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CYLINDERS IN DEL PEZZO SURFACES

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Abstract. On del Pezzo surfaces, we study effective ample $\mathbb{R}$-divisors such that the complements of their supports are isomorphic to $\mathbb{A}^1$-bundles over smooth affine curves.

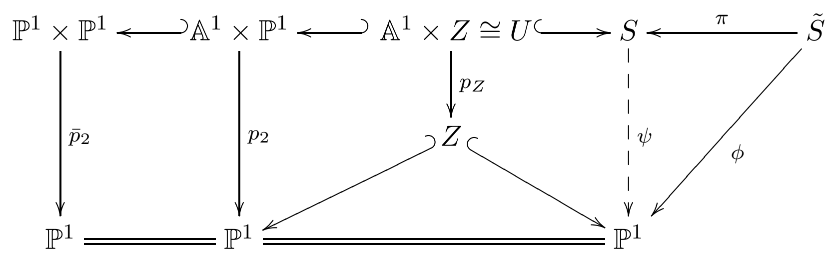
All considered varieties are assumed to be algebraic and defined over an algebraically closed field of characteristic 0 throughout this article.

1. Introduction

1.1. **Cylinders.** The purpose of this article is to study cylinders in rational surfaces and, more specially, in del Pezzo surfaces. A cylinder in a projective variety X is a Zariski open subset that is isomorphic to $\mathbb{A}^1 \times Z$ for some affine variety Z. So, if X is a rational surface, then Z is just the projective line with finitely many missing points.

One can easily see that every smooth rational surface contains cylinders ([19, Proposition 3.13]). However, this is no longer true for singular rational surfaces, i.e., there are plenty of singular rational surfaces without any cylinder. Let us explain how to find such rational surfaces. First, let S be a rational surface with quotient singularities and suppose that S has a cylinder U, i.e., a Zariski open subset in S such that $U \cong \mathbb{A}^1 \times Z$ for some affine curve Z. Consider the following commutative diagram

(1.1.1)



where p_Z, p_2, and $\bar{p}_2$ are the natural projections to the second factors, ψ is the rational map induced by p_Z, π is a birational morphism resolving the indeterminacy of ψ, and ϕ is a morphism. By construction, a general fiber of ϕ is $\mathbb{P}^1$. Let $C_1, \ldots, C_n$ be the irreducible curves in S such that

$$S \setminus U = \bigcup_{i=1}^{n} C_i.$$

Then the curves $C_1, \ldots, C_n$ generate the divisor class group $\mathrm{Cl}(S)$ of the surface S because $\mathrm{Cl}(U) = 0$. In particular, one has

$$n \geqslant \operatorname{rank} \mathrm{Cl}(S). \qquad (1.1.2)$$

Let $E_1, \ldots, E_r$ be the π-exceptional curves, if any, and let Γ be the section of $\bar{p}_2$ that is the complement of $\mathbb{A}^1 \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\tilde{C}_1, \ldots, \tilde{C}_n$, and $\tilde{\Gamma}$ the proper transforms of the curves $C_1, \ldots, C_n$, and Γ on the surface $\tilde{S}$, respectively. Then $\tilde{\Gamma}$ is a section of ϕ. Moreover, the curve $\tilde{\Gamma}$ is one of the curves $\tilde{C}_1, \ldots, \tilde{C}_n$ and $E_1, \ldots, E_r$. Furthermore, all the other curves among $\tilde{C}_1, \ldots, \tilde{C}_n$ and $E_1, \ldots, E_r$ are irreducible components of some fibers of ϕ. We may assume either $\tilde{\Gamma} = \tilde{C}_1$ or $\tilde{\Gamma} = E_r$.

If $\tilde{\Gamma} = \tilde{C}_1$, then ψ is a morphism. Conversely, if ψ is a morphism, then $\tilde{\Gamma} = \tilde{C}_1$.

Let $\lambda_1, \ldots, \lambda_n$ be arbitrary real numbers. Then

$$K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i + \sum_{i=1}^{r} \mu_i E_i = \pi^*\Big(K_S + \sum_{i=1}^{n} \lambda_i C_i\Big)$$

for some real numbers $\mu_1, \ldots, \mu_r$. Let $\tilde{F}$ be a general fiber of ϕ. Then $K_{\tilde{S}} \cdot \tilde{F} = -2$ by the adjunction formula. Put $F = \pi(\tilde{F})$.

If $\tilde{\Gamma} = E_r$, then

$$-2+\mu_r = \left(K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i + \sum_{i=1}^{r} \mu_i E_i\right) \cdot \tilde{F} = \pi^* \left(K_S + \sum_{i=1}^{n} \lambda_i C_i\right) \cdot \tilde{F} = \left(K_S + \sum_{i=1}^{n} \lambda_i C_i\right) \cdot F$$

If $\tilde{\Gamma} = C_1$, then

$$-2+\lambda_1 = \left(K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i + \sum_{i=1}^{r} \mu_i E_i\right) \cdot \tilde{F} = \pi^* \left(K_S + \sum_{i=1}^{n} \lambda_i C_i\right) \cdot \tilde{F} = \left(K_S + \sum_{i=1}^{n} \lambda_i C_i\right) \cdot F.$$

On the other hand, if $K_S + \sum_{i=1}^{n} \lambda_i C_i$ is pseudo-effective, then

$$\Big(K_S + \sum_{i=1}^{n} \lambda_i C_i\Big) \cdot F \geqslant 0$$

because $\tilde{F}$ is a general fiber of ϕ.

Remark 1.1.3. We are therefore able to draw the following conclusions:

- if $K_S + \sum_{i=1}^{n} \lambda_i C_i$ is pseudo-effective, the log pair $(S, \sum_{i=1}^{n} \lambda_i C_i)$ is not log canonical;
- if $K_S + \sum_{i=1}^{n} \lambda_i C_i$ is pseudo-effective for some real numbers $\lambda_i < 2$, the rational map ψ cannot be a morphism.

This observation is originated from [19, Lemma 4.11], [19, Lemma 4.14], and [21, Lemma 5.3].

From Remark 1.1.3 it immediately follows that if a rational surface with pseudo-effective canonical class has only quotient singularities then it cannot contain any cylinder. We will present various examples of such surfaces in Section 3.

1.2. **Polar cylinders.** In general, it seems hopeless to determine which singular rational surfaces have cylinders and which do not have cylinders. This is simply because we do not have any reasonable classification of singular del Pezzo surfaces. Instead of this, we are to consider a similar problem for polarized surfaces, which has a significant application to theory of unipotent group actions in affine geometry (for instance, see [19], [20], [21]). To do this, let S be a rational surface with at most quotient singularities.

Definition 1.2.1 ([19])**.** Let M be an $\mathbb{R}$-divisor on S. An M-polar cylinder in S is a Zariski open subset U of S such that

- $U = \mathbb{A}^1 \times Z$ for some affine curve Z, i.e., U is a cylinder in S,
- there is an effective $\mathbb{R}$-divisor D on S with $D \equiv M$ and $U = S \setminus \mathrm{Supp}(D)$.

With the notation at the beginning, the second condition can be rephrased as follows:

$$M \equiv \sum_{i=1}^{n} \lambda_i C_i \tag{1.2.2}$$

for some positive real numbers $\lambda_1, \ldots, \lambda_n$. We here remark that on a log del Pezzo surface, numerical equivalence for $\mathbb{Q}$-divisors coincides with $\mathbb{Q}$-linear equivalence, i.e., if S is a log del

Pezzo surface and M is a $\mathbb{Q}$-divisor, then (1.2.2) can be rewritten as

$$M \sim_{\mathbb{Q}} \sum_{i=1}^{n} \lambda_i C_i$$

for some positive rational numbers $\lambda_1, \ldots, \lambda_n$.

Let $\mathrm{Amp}(S)$ be the ample cone of S in $\mathrm{Pic}(S) \otimes \mathbb{R}$. Denote by $\mathrm{Amp}^{cyl}(S)$ the set

$$\{H \in \mathrm{Amp}(S) : \text{ there is an } H\text{-polar cylinder on } S\}.$$

This set will be called the cone of cylindrical ample divisors of S. We have seen that the set $\mathrm{Amp}^{cyl}(S)$ can be empty. On the other hand, one can show that $\mathrm{Amp}^{cyl}(S) \neq \varnothing$ provided that S is smooth (see [19, Proposition 3.13]).

For smooth del Pezzo surfaces, [5] and [21] have achieved the following:

Theorem 1.2.3. *Let S_d be a smooth del Pezzo surface of degree d. Then the set $\mathrm{Amp}^{cyl}(S_d)$ contains the anticanonical class if and only if $d \geqslant 4$.*

Theorem 1.2.3 has been generalised in [6] as follows:

Theorem 1.2.4. *Let S_d be a del Pezzo surface of degree d with at most du Val singularities. The set $\mathrm{Amp}^{cyl}(S_d)$ contains the anticanonical class if and only if one of the following conditions holds:*

- *$d \geqslant 4$,*
- *$d = 3$ and S_d is singular,*
- *$d = 2$ and S_d has a singular point that is not of type A_1,*
- *$d = 1$ and S_d has a singular point that is not of type A_1, A_2, A_3, or D_4.*

In [5] and [6] we have witnessed several vague pieces of evidence for the supposition that a cylinder polarized by an ample divisor can be obtained by manipulating an anticanonically polarized cylinder, if any, on a log del Pezzo surface.

