, \operatorname{Re} v\rangle + \langle (Q_\delta^\rho)^4, \operatorname{Re}(v^2) + \tfrac{3}{2}|v|^2\rangle \\
&\quad\quad + \langle (Q_\delta^\rho)^3, \tfrac{1}{3}\operatorname{Re}(v^3) + 3|v|^2 \operatorname{Re} v\rangle + \langle (Q_\delta^\rho)^2, |v|^2(\operatorname{Re}(v^2) + \tfrac{3}{2}|v|^2)\rangle \\
&\quad\quad + \langle Q_\delta^\rho, |v|^4 \operatorname{Re} v\rangle + \frac{1}{6}\int_{\mathbb{T}} |v|^6 \,\mathrm{d}x.
\end{aligned} \tag{6.6.3}$$

From (6.6.2) and (6.6.3) with (6.1.3) and (6.1.7), we have

$$\begin{aligned}
F(v) &= \frac{1}{6}\int_{\mathbb{T}} (Q_\delta^\rho)^6 \,\mathrm{d}x - \frac{1}{2}\|Q_\delta^\rho\|_{H^1(\mathbb{T})}^2 \\
&\quad + \langle (\rho^5 - \rho) Q_\delta^5, \operatorname{Re} v\rangle + 2\langle (\partial_x \rho)(\partial_x Q_\delta), \operatorname{Re} v\rangle + \langle (\partial_x^2 \rho) Q_\delta, \operatorname{Re} v\rangle \\
&\quad + (2\delta^{-2} - 1)\langle Q_\delta^\rho, \operatorname{Re} v\rangle + \langle (Q_\delta^\rho)^4, 2(\operatorname{Re} v)^2 + \tfrac{1}{2}|v|^2\rangle \\
&\quad + \langle (Q_\delta^\rho)^3, \tfrac{1}{3}\operatorname{Re}(v^3) + 3|v|^2 \operatorname{Re} v\rangle + \langle (Q_\delta^\rho)^2, |v|^2(\operatorname{Re}(v^2) + \tfrac{3}{2}|v|^2)\rangle \\
&\quad + \langle Q_\delta^\rho, |v|^4 \operatorname{Re} v\rangle + \frac{1}{6}\int_{\mathbb{T}} |v|^6 \,\mathrm{d}x,
\end{aligned} \tag{6.6.4}$$

where we used $\operatorname{Re}(v^2) = 2(\operatorname{Re} v)^2 - |v|^2$ to get the seventh term on the right-hand side.

By the sharp Gagliardo-Nirenberg-Sobolev inequality (Proposition 3.1), we have $\|Q_\delta\|_{L^6(\mathbb{R})}^6 = 3\|Q'_\delta\|_{L^2(\mathbb{R})}^2$ and thus we have

$$\bigg| \frac{1}{6}\int_{\mathbb{T}} (Q_\delta^\rho)^6 \,\mathrm{d}x - \frac{1}{2}\|Q_\delta^\rho\|_{H^1(\mathbb{T})}^2 \bigg| \le C, \tag{6.6.5}$$

uniformly in $0 < \delta \le 1$, thanks to the exponential decay of Q (as in (6.1.8)). Moreover, recalling from (6.1.6) that $\rho \equiv 1$ on $[-\frac{1}{8}, \frac{1}{8}]$ and using the exponential decay of Q again with $\|v\|_{L^2(\mathbb{T})} \le 1$, we obtain

$$\Big| \langle (\rho^5 - \rho) Q_\delta^5, \operatorname{Re} v\rangle + 2\langle (\partial_x \rho)(\partial_x Q_\delta), \operatorname{Re} v\rangle + \langle (\partial_x^2 \rho) Q_\delta, \operatorname{Re} v\rangle \Big| \le C, \tag{6.6.6}$$

uniformly in $0 \le \delta \le 1$. By Young's inequality, the sum of the last four terms in (6.6.4) is bounded by

$$\eta \int_{\mathbb{T}} (Q_\delta^\rho)^4 \big(2(\operatorname{Re} v)^2 + \tfrac{1}{2}|v|^2\big) \,\mathrm{d}x + C_\eta \int_{\mathbb{T}} |v|^6 \,\mathrm{d}x \tag{6.6.7}$$

for any $0 < \eta \ll 1$ and a (large) constant $C_\eta > 0$. Hence, from (6.6.1), (6.6.4), (6.6.5), (6.6.6), and (6.6.7) together with Proposition 6.4, we have

$$\begin{aligned}
&\int_{U_\varepsilon} \exp\bigg(\frac{1}{6}\int_{\mathbb{T}} |u|^6 \,\mathrm{d}x\bigg) \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})} \le K\}} \,\mu(\mathrm{d}u) \\
&\lesssim \int_0^{\delta^*} \int_{\{\|v\|_{L^2(\mathbb{T})} \le \varepsilon_1\}} e^{G(v)} e^{C_\eta \int |v|^6 \,\mathrm{d}x} \,\mathbf{1}_{\{\|Q_\delta^\rho + v\|_{L^2(\mathbb{T})} \le K\}} \,\mu_\delta^\perp(\mathrm{d}v) \sigma(\mathrm{d}\delta),
\end{aligned} \tag{6.6.8}$$

[14] In view of (1.18), we did not need to use the real part symbol in (6.6.3). We, however, chose to use the real part symbol for clarity.

where $G(v)$ is defined by

$$G(v) = (2\delta^{-2} - 1)\langle Q_\delta^\rho, \operatorname{Re} v\rangle + (1+\eta)\langle (Q_\delta^\rho)^4, 2(\operatorname{Re} v)^2 + \tfrac{1}{2}|v|^2\rangle. \tag{6.6.9}$$

Here, $\varepsilon_1 > 0$ is a small number to be chosen later (see Lemma 6.12 below), which also appears Proposition 6.4, determining small $\varepsilon = \varepsilon(\varepsilon_1) > 0$ and $\delta^* = \delta^*(\varepsilon_1) > 0$.

By a slight modification of the argument presented in Subsection 4.1, we have the following integrability result.

Lemma 6.12. *Given any $C'_\eta > 0$, there exists small $\varepsilon_1 > 0$ such that*

$$\int_{\{\|v\|_{L^2(\mathbb{T})} \le \varepsilon_1\}} \exp\bigg(C'_\eta \int_{\mathbb{T}} |v|^6 \, dx\bigg) \mu_\delta^\perp(dv) < \infty, \tag{6.6.10}$$

uniformly in $0 < \delta \ll 1$.

Proof. Let W be a finite-dimensional subspace of $H^1(\mathbb{T})$ of dimension n with an orthonormal basis $\{w_1, \dots, w_n\} \subset H^2(\mathbb{T})$ (with respect to the $H^1(\mathbb{T})$-inner product). Define the projector $P_{W^\perp}$ by

$$P_{W^\perp} u = u - \sum_{j=1}^n \langle u, (1 - \partial_x^2) w_j\rangle w_j, \tag{6.6.11}$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\mathbb{T})}$. On $H^1(\mathbb{T})$, this is nothing but the usual H^1-orthogonal projection onto $W^\perp$. The definition (6.6.11) allows us to extend $P_{W^\perp}$ to $L^2(\mathbb{T})$. Then, by the definition of $\mu_\delta^\perp = \mu_{\delta,0}^\perp$, it suffices to show that given $C'_\eta > 0$, there exist $M \gg 1$ and $\varepsilon_1 > 0$, depending only on $n = \dim W$, such that

$$\int_{\{\|P_{W^\perp} u\|_{L^2(\mathbb{T})} \le \varepsilon_1\}} \exp\bigg(C'_\eta \int_{\mathbb{T}} |P_{W^\perp} u|^6 \, dx\bigg) \mu(du) \le M. \tag{6.6.12}$$

Indeed, in view of (6.3.3) and the definition of $\mu_\delta^\perp = \mu_{\delta,0}^\perp$ with the Cameron-Martin space $V_{\delta,0,0}$, by simply setting $w_1 = \frac{\partial_\delta Q_\delta^\rho}{\|\partial_\delta Q_\delta^\rho\|_{H^1(\mathbb{T})}}$, $w_2 = \frac{\partial_{x_0} Q_\delta^\rho}{\|\partial_{x_0} Q_\delta^\rho\|_{H^1(\mathbb{T})}}$, and $w_3 = \frac{i Q_\delta^\rho}{\|Q_\delta^\rho\|_{H^1(\mathbb{T})}}$ with $n = 3$, the desired bound (6.6.10) follows from (6.6.12).

The proof of the inequality (6.6.12) follows closely the argument presented in Subsection 4.1. In the following, we point out the modifications required to obtain (6.6.12). First of all, we replace the definition (4.1) of the set E_k by

$$\begin{aligned} E_k &= \big\{\|(P_{W^\perp} u)_{\ge 0}\|_{L^p} > \lambda, \dots, \|(P_{W^\perp} u)_{\ge k-1}\|_{L^p} > \lambda, \|(P_{W^\perp} u)_{\ge k}\|_{L^p} \le \lambda\big\} \\ &\subset \big\{\|(P_{W^\perp} u)_{\ge k-1}\|_{L^p} > \lambda, \|(P_{W^\perp} u)_{\ge k}\|_{L^p} \le \lambda\big\}. \end{aligned} \tag{6.6.13}$$

Namely, in the definition (4.1) of E_k, we replace u with $P_{W^\perp} u$. Arguing as in (4.2) with (6.6.13), it suffices to show that

$$\begin{aligned} \mathbf{E}\Big[& e^{C'_\eta \int_{\mathbb{T}} |P_{W^\perp} u(x)|^6 \, dx}, \|(P_{W^\perp} u)_{\ge k-1}\|_{L^p(\mathbb{T})} > \lambda, \\ & \qquad \|(P_{W^\perp} u)_{\ge k}\|_{L^p(\mathbb{T})} \le \lambda, \, \|P_{W^\perp} u\|_{L^2(\mathbb{T})} \le \varepsilon_1\Big] \end{aligned} \tag{6.6.14}$$

is summable in $k \in \mathbb{N}$. Proceeding in the same way as in (4.4) and (4.5), we obtain the following analogue of (4.6), bounding (6.6.14):

$$\begin{aligned} e^{C_6(\varepsilon)\lambda^6}\mathbf{E}\Big[& e^{C'_\eta(1+\delta_0)\int_{\mathbb{T}}|(P_{W^\perp}u)_{\le k-1}(x)|^6\,dx}, \\ & \|(P_{W^\perp}u)_{\ge k-1}\|_{L^p(\mathbb{T})} > \lambda, \|P_{W^\perp}u\|_{L^2(\mathbb{T})} \le \varepsilon_1\Big], \end{aligned} \tag{6.6.15}$$

where we used δ_0 for the constant $\delta = \delta(6,\varepsilon)$ in (4.5) to avoid confusion with the dilation parameter δ. Note that from (3.5) in Lemma 3.3 with (6.6.11), we have

$$\begin{aligned} \|(P_{W^\perp}u)_{\le k-1}\|^6_{L^6(\mathbb{T})} & \lesssim \|(P_{W^\perp}u)_{\le k-1}\|^2_{H^1(\mathbb{T})}\|P_{W^\perp}u\|^4_{L^2(\mathbb{T})} \\ & \lesssim \Big(\|u_{\le k-1}\|^2_{H^1(\mathbb{T})} + \sum_{j=1}^n |\langle u,(1-\partial_x^2)w_j\rangle|^2\|(w_j)_{\le k-1}\|^2_{H^1(\mathbb{T})}\Big)\|P_{W^\perp}u\|^4_{L^2(\mathbb{T})} \\ & \le \Big(\|u_{\le k-1}\|^2_{H^1(\mathbb{T})} + \sum_{j=1}^n |\langle u,(1-\partial_x^2)w_j\rangle|^2\Big)\|P_{W^\perp}u\|^4_{L^2(\mathbb{T})}, \end{aligned} \tag{6.6.16}$$

where the implicit constant depends only on $n = \dim W$.