Conjecture 1.2.5. *A log del Pezzo surface S has a $(-K_S)$-polar cylinder if and only if $\mathrm{Amp}^{cyl}(S) = \mathrm{Amp}(S)$.*

2. Main Results

2.1. **Fujita invariant.** To investigate the cones of cylindrical ample divisors on log del Pezzo surfaces, we adopt the concept, so-called, the Fujita invariant of a log pair defined in [15, Definition 2.2]. This was originally introduced by T. Fujita, disguised as its negative value and under the name Kodaira energy ([10], [11], [12], [13]). This plays essential roles in Manin's conjecture (see, for example, [1], [15]).

Let S be a log del Pezzo surface and A be a big $\mathbb{R}$-divisor on S. It follows from Cone Theorem (see, for instance, [23, Theorem 3.7]) that the Mori cone $\overline{\mathbb{NE}}(S)$ of the surface S is polyhedral.

Definition 2.1.1. For the log pair (S, A), we define the Fujita invariant of (S, A) by

$$\mu_A := \inf\left\{\lambda \in \mathbb{R}_{>0} \;\middle|\; \text{the } \mathbb{R}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective}\right\}.$$

The smallest extremal face Δ_A of the Mori cone $\overline{\mathbb{NE}}(S)$ that contains $K_S + \mu_A A$ is called the Fujita face of A. The Fujita rank of (S, A) is defined by $r_A := \dim \Delta_A$. Note that $r_A = 0$ if and only if $-K_S \equiv \mu_A A$.

Remark 2.1.2. In [15, Definition 2.2], B. Hassett, Sh. Tanimoto, and Yu. Tschinkel define the Fujita invariants only for $\mathbb{Q}$-factorial varieties with canonical singularities. For general varieties, they define the Fujita invariants by taking the pull-backs to smooth models ([28]).

Let $\phi_A\colon S \to Z$ be the contraction given by the Fujita face Δ_A of the divisor A. Then either ϕ_A is a birational morphism or a conic bundle with $Z \cong \mathbb{P}^1$ (see, for instance, [7, Subsection 8.2.6]). In the former case, the $\mathbb{R}$-divisor A is said to be of type $B(r_A)$ and in the latter case it is said to be of type $C(r_A)$.

Now we suppose that S is a smooth del Pezzo surface of degree $d \leqslant 7$. Then the Mori cone $\overline{\mathbb{NE}}(S)$ of the surface S is generated by all the (-1)-curves in S (see [7, Theorem 8.2.23]). Let H be an ample $\mathbb{R}$-divisor on S.

If H is of type $B(r_H)$, then its Fujita face Δ_H is generated by r_H disjoint (-1)-curves contracted by ϕ_H, where $r_H \leqslant 9-d$. If H is of type $C(r_H)$, then Δ_H is generated by the (-1)-curves in the $(8-d)$ reducible fibers of ϕ_H. Each reducible fiber consists of two (-1)-curves that intersect transversally at a single point.

Suppose that H is of type $B(r_H)$. Let $E_1, \ldots, E_{r_H}$ be all the (-1)-curves contained in Δ_H. These are disjoint and generate the Fujita face Δ_H. Therefore,

$$K_S + \mu_H H \equiv \sum_{i=1}^{r_H} a_i E_i \tag{2.1.3}$$

for some positive real numbers $a_1, \ldots, a_{r_H}$. We have $a_i < 1$ for every i because $H \cdot E_i > 0$. Vice versa, for every positive real numbers $b_1, \ldots, b_{r_H} < 1$, the divisor

$$-K_S + \sum_{i=1}^{r_H} b_i E_i$$

is ample. The set of all ample $\mathbb{R}$-divisor classes of type $B(r_H)$ in $\mathrm{Pic}(S) \otimes \mathbb{R}$ is denoted by $\mathrm{Amp}^B_{r_H}(S)$.

Suppose that H is of type $C(r_H)$. Note that $r_H = 9-d$. There are a 0-curve B and $(8-d)$ disjoint (-1)-curves $E_1, E_2, E_3, \ldots, E_{8-d}$, each of which is contained in a distinct fiber of ϕ_H, such that

$$K_S + \mu_H H \equiv aB + \sum_{i=1}^{8-d} a_i E_i \tag{2.1.4}$$

for some positive real number a and non-negative real numbers $a_1, \ldots, a_{8-d} < 1$. In particular, these curves generate the Fujita face Δ_H. Vice versa, for every positive real number b and non-negative real numbers $b_1, \ldots, b_{8-d} < 1$ the divisor

$$-K_S + bB + \sum_{i=1}^{8-d} b_i E_i$$

is ample.

In the case of type $C(r_H)$, put $\ell_H = |\{a_i | a_i \neq 0\}|$. The $\mathbb{R}$-divisor H is said to be of length ℓ_H. The set of all ample $\mathbb{R}$-divisor classes of type $C(r_H)$ with length ℓ_H in $\mathrm{Pic}(S) \otimes \mathbb{R}$ is denoted by $\mathrm{Amp}^C_{\ell_H}(S)$. It is clear that

$$\mathrm{Amp}(S) = \bigcup_{\ell=0}^{8-d} \mathrm{Amp}^C_{\ell}(S) \cup \bigcup_{r=0}^{9-d} \mathrm{Amp}^B_{r}(S).$$

Note that $\mathrm{Amp}^B_0(S)$ is the ray generated by the anticanonical class $[-K_S]$.

2.2. **Main Theorems.** The goal of the present article is to study the cones of cylindrical ample divisors of smooth del Pezzo surfaces. This continues the work of T. Kishimoto, Yu. Prokhorov, and M. Zaidenberg in [21] and the work of I. Cheltsov, J. Park, and J. Won in [5] and [6].

Theorem 2.2.1. *Let S_d be a smooth del Pezzo surface of degree d.*

(1) For $4 \leqslant d \leqslant 9$,

$$\mathrm{Amp}^{cyl}(S_d) = \mathrm{Amp}(S_d).$$

(2) For $d = 3$,

$$\mathrm{Amp}^{cyl}(S_3) = \mathrm{Amp}(S_3) \setminus \mathrm{Amp}_0^B(S_3),$$

that is, any ample polarization H *of* S_3 *admits an* H*-polar cylinder unless* $H \equiv \alpha(-K_{S_3})$ *for some* $\alpha > 0$.

A. Perepechko verified that $\mathrm{Amp}^{cyl}(S_d) = \mathrm{Amp}(S_d)$ for $d \geqslant 5$ by showing that the ample cones of smooth del Pezzo surfaces of degrees at least 5 are contained in the cones generated by components of a certain effective divisor the complement of the support of which is a cylinder ([36, Subsection 3.2]). However, his method cannot be fully generalised to del Pezzo surfaces of lower degrees. Indeed, he yielded partial description of $\mathrm{Amp}^{cyl}(S_4)$ ([36, Theorem 7]) and thereafter J. Park and J. Won showed that $\mathrm{Amp}^{cyl}(S_4) = \mathrm{Amp}(S_4)$ using the same idea as in the present article ([35]).

The proof of Theorem 2.2.1 is given in Subsection 4.2.

Corollary 2.2.2. *Conjecture 1.2.5 holds for smooth del Pezzo surfaces.*

In order to analyze Conjecture 1.2.5 for smooth del Pezzo surfaces we do not have to study the sets $\mathrm{Amp}^{cyl}(S)$ for del Pezzo surfaces of degrees $\leqslant 3$ since these surfaces do not have $(-K_S)$-polar cylinders already. Only Theorem 2.2.1 (1) is required. Meanwhile, Theorem 2.2.1 (2) completely describes the set $\mathrm{Amp}^{cyl}(S)$ for smooth del Pezzo surfaces of degree 3. After [5] resolved the question whether the anticanonical class lies in $\mathrm{Amp}^{cyl}(S)$ or not for a smooth cubic surface S, Yu. Prokhorov proposed a more general problem:

To describe the cones of cylindrical ample divisors of smooth cubic surfaces.

Theorem 2.2.1 (2) gives the complete answer to Yu. Prokhorov's problem. It is however natural that Yu. Prokhorov's problem should be extended to smooth del Pezzo surfaces of degrees $\leqslant 2$. In the present article we also give some partial descriptions for them as follows:

Theorem 2.2.3. *Let* S_d *be a smooth del Pezzo surface of degree* $d \leqslant 3$ *and* H *be an ample* $\mathbb{R}$*-divisor of Fujita rank* r_H *on* S_d*. If* $r_H \leqslant 3 - d$*, then no* H*-polar cylinder exists on* S_d.

Theorem 2.2.3 immediately shows that

$$\mathrm{Amp}^{cyl}(S_d) \bigcap \left\{ \bigcup_{r=0}^{3-d} \mathrm{Amp}_r^B(S_d) \right\} = \emptyset$$

for a smooth del Pezzo surface S_d of degree $d \leqslant 3$. The proof of the theorem follows from Theorems 5.2.3 and 5.2.4.

Theorem 2.2.4. *Let* S *be a smooth del Pezzo surface of degree* 2*. Then*

(1)

$$\mathrm{Amp}^{cyl}(S) \supset \bigcup_{r=3}^{7} \mathrm{Amp}_r^B(S).$$

(2)

$$\mathrm{Amp}^{cyl}(S) \bigcap \mathrm{Amp}_2^B(S) \neq \emptyset.$$

(3)

$$\mathrm{Amp}^{cyl}(S) \supset \bigcup_{\ell=3}^{6} \mathrm{Amp}_\ell^C(S).$$

(4) For each $0 \leqslant \ell \leqslant 6$
$$\mathrm{Amp}^{cyl}(S) \bigcap \mathrm{Amp}^{C}_{\ell}(S) \neq \emptyset.$$

In Theorem 2.2.4, (1) follows from Theorem 6.2.2, (2) from Theorem 6.2.3, (3) from Theorem 6.2.10, and (4) from Theorem 6.2.11.

Theorem 2.2.5. *Let* S *be a smooth del Pezzo surface of degree* 1. *Then*

(1) For each $3 \leqslant r \leqslant 8$
$$\mathrm{Amp}^{cyl}(S) \bigcap \mathrm{Amp}^{B}_{r}(S) \neq \emptyset.$$

(2) For each $0 \leqslant \ell \leqslant 7$
$$\mathrm{Amp}^{cyl}(S) \bigcap \mathrm{Amp}^{C}_{\ell}(S) \neq \emptyset.$$

The statement follows from Propositions 6.3.1 and 6.3.2.

3. Rational singular surfaces without any cylinder

3.1. **Examples.** Before we proceed, let us remind that a normal surface singularity is Kawamata log terminal if and only if it is a quotient singularity ([17, Corollary 1.9]). A projective surface with quotient singularities is always $\mathbb{Q}$-factorial.

Definition 3.1.1. A normal projective surface with quotient singularities is called a log Enriques surface if its canonical class is numerically trivial and its irregularity is zero. It is called a log del Pezzo surface if its anticanonical class is ample ([29]).

We are now ready to present several examples of rational singular surfaces without any cylinder.

Example 3.1.2. J. Kollár has constructed a series of rational surfaces with ample canonical classes in [22]. The following is an easy example based on his construction.

Let a_1, a_2, a_3, a_4; w_1, w_2, w_3, w_4 be positive integers such that

- $a_1w_1 + w_2 = a_2w_2 + w_3 = a_3w_3 + w_4 = a_4w_4 + w_1$;
- $\gcd(w_1, w_3) = 1$, $\gcd(w_2, w_4) = 1$.

From the first condition above we obtain
$$w_1 = (a_2a_3a_4 - a_3a_4 + a_4 - 1), \quad w_2 = (a_1a_3a_4 - a_1a_4 + a_1 - 1),$$
$$w_3 = (a_1a_2a_4 - a_1a_2 + a_2 - 1), \quad w_4 = (a_1a_2a_3 - a_2a_3 + a_3 - 1).$$
Let S be the Klein-type hypersurface in $\mathbb{P}(w_1, w_2, w_3, w_4)$ defined by the quasi-homogeneous equation of degree $(a_1a_2a_3a_4 - 1)$
$$x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_4 + x_4^{a_4}x_1 = 0.$$
By the conditions $\gcd(w_1, w_3) = 1$ and $\gcd(w_2, w_4) = 1$ we can easily see that S is well-formed. Therefore,
$$K_S = \mathcal{O}_S\left(a_1a_2a_3a_4 - w_1 - w_2 - w_3 - w_4 - 1\right).$$
If a_1, a_2, a_3, $a_4 \geqslant 4$, then $a_1a_2a_3a_4 - w_1 - w_2 - w_3 - w_4 - 1 > 0$, and hence K_S is ample. By [22, Theorem 39] S is a rational surface of Picard rank three with four cyclic quotient singularities. Therefore, the surface S cannot contain any cylinder. In [16] D. Hwang and J. Keum have constructed another types of singular rational surfaces with ample canonical divisors.

Example 3.1.3. In order to construct smooth surface of general type with $p_g = q = 0$ for a given self-intersection number of the canonical class, Y. Lee, H. Park, J. Park, and D. Shin generate rational elliptic surfaces with nef canonical classes and quotient singularities of class T and then take their $\mathbb{Q}$-smoothings in [26], [33], [34]. Many such surfaces are presented in [27]. These rational surfaces meet our conditions not to have any cylinder.

Example 3.1.4 (cf. [30])**.** Let
$$E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$$
be the elliptic curve of period $\tau = e^{\frac{2}{3}\pi}$. Its j-invariant is 0 and it is isomorphic to the Fermat cubic curve. It is the unique elliptic curve admitting an automorphism σ of order three such that $\sigma^*(\omega) = \tau\omega$, where ω is a non-zero regular 1-form on E. Let S be the quotient surface
$$E \times E/\langle \mathrm{diag}(-\sigma, -\sigma)\rangle.$$
Then, $6K_S$ is linearly trivial. Since there is no non-zero regular 1-form on $E \times E$ invariant by $\mathrm{diag}(-\sigma, -\sigma)$, we obtain $h^1(S, \mathcal{O}_S) = 0$. Therefore, the surface S is a rational surface with quotient singularities whose canonical class is numerically trivial, i.e., it is a log Enriques surface, and hence it cannot contain any cylinder.

Example 3.1.5. This construction is due to K. Oguiso and D.-Q. Zhang ([31, Example 1]). The most extremal rational log Enriques surface of type D_{19} is defined as follows. Let $\overline{S}'$ be the quotient surface
$$E \times E/\langle \mathrm{diag}(\sigma, \sigma^2)\rangle,$$
where the notations are the same as in Example 3.1.4. Note that the automorphism σ on E has exactly three fixed points P_1, P_2, P_3, respectively. They correspond to 0, $\frac{2}{3} + \frac{1}{3}\tau$, and $\frac{1}{3} + \frac{2}{3}\tau$. The action by $\langle \mathrm{diag}(\sigma, \sigma^2)\rangle$ on $E \times E$ has nine fixed points and these nine fixed points become du Val singular points of type A_2 on $\overline{S}'$. Let S' be the minimal resolution of $\overline{S}'$. It is a K3 surface with twenty four smooth rational curves. Six of them come from the six fixed curves, $\{P_i\} \times E$, $E \times \{P_i\}$ on $E \times E$. The others come from the nine singular points of type A_2. Let σ' be the automorphism of S' induced by the automorphism $\mathrm{diag}(\sigma, 1)$ on $E \times E$. Our twenty four smooth rational curves on S' are σ'-invariant. Among these twenty four curves we can find a rational tree of type D_{19}. Let $S' \to \hat{S}$ be the contraction of this tree. Then σ' acts on $\hat{S}$ and it fixes two points. The quotient surface $\hat{S}/\langle\sigma'\rangle$ is a rational log Enriques surface. It follows from Remark 1.1.3 that any rational log Enriques surface cannot contain a cylinder.

Rational log Enriques surfaces of ranks 19 and 18 are completely classified in [32], [37], respectively.

An effective $\mathbb{Q}$-divisor D on a proper normal variety X is called a tiger if it is numerically equivalent to $-K_X$ and (X, D) is not log canonical. The tiger was introduced by S. Keel and J. Mckernan in their study of log del Pezzo surfaces of Picard rank one ([18]).

Example 3.1.6. M. Miyanishi proposed a conjecture (see [14]) that for a log del Pezzo surface of Picard rank one, its smooth locus has a finite unramified covering that contains a cylinder. It however turned out that the conjecture is answered in negative. S. Keel and J. Mckernan have constructed log del Pezzo surfaces of Picard rank one such that

- they have no tigers;
- their smooth loci have trivial algebraic fundamental groups.

For their construction, see [18, Eample 21.3.3]. If such a surface S contains a cylinder U, then we are able to obtain an effective divisor D such that $S \setminus U = \mathrm{Supp}(D)$. Since S is a log del Pezzo surface of Picard rank one, for some positive rational number λ the divisor λD is linearly equivalent to $-K_S$. It immediately follows from Remark 1.1.3 that λD is a tiger. This is a contradiction. Therefore, the surface S cannot contain any cylinder at all.

The rational surfaces in Exmple 3.1.6 are overqualified to be free from cylinders. We may give away the condition of the algebraic fundamental group since we are not considering cylinders in étale covers. On del Pezzo surfaces of Picard rank one with du Val singularities, instead of non-existence of tigers, we can apply a finer condition than Remark 1.1.3 that there is a tiger that does not contain the support of any effective anticanonical divisor (see [6, Remark 3.8]).

Example 3.1.7. Let S be a del Pezzo surface of degree 1 with one of the following types of singularities

$$2\mathrm{D}_4, \quad 2\mathrm{A}_3+2\mathrm{A}_1, \quad 4\mathrm{A}_2.$$

Then its divisor class group is generated by the anticanonical class over $\mathbb{R}$. Therefore, Theorem 1.2.4 implies that S cannot contain any cylinder at all. The smooth loci of these surfaces are not simply connected ([29]).