Using the inequality (6.6.16) instead of (3.5), and fixing $\lambda = 1$, we obtain

$$\begin{aligned} (6.6.15) \le e^{C_6(\varepsilon)}\mathbf{E}\bigg[& \exp\bigg(CC'_\eta(1+\delta_0)\varepsilon_1^4\Big(\|u_{\le k-1}\|^2_{H^1(\mathbb{T})} + \sum_{j=1}^n |\langle u,(1-\partial_x^2)w_j\rangle|^2\Big)\bigg), \\ & \|(P_{W^\perp}u)_{\ge k-1}\|_{L^p(\mathbb{T})} > 1\bigg] \end{aligned}$$

for some $C = C(n) > 0$. Then, by Hölder's inequality, this implies the following analogue of (4.7):

$$\begin{aligned} (6.6.15) \le e^{C_6(\varepsilon)} & \Big\{\mathbf{E}\Big[\exp\Big((n+2)CC'_\eta(1+\delta_0)\varepsilon_1^4\|u_{\le k-1}\|^2_{H^1(\mathbb{T})}\Big)\Big]\Big\}^{\frac{1}{n+2}} \\ & \times \prod_{j=1}^n \Big\{\mathbf{E}\Big[\exp\Big((n+2)CC'_\eta(1+\delta_0)\varepsilon_1^4|\langle u,(1-\partial_x^2)w_j\rangle|^2\Big)\Big]\Big\}^{\frac{1}{n+2}} \\ & \times \Big\{\mathbf{P}\big(\|(P_{W^\perp}u)_{\ge k-1}\|_{L^6(\mathbb{T})} > 1\big)\Big\}^{\frac{1}{n+2}}. \end{aligned} \tag{6.6.17}$$

We note from (6.0.1) that $\langle u,(1-\partial_x^2)w_j\rangle$ is a mean-zero Gaussian random variable with $\mathbf{E}\big[|\langle u,(1-\partial_x^2)w_j\rangle|^2\big] = \|w_j\|^2_{H^1(\mathbb{T})} = 1$. In particular, if $(n+2)CC'_\eta(1+\delta_0)\varepsilon_1^4 < \frac{1}{2}$, it follows from (4.9) that

$$\mathbf{E}\Big[\exp\Big((n+2)CC'_\eta(1+\delta_0)\varepsilon_1^4|\langle u,(1-\partial_x^2)w\rangle|^2\Big)\Big] < \infty. \tag{6.6.18}$$

Moreover, by Bernstein's inequality with $\|w_j\|^2_{H^1(\mathbb{T})} = 1$, there exists $c = c(n) > 0$ such that

$$\begin{aligned} &\mathbf{P}\bigg(\|(\langle u, (1-\partial_x^2)w_j\rangle w)_{\geq k-1}\|_{L^6(\mathbb{T})} > \frac{1}{n+1}\bigg) \\ &\quad = \mathbf{P}\bigg(|\langle u, (1-\partial_x^2)w_j\rangle| > \frac{1}{(n+1)\|(w_j)_{\geq k-1}\|_{L^6(\mathbb{T})}}\bigg) \\ &\quad \leq \mathbf{P}\bigg(|\langle u, (1-\partial_x^2)w_j\rangle| \gtrsim \frac{2^{\frac{2}{3}k}}{\|w_j\|_{H^1(\mathbb{T})}}\bigg) \\ &\quad \lesssim \exp(-c2^{\frac{4}{3}k}), \end{aligned} \tag{6.6.19}$$

uniformly in $j = 1, \dots, n$. Hence, from (6.6.11), (2.8), and (6.6.19), we obtain, for some $c' = c'(n) > 0$,

$$\begin{aligned} &\mathbf{P}\big(\|(P_{W^\perp}u)_{\geq k-1}\|_{L^6(\mathbb{T})} > 1\big) \\ &\quad \leq \mathbf{P}\bigg(\|u_{\geq k-1}\|_{L^6(\mathbb{T})} > \frac{1}{n+1}\bigg) + \sum_{j=1}^{n} \mathbf{P}\bigg(\|(\langle u, (1-\partial_x^2)w_j\rangle w_j)_{\geq k-1}\|_{L^6(\mathbb{T})} > \frac{1}{n+1}\bigg) \\ &\quad \lesssim \exp(-c'2^{\frac{4}{3}k}), \end{aligned} \tag{6.6.20}$$

which is an analogue of (4.8). In addition, if $(n+2)CC'_\eta(1+\delta_0)\varepsilon_1^4 < \frac{1}{2}$, then using (4.9), we can repeat the computations in (4.10) and obtain

$$\begin{aligned} &\mathbf{E}\Big[\exp\Big((n+2)CC'_\eta(1+\delta_0)\varepsilon_1^4\|u_{\leq k-1}\|^2_{H^1(\mathbb{T})}\Big)\Big] \\ &\quad \leq (1-2(n+2)CC'_\eta(1+\delta_0)\varepsilon_1^4)^{-2^k}. \end{aligned} \tag{6.6.21}$$

Hence, by applying (6.6.21), (6.6.18), and (6.6.20) into (6.6.17) and proceeding as in (4.11), we see that (6.6.14) is summable in $k \in \mathbb{N}$. Therefore, we conclude that

$$\int_{\{\|P_{W^\perp}u\|_{L^2(\mathbb{T})} \leq \varepsilon_1\}} \exp\bigg(C'_\eta \int_{\mathbb{T}} |P_{W^\perp}u|^6 \, dx\bigg)\mu(du) \lesssim 1,$$

where the implicit constant depends only on the dimension n of W. □

By applying Hölder's inequality and Lemma 6.12 to (6.6.8), we obtain

$$\begin{aligned} &\int_{U_\varepsilon} \exp\bigg(\frac{1}{6}\int_{\mathbb{T}} |u|^6 \, dx\bigg) \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})} \leq K\}} \, \mu(du) \\ &\quad \lesssim \int_0^{\delta^*} \bigg(\int_{\{\|v\|_{L^2(\mathbb{T})} \leq \varepsilon_1\}} e^{(1+\eta)G(v)} \mathbf{1}_{\{\|Q_\delta^\rho + v\|_{L^2(\mathbb{T})} \leq K\}} \mu_\delta^\perp(dv)\bigg)^{\frac{1}{1+\eta}} \sigma(d\delta). \end{aligned}$$

Our goal in the remaining part of this section is to bound the inner integral on the right-hand side.

Next, we decompose the subspace[15] $V_{\delta,0,0} = V_{\delta,0,0}(\mathbb{T}) \subset H^1(\mathbb{T})$ in (6.3.3) as

$$V_{\delta,0,0} = V' \oplus \text{span}\{e\}, \tag{6.6.22}$$

[15] Recall the convention $V_{\delta,0,0} = V_{\delta,0,0} \cap H^1(\mathbb{T})$ introduced at the beginning of Subsection 6.5.

where $\|e\|_{H^1(\mathbb{T})} = 1$ and e is orthogonal in $H^1(\mathbb{T})$ to

$$V' = \big\{ w \in V_{\delta,0,0} : \langle w, Q_\delta^\rho \rangle = 0 \big\}. \tag{6.6.23}$$

Denote by $P_W^{H^1}$ the H^1-orthogonal projection on a given subspace $W \subset H^1(\mathbb{T})$. Then, by noting the orthogonality of Q_δ^ρ with $\partial_{x_0} Q_\delta^\rho$ and iQ_δ^ρ in $L^2(\mathbb{T})$ and by directly computing $\langle Q_\delta^\rho, \partial_\delta Q_\delta^\rho \rangle$ with (6.1.9) and integrating by parts, we have

$$\begin{aligned} e &= \frac{P_{V_{\delta,0,0}}^{H^1} (1-\partial_x^2)^{-1} Q_\delta^\rho}{\|P_{V_{\delta,0,0}}^{H^1} (1-\partial_x^2)^{-1} Q_\delta^\rho\|_{H^1(\mathbb{T})}} \\ &= \frac{(1-\partial_x^2)^{-1} Q_\delta^\rho}{\|(1-\partial_x^2)^{-1} Q_\delta^\rho\|_{H^1(\mathbb{T})}} + O_{L^2(\mathbb{T})}(\exp(-c\delta^{-1})) \end{aligned} \tag{6.6.24}$$

for $0 < \delta \ll 1$. Corresponding to the decomposition (6.6.22) of the Cameron-Martin space for $\mu_\delta^\perp$, we have the following decomposition of the measure:

$$\mathrm{d}\mu_\delta^\perp(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}g^2} \mathrm{d}g \mathrm{d}\mu_\delta^{\perp\perp}(w)$$

with

$$v = ge + w, \quad w \in V'. \tag{6.6.25}$$

Lemma 6.13. *Let $G(v)$ be as in* (6.6.9)*. Then, we have*

$$\begin{aligned} &\int_{\{\|v\|_{L^2(\mathbb{T})} \le \varepsilon_1\}} e^{(1+\eta)G(v)} \mathbf{1}_{\{\|Q_\delta^\rho + v\|_{L^2(\mathbb{T})} \le K\}} \mu_\delta^\perp(\mathrm{d}v) \\ &\qquad \lesssim \int \exp\Big(-(1-\eta^2) \widetilde{H}_\delta(w) \Big) \mu_\delta^{\perp\perp}(\mathrm{d}w), \end{aligned} \tag{6.6.26}$$

uniformly in $0 < \delta \ll 1$, where $\widetilde{H}_\delta(w)$ is given by

$$\widetilde{H}_\delta(w) = \delta^{-2} \int_{\mathbb{T}} |w|^2 \, \mathrm{d}x - (1+5\eta) \int_{\mathbb{T}} (Q_\delta^\rho)^4 \big(2(\operatorname{Re} w)^2 + \tfrac{1}{2}|w|^2 \big) \mathrm{d}x.$$

Proof. By expanding $\langle Q_\delta^\rho + v, Q_\delta^\rho + v \rangle$ with the decomposition (6.6.25), we have

$$\begin{aligned} \|Q\|_{L^2(\mathbb{R})}^2 = K^2 &\ge \langle Q_\delta^\rho + v, Q_\delta^\rho + v \rangle \\ &= \|Q_\delta^\rho\|_{L^2(\mathbb{T})}^2 + 2g \langle Q_\delta^\rho, e \rangle + \|v\|_{L^2(\mathbb{T})}^2 \\ &= \|Q_\delta^\rho\|_{L^2(\mathbb{T})}^2 + 2g \langle Q_\delta^\rho, e \rangle + \|ge\|_{L^2(\mathbb{T})}^2 + \|w\|_{L^2(\mathbb{T})}^2 + 2g \langle e, w \rangle. \end{aligned}$$

Together with (6.1.8), we obtain

$$\begin{aligned} &2g \langle Q_\delta^\rho, e \rangle + \|v\|_{L^2(\mathbb{T})}^2 \\ &\quad = 2g \langle Q_\delta^\rho, e \rangle + \|ge\|_{L^2(\mathbb{T})}^2 + \|w\|_{L^2(\mathbb{T})}^2 + 2g \langle e, w \rangle \\ &\quad \le C \exp(-c\delta^{-1}). \end{aligned} \tag{6.6.27}$$

We also note from (6.6.24) that

$$\langle Q_\delta^\rho, e \rangle = \|Q_\delta^\rho\|_{H^{-1}(\mathbb{T})} + O(\exp(-c\delta^{-1})) \sim \|Q_\delta\|_{H^{-1}(\mathbb{R})} \sim \delta \sim \|e\|_{L^2(\mathbb{T})}, \tag{6.6.28}$$

uniformly in $0 < \delta \ll 1$.

We also note that if $\|v\|_{L^2(\mathbb{T})} \le \varepsilon \le \varepsilon_1$, then we have

$$\|ge\|_{L^2(\mathbb{T})} + \|w\|_{L^2(\mathbb{T})} \lesssim \varepsilon. \tag{6.6.29}$$

Indeed, by observing

$$\begin{aligned}
&\Big| \|Q_\delta^\rho\|_{L^2(\mathbb{T})}^2 - \|Q_\delta^\rho + v\|_{L^2(\mathbb{T})}^2 \Big| \\
&\quad = \Big| \|Q_\delta^\rho\|_{L^2(\mathbb{T})} - \|Q_\delta^\rho + v\|_{L^2(\mathbb{T})} \Big| \Big(\|Q_\delta^\rho\|_{L^2(\mathbb{T})} + \|Q_\delta^\rho + v\|_{L^2(\mathbb{T})} \Big) \\
&\quad \lesssim \|v\|_{L^2(\mathbb{T})} \Big(\|Q_\delta^\rho\|_{L^2(\mathbb{T})} + \|v\|_{L^2(\mathbb{T})} \Big) \\
&\quad \lesssim \varepsilon,
\end{aligned}$$

we have

$$\begin{aligned}
|2g\langle Q_\delta^\rho, e\rangle| &\le \Big| 2g\langle Q_\delta^\rho, e\rangle + \|v\|_{L^2(\mathbb{T})}^2 \Big| + \|v\|_{L^2(\mathbb{T})}^2 \\
&= \Big| \|Q_\delta^\rho\|_{L^2(\mathbb{T})}^2 - \|Q_\delta^\rho + v\|_{L^2(\mathbb{T})}^2 \Big| + O(\varepsilon^2) \\
&\lesssim \varepsilon.
\end{aligned} \tag{6.6.30}$$

Then, from (6.6.28) and (6.6.30), we conclude that $\|ge\|_{L^2(\mathbb{T})} \sim |g\langle Q_\delta^\rho, e\rangle| \lesssim \varepsilon$. By the triangle inequality with $w = v - ge$, we obtain $\|w\|_{L^2(\mathbb{T})} \lesssim \varepsilon$. This proves (6.6.29).

Now, from (6.6.9), the decomposition (6.6.25), Cauchy's inequality, and $\|Q_\delta^\rho\|_{L^\infty(\mathbb{T})} \sim \delta^{-\frac{1}{2}}$, we have

$$\begin{aligned}
G(v) &\le (2\delta^{-2} - 1)\langle Q_\delta^\rho, ge\rangle + C_\eta \int_{\mathbb{T}} (Q_\delta^\rho)^4 |ge|^2 \mathrm{d}x \\
&\qquad + (1 + 2\eta) \int_{\mathbb{T}} (Q_\delta^\rho)^4 \big(2(\operatorname{Re} w)^2 + \tfrac{1}{2}|w|^2\big) \mathrm{d}x \\
&\le \delta^{-2} \big((2 - \delta^2)\langle Q_\delta^\rho, ge\rangle + C'_\eta \|ge\|_{L^2(\mathbb{T})}^2 \big) \\
&\qquad + (1 + 2\eta) \int_{\mathbb{T}} (Q_\delta^\rho)^4 \big(2(\operatorname{Re} w)^2 + \tfrac{1}{2}|w|^2\big) \mathrm{d}x.
\end{aligned} \tag{6.6.31}$$

Noting that $(1+\eta)(1+2\eta) \le (1-\eta^2)(1+5\eta)$ for any $0 < \eta \ll 1$, in order to obtain (6.6.26), we only need to bound the first term on the right-hand side of (6.6.31).

• **Case 1:** $g \ge 0$.

In this case, from (6.6.27) and (6.6.28) (which implies $\langle Q_\delta^\rho, e\rangle > 0$), we have

$$\|v\|_{L^2(\mathbb{T})}^2 \le g\langle Q_\delta^\rho, e\rangle + \|v\|_{L^2(\mathbb{T})}^2 = O(\exp(-c\delta^{-1})). \tag{6.6.32}$$

Then, for sufficiently small $\delta > 0$, (6.6.32) shows that the hypothesis of (6.6.29) on the size of $\|v\|_{L^2(\mathbb{T})}$ is satisfied with $\varepsilon \sim \exp(-c\delta^{-1})$. Thus, from (6.6.29), we have

$$\|ge\|_{L^2(\mathbb{T})} + \|w\|_{L^2(\mathbb{T})} = O(\exp(-c\delta^{-1})), \tag{6.6.33}$$

provided that $\delta = \delta(\eta, \varepsilon_1) > 0$ is sufficiently small. Hence, from (6.6.32) and (6.6.33) with (6.6.25), we obtain, for some $C''_\eta > 0$,

$$(2 - \delta^2)\langle Q_\delta^\rho, ge\rangle + C'_\eta \|ge\|_{L^2(\mathbb{T})}^2 \le -(1 - \eta)\|w\|_{L^2(\mathbb{T})}^2 + C''_\eta \exp(-c\delta^{-1}). \tag{6.6.34}$$

Therefore, from (6.6.31) and (6.6.34), the contribution to the left-hand side of (6.6.26) in this case is bounded by

$$\begin{aligned}&\lesssim \int\int_{\{0\le g\lesssim \exp(-c\delta^{-1})\}} \exp\Big(-(1-\eta^2)\widetilde{H}_\delta(w)\Big)\, e^{-\frac12 g^2}\, dg\, d\mu_\delta^{\perp\perp}(w)\\ &\lesssim \int \exp\Big(-(1-\eta^2)\widetilde{H}_\delta(w)\Big)\,\mu_\delta^{\perp\perp}(dw).\end{aligned}$$

• **Case 2:** $g<0$.

Under the condition $\|v\|_{L^2}\le\varepsilon_1$, it follows from (6.6.27), Cauchy's inequality, (6.6.28) (which implies $\|ge\|_{L^2(\mathbb{T})}\sim -g\langle Q_\delta^\rho, e\rangle$), and (6.6.29) that

$$\begin{aligned}2g\langle Q_\delta^\rho, e\rangle &\le -\big(1-\tfrac{\eta}{4}\big)\|w\|_{L^2(\mathbb{T})}^2 + K_\eta\|ge\|_{L^2(\mathbb{T})}^2 + O(\exp(-c\delta^{-1}))\\ &\le -\big(1-\tfrac{\eta}{4}\big)\|w\|_{L^2(\mathbb{T})}^2 - \varepsilon_1 K'_\eta g\langle Q_\delta^\rho, e\rangle + O(\exp(-c\delta^{-1})).\end{aligned}$$

By choosing $\varepsilon_1=\varepsilon_1(\eta)>0$ ands $\delta=\delta(\eta)>0$ sufficiently small, we then have

$$(2-\delta^2)g\langle Q_\delta^\rho, e\rangle \le -\big(1-\tfrac{\eta}{2}\big)\|w\|_{L^2(\mathbb{T})}^2 + O(\exp(-c\delta^{-1})).$$

Hence, we obtain

$$\begin{aligned}(2-\delta^2)g\langle Q_\delta^\rho, e\rangle + C'_\eta\|ge\|_{L^2(\mathbb{T})}^2 &\le \big(2-\delta^2-O(\varepsilon_1)\big)g\langle Q_\delta^\rho, e\rangle\\ &\le -(1-\eta)\|w\|_{L^2(\mathbb{T})}^2 + O(\exp(-c\delta^{-1})).\end{aligned}$$

Proceeding as in Case 1, we also obtain (6.6.26) in this case. □

6.7. **Spectral analysis.** Given small $\delta,\eta>0$, define an operator $A=A(\delta,\eta)$ on $H^1(\mathbb{T})$ by

$$Aw = P_{V'}^{H^1}(1-\partial_x^2)^{-1}\bigg(\delta^{-2}P_{V'}^{H^1}w - (1+5\eta)(Q_\delta^\rho)^4\Big(2\,\mathrm{Re}(P_{V'}^{H^1}w)+\frac12 P_{V'}^{H^1}w\Big)\bigg), \tag{6.7.1}$$

where V' is as in (6.6.23). Then, the integrand of the Gaussian integral on the right-hand side of (6.6.26) can be written as

$$\exp\big(-(1-\eta^2)\langle Aw, w\rangle_{H^1(\mathbb{T})}\big)\,, \tag{6.7.2}$$

The main goal of this subsection is to establish the following integrability result.

Proposition 6.14. *Given any sufficiently small $\eta>0$, there exists a constant $c=c(\eta)>0$ such that*

$$\int \exp\big(-(1-\eta^2)\langle Aw, w\rangle_{H^1(\mathbb{T})}\big)d\mu_\delta^{\perp\perp}(w)\lesssim \exp(-c\delta^{-1}), \tag{6.7.3}$$

uniformly in $0<\delta\ll 1$.