Remark 3.1.8. The surfaces in Example 3.1.7 are the only del Pezzo surfaces with du Val singularities that have no cylinder. The other del Pezzo surfaces with du Val singularities contain cylinders. Indeed, del Pezzo surfaces with one of the types of singularities $2\mathrm{D}_4$, $2\mathrm{A}_3$+$2\mathrm{A}_1$, $4\mathrm{A}_2$ are the only ones that contain no $(-K_S)$-cylinder and have Picard rank one (see [6]). Furthermore, since their divisor class groups are generated by the anticanonical classes over $\mathbb{R}$, they cannot contain any cylinder at all. In [6], all the del Pezzo surfaces with du Val singularities that have no anticanonically polarized cylinders are completely classified. Among them, the surfaces of the three types of singularities above are the only ones of Picard rank one. For those of higher Picard rank without anticanonically polarized cylinders one can always construct cylinders polarized by ample $\mathbb{Q}$-divisors. These constructions can be made by manipulating various anticanonically polarized cylinders that appear in [6].

4. Cylinders in del Pezzo surfaces of big degree

4.1. **Basic cylinders.** We here present many examples of cylinders on smooth del Pezzo surfaces. These will be building blocks from which we are able to construct cylinders polarized by various ample divisors.

Before we proceed, note that an (n)-curve on a smooth surface is an integral curve isomorphic to $\mathbb{P}^1$ with self-intersection number n.

Example 4.1.1. On $\mathbb{P}^2$, let L_i, $i=1,\ldots,r$, be lines meeting altogether at a single point. The complement of the union of these r lines is an $\mathbb{A}^1$-bundle over an $(r-1)$-punctured affine line. Therefore, this is a cylinder in $\mathbb{P}^2$.

Example 4.1.2. Let L and M be a line and a conic on $\mathbb{P}^2$ intersecting tangentially at a point. Each divisor $aL+bM$ with positive real numbers a and b defines a cylinder isomorphic to an $\mathbb{A}^1$-bundle over a simply punctured affine line $\mathbb{A}^1_*$.

Example 4.1.3. Let C be a cuspidal cubic curve on $\mathbb{P}^2$ and T be the Zariski tangent line at its cuspidal point P. Their complement is a cylinder isomorphic to $\mathbb{A}^1\times\mathbb{A}^1_*$. To see this, take the blow up $\pi_1:\mathbb{F}_1\to\mathbb{P}^2$ at the point P and then take the blow up $\pi_2:\tilde{S}_7\to\mathbb{F}_1$ at the intersection point of the proper transforms of C and T. Let E_1 be the exceptional curve on $\tilde{S}_7$ contracted by π_1 and E_2 be the exceptional curve on $\tilde{S}_7$ contracted by π_2. In addition, let $\tilde{C}$ and $\tilde{T}$ be the proper transforms of C and T by $\pi_1\circ\pi_2$. Now we contract $\tilde{T}$, which is a (-1)-curve on $\tilde{S}_7$. This gives us a birational morphism $\pi:\tilde{S}_7\to\mathbb{F}_2$. Then $\pi(E_1)$ is the exceptional section of $\mathbb{F}_2$ with self-intersection number -2 and $\pi(E_2)$ is a fiber of $\mathbb{F}_2$. The curve $\pi(\tilde{C})$ is linearly equivalent to $\pi(E_1)+3\pi(E_2)$ and is a section of $\mathbb{F}_2$ with the self-intersection number 4. The three curves $\pi(E_1)$, $\pi(E_2)$, and $\pi(\tilde{C})$ on $\mathbb{F}_2$ meet transversally at a single point, and hence their complement is isomorphic to $\mathbb{A}^1\times\mathbb{A}^1_*$. Since

$$\mathbb{P}^2\setminus(C\cup T)\cong\tilde{S}_7\setminus(\tilde{C}\cup\tilde{T}\cup E_1\cup E_2)\cong\mathbb{F}_2\setminus(\pi(\tilde{C})\cup\pi(E_1)\cup\pi(E_2))$$

the complement of C and T is a cylinder.

Example 4.1.4. On $\mathbb{P}^1\times\mathbb{P}^1$, let E be a curve of bidegree $(1,0)$ and F_i, $i=1,\ldots,r$, be curves of bidegree $(0,1)$. The complement of these curves is isomorphic to an $\mathbb{A}^1$-bundle over a $(r-1)$-punctured affine line. By blowing up $\mathbb{P}^1\times\mathbb{P}^1$ at the intersection point of E and F_1 and then contracting the proper transforms of E and F_1 to $\mathbb{P}^2$, we may encounter the cylinder described in Example 4.1.1.

Example 4.1.5 ([19, Theorem 3.19])**.** Let S_d be a smooth del Pezzo surface of degree $d \geqslant 2$, not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then it can be obtained by blowing up $\mathbb{P}^2$ at $(9-d)$ points in general position. Let $\sigma : S_d \to \mathbb{P}^2$ be such a blow up and let E_i be the exceptional curve, where $i = 1, \ldots, 9-d$. Put $P_i = \sigma(E_i)$. Suppose that these $(9-d)$ points $P_1, \ldots, P_{9-d}$ lie on the union of a line L and a conic C that intersect tangentially at a single point. Note

$$S_d \setminus (\tilde{L} \cup \tilde{C} \cup E_1 \cup \ldots \cup E_{9-d}) \cong \mathbb{P}^2 \setminus (L \cup C) \cong \mathbb{A}^1 \times \mathbb{A}^1_*,$$

where $\tilde{L}$ and $\tilde{C}$ are the proper transforms of L and C by σ.

We now suppose that $d \geqslant 4$. There is a conic passing through all the points $P_1, \ldots, P_{9-d}$. So we may assume that $P_1, \ldots, P_{9-d}$ lie on C and that L meets C tangentially at a point other than $P_1, \ldots, P_{9-d}$. Then, for a real number $0 < \varepsilon < \frac{1}{2}$, the $\mathbb{R}$-divisor

$$(1+\varepsilon)\tilde{C} + (1-2\varepsilon)\tilde{L} + \varepsilon \sum_{i=1}^{9-d} E_i$$

is effective and numerically equivalent to $-K_{S_d}$. Therefore, the cylinder

$$S_d \setminus (\tilde{L} \cup \tilde{C} \cup E_1 \cup \ldots \cup E_{9-d})$$

is $(-K_{S_d})$-polar.

Example 4.1.6. Let C be an irreducible curve of bidegree $(1,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. There are two curves of bidegree $(1,0)$ that intersect C tangentially. Let T be one of them and P be the intersection point of C and T. In addition, let L be the curve of bidegree $(0,1)$ passing through the point P. Then the open set

$$\mathbb{P}^1 \times \mathbb{P}^1 \setminus (C \cup L \cup T)$$

is a cylinder. To see this, let $\pi : S_7 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow up at P and E be its exceptional curve. Denote the proper transforms of the curves C, T, and L by $\tilde{C}$, $\tilde{T}$, and $\tilde{L}$, respectively. The curves $\tilde{T}$ and $\tilde{L}$ are disjoint (-1)-curves. Let $\psi : S_7 \to \mathbb{P}^2$ be the contraction of $\tilde{L}$ and $\tilde{T}$. Then, $\psi(E)$ is a line, $\psi(C)$ is a conic, and they meet tangentially. Therefore, the open set above is a cylinder since

$$\mathbb{P}^1 \times \mathbb{P}^1 \setminus (C \cup L \cup T) \cong S \setminus (\tilde{C} \cup \tilde{L} \cup \tilde{T} \cup E) \cong \mathbb{P}^2 \setminus (\psi(\tilde{C}) \cup \psi(E)) \cong \mathbb{A}^1 \times \mathbb{A}^1_*.$$

In particular, for a real number $0 < \varepsilon < 1$,

$$(1-\varepsilon)C + (1+\varepsilon)T + 2\varepsilon L \equiv -K_{\mathbb{P}^1 \times \mathbb{P}^1},$$

and hence the cylinder is $(-K_{\mathbb{P}^1 \times \mathbb{P}^1})$-polar.

Example 4.1.7. Let C_1 and C_2 be irreducible curves of bidegree $(1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that these two curves meet tangentially at a single point P. Let L_1 and L_2 be the curves of bidegrees $(1,0)$ and $(0,1)$, respectively, that pass through the point P. We claim that the open set

$$\mathbb{P}^1 \times \mathbb{P}^1 \setminus (C_1 \cup C_2 \cup L_1 \cup L_2)$$

is a cylinder. Let $\pi : S_7 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow up at P and E be its exceptional curve. Denote the proper transforms of C_i and L_i by $\tilde{C}_i$ and $\tilde{L}_i$, respectively, $i = 1, 2$. The curves $\tilde{L}_1$ and $\tilde{L}_2$ are disjoint (-1)-curves. Therefore, we obtain a birational morphism $\psi : S_7 \to \mathbb{P}^2$ by contracting $\tilde{L}_1$ and $\tilde{L}_2$. The three curves $\psi(E)$, $\psi(\tilde{C}_1)$, $\psi(\tilde{C}_2)$ are lines intersecting at a single point. Since

$$\mathbb{P}^1 \times \mathbb{P}^1 \setminus (C_1 \cup C_2 \cup L_1 \cup L_2) \cong S_7 \setminus (\tilde{C}_1 \cup \tilde{C}_2 \cup \tilde{L}_1 \cup \tilde{L}_2 \cup E) \cong \mathbb{P}^2 \setminus (\psi(\tilde{C}_1) \cup \psi(\tilde{C}_2) \cup \psi(E)) \cong \mathbb{A}^1 \times \mathbb{A}^1_{**},$$

where $\mathbb{A}^1_{**}$ is a 2-punctured affine line, our claim is confirmed. In particular, the cylinder is $(-K_{\mathbb{P}^1\times\mathbb{P}^1})$-polar because the divisor

$$(1+\varepsilon)C_1+(1-2\varepsilon)C_2+\varepsilon L_1+\varepsilon L_2$$

is effective for an arbitrary real number $0<\varepsilon<\frac{1}{2}$ and nemerically equivalent to $-K_{\mathbb{P}^1\times\mathbb{P}^1}$.