From (6.7.1) with $\|Q_\delta^\rho\|_{L^\infty(\mathbb{T})}\sim\delta^{-\frac12}$, we have

$$|\langle Aw, w\rangle_{H^1(\mathbb{T})}|\le C\delta^{-2}\|w\|_{L^2(\mathbb{T})}^2.$$

Thus, by Rellich's lemma, we see that A is a compact operator on $V'\subset H^1(\mathbb{T})$ and thus the spectrum of A consists of eigenvalues. Recalling that $V'\subset H^1(\mathbb{T})$ in (6.6.23) is the Cameron-Martin space for $\mu_\delta^{\perp\perp}$, we see that evaluating the integral in (6.7.3) is equivalent to estimating the product of the eigenvalues of $\frac12\,\mathrm{id}+(1-\eta^2)A$ on V'. Thus, the rest of this subsection is devoted to studying the eigenvalues of A. We present the proof of Proposition 6.14 at the end of this subsection.

Since A preserves the subspace of $\mathrm{Re}\, H^1(\mathbb{T})$ consisting of real-valued functions and also the subspace $\mathrm{Im}\, H^1(\mathbb{T})$ consisting of purely-imaginary-valued functions, in studying the spectrum of A, we split the analysis onto these two subspaces. On the real subspace $\mathrm{Re}\, H^1(\mathbb{T})$, A coincides with the operator

$$A_1 = P^{H^1}_{V_1(\delta)}(1-\partial_x^2)^{-1}\left(\delta^{-2} - \frac{5}{2}(1+5\eta)(Q^\rho_\delta)^4\right)P^{H^1}_{V_1(\delta)}, \tag{6.7.4}$$

where

$$\begin{aligned} V_1(\delta) = \big\{w \in \mathrm{Re}\, H^1(\mathbb{T}) : &\langle w, (1-\partial_x^2)\partial_\delta Q^\rho_\delta\rangle = 0, \\ &\langle w, (1-\partial_x^2)\partial_{x_0} Q^\rho_\delta\rangle = 0,\ \langle w, Q^\rho_\delta\rangle = 0\big\}. \end{aligned} \tag{6.7.5}$$

On the other hand, identifying $\mathrm{Im}\, H^1(\mathbb{T})$ with $\mathrm{Re}\, H^1(\mathbb{T})$ by the multiplication with i, we see that, on the imaginary subspace, A coincides with the operator

$$A_2 = P^{H^1}_{V_2(\delta)}(1-\partial_x^2)^{-1}\left(\delta^{-2} - \frac{1}{2}(1+5\eta)(Q^\rho_\delta)^4\right)P^{H^1}_{V_2(\delta)}, \tag{6.7.6}$$

where

$$V_2(\delta) = \big\{w \in \mathrm{Re}\, H^1(\mathbb{T}) : \langle w, (1-\partial_x^2)Q^\rho_\delta\rangle = 0\big\}. \tag{6.7.7}$$

In the following, we proceed to analyze the eigenvalues of A_1 and A_2. In Proposition 6.15, we provide a lower bound for the eigenvalues of $\frac{1}{2}\,\mathrm{id} + (1-\eta^2)A$. Then, by comparing A_j with simple operators whose spectrum can be determined explicitly (Lemma 6.19), we establish asymptotic bounds on the eigenvalues in Proposition 6.18.

Proposition 6.15. *There exists a constant $\varepsilon_0 > 0$ such that for any $\delta, \eta > 0$ sufficiently small, the smallest eigenvalue of $A = A(\delta,\eta)$ on V', defined in* (6.6.23), *is greater than $-\frac{1}{2} + \varepsilon_0$.*

Proof. Fix $0 < \varepsilon_0 \ll 1$. Suppose by contradiction that there exists a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of positive numbers tending to 0 and $w^j_n \in V_j(\delta_n)$, $j = 1, 2$, such that

$$\begin{aligned} &\Big(-\frac{1}{2} + \varepsilon_0\Big)\int_{\mathbb{T}} (1-\partial_x^2)w^j_n \cdot w^j_n \mathrm{d}x \\ &\qquad \geq \langle A_j w^j_n, w^j_n\rangle_{H^1(\mathbb{T})} = \delta_n^{-2}\int_{\mathbb{T}} (w^j_n)^2 \mathrm{d}x - c_j(1+5\eta)\int_{\mathbb{T}} (Q^\rho_{\delta_n})^4 (w^j_n)^2 \mathrm{d}x \end{aligned} \tag{6.7.8}$$

for $j = 1, 2$, where $c_1 = \frac{5}{2}$ and $c_2 = \frac{1}{2}$. Let $\widetilde{A}^j_n$ denote the Schrödinger operator given by

$$\begin{aligned} \widetilde{A}^j_n &= P_{V_j(\delta_n)} T^j_n P_{V_j(\delta_n)} \\ &:= P_{V_j(\delta_n)}\bigg(-\Big(\frac{1}{2} - \varepsilon_0\Big)\partial_x^2 + \delta_n^{-2} + \Big(\frac{1}{2} - \varepsilon_0\Big) - c_j(1+5\eta)(Q^\rho_{\delta_n})^4\bigg)P_{V_j(\delta_n)}, \end{aligned} \tag{6.7.9}$$

where $P_{V_j(\delta_n)}$ denotes the projection onto $V_j(\delta_n)$ in $\mathrm{Re}\, L^2(\mathbb{T})$. Then, by the min-max principle (see [86, Theorem XIII.1]), the minimum eigenvalue $\widetilde{\lambda}^j_n$ is non-positive. We denote by $\widetilde{w}^j_n \in V_j(\delta_n) \subset L^2(\mathbb{T})$ an L^2-normalized eigenvector associated with this minimum eigenvalue $\widetilde{\lambda}^j_n$:

$$\widetilde{A}^j_n \widetilde{w}^j_n = \widetilde{\lambda}^j_n \widetilde{w}^j_n. \tag{6.7.10}$$

Recalling $\|Q^\rho_\delta\|_{L^\infty(\mathbb{T})} \sim \delta^{-\frac12}$, we have

$$\langle T^j_n w, w\rangle \gtrsim -\delta_n^{-2}\|w\|^2_{L^2(\mathbb{T})}$$

for any $w \in V_j(\delta_n)$, which shows that $\widetilde{A}^j_n$ is semi-bounded on $V_j(\delta_n)$ with a constant of order δ_n^{-2}. In particular, we have

$$-C\delta_n^{-2} \le \widetilde{\lambda}^j_n \le 0. \tag{6.7.11}$$

When $j = 1$, the condition $\langle \widetilde{w}^1_n, Q^\rho_{\delta_n}\rangle = 0$ in (6.7.5) together with the positivity of Q and the definition $Q^\rho_{\delta_n} = \rho Q_{\delta_n}$ implies that $\widetilde{w}^1_n(x_n) = 0$ for some $x_n \in \mathbb{T}$. We define a sequence $\{v^1_n\}_{n\in\mathbb{N}}$ of L^2-normalized functions on $\mathbb{R}$ by

$$v^1_n(x) = \begin{cases} \delta_n^{\frac12}\widetilde{w}^1_n(\delta_n x + x_n - \frac12), & \text{for } |x| \le \frac12\delta_n^{-1},\\ 0, & \text{for } |x| > \frac12\delta_n^{-1}. \end{cases} \tag{6.7.12}$$

where the addition is understood mod 1. When $j = 2$, it follows from the continuity of $\widetilde{w}^2_n$ and $\|\widetilde{w}^2_n\|_{L^2(\mathbb{T})} = 1$ that $|\widetilde{w}^2(x_n)| \le 1$ for some $x_n \in \mathbb{T}$. Then, we define a sequence $\{v^2_n\}_{n\in\mathbb{N}}$ of functions on $\mathbb{R}$ by

$$v^2_n(x) = \begin{cases} \delta_n^{\frac12}\widetilde{w}^2_n(\delta_n x + x_n - \frac12), & \text{for } |x| \le \frac12\delta_n^{-1},\\ 0, & \text{for } |x| > \frac12\delta_n^{-1} + \delta_n^{\frac12}|\widetilde{w}^2(x_n)|, \end{cases} \tag{6.7.13}$$

and by linear interpolation for $\frac12\delta_n^{-1} < |x| \le \frac12\delta_n^{-1} + \delta_n^{\frac12}|w(x_n)|$. In both cases, we have $v^j_n \in H^1(\mathbb{R})$. In the remaining part of the proof, we drop the superscript j for simplicity of notations, when there is no confusion.

Define $\widetilde{T}^j_n$ by

$$\widetilde{T}^j_n = -\Big(\frac12 - \varepsilon_0\Big)\partial_x^2 + 1 + \Big(\frac12 - \varepsilon_0\Big)\delta_n^2 - c_j(1+5\eta)\big(\tau_{-\delta_n^{-1}(x_n - \frac12)}\rho(\delta_n\cdot)Q\big)^4, \tag{6.7.14}$$

where τ_{x_0} denotes the translation by x_0 as in (6.1.4). Then, from (6.7.9), (6.7.12), (6.7.13), and (6.7.14), a direct computation shows that

$$\widetilde{T}^j_n v_n(x) = \delta_n^{\frac52} T^j_n \widetilde{w}_n(y) \tag{6.7.15}$$

with $y = \delta_n x + x_n - \frac12$, as long as $|x| \le \frac12\delta_n^{-1}$.

Define $\widetilde{V}_j \subset \operatorname{Re} H^1(\mathbb{R})$, $j = 1, 2$, by

$$\begin{aligned} \widetilde{V}_1 = \big\{v \in \operatorname{Re} H^1(\mathbb{R}) : \langle v, \partial_\delta Q\rangle_{\dot H^1(\mathbb{R})} = 0, \langle v, \partial_{x_0} Q\rangle_{\dot H^1(\mathbb{R})} = 0,\\ \langle v, Q\rangle_{L^2(\mathbb{R})} = 0\big\}, \end{aligned} \tag{6.7.16}$$

$$\widetilde{V}_2 = \big\{v \in \operatorname{Re} H^1(\mathbb{R}) : \langle v, Q\rangle_{\dot H^1(\mathbb{R})} = 0\big\}. \tag{6.7.17}$$

Noting that $\langle v, w\rangle_{\dot H^1(\mathbb{R})} = \langle v, -\partial_x^2 w\rangle_{L^2(\mathbb{R})}$ and that Q is a smooth function, we can view $\widetilde{V}_j$ as a subspace of $\operatorname{Re} L^2(\mathbb{R})$. We now define the operator $\widetilde{T}_j$, $j = 1, 2$, on $\operatorname{Re} L^2(\mathbb{R})$ by

$$\begin{aligned} \widetilde{T}_j &= P^{L^2(\mathbb{R})}_{\widetilde{V}_j} S_j P^{L^2(\mathbb{R})}_{\widetilde{V}_j}\\ &:= P^{L^2(\mathbb{R})}_{\widetilde{V}_j}\bigg(-\Big(\frac12 - \varepsilon_0\Big)\partial_x^2 + 1 - c_j(1+5\eta)Q^4\bigg)P^{L^2(\mathbb{R})}_{\widetilde{V}_j}. \end{aligned} \tag{6.7.18}$$

Here, $P^{L^2(\mathbb{R})}_{\widetilde{V}_j}$ denotes the projection onto $\widetilde{V}_j$ in $\operatorname{Re} L^2(\mathbb{R})$. Then, from (6.7.10), (6.7.14), (6.7.15), and (6.7.18) along with the smoothness and exponential decay of Q and its derivatives, we have

$$\langle \widetilde{T}_j \tau_{\delta_n^{-1}(x_n-\frac{1}{2})} v_n - \lambda_n \tau_{\delta_n^{-1}(x_n-\frac{1}{2})} v_n, \tau_{\delta_n^{-1}(x_n-\frac{1}{2})} v_n \rangle_{L^2(\mathbb{R})} \longrightarrow 0 \tag{6.7.19}$$

as $n \to \infty$, where $\lambda_n = \delta_n^2 \widetilde{\lambda}_n \le 0$ Moreover, (6.7.5), (6.7.7), (6.7.12), (6.7.13), (6.7.16), and (6.7.17) along with (6.1.6) and the exponential decay of Q and its derivatives once again, we have

$$\| P^{L^2(\mathbb{R})}_{\widetilde{V}_j} \tau_{\delta_n^{-1}(x_n-\frac{1}{2})} v_n \|_{L^2} \gtrsim 1. \tag{6.7.20}$$

Note that from (6.7.11), we have

$$-C \le \lambda_n \le 0$$

for any $n \in \mathbb{N}$. Thus, passing to a subsequence, we may assume that $\lambda_n \to \lambda$ for some $\lambda \le 0$ and thus from (6.7.19), we obtain

$$\langle \widetilde{T}_j \tau_{\delta_n^{-1}(x_n-\frac{1}{2})} v_n - \lambda \tau_{\delta_n^{-1}(x_n-\frac{1}{2})} v_n, \tau_{\delta_n^{-1}(x_n-\frac{1}{2})} v_n \rangle_{L^2(\mathbb{R})} \longrightarrow 0. \tag{6.7.21}$$

Noting that $\widetilde{T}_j$ is semi-bounded, let λ_0 be the minimum of the spectrum of $\widetilde{T}_j$ on $\widetilde{V}_j$. By the min-max principle (see [86, Theorem XIII.1]), we have

$$\langle \widetilde{T}_j v - \lambda v, v \rangle_{L^2(\mathbb{R})} \ge (\lambda_0 - \lambda) \| P^{L^2(\mathbb{R})}_{\widetilde{V}_j} v \|^2_{L^2(\mathbb{R})}$$

for every $v \in H^1(\mathbb{R})$. Therefore, from (6.7.21) and (6.7.20), we obtain that $\lambda_0 \le \lambda \le 0$.

Since Q is a Schwartz function, the essential spectrum of the Schrödinger operator S_j in (6.7.18) is equal to $[1, \infty)$; see [41, Theorem V-5.7]. Moreover, $\widetilde{T}_j$ in (6.7.18) differs from S_j by a finite rank perturbation and thus its essential spectrum is $[1, \infty)$; see [41, Theorem IV-5.35]. In particular, $\lambda_0 \le 0$ does *not* lie in the essential spectrum of $\widetilde{T}_j$. Namely, λ_0 belongs to the discrete spectrum of $\widetilde{T}_j$. Hence, there exists $v \in H^2(\mathbb{R})$ such that

$$\widetilde{T}_j v = \lambda_0 v. \tag{6.7.22}$$

In order to derive a contradiction, we invoke the following lemma.

Lemma 6.16. *For any sufficiently small $\varepsilon_0 > 0$ and $\eta > 0$, the following statements hold. The operator*

$$B_1(\varepsilon_0, \eta) = -\Big(\frac{1}{2} - \varepsilon_0\Big)\partial_x^2 + 1 - \frac{5}{2}(1 + 5\eta) Q^4,$$

viewed as an operator on $L^2(\mathbb{R})$, is strictly positive on $\widetilde{V}_1$ defined in (6.7.16). *Similarly, the operator*

$$B_2(\varepsilon_0, \eta) = -\Big(\frac{1}{2} - \varepsilon_0\Big)\partial_x^2 + 1 - \frac{1}{2}(1 + 5\eta) Q^4,$$

viewed as an operator on $L^2(\mathbb{R})$, is strictly positive on $\widetilde{V}_2$ defined in (6.7.17).

This lemma shows that (6.7.22) can not hold for $\lambda \le 0$. Therefore, we arrive at a contradiction to (6.7.8). This concludes the proof of Proposition 6.15 (modulo the proof of Lemma 6.16 which we present below). □

We now present the proof of Lemma 6.16.

Proof of Lemma 6.16. In the following, we only prove the strict positivity of $B_j := B_j(0,0)$ on $\widetilde{V}_j$, $j = 1, 2$, when $\varepsilon_0 = \eta = 0$. Namely, we show that there exists $\theta > 0$ such that

$$\langle B_j v, v\rangle_{L^2(\mathbb{R})} \geq \theta \|v\|^2_{L^2(\mathbb{R})} \tag{6.7.23}$$

for any $v \in \widetilde{V}_j$. Then, by writing

$$B_j(\varepsilon_0, \eta) = (1 - 2\varepsilon_0)B_j(0,0) + 2\varepsilon_0\big(1 - c_j Q^4\big) - 5c_j\eta Q^4$$

with $c_1 = \frac{5}{2}$ and $c_2 = \frac{1}{2}$, the strict positivity of $B_j(\varepsilon, \eta)$ for sufficiently small $\varepsilon_0, \eta > 0$ follows from (6.7.23).

Consider the Hamiltonian $H(u) = H_\mathbb{R}(u)$:

$$H(u) = \frac{1}{2}\int_\mathbb{R} |u'|^2 \mathrm{d}x - \frac{1}{6}\int_\mathbb{R} |u|^6 \mathrm{d}x.$$

By the sharp Gagliardo-Nirenberg-Sobolev inequality (Proposition 3.1), we know that $H : H^1(\mathbb{R}) \to \mathbb{R}$ has a global minimum at Q, when restricted to the manifold $\|u\|_{L^2(\mathbb{R})} = \|Q\|_{L^2(\mathbb{R})}$. In view of (6.1.2), we see that $u = Q$ and $\lambda = -1$ satisfy the following Lagrange multiplier problem:

$$dH|_u(v) = \lambda dG|_u(v) \tag{6.7.24}$$

for any $v \in H^1(\mathbb{T})$, where $G(u) = \|u\|^2_{L^2(\mathbb{R})} - \|Q\|^2_{L^2(\mathbb{R})}$. By a direct computation, the second variation of $H(u) + G(u)$ at Q in the direction v is given by $2\langle B_1 \operatorname{Re} v, \operatorname{Re} v\rangle_{L^2(\mathbb{R})} + 2\langle B_2 \operatorname{Im} v, \operatorname{Im} v\rangle_{L^2(\mathbb{R})}$, while the constraint $G(u) = 0$ gives $\langle v, Q\rangle_{L^2(\mathbb{R})} = 0$. Then, from the second derivative test for constrained minima, we obtain that

$$\begin{aligned} &\langle B_1 w, w\rangle_{L^2(\mathbb{R})} \geq 0 \quad \text{on } \big\{w \in \operatorname{Re} H^1(\mathbb{R}) : \langle w, Q\rangle_{L^2(\mathbb{R})} = 0\big\}, \\ &\langle B_2 w, w\rangle_{L^2(\mathbb{R})} \geq 0 \quad \text{for every } w \in \operatorname{Re} H^1(\mathbb{R}). \end{aligned} \tag{6.7.25}$$

From (3.3), we have $B_2 Q = 0$. Also by differentiating (6.1.2), we obtain $B_1 Q' = 0$. Moreover, by the elementary theory of Schrödinger operators,[16] the kernel of the operator B_j, $j = 1, 2$, has dimension 1. In particular, when $j = 2$, $B_2 Q = 0$ implies (6.7.23) for any $v \in \widetilde{V}_2$. Hereafter, we use the fact that the essential spectrum of B_j is $[1, \infty)$ and that the spectrum of B_j on $(-\infty, 1)$ consists only of isolated eigenvalues of finite multiplicities.[17]

In the following, we focus on B_1. With $Q^\perp = (\operatorname{span}(Q))^\perp$, let $P_{Q^\perp}$ denote the $L^2(\mathbb{R})$-projection onto $Q^\perp$ given by

$$P_{Q^\perp} v = v - \frac{\langle Q, v\rangle_{L^2(\mathbb{R})}}{\|Q\|^2_{L^2(\mathbb{R})}} Q.$$

Then, from (6.7.25), we have

$$\big\langle P_{Q^\perp} B_1 P_{Q^\perp} v, v\big\rangle_{L^2(\mathbb{R})} \geq 0. \tag{6.7.26}$$

Obviously, the quadratic form (6.7.26) vanishes for $v = Q$. On $Q^\perp$, we have $P_{Q^\perp} B_1 v = 0$ if and only if

$$B_1 v \in \ker P_{Q^\perp} = \operatorname{span}(Q). \tag{6.7.27}$$

[16] If there is a linearly independent solution v to $B_j v = 0$, then by considering the Wronskian, we obtain that $v'(x) \not\to 0$ as $|x| \to \infty$. This in particular implies $v \notin L^2(\mathbb{R}) \cup \dot{H}^1(\mathbb{R})$.

[17] This claim follows from Weyl's criterion ([85, Theorem VII.12]).

Recalling that the restriction of B_1 to $(Q')^\perp$ is invertible, we see that the condition (6.7.27) holds at most for a two-dimensional space. Recall from Remark 6.1 that $Q' \,(= -\partial_{x_0} Q)$ is orthogonal to Q in $L^2(\mathbb{R})$. Moreover, by differentiating (6.1.2) in δ, we have

$$B_1 \partial_\delta Q = 2Q \in \ker P_{Q^\perp},$$

while $\partial_\delta Q \perp Q$ in $L^2(\mathbb{R})$. Hence, we have $P_{Q^\perp} B_1 P_{Q^\perp} v = 0$ on the three-dimensional subspace $\operatorname{span}\{Q, \partial_{x_0} Q, \partial_\delta Q\}$. As a result, it follows from (6.7.16) that there exists $\theta > 0$ such that

$$\langle B_1 v, v\rangle_{L^2(\mathbb{R})} \ge \theta \|v\|^2_{L^2(\mathbb{R})}$$

for any $v \in \widetilde{V}_1$. □

Remark 6.17. The spectral analysis of the operators B_1 and B_2 defined in Lemma 6.16 resembles closely the analysis of similar Schrödinger operators that is at the basis of the results in [65, 67, 68, 69, 44, 45, 30, 46, 47, 48]. For example, focusing on the operator $B_1(0,0)$, we have the following picture:

- $B_1(0,0)$ has one negative eigenvalue, whose existence can be inferred from

$$\langle B_1(0,0)Q, Q\rangle_{L^2(\mathbb{R})} = -2\int Q^6 < 0.$$

- the eigenvalue $0 \in \sigma(B_1(0,0))$ has multiplicity 2, corresponding to the directions on the tangent space, $\operatorname{span}\{\partial_\delta Q, \partial_{x_0} Q\}$, to the soliton manifold .
- On the resulting space of codimension 3, corresponding to the orthogonal complement to the eigenfunctions described above, $B_1(0,0)$ is strictly positive.