Example 4.1.8. Let C be a cuspidal rational curve in the anticanonical linear system of the Hirzebruch surface $\mathbb{F}_1$. Let M be the 0-curve that passes through the cuspidal point P. There is a unique 1-curve T that intersects C only at the point P. For an arbitrary real number $0<\varepsilon<1$, the divisor

$$(1-\varepsilon)C+\varepsilon M+2\varepsilon T$$

defines a $(-K_{\mathbb{F}_1})$-polar cylinder. To see this, take the blow up $\phi: S_7\to\mathbb{F}_1$ at the point P. Let E be the exceptional curve. Denote the proper transforms of the curves C, M, and T on S_7 by $\tilde{C}$, $\tilde{M}$, and $\tilde{T}$, respectively. The $\mathbb{R}$-divisor $(1-\varepsilon)\tilde{C}+\varepsilon\tilde{M}+2\varepsilon\tilde{T}+(1+\varepsilon)E$ is numerically equivalent to $-K_{S_7}$. Let $\psi: S_7\to\mathbb{P}^1\times\mathbb{P}^1$ be the contraction of the (-1)-curve $\tilde{M}$. Then the $\mathbb{R}$-divisor $(1-\varepsilon)\psi(\tilde{C})+2\varepsilon\psi(\tilde{T})+(1+\varepsilon)\psi(E)$ is the divisor in Example 4.1.6 that defines a $(-K_{\mathbb{P}^1\times\mathbb{P}^1})$-polar cylinder.

Example 4.1.9. On the Hirzebruch surface $\mathbb{F}_1$, let M be a 0-curve and let T be a 1-curve. Let P be the intersection point of T and M. There is a 3-curve C that is tangent to the curve T at the point P. Then for an arbitrary real number $0<\varepsilon<1$, the divisor

$$(1-\varepsilon)C+(1+\varepsilon)T+\varepsilon M$$

defines a $(-K_{\mathbb{F}_1})$-polar cylinder. Take the blow up $\phi: S_7\to\mathbb{F}_1$ at the point P. Let E be its exceptional curve. Denote the proper transforms of the curves C, M, and T on S_7 by $\tilde{C}$, $\tilde{M}$, and $\tilde{T}$, respectively. The $\mathbb{R}$-divisor $(1-\varepsilon)\tilde{C}+(1+\varepsilon)\tilde{T}+\varepsilon\tilde{M}+(1+\varepsilon)E$ is numerically equivalent to $-K_{S_7}$. Let $\psi: S_7\to\mathbb{P}^1\times\mathbb{P}^1$ be the contraction of the (-1)-curve $\tilde{M}$. Then the $\mathbb{R}$-divisor $(1-\varepsilon)\psi(\tilde{C})+(1+\varepsilon)\psi(\tilde{T})+(1+\varepsilon)\psi(E)$ is numerically equivalent to $-K_{\mathbb{P}^1\times\mathbb{P}^1}$. The curve $\psi(\tilde{C})$ is an irreducible curve of bidegree $(1,1)$, $\psi(\tilde{T})$ is of bidegree $(1,0)$, and $\psi(\tilde{E})$ is of bidegree $(0,1)$. Moreover, these three curves meet at a single point. This easily shows that the divisor $(1-\varepsilon)C+(1+\varepsilon)T+\varepsilon M$ defines a $(-K_{\mathbb{F}_1})$-polar cylinder.

Example 4.1.10. Let C be a cuspidal rational curve in the anticanonical linear system of the smooth del Pezzo surface S_7 of degree 7. There are exactly two 0-curves M_1, M_2 passing through the cuspidal point P of C. There is a unique 1-curve T that meets C only at the point P. For an arbitrary real number $0<\varepsilon<1$, the divisor

$$(1-\varepsilon)C+\varepsilon M_1+\varepsilon M_2+\varepsilon T$$

defines a $(-K_{S_7})$-polar cylinder. To see this, take the blow up $\phi: S_6\to S_7$ at the point P. Let E be the exceptional divisor. The proper transforms of M_1 and M_2 are disjoint (-1)-curves on S_6. Let $\psi: S_6\to\mathbb{F}_1$ be the birational morphism obtained by contracting these two curves. Then the curve E becomes a 1-curve and the curve T becomes a 0-curve on $\mathbb{F}_1$. They intersect at a single point Q. The curve C becomes a 3-curve tangent to $\psi(E)$ at Q. Then Example 4.1.9 shows that the divisor $(1-\varepsilon)C+\varepsilon M_1+\varepsilon M_2+\varepsilon T$ defines a $(-K_{S_7})$-polar cylinder.

Example 4.1.11. Let S_6 be a smooth del Pezzo surface of degree 6. Let C be a cuspidal rational curve in the anticanonical linear system and let P be its cuspidal point. There are three disjoint (-1)-curves E_1,E_2,E_3 on S_6. Each of them meets C at a single smooth point of C. Let $\phi: S_6\to\mathbb{P}^2$ be the birational morphism obtained by contracting E_1,E_2,E_3. Let T' be the Zariski tangent line to the cuspidal rational curve $\phi(C)$ at the point $\phi(P)$. There is a unique conic curve T_0' such that it is tangent to T' at the point $\phi(P)$ and passes through the points $\phi(E_1),\phi(E_2),\phi(E_3)$. Let M_i' be the line through $\phi(P)$ and $\phi(E_i)$. Let T, T_0, and M_i be the proper transforms of T', T_0', and M_i by the birational morphism ϕ. For an arbitrary real number $0 < \varepsilon < \frac{1}{2}$, the $\mathbb{R}$-divisor

$$(1-2\varepsilon)C + \varepsilon T + \varepsilon T_0 + \varepsilon M_1 + \varepsilon M_2 + \varepsilon M_3$$

is numerically equivalent to the anticanonical class on S_6. We take the blow up $\phi : S_5 \to S_6$ at the point P. Let E be the exceptional curve. The proper transforms of M_1, M_2, and M_3 are disjoint (-1)-curves on S_5. Let $\psi : S_5 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism obtained by contracting these three curves. Then Example 4.1.7 shows that the $\mathbb{R}$-divisor above defines a $(-K_{S_6})$-polar cylinder on S_6.

Example 4.1.12. Let S_5 be a smooth del Pezzo surface of degree 5. In addition, let H be an effective anticanonical divisor on S_5 that consists of one 1-curve C and one 0-curve M meeting tangentially at a single point P. Then there are four 0-curves M_1, M_2, M_3, M_4, other than M, passing through the point P. They intersect each other only at the point P. For an arbitrary real number $0 < \varepsilon < \frac{1}{2}$, the divisor

$$(1-\varepsilon)M + (1-2\varepsilon)C + \varepsilon M_1 + \varepsilon M_2 + \varepsilon M_3 + \varepsilon M_4$$

defines a $(-K_{S_5})$-polar cylinder. Indeed, to see this, consider the blow up $\phi : S_4 \to S_5$ at the point P, and then contract the proper transforms of M_i's and M by ϕ to $\mathbb{P}^2$.

Example 4.1.13. Let S_5 be a smooth del Pezzo surface of degree 5. Let C be a cuspidal rational curve in the anticanonical linear system and let P be its cuspidal point. We have four disjoint (-1)-curves $E_1, \ldots, E_4$ on S_5. Each of them intersects C at a single smooth point of C. Let $\phi : S_5 \to \mathbb{P}^2$ be the birational morphism obtained by contracting $E_1, \ldots, E_4$. Let M_0' be the conic on $\mathbb{P}^2$ determined by the points $\phi(E_1), \ldots, \phi(E_4)$, and $\phi(P)$. For $i = 1, 2, 3, 4$, let M_i' be the line passing through the points $\phi(E_i)$ and $\phi(P)$. Let M_i be the proper transform of M_i' by the birational morphism ϕ. These five curves are 0-curves on S_5 passing through P. For an arbitrary real number $0 < \varepsilon < \frac{1}{2}$, the divisor

$$(1-2\varepsilon)C + \varepsilon M_0 + \varepsilon M_1 + \varepsilon M_2 + \varepsilon M_3 + \varepsilon M_4$$

defines a $(-K_{S_5})$-polar cylinder. We consider the blow up $\psi : S_4 \to S_5$ at P. The proper transforms of M_i by ψ are mutually disjoint (-1)-curves. We contract these five (-1)-curves to $\mathbb{P}^2$. Then C and the exceptional curve of ψ become a line and a conic meeting tangentially on $\mathbb{P}^2$. Therefore, the effective $\mathbb{R}$-divisor above defines a cylinder.

Example 4.1.14. We here use the same notations S_5, C, P, $E_1, \ldots, E_4$, and $\phi : S_5 \to \mathbb{P}^2$ as in Example 4.1.13. Let T_0' be the Zariski tangent line to the cuspidal rational curve $\phi(C)$ at the point $\phi(P)$. For $i = 1, 2, 3, 4$, there is a unique conic curve T_i' such that it is tangent to T_0' at the point $\phi(P)$ and passes through the points $\phi(E_1), \ldots, \phi(E_4)$ except $\phi(E_i)$. Let T_0 and T_i be the proper transforms of T_0' and T_i' by the birational morphism ϕ. Then, for an arbitrary real number $0 < \varepsilon < \frac{1}{3}$, the divisor

$$(1-3\varepsilon)C + \varepsilon T_0 + \varepsilon T_1 + \varepsilon T_2 + \varepsilon T_3 + \varepsilon T_4$$

defines a $(-K_{S_5})$-polar cylinder. To see this, let $\phi_1 : S_4 \to S_5$ be the blow up at the point P. Then the exceptional curve B_1 of ϕ_1 intersects the proper transform of C tangentially at a single point Q. Let $\phi_2 : S_3 \to S_4$ be the blow up at the point Q and denote its exceptional curve by B_2. Let $\tilde{C}$ and $\tilde{T}_i$ be the proper transforms of C and T_i by the morphism $\phi_1 \circ \phi_2$, where $i = 0, \ldots, 4$. In addition, let $\tilde{B}_1$ be the proper transform of B_1 by ϕ_2. Then the $\mathbb{R}$-divisor

$$(1-3\varepsilon)\tilde{C} + (1-\varepsilon)\tilde{B}_1 + (1+\varepsilon)B_2 + \varepsilon\tilde{T}_0 + \varepsilon\tilde{T}_1 + \varepsilon\tilde{T}_2 + \varepsilon\tilde{T}_3 + \varepsilon\tilde{T}_4$$

is numerically equivalent to the anticanonical class on S_3. Now, contracting the five disjoint (-1)-curves $\tilde{T}_0, \ldots, \tilde{T}_4$ to the Hirzebruch surface $\mathbb{F}_2$, we obtain a birational morphism $\psi_1 : S_3 \to \mathbb{F}_2$.

Put $\overline{C}=\psi_1(\tilde{C})$, $\overline{B}_1=\psi_1(\tilde{B}_1)$, and $\overline{B}_2=\psi_1(B_2)$. Then

$$(1-3\varepsilon)\overline{C}+(1-\varepsilon)\overline{B}_1+(1+\varepsilon)\overline{B}_2$$

is numerically equivalent to $-K_{\mathbb{F}_2}$. Note that $\overline{B}_1$ is the (-2)-curve on $\mathbb{F}_2$ and $\overline{C}$ is a 0-curve. We take the blow up $\phi_3: S\to\mathbb{F}_2$ at a general point of $\overline{C}$. Let B_3 be its exceptional curve. In addition, let $\hat{C}$, $\hat{B}_1$, and $\hat{B}_2$ be the proper transforms of $\overline{C}$, $\overline{B}_1$, and $\overline{B}_2$ by ϕ_3. Finally, by contracting $\hat{C}$ and $\hat{B}_1$ in turn, we obtain a birational morphism $\psi_2: S\to\mathbb{P}^2$. We immediately see that $\psi_3(\hat{B}_2)$ is a cuspidal rational curve and $\psi_3(B_3)$ is the Zariski tangent line to $\psi_3(\hat{B}_2)$ at its cuspidal point. Even though

$$(1+\varepsilon)\psi_3(\hat{B}_2)-3\varepsilon\psi_3(B_3)$$

is a non-effective divisor on $\mathbb{P}^2$, the original divisor on S_5 defines a $(-K_{S_5})$-polar cylinder since

$$S_5\setminus(C\cup T_0\cup\ldots\cup T_4)\cong\mathbb{P}^2\setminus(\psi_3(\hat{B}_2)\cup\psi_3(B_3))$$

(see Example 4.1.3).

Example 4.1.15. Let S_4 be a smooth del Pezzo surface of degree 4. In addition, let H be an effective anticanonical divisor on S_4 that consists of one 1-curve C and one (-1)-curve L meeting tangentially at a single point P.

By contracting L, we see from Example 4.1.14 that there are five 0-curves $T_1,\ldots,T_5$ passing through the point P such that they intersect each other only at P and meet L and C only at P. Example 4.1.14 immediately shows that

$$(1-3\varepsilon)C+\ (1-\varepsilon)L+\varepsilon\sum_{i=1}^{5}T_i$$

defines a $(-K_{S_4})$-polar cylinder on S_4 for $0<\varepsilon<\frac{1}{3}$.

4.2. **Proof of Theorem 2.2.1.** Let S_d be a smooth del Pezzo surface of degree $d\geqslant 3$ and let H be an ample $\mathbb{R}$-divisor on S_d. It is easy to check that S_d always contains an H-polar cylinder isomorphic to $\mathbb{A}^2$ if $d\geqslant 8$. For this reason, in order to prove Theorem 2.2.1, we assume that $d\leqslant 7$.

Let μ be the Fujita invariant of H, Δ be the Fujita face of H, and r be the Fujita rank of H. Let $\phi: S_d\to Z$ be the contraction given by Δ.

Lemma 4.2.1 (cf. Example 4.1.5)**.** *Suppose that H is of type $B(r)$ and $Z\not\cong\mathbb{P}^1\times\mathbb{P}^1$. If $(d,r)\neq(3,0)$, then S_d contains an H-polar cylinder.*

Proof. For the case $r=0$ and $d\geqslant 4$, Theorem 1.2.3 implies the statement. Therefore, we may assume that $r>0$.

Let $E_1,\ldots,E_r$ be the (-1)-curves that generate the face Δ. Then

$$K_{S_d}+\mu H\equiv\sum_{i=1}^{r}a_iE_i$$

for some positive real numbers $a_1,\ldots,a_r$. Note that $r\leqslant 9-d$ and $E_1,\ldots,E_r$ are disjoint.

The surface Z is a smooth del Pezzo surface of degree $(d+r)$. Since $Z\not\cong\mathbb{P}^1\times\mathbb{P}^1$, either $Z=\mathbb{P}^2$ or Z is a blow up of $\mathbb{P}^2$ in $(9-d-r)$ points in general position. Let $\psi\colon Z\to\mathbb{P}^2$ be the blow up. Put $k=9-d$ and $\sigma=\psi\circ\phi$. If $k>r$, denote the proper transforms of these ψ-exceptional curves on S_d by $E_{r+1},\ldots,E_k$. Put $P_i=\sigma(E_i)$.

Let C be an irreducible conic in $\mathbb{P}^2$ passing through the points $P_2,\ldots,P_k$. Such a conic exists because $k\leqslant 6$. Let L be a line in $\mathbb{P}^2$ passing through the point P_1 and tangent to the conic C. Note that L may be tangent to C at one of the points $P_2,\ldots,P_k$, say P_2.

For a positive real number ε we have $-K_{\mathbb{P}^2}\equiv(1+\varepsilon)C+(1-2\varepsilon)L$. Hence,

$$-K_{S_d} \sim \sigma^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{k} E_i \equiv \begin{cases} (1+\varepsilon)\tilde{C} + (1-2\varepsilon)\tilde{L} - 2\varepsilon E_1 + (1-\varepsilon)E_2 + \varepsilon \sum_{i=3}^{k} E_i \\ \text{if } L \text{ meets } C \text{ at } P_2; \\ (1+\varepsilon)\tilde{C} + (1-2\varepsilon)\tilde{L} - 2\varepsilon E_1 + \varepsilon \sum_{i=2}^{k} E_i \\ \text{otherwise,} \end{cases}$$

where $\tilde{C}$ and $\tilde{L}$ are the proper transforms of C and L, respectively, by σ. Thus, we have

$$H \equiv \begin{cases} \frac{1}{\mu}\left((1+\varepsilon)\tilde{C} + (1-2\varepsilon)\tilde{L} + (a_1-2\varepsilon)E_1 + (a_2+1-\varepsilon)E_2 + \sum_{i=3}^{r}(a_i+\varepsilon)E_i + \varepsilon \sum_{i=r+1}^{k} E_i\right) \\ \text{if } L \text{ meets } C \text{ at } P_2; \\ \frac{1}{\mu}\left((1+\varepsilon)\tilde{C} + (1-2\varepsilon)\tilde{L} + (a_1-2\varepsilon)E_1 + \sum_{i=2}^{r}(a_i+\varepsilon)E_i + \varepsilon \sum_{i=r+1}^{k} E_i\right) \\ \text{otherwise.} \end{cases}$$

For $0 < \varepsilon < \frac{a_1}{2}$, this defines an H-polar cylinder because

$$S_d \setminus (\tilde{C} \cup \tilde{L} \cup E_1 \cup \ldots \cup E_k) \cong \mathbb{P}^2 \setminus (C \cup L).$$

□

As in the case where L meets C at P_2 in the proof of Lemma 4.2.1, it can happen that we should separately deal with the case where two curves meet at one of the centers of blow ups. However, in such a case, we always obtain a bigger coefficient for the exceptional curve over the center than when it is not the case. For this reason, in the sequel, we always omit the proof for such a special case. The proof for a non-special case works almost verbatim for such a special case.

Lemma 4.2.2. *Suppose that H is of type $B(8-d)$ and $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then S_d contains an H-polar cylinder.*

Proof. Let $E_1, \ldots, E_r$ be the (-1)-curves that generate the face Δ. Note that $r = 8-d$. Then

$$K_{S_d} + \mu H \equiv \sum_{i=1}^{r} a_i E_i$$

for some positive real numbers $a_1, \ldots, a_r$. The (-1)-curves $E_1, \ldots, E_r$ are disjoint. Put $P_i = \phi(E_i)$.