Compare this result with [66, Lemma 3.2] and [30, Lemma 3.7], which depict a very similar picture in their respective cases. We would like to point out that, in the current work, we make use of the orthogonal coordinate system around the soliton manifold to remove the complications coming from the 0 eigenvalues, while the papers cited above deal with the problem differently. In particular, we require the strict positivity (see (6.7.28)) of the operators A_j defined in (6.7.4) and (6.7.6) in order to guarantee that the denominator appearing in (6.7.50) does not vanish. Moreover, in the same expression, we crucially use the asymptotics of the positive eigenvalues provided by (6.7.29), while this finer analysis does not seem to be present in the aforementioned works.

Next, we establish asymptotic bounds on the eigenvalues of A_j, $j = 1, 2$. We achieve this goal by comparing A_j to a simpler operator whose spectrum is studied in Lemma 6.19 below.

Proposition 6.18. *Let $j = 1, 2$. The spectrum of the operators A_j defined in* (6.7.4) *or* (6.7.6) *consists of a countable collection of eigenvalues. Denoting by $\{-\lambda_n^j\}_{n\in\mathbb{N}}$ the negative eigenvalues of A_j and by $\{\mu_n^j\}_{n\in\mathbb{N}}$ the positive eigenvalues, we have*

$$1 - 2\lambda_n^j \ge 2\varepsilon_0 > 0, \tag{6.7.28}$$

$$\lambda_n^j \lesssim \frac{1}{n^2}, \qquad \text{and} \qquad \mu_n^j \gtrsim \frac{\delta^{-2}}{n^2}. \tag{6.7.29}$$

Proof. The first bound (6.7.28) is just a restatement of Proposition 6.15. In the following, we establish the asymptotic behavior (6.7.29) of λ_n^j and μ_n^j.

For $j = 1, 2$, define an operator T_j by

$$T_j = (1 - \partial_x^2)^{-1}\big(\delta^{-2} - c_j(1 + 5\eta)(Q_\delta^\rho)^4\big), \tag{6.7.30}$$

where $c_1 = \frac{5}{2}$ and $c_2 = \frac{1}{2}$. Since T_j is the composition of a bounded multiplication operator and $(1-\partial_x^2)^{-1}$, which is a compact operator on $H^1(\mathbb{T})$, it is compact and thus has a countable sequence of eigenvalues accumulating only at zero. We label the negative eigenvalues as $-\bar\lambda_1^j \le -\bar\lambda_2^j \le \cdots \le 0$, while the positive eigenvalues are labeled as $\bar\mu_1^j \ge \bar\mu_2^j \ge \cdots \ge 0$. Since A_j is the composition of T_j with a projection, A_j is also compact. Then, by the min-max principle, we have

$$-\bar\lambda_n^j \le -\lambda_n^j \le -\bar\lambda_{n+3}^j \qquad \text{and} \qquad \bar\mu_n^j \le \mu_n^j \le \bar\mu_{n+3}^j. \tag{6.7.31}$$

Hence, it suffices to prove the asymptotic bounds (6.7.29) for the eigenvalues $-\bar\lambda_n^j$ and $\bar\mu_n^j$ of T_j.

We estimate the eigenvalues $\bar\lambda_n^j$ and $\bar\mu_n^j$ for large n by comparing T_j with an operator with piecewise constant coefficients, whose spectrum can be computed explicitly. Fix $a > 0$ such that $10Q^4(a) \le 1$. Given small $\delta > 0$, define the function $S = S_\delta$ on $[-\frac{1}{2}, \frac{1}{2})$ by setting

$$S(x) = \begin{cases} \delta^{-2}(2 - 5\|Q\|_{L^\infty(\mathbb{R})}^4), & \text{for } |x| \le a\delta, \\ \frac{1}{2}\delta^{-2}, & \text{for } a\delta < |x| \le \frac{1}{2}. \end{cases} \tag{6.7.32}$$

Recalling the explicit formula for the ground state $Q(x) = \frac{6^{\frac{1}{4}}}{\cosh^{\frac{1}{2}}(2^{\frac{3}{2}}x)}$, we have $\|Q\|_{L^\infty(\mathbb{R})}^4 = Q^4(0) = 6$. Hence, for any sufficiently small $\eta > 0$, we have $S \le \delta^{-2} - \frac{5}{2}(1+\eta)(Q_\delta^\rho)^4$ pointwise. Therefore, from (6.7.30), we have

$$(1 - \partial_x^2)^{-1} S \le T_j$$

for $j = 1, 2$ in the sense of operators on $H^1(\mathbb{T})$. By the min-max principle, it then suffices to establish the asymptotic behavior (6.7.29) for the eigenvalues of $(1-\partial_x^2)^{-1}S$. This is an explicit computation which we carry out in the following lemma.

Lemma 6.19. *Given $0 < \delta \ll 1$, define the operator $R = R_\delta$ by*

$$R = (1 - \partial_x^2)^{-1} S$$

on $H^1(\mathbb{T})$, where S is as in (6.7.32). Then, R has a countable sequence of eigenvalues $\{-\widetilde\lambda_n\}_{n\in\mathbb{N}} \cup \{\widetilde\mu_n\}_{n\in\mathbb{N}}$ with $\widetilde\lambda_n$, $\widetilde\mu_n \ge 0$. Moreover, we have the following asymptotics for the eigenvalues:

$$\widetilde\lambda_n \sim \frac{1}{n^2} \qquad \textit{and} \qquad \widetilde\mu_n \sim \frac{\delta^{-2}}{n^2}, \tag{6.7.33}$$

where the implicit constants are independent of $0 < \delta \ll 1$.

We first complete the proof of Proposition 6.18, assuming Lemma 6.19. It follows from the asymptotics (6.7.33) and the min-max principle that

$$\bar\lambda_n \lesssim \frac{1}{n^2} \qquad \text{and} \qquad \bar\mu_n \gtrsim \frac{\delta^{-2}}{n^2} \tag{6.7.34}$$

for the eigenvalues $-\bar{\lambda}_n^j$ and $\bar{\mu}_n^j$ of T_j. Then, the asymptotic bounds (6.7.29) for the eigenvalues of $-\lambda_n^j$ and μ_n^j of A_j follows from (6.7.31) and (6.7.34). □

We now present the proof of Lemma 6.19.

Proof of Lemma 6.19. Noting that S defined in (6.7.32) is a bounded operator, we see that $R = (1-\partial_x^2)^{-1}S$ is a compact operator on $H^1(\mathbb{T})$ and thus has a countable sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ accumulating only at 0 with the associated eigenfunctions $\{f_n\}_{n\in\mathbb{N}}$ forming a complete orthonormal system in $H^1(\mathbb{T})$. Moreover, since the operator R has even and odd functions as invariant subspaces, we assume that any such eigenfunction f_n is even or odd. The eigenvalue equation:

$$(1-\partial_x^2)^{-1}Sf_n = \lambda_n f_n \tag{6.7.35}$$

shows that the second derivative of any eigenfunction is piecewise continuous. With $C_0 = 5\|Q\|^4_{L^\infty(\mathbb{R})} - 2 = 5\cdot 6 - 2 = 28$, define

$$\begin{aligned} A_0 &= \lambda^{-1}\delta^{-2}C_0 + 1,\\ B_0 &= \tfrac{1}{2}\lambda^{-1}\delta^{-2} - 1. \end{aligned} \tag{6.7.36}$$

Then, by dropping the subscript n, we can rewrite the eigenvalue equation (6.7.35) as

$$\begin{aligned} f'' &= A_0 f, && \text{for } x\in[-a\delta, a\delta],\\ f'' &= -B_0 f, && \text{for } x\in[-\tfrac{1}{2}, -a\delta)\cup(a\delta,\tfrac{1}{2}]. \end{aligned} \tag{6.7.37}$$

Let $A = \sqrt{|A_0|}$ and $B = \sqrt{|B_0|}$. In the following, we carry out case-by-case analysis, depending on the signs of A_0 and B_0. It follows from (6.7.36) that $\lambda > 0$ implies $A_0 > B_0$, while $\lambda < 0$ implies $B_0 < 0$. This in particular implies that the case $A_0 < 0 < B_0$ can not happen for any parameter choice.

- **Case 1**: $A_0, B_0 < 0$ and f is even. Without loss of generality, we may assume that

$$f(0) = 1, \quad f'(0) = 0, \quad \text{and} \quad f'(\tfrac{1}{2}) = 0.$$

From solving (6.7.37) on $[-a\delta, a\delta]$, we have

$$f(x) = \cos(Ax).$$

On the other hand, the general solution to (6.7.37) on $(a\delta, \frac{1}{2}]$ is given by

$$f(x) = \alpha\cosh(B(x-a\delta)) + \beta\sinh(B(x-a\delta)), \qquad x\in(a\delta,\tfrac{1}{2}].$$

Thus, with the notations:

$$g(x+) := \lim_{\varepsilon\downarrow 0} g(x+\varepsilon) \qquad \text{and} \qquad g(x-) := \lim_{\varepsilon\downarrow 0} g(x-\varepsilon),$$

we have the following "transmission conditions" at $x = a\delta$:

$$\begin{aligned} f(a\delta-) &= \cos(Aa\delta) = f(a\delta+) = \alpha,\\ f'(a\delta-) &= -A\sin(Aa\delta) = f'(a\delta+) = \beta B. \end{aligned}$$

Hence, on $(a\delta, \frac{1}{2}]$, we have

$$f(x) = \cos(Aa\delta)\cosh(B(x-a\delta)) - \frac{A}{B}\sin(Aa\delta)\sinh(B(x-a\delta)).$$

To enforce the periodicity condition $f'(\frac{1}{2}) = 0$, we need

$$\begin{aligned} 0 &= f'(\tfrac{1}{2}) \\ &= B\cos(Aa\delta)\sinh(B(2^{-1}-a\delta)) - A\sin(Aa\delta)\cosh(B(2^{-1}-a\delta)). \end{aligned}$$

By rearranging, we obtain the condition

$$\frac{A}{B}\tan(Aa\delta)\coth(B(2^{-1}-a\delta)) = 1. \tag{6.7.38}$$

Our goal is to show that for every $k \in \mathbb{Z}_{\geq 0}$, there exists exactly one A_k^1 such that (6.7.38) is satisfied and

$$\max\Big(k\pi - \frac{\pi}{2}, 0\Big) < A_k^1 a\delta < k\pi + \frac{\pi}{2}. \tag{6.7.39}$$

From (6.7.36), we have

$$B = \sqrt{1 + \frac{A^2+1}{2C_0}}. \tag{6.7.40}$$

Thus, we can rewrite (6.7.38) as $F_1(A) = 1$, where $F_1(A)$ is given by

$$F_1(A) = \frac{A}{\sqrt{1+\frac{A^2+1}{2C_0}}}\tan(Aa\delta)\coth\Bigg(\sqrt{1+\frac{A^2+1}{2C_0}}(2^{-1}-a\delta)\Bigg).$$

The existence of A_k^1 satisfying $F_1(A_k^1) = 1$ follows from continuity of F_1, together with the limits

$$F_1(0) = 0, \quad F_1\Big(\frac{(2k-1)\pi}{2a\delta}+\Big) = -\infty, \quad \text{and} \quad F_1\Big(\frac{(2k+1)\pi}{2a\delta}-\Big) = \infty.$$

In order to show uniqueness, it suffices to show that for every A with $F_1(A) = 1$, we have $F_1'(A) > 0$. We first note that, for $0 \leq A \ll \delta^{-1}$, we have

$$F_1(A) \lesssim A^2\delta.$$

Thus, in order to have $F_1(A) = 1$, we must have $A \gtrsim \delta^{-\frac{1}{2}}$. Moreover, for $A \gtrsim \delta^{-\frac{1}{2}}$, we have

$$F_1(A) = \sqrt{2C_0}\tan(Aa\delta)(1+O(A^{-1})),$$

from which we obtain $A \gtrsim \delta^{-1}$. Namely, $F_1(A) = 1$ implies $A \gtrsim \delta^{-1}$.