Since $r \leqslant 5$, there is an irreducible curve C of bidegree $(2,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the points $P_1, \ldots, P_r$. Let L be a curve of bidegree $(0,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ that is tangent to the curve C. Let P be the intersection point of C and L. Then there is a unique curve M of bidegree $(1,0)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the point P.

For a positive real number ε we have $-K_{\mathbb{P}^1\times\mathbb{P}^1} \equiv (1-\varepsilon)C + (1+\varepsilon)L + 2\varepsilon M$. Hence,

$$-K_{S_d} \sim \phi^*(-K_{\mathbb{P}^1\times\mathbb{P}^1}) - \sum_{i=1}^{r} E_i \equiv (1-\varepsilon)\tilde{C} + (1+\varepsilon)\tilde{L} + 2\varepsilon\tilde{M} - \varepsilon \sum_{i=1}^{r} E_i,$$

where $\tilde{C}$, $\tilde{L}$, and $\tilde{M}$ are the proper transforms of C, L, and M, respectively, by ϕ. Thus, we have

$$H \equiv \frac{1}{\mu}\left((1-\varepsilon)\tilde{C} + (1+\varepsilon)\tilde{L} + 2\varepsilon\tilde{M} + \sum_{i=1}^{r}(a_i-\varepsilon)E_i\right).$$

Furthermore, we see immediately that

$$S_d \setminus (\tilde{C} \cup \tilde{L} \cup \tilde{M} \cup E_1 \cup \ldots \cup E_r) \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus (C \cup L \cup M).$$

By taking $0 < \varepsilon < \min\{a_1, \ldots, a_r\}$ we obtain an H-polar cylinder on S_d (see Example 4.1.6). □

Lemma 4.2.3. *Suppose that H is of type $C(9-d)$. Then S_d contains an H-polar cylinder.*

Proof. If the contraction ϕ is a conic bundle, then, as in (2.1.4), we may write

$$K_{S_d} + \mu H \equiv aB + \sum_{i=1}^{m} a_i E_i$$

where B is an irreducible fiber of ϕ, E_i's are disjoint (-1)-curves in fibers of ϕ, a is a positive real number, a_i's are non-negative real numbers, and $m = 8-d$. We may assume that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_m$. Let $\phi_1 : S_d \to W$ be the birational morphism obtained by contracting the disjoint (-1)-curves $E_1, \ldots, E_m$. Thus W is a smooth del Pezzo surface of degree 8, hence either $W \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $W \cong \mathbb{F}_1$.

Case 1. $a_m \neq 0$ and $W \cong \mathbb{F}_1$.

There is a (-1)-curve E on S_d whose image by ϕ_1 is the unique (-1)-curve on W. Let $\psi : W \to \mathbb{P}^2$ be the birational morphism given by contracting $\phi_1(E)$. Put $\sigma = \psi \circ \phi_1$. Denote the points $\sigma(E_i)$ by P_i, $i = 1, \ldots, m$, the point $\sigma(E)$ by P, and the line $\sigma(B)$ by M. Note that the line M passes through the point P.

Let C be the conic passing through the points $P_1, \ldots, P_m$. Such a conic exists because $m \leqslant 5$. Let L be a line that passes through the point P and that is tangent to the conic C. We may assume that the line L is different from M.

For any real number ε we have $-K_{\mathbb{P}^2} \equiv (1-\varepsilon)C + (1+2\varepsilon+a)L - aM$. Hence,

$$\begin{aligned} -K_{S_d} &\sim \sigma^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{m} E_i - E \\ &\equiv (1-\varepsilon)\tilde{C} + (1+2\varepsilon+a)\tilde{L} + 2\varepsilon E - aB - \varepsilon\sum_{i=1}^{m} E_i, \end{aligned}$$

where $\tilde{C}$ and $\tilde{L}$ are the proper transforms of C and L, respectively, by σ. Thus, we have

$$H \equiv \frac{1}{\mu}\left((1-\varepsilon)\tilde{C} + (1+2\varepsilon+a)\tilde{L} + 2\varepsilon E + \sum_{i=1}^{m}(a_i-\varepsilon)E_i\right).$$

By taking a sufficiently small positive real number ε we obtain an H-polar cylinder on S_d.

Case 2. Either $a_m \neq 0$ with $W \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $a_m = 0$.

We first assume that $W \cong \mathbb{P}^1 \times \mathbb{P}^1$. Denote the points $\phi_1(E_i)$ by P_i, $i = 1, \ldots, m-1$, the point $\phi_1(E_m)$ by P, and the curve $\phi_1(B)$ by M. The curve M is a curve of bidegree $(0,1)$ or $(1,0)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We may assume that it is of bidegree $(0,1)$.

There is a unique curve C of bidegree $(1,2)$ passing through the points $P, P_1, \ldots, P_{m-1}$. There is a curve L of bidegree $(1,0)$ that is tangent to C. Let Q be the point at which L meets C and let N be the curve of bidegree $(0,1)$ passing through the point Q.

For an arbitrary real number ε we have $-K_{\mathbb{P}^1\times\mathbb{P}^1} \equiv (1+\varepsilon)C + (1-\varepsilon)L + (a-2\varepsilon)N - aM$. Hence,

$$\begin{aligned}-K_{S_d} &\sim \phi_1^*(-K_{\mathbb{P}^1\times\mathbb{P}^1}) - E_m - \sum_{i=1}^{m-1} E_i\\ &\equiv (1+\varepsilon)\tilde{C} + (1-\varepsilon)\tilde{L} + (a-2\varepsilon)\tilde{N} - aB + \varepsilon E_m + \sum_{i=1}^{m-1} \varepsilon E_i,\end{aligned}$$

where $\tilde{C}$, $\tilde{L}$, and $\tilde{N}$ are the proper transforms of C, L, N, respectively, by ϕ_1. Thus, we have

$$H \equiv \frac{1}{\mu}\left((1+\varepsilon)\tilde{C} + (1-\varepsilon)\tilde{L} + (a-2\varepsilon)\tilde{N} + (a_m+\varepsilon)E_m + \sum_{i=1}^{m-1}(a_i+\varepsilon)E_i,\right).$$

By taking a sufficiently small positive real number ε we obtain an H-polar cylinder on S_d (see Example 4.1.6).

Now we assume that $W \cong \mathbb{F}_1$ and $a_m = 0$. Let E'_m be the other (-1)-curve in the fiber of ϕ containing the (-1)-curve E_m. Let $\phi_2 : S_d \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism given by contracting the (-1)-curves $E_1, \ldots, E_{m-1}, E'_m$. Since $a_m = 0$, after replacing ϕ_1 and E_m by ϕ_2 and E'_m, we see immediately that the previous argument also works for this case. □

Theorem 2.2.1 immediately follows from Theorem 1.2.3 and Lemmas 4.2.1, 4.2.2, and 4.2.3.

5. Absence of cylinders

5.1. **The main obstruction.** We here refine Remark 1.1.3 for a smooth rational surface as below.

Let S be a smooth rational surface and let A be a big $\mathbb{R}$-divisor on S. Suppose that S contains an A-polar cylinder, i.e., there is an open affine subset $U \subset S$ and an effective $\mathbb{R}$-divisor D such that $D \equiv A$, $U = S \setminus \mathrm{Supp}(D)$, and $U \cong \mathbb{A}^1 \times Z$ for some smooth rational affine curve Z. Put $D = \sum_{i=1}^n a_i C_i$, where each C_i is an irreducible reduced curve and each a_i is a positive real number. Let μ be the Fujita invariant of A.

As in (1.1.1), the natural projection $p_Z : U \cong \mathbb{A}^1 \times Z \to Z$ induces a rational map $\psi\colon S \dashrightarrow \mathbb{P}^1$ given by a pencil $\mathcal{L}$ on the surface S. If the base locus $\mathrm{Bs}(\mathcal{L})$ of the pencil is non-empty, then it must consist of a single point because $\psi^*(Q) \cong \mathbb{P}^1$ for a general point Q of $\mathbb{P}^1$ and $\mathrm{Supp}(\psi^*(Q)) \setminus \mathrm{Supp}(\mathrm{Bs}(\mathcal{L}))$ contains an affine line.

Theorem 5.1.1. *Suppose that the base locus of $\mathcal{L}$ consists of a point P on S. Then, for every effective $\mathbb{R}$-divisor B on S such that $\mathrm{Supp}(B) \subset \mathrm{Supp}(D)$ and $K_S + B$ is pseudo-effective, the log pair (S, B) is not log canonical at P. In particular, $(S, \mu D)$ is not log canonical at P.*

Proof. The proof is the same as the explanation for Remark 1.1.3. We may assume that the exceptional divisor of π lies over the point P. Since $\mathrm{Supp}(B) \subset \mathrm{Supp}(D)$, the divisor B can be written as $B = \sum_{i=1}^n b_i C_i$, where b_i's are non-negative real numbers. The remaining parts are exactly the same as in the explanation for Remark 1.1.3. □

Theorem 5.1.2. *Let S_d be a smooth del Pezzo surface of degree $d \leqslant 3$ and let D be an effective $\mathbb{R}$-divisor on S_d such that $D \equiv -K_{S_d}$. If the log pair (S_d, D) is not log canonical at a point P, then there exists a divisor T in the anticanonical linear system $|-K_{S_d}|$ such that the log pair (S_d, T) is not log canonical at the point P and $\mathrm{Supp}(T) \subset \mathrm{Supp}(D)$.*

Proof. See [5, Theorem 1.12] for an effective $\mathbb{Q}$-divisor. The proof works verbatim for an effective $\mathbb{R}$-divisor. □

5.2. **Proof of Theorem 2.2.3.** Before we prove Theorem 2.2.3, we introduce two easy results that we use for the proof.