From (6.7.40), we have

$$\frac{A}{B} = \sqrt{2C_0}\big(1+O(A^{-2})\big) \qquad \text{and} \qquad \frac{B}{A} = \frac{1}{\sqrt{2C_0}}\big(1+O(A^{-2})\big) \tag{6.7.41}$$

for $A \gg 1$. Therefore, whenever $F_1(x) = 1$, we have

$$
\begin{aligned}
F_1'(A) &= \frac{A}{\sqrt{1+\frac{A^2+1}{2C_0}}} \coth\left(\sqrt{1+\frac{A^2+1}{2C_0}}(2^{-1}-a\delta)\right) a\delta\big(1+(\tan(Aa\delta))^2\big) \\
&\quad + \frac{1}{\sqrt{1+\frac{A^2+1}{2C_0}}} \coth\left(\sqrt{1+\frac{A^2+1}{2C_0}}(2^{-1}-a\delta)\right)\tan(Aa\delta) \\
&\quad - \frac{A^2}{2C_0\big(1+\frac{A^2+1}{2C_0}\big)^{\frac{3}{2}}} \coth\left(\sqrt{1+\frac{A^2+1}{2C_0}}(2^{-1}-a\delta)\right)\tan(Aa\delta) \\
&\quad + \frac{A}{\sqrt{1+\frac{A^2+1}{2C_0}}} \frac{d}{dA}\left(\coth\left(\sqrt{1+\frac{A^2+1}{2C_0}}(2^{-1}-a\delta)\right)\right)\tan(Aa\delta) \\
&= a\delta\left(\sqrt{2C_0}+\frac{1}{\sqrt{2C_0}}\right) + O(A^{-2}) > 0
\end{aligned}
\tag{6.7.42}
$$

for $A \gtrsim \delta^{-1}$. Note that we used the relation (6.7.38) in handling the first, second, and third terms after the first equality in (6.7.42).

• **Case 2:** $A_0, B_0 < 0$ and f is odd. Without loss of generality, we may assume that

$$
f(0) = 0, \quad f'(0) = A, \quad \text{and} \quad f(\tfrac{1}{2}) = 0.
$$

By solving the eigenfunction equation (6.7.37) and solving for the transmission conditions as in Case 1, we have

$$
f(x) = \sin(A\alpha\delta)\cosh(B(x-a\delta)) + \frac{A}{B}\cos(Aa\delta)\sinh(B(x-a\delta))
$$

for $x \in (a\delta, \frac{1}{2}]$. Therefore, the periodicity condition $f(\frac{1}{2}) = 0$ becomes

$$
F_2(A) := \frac{A}{B}\cot(A\alpha\delta)\tanh(B(2^{-1}-a\delta)) = -1, \tag{6.7.43}
$$

where $B = B(A)$ is as in (6.7.40). As in Case 1, we want to show that for every $k \in \mathbb{Z}_{\geq 0}$, there exists exactly one value A_k^2 that satisfies (6.7.43) with

$$
k\pi < A_k^2 a\delta < (k+1)\pi. \tag{6.7.44}
$$

From (6.7.43), we have $\cot(Aa\delta) < 0$, which implies $Aa\delta > \frac{\pi}{2}$. Then, using (6.7.41) and (6.7.44) with $A \gtrsim \delta^{-1}$, we can proceed as in Case 1 and show existence and uniqueness of such A_k^2, $k \in \mathbb{Z}_{\geq 0}$. Note that in this case, we show $F_2'(A) < 0$ whenever $F_2(A) = -1$.

• **Case 3:** $B_0 < 0 < A_0$ and f is even. By solving the eigenfunction equation (6.7.37), we obtain

$$
f(x) = \cosh(Aa\delta)\cosh(B(x-a\delta)) + \frac{A}{B}\sinh(Aa\delta)\sinh(B(x-a\delta))
$$

for $x \in (a\delta, \frac{1}{2}]$. By imposing the periodicity condition $f'(\frac{1}{2}) = 0$, we obtain

$$
\frac{A}{B}\tanh(Aa\delta)\coth(B(2^{-1}-a\delta)) = -1.
$$

However, since $A, B, a > 0$, we see that this condition can never be satisfied for $0 < \delta \ll 1$.

• **Case 4:** $B_0 < 0 < A_0$ and f is odd. In this case, we have

$$f(x) = \sinh(Aa\delta)\cosh(B(x-a\delta)) + \frac{A}{B}\cosh(Aa\delta)\sinh(B(x-a\delta))$$

for $x \in (a\delta, \frac{1}{2}]$. By imposing the periodicity condition $f(2^{-1}) = 0$, we obtain

$$\frac{A}{B}\coth(Aa\delta)\tanh(B(2^{-1}-a\delta)) = -1.$$

Since $A, B, a > 0$, we once again see that this condition can not be satisfied for $0 < \delta \ll 1$.

• **Case 5:** $A_0, B_0 > 0$ and f is even. By solving the eigenfunction equation (6.7.37), we have

$$f(x) = \cosh(Aa\delta)\cos(B(x-a\delta)) + \frac{A}{B}\sinh(Aa\delta)\sin(B(x-a\delta))$$

for $x \in (a\delta, \frac{1}{2}]$. Then, by imposing the periodicity condition $f'(\frac{1}{2}) = 0$, we obtain

$$B\tan(B(2^{-1}-a\delta)) - A\tanh(Aa\delta) = 0. \tag{6.7.45}$$

By writing

$$A = \sqrt{2(B^2+1)C_0+1}, \tag{6.7.46}$$

we can rewrite (6.7.45) as $G_1(B) = 0$, where

$$\begin{aligned} G_1(B) &= B\tan(B(2^{-1}-a\delta)) \\ &\quad - \sqrt{2(B^2+1)C_0+1}\tanh\big(\sqrt{2(B^2+1)C_0+1}\,a\delta\big). \end{aligned}$$

As in Case 1, we need to show that for every $k \in \mathbb{Z}_{\geq 0}$, there exists exactly one value B_k^1 such that $G(B_k^1) = 0$ with

$$\max\Big(k\pi - \frac{\pi}{2}, 0\Big) < B_k^1(2^{-1}-a\delta) < k\pi + \frac{\pi}{2}. \tag{6.7.47}$$

Existence follows from continuity of G_1, together with the limits

$$G_1(0) < 0, \quad G_1\Big(\frac{2k-1}{1-2a\delta}+\Big) = -\infty, \quad \text{and} \quad G_1\Big(\frac{2k+1}{1-2a\delta}-\Big) = \infty.$$

As for uniqueness of B_k^1, it suffices to show $G_1'(B) > 0$ whenever $G_1(B) = 0$. Using (6.7.45), we have

$$\begin{aligned} G_1'(B) &= \tan(B(2^{-1}-a\delta)) + B(2^{-1}-a\delta)\big(1 + (\tan(B(2^{-1}-a\delta)))^2\big) \\ &\quad - \frac{2C_0B}{\sqrt{2(B^2+1)C_0+1}}\tanh\big(\sqrt{2(B^2+1)C_0+1}\,a\delta\big) \\ &\quad - 2a\delta C_0 B + 2a\delta C_0 B\big(\tanh(\sqrt{2(B^2+1)C_0+1}\,a\delta)\big)^2 \\ &= \tan(B(2^{-1}-a\delta)) + B(2^{-1}-a\delta)\big(1 + (\tan(B(2^{-1}-a\delta)))^2\big) \\ &\quad - \frac{2C_0B^2}{2(B^2+1)C_0+1}\tan(B(2^{-1}-a\delta)) + O(B\delta) \\ &\geq B(2^{-1}-a\delta) + O(B\delta) > 0 \end{aligned}$$

for $0 < \delta \ll 1$.

• **Case 6:** $A_0, B_0 > 0$ and f is odd. By solving the eigenfunction equation (6.7.37), we have that

$$f(x) = \sinh(Aa\delta)\cos(B(x-a\delta)) + \frac{A}{B}\cosh(Aa\delta)\sin(B(x-a\delta))$$

for $x \in (a\delta, \frac{1}{2}]$. Then, by imposing the periodicity condition $f(\frac{1}{2}) = 0$, we obtain

$$B \cot(B(2^{-1} - a\delta)) \tanh(Aa\delta) + A = 0. \tag{6.7.48}$$

In view of (6.7.46), define $G_2(B)$ by

$$\begin{aligned} G_2(B) &= B \cot(B(2^{-1} - a\delta)) \tanh\big(\sqrt{2(B^2+1)C_0+1}\, a\delta\big) \\ &\quad + \sqrt{2(B^2+1)C_0+1}. \end{aligned}$$

As in the previous cases, we show that for every $k \in \mathbb{Z}_{\geq 0}$, there exists exactly one value B_k^2 such that $G_2(B_k^2) = 0$ and

$$k\pi < B_k^2(2^{-1} - a\delta) < (k+1)\pi. \tag{6.7.49}$$

Noting $G_2(0) > 0$, existence follows from the same argument as in the previous cases. As for uniqueness, we show that $G_2'(B) < 0$ whenever $G_2(B) = 0$. Using (6.7.48) (which implies $\cot(B(2^{-1} - a\delta)) < 0$) and noting that $\tanh\big(\sqrt{2(B^2+1)C_0+1}\, a\delta\big)$ is strictly increasing in B, we have

$$\begin{aligned} G_2'(B) &\leq \cot(B(2^{-1} - a\delta)) \tanh\big(\sqrt{2(B^2+1)C_0+1}\, a\delta\big) \\ &\quad - (2^{-1} - a\delta) B \tanh\big(\sqrt{2(B^2+1)C_0+1}\, a\delta\big)\big(1 + (\cot(B(2^{-1} - a\delta)))^2\big) \\ &\quad + \frac{2C_0 B}{\sqrt{2(B^2+1)C_0+1}} \\ &= -\frac{\sqrt{2(B^2+1)C_0+1}}{B} + \frac{2C_0 B}{\sqrt{2(B^2+1)C_0+1}} \\ &\quad - (2^{-1} - a\delta) B \tanh\big(\sqrt{2(B^2+1)C_0+1}\, a\delta\big)\big(1 + (\cot(B(2^{-1} - a\delta)))^2\big) \\ &\leq -(2^{-1} - a\delta) B \tanh\big(\sqrt{2(B^2+1)C_0+1}\, a\delta\big)\big(1 + (\cot(B(2^{-1} - a\delta)))^2\big) \\ &< 0 \end{aligned}$$

for $0 < \delta \ll 1$.

Conclusion: Fix $0 < \delta \ll 1$. Putting all the cases together, we conclude that every λ such that in (6.7.36), we have $A_0 = -(A_k^1)^2$, $A_0 = -(A_k^2)^2$, $B_0 = (B_k^1)^2$, or $B_0 = (B_k^2)^2$ for some $k \in \mathbb{Z}_{\geq 0}$ corresponds to an eigenvalue for R. Moreover, this exhausts all the possibilities for the eigenvalues of R, with possible exceptions of λ such that $A_0 = 0$ or $B_0 = 0$. By inverting the formulas of $A_0 = -(A_k^j)^2$ and $B_0 = (B_k^j)^2$ in (6.7.36) for λ together with (6.7.39), (6.7.44), (6.7.47), (6.7.49), and $A_k^j \gtrsim \delta^{-1}$, $j = 1, 2$, we obtain the asymptotics:

$$\widetilde{\lambda}_n \sim \frac{1}{n^2} \qquad \text{and} \qquad \widetilde{\mu}_n \sim \frac{\delta^{-2}}{n^2}.$$

This completes the proof of Lemma 6.19. □

Finally, we conclude this subsection by presenting the proof of Proposition 6.14.

Proof of Proposition 6.14. Given $N \in \mathbb{N}$, let W_N be the subspace spanned by the eigenvectors of A corresponding to the eigenvalues λ_n^j and μ_n^j, $1 \leq n \leq N$, $j = 1, 2$. Define

$A_N = AP_N$, where $P_N = P^{H^1}_{W_N}$ is the projection onto W_N in $H^1(\mathbb{T})$. Then, by Proposition 6.18 with $\mathrm{d}\alpha\mathrm{d}\beta = \prod_{j=1}^2 \prod_{n=1}^N \mathrm{d}\alpha_n^j \mathrm{d}\beta_n^j$ and (4.9), we have

$$\begin{aligned}
&\int \exp(-(1-\eta^2)\langle A_N w, w\rangle_{H^1(\mathbb{T})}) \mathrm{d}\mu_\delta^{\perp\perp}(w) \\
&= \frac{1}{(2\pi)^{2N}} \int_{\mathbb{R}^{4N}} \exp\bigg(- \frac{1}{2}\sum_{j=1}^2 \sum_{n=1}^N (1 - 2(1-\eta^2)\lambda_n^j)(\alpha_n^j)^2 \\
&\qquad\qquad - \frac{1}{2}\sum_{j=1}^2 \sum_{n=1}^N (1 + 2(1-\eta^2)\mu_n^j)(\beta_n^j)^2 \bigg) \mathrm{d}\alpha\mathrm{d}\beta \\
&= \bigg(\prod_{j=1}^2 \prod_{n=1}^N \frac{1}{\sqrt{1 - 2(1-\eta^2)\lambda_n^j}} \bigg) \bigg(\prod_{j=1}^2 \prod_{n=1}^N \frac{1}{\sqrt{1 + 2(1-\eta^2)\mu_n^j}} \bigg) \\
&\le C_\eta \exp\bigg(- \frac{1}{2} \sum_{j=1}^2 \sum_{\delta^{-1} \ll n \le N} \log\big(1 + 2(1-\eta^2)\mu_n^j\big) \bigg) \\
&\le C_\eta \exp\bigg(- c \sum_{n \gg \delta^{-1}} \frac{\delta^{-2}}{n^2} \bigg) \lesssim \exp(-c'\delta^{-1}),
\end{aligned} \tag{6.7.50}$$

where we used $\log(1+x) \ge x - \frac{x^2}{2} \gtrsim x$ for $|x| \ll 1$ in the penultimate step.

For $v, w \in L^2(\mathbb{T})$, we have

$$\sup_{N\in\mathbb{N}} |\langle A_N v, w\rangle_{H^1(\mathbb{T})}| + |\langle Av, w\rangle_{H^1(\mathbb{T})}| \le C\delta^{-2} \|v\|_{L^2(\mathbb{T})} \|w\|_{L^2(\mathbb{T})}.$$

Then, by a density argument, we see that $\langle A_N w, w\rangle$ converges to $\langle Aw, w\rangle$ as $N \to \infty$ for each $w \in L^2(\mathbb{T})$. Hence, the estimate (6.7.3) follows from (6.7.50) and Fatou's lemma. □

6.8. **Proof of Theorem 1.4.** We now put all the steps together and present the proof of of Theorem 1.4. Let $K = \|Q\|_{L^2(\mathbb{R})}$. By Lemma 6.3, we have

$$\begin{aligned}
Z_{6,K=\|Q\|_{L^2(\mathbb{R})}} &= \mathbf{E}\Big[e^{\frac{1}{p}\int_\mathbb{T} |u|^6 \,\mathrm{d}x} \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le K\}} \Big] \\
&\le \mathbf{E}\Big[e^{\frac{1}{6}\int_\mathbb{T} |u(x)|^6 \,\mathrm{d}x}, S_\gamma \Big] + \mathbf{E}\Big[e^{\frac{1}{6}\int_\mathbb{T} |u(x)|^6 \,\mathrm{d}x} \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\le K\}}, U_\varepsilon(\delta^*) \Big] \\
&=: \mathrm{I}(\gamma) + \mathrm{II}(\varepsilon, \delta^*),
\end{aligned} \tag{6.8.1}$$

where S_γ and $U_\varepsilon(\delta^*)$ are as in (6.2.1) and (6.3.2). From (6.2.3), we have

$$\mathrm{I}(\gamma) < \infty \tag{6.8.2}$$

for any $\gamma > 0$. As for $\text{II}(\varepsilon, \delta^*)$, it follows from (6.6.8), Hölder's inequality, and Lemma 6.12 that

$$\begin{aligned} \text{II}(\varepsilon, \delta^*) &\lesssim \int_0^{\delta^*} \bigg(\int_{\{\|v\|_{L^2(\mathbb{T})} \le \varepsilon_1\}} \exp\Big(C'_\eta \int_{\mathbb{T}} |v|^6 \, \mathrm{d}x \Big) \mu_\delta^\perp(\mathrm{d}v) \bigg)^{\frac{\eta}{1+\eta}} \\ &\quad \times \bigg(\int_{\{\|v\|_{L^2(\mathbb{T})} \le \varepsilon_1\}} e^{(1+\eta)G(v)} \mathbf{1}_{\{\|Q_\delta^\rho + v\|_{L^2(\mathbb{T})} \le K\}} \mu_\delta^\perp(\mathrm{d}v) \bigg)^{\frac{1}{1+\eta}} \sigma(\mathrm{d}\delta) \\ &\lesssim \int_0^{\delta^*} \bigg(\int_{\{\|v\|_{L^2(\mathbb{T})} \le \varepsilon_1\}} e^{(1+\eta)G(v)} \mathbf{1}_{\{\|Q_\delta^\rho + v\|_{L^2(\mathbb{T})} \le K\}} \mu_\delta^\perp(\mathrm{d}v) \bigg)^{\frac{1}{1+\eta}} \sigma(\mathrm{d}\delta), \end{aligned}$$

provided that $\varepsilon_1 = \varepsilon_1(\eta) > 0$ is sufficiently small. Note that in obtaining (6.6.8), we applied Proposition 6.4 which gives an orthogonal coordinate system in a neighborhood of the soliton manifold. Then, from Lemma 6.13, (6.7.2), and Proposition 6.14, we obtain

$$\begin{aligned} \text{II}(\varepsilon, \delta^*) &\lesssim \int_0^{\delta^*} \bigg(\int \exp\big(-(1-\eta^2)\langle Aw, w\rangle_{H^1(\mathbb{T})}\big) \mathrm{d}\mu_\delta^{\perp\perp}(w) \bigg)^{\frac{1}{1+\eta}} \sigma(\mathrm{d}\delta) \\ &\lesssim \int_0^{\delta^*} \exp(-c\delta^{-1}) \sigma(\mathrm{d}\delta) < \infty, \end{aligned} \tag{6.8.3}$$

provided that $\eta > 0$ is sufficiently small, where we used Lemma 6.10 in the last step. Therefore, from (6.8.1), (6.8.2), and (6.8.3), we conclude that

$$Z_{6, K = \|Q\|_{L^2(\mathbb{R})}} < \infty.$$

For readers' convenience, we go over how we choose the parameters. We first choose $\eta > 0$ in (6.6.7) sufficiently small such that Proposition 6.14 holds. Next, we fix small $\varepsilon_1 = \varepsilon_1(\eta) > 0$ such that Lemma 6.12 holds. Then, Proposition 6.4 determines small $\varepsilon = \varepsilon(\varepsilon_1) > 0$ and $\delta^* = \delta^*(\varepsilon_1) > 0$. See also Lemma 6.13, where we need smallness of $\delta = \delta(\eta, \varepsilon_1) > 0$. Finally, we fix $\gamma = \gamma(\varepsilon) > 0$ by applying Lemma 6.3. This completes the proof of Theorem 1.4.

Remark 6.20. In this section, we presented the proof of Theorem 1.4, where the base Gaussian process is given by the Ornstein-Uhlenbeck loop in (6.0.1). When the base Gaussian process is given by the mean-zero Brownian loop in (1.3), the same but simpler argument gives Theorem 1.4. For example, in the case of the mean-zero Brownian loop, we can omit (6.2.2) and the reduction to the mean-zero case at the beginning of the proof of Lemma 6.3. In the proof of Proposition 6.4, we introduced $V^{\gamma_0}_{\delta, x_0, \theta}$ in (6.3.4) in order to use a scaling argument in the non-homogeneous setting. In the case of the mean-zero Brownian loop, we can simply use $V^0_{\delta, x_0, \theta}$, i.e. $\gamma_0 = 0$. The rest of the proof remains essentially the same.

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Tadahiro Oh, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom
Email address: hiro.oh@ed.ac.uk

Philippe Sosoe, Department of Mathematics, Cornell University, 310 Malott Hall, Cornell University, Ithaca, New York 14853, USA
Email address: psosoe@math.cornell.edu

Leonardo Tolomeo, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom and Mathematical Institute, Hausdorff Center for Mathematics, Universität Bonn, Bonn, Germany
Email address: tolomeo@math.uni-bonn.de