Lemma 5.2.1. *Let S be a smooth surface and let D be an effective $\mathbb{R}$-divisor on S. If the log pair (S,D) is not log canonical at a point P, then* $\mathrm{mult}_P(D)>1$.

Proof. For instance, see [25, Proposition 9.5.13]. ☐

Lemma 5.2.2. *Let S be a smooth del Pezzo surface of degree $d\leqslant 3$ and let*

$$D=\sum_{i=1}^{n}a_iD_i$$

be an effective $\mathbb{R}$-divisor on S such that $D\equiv -K_S$, where $D_1,\ldots,D_n$ are irreducible curves and $a_1,\ldots,a_n$ are positive real numbers. Then $a_i\leqslant 1$ for each $i=1,\ldots,n$.

Proof. For the case where $d=1$, the statement follows from

$$1=K_S^2=\sum_{i=1}^{n}a_iD_i\cdot(-K_S)\geqslant a_iD_i\cdot(-K_S)\geqslant a_i.$$

For the cases where $d=2$ and 3, see [5, Lemmas 3.1 and 4.1], respectively. Their proofs work verbatim for an effective $\mathbb{R}$-divisor. ☐

Theorem 2.2.3 immediately follows from the following two statements.

Theorem 5.2.3. *Let S be a smooth del Pezzo surface of degree $d\leqslant 2$. Let E be a (-1)-curve on S. For a positive real number a the surface S does not contain any $(-K_S+aE)$-polar cylinder.*

Proof. Suppose that there exists an effective $\mathbb{R}$-divisor D such that $D\equiv -K_S+aE$ and $S\setminus \mathrm{Supp}(D)$ is isomorphic to $\mathbb{A}^1\times Z$ for some affine variety Z.

Let $f\colon S\to\overline{S}$ be the contraction of the curve E. Put $\overline{D}=f(D)$. Then $\overline{S}$ is a smooth del Pezzo surface of degree $d+1\leqslant 3$. Moreover, we have $\overline{D}\equiv -K_{\overline{S}}$.

If $E\subset\mathrm{Supp}(D)$, then

$$\overline{S}\setminus\mathrm{Supp}(\overline{D})\cong S\setminus\mathrm{Supp}(D)\cong\mathbb{A}^1\times Z,$$

which implies that $\overline{S}\setminus\mathrm{Supp}(\overline{D})$ is a $(-K_{\overline{S}})$-polar cylinder on $\overline{S}$. This contradicts Theorem 1.2.3. Therefore, $E\not\subset\mathrm{Supp}(D)$. In particular, $D\cdot E\geqslant 0$ and $a\leqslant 1$.

Put $D=\sum_{i=1}^{n}a_iD_i$, where $D_1,\ldots,D_n$ are irreducible curves and $a_1,\ldots,a_n$ are positive real numbers. None of the curves $D_1,\ldots,D_n$ are contracted by the morphism f and

$$\sum_{i=1}^{n}a_if(D_i)=\overline{D}\equiv -K_{\overline{S}}.$$

Therefore, we have $a_i\leqslant 1$ for each $i=1,\ldots,n$ by Lemma 5.2.2. Since $a\leqslant 1$ too, by the second case in Remark 1.1.3 the linear system $\mathcal{L}$ associated with the cylinder $S\setminus\mathrm{Supp}(D)$ has a base point, say P. Due to Theorem 5.1.1 for every effective $\mathbb{R}$-divisor B on S such that K_S+B is pseudo-effective and $\mathrm{Supp}(B)\subset\mathrm{Supp}(D)$, the log pair (S,B) is not log canonical at P. In particular, (S,D) is not log canonical at the point P.

The inequality

$$1>1-a=(-K_S+aE)\cdot E=D\cdot E\geqslant \mathrm{mult}_P(D)\mathrm{mult}_P(E)$$

and Lemma 5.2.1 show that P lies outside E. Therefore, $(\overline{S},\overline{D})$ is not log canonical at $f(P)$.

Let $\overline{T}$ be the unique divisor in $|-K_{\overline{S}}|$ that is singular at $f(P)$. Denote by T its proper transform on the surface S. Since $\overline{D}\equiv -K_{\overline{S}}$ and $(\overline{S},\overline{D})$ is not log canonical at the point $f(P)$, it follows from Theorem 5.1.2 that $(\overline{S},\overline{T})$ is not log canonical at $f(P)$ and $\mathrm{Supp}(\overline{T})\subset\mathrm{Supp}(\overline{D})$. Hence, $\mathrm{Supp}(T)\subset\mathrm{Supp}(D)$.

For every non-negative real number μ, put $D_\mu = (1+\mu)D - \mu T$ and $\overline{D}_\mu = (1+\mu)\overline{D} - \mu\overline{T}$. Since $-K_{\overline{S}} \cdot \overline{T} = K_{\overline{S}}^2 \leqslant 3$, the divisor T consists of at most 3 irreducible components. Therefore, $D \neq T$ because the divisor D has at least 8 components by (1.1.2). Put

$$\nu = \sup\Big\{\mu \in \mathbb{R}_{\geqslant 0} \ \Big|\ D_\mu \text{ is effective}\Big\}.$$

Then $\mathrm{Supp}(T) \not\subset \mathrm{Supp}(D_\nu)$ and $\mathrm{Supp}(\overline{T}) \not\subset \mathrm{Supp}(\overline{D}_\nu)$. In particular, we have $\nu > 0$ since $\mathrm{Supp}(T) \subset \mathrm{Supp}(D)$.

We have $\overline{D}_\mu \equiv \overline{D} \equiv \overline{T} \equiv -K_{\overline{S}}$ for each real number μ. This implies that

$$D_\mu \equiv -K_S + a_\mu E$$

for some real number a_μ. Note that a_μ is either linear or constant in μ.

Suppose that $a_\nu \geqslant 0$. Then $K_S + D_\nu$ is pseudo-effective. Therefore, the log pair (S, D_ν) is not log canonical at the point P by Theorem 5.1.1. Then $(\overline{S}, \overline{D}_\nu)$ is not log canonical at $f(P)$. The latter contradicts Theorem 5.1.2 because $\mathrm{Supp}(\overline{T}) \not\subset \mathrm{Supp}(\overline{D}_\nu)$ by the choice of ν.

Suppose that $a_\nu < 0$. Since $a_0 = a > 0$, there exists a positive real number $\lambda \in (0, \nu)$ such that $a_\lambda = 0$. It follows from $\lambda < \nu$ that $\mathrm{Supp}(T) \subset \mathrm{Supp}(D_\lambda)$ and $\mathrm{Supp}(D_\lambda) = \mathrm{Supp}(D)$. Therefore,

$$S \setminus \mathrm{Supp}(D_\lambda) \cong S \setminus \mathrm{Supp}(D) \cong \mathbb{A}^1 \times Z$$

is a cylinder. However, this contradicts Theorem 1.2.3 because $a_\lambda = 0$, i.e., $D_\lambda \equiv -K_S$. □

Theorem 5.2.4. *Let S be a smooth del Pezzo surface of degree 1. Let E and F be two disjoint (-1)-curves on S. The surface S contains no $(-K_S + aE + bF)$-polar cylinder for any positive real numbers a and b.*

Proof. Suppose that there exists an effective $\mathbb{R}$-divisor D such that $D \equiv -K_S + aE + bF$ and such that $S \setminus \mathrm{Supp}(D)$ is isomorphic to $\mathbb{A}^1 \times Z$ for some affine variety Z. In the following we seek for a contradiction.

Let $g\colon S \to \hat{S}$ be the contraction of the curve E. Put $\hat{D} = g(D)$ and $\hat{F} = g(F)$. Then $\hat{S}$ is a smooth del Pezzo surface of degree 2, $\hat{F}$ is a (-1)-curve, and $\hat{D} \equiv -K_{\hat{S}} + b\hat{F}$. This implies that $E \not\subset \mathrm{Supp}(D)$. Indeed, if $E \subset \mathrm{Supp}(D)$, then

$$\hat{S} \setminus \mathrm{Supp}(\hat{D}) \cong S \setminus \mathrm{Supp}(D) \cong \mathbb{A}^1 \times Z$$

is a $\hat{D}$-polar cylinder on $\hat{S}$. This is impossible by Theorem 5.2.3. Similarly, we see that $F \not\subset \mathrm{Supp}(D)$. Therefore, $D \cdot E \geqslant 0$, $D \cdot F \geqslant 0$ and $a \leqslant 1$, $b \leqslant 1$.

Write $D = \sum_{i=1}^n a_i D_i$, where $D_1, \ldots, D_n$ are irreducible curves and $a_1, \ldots, a_n$ are positive real numbers.

Let $f\colon S \to \overline{S}$ be the contraction of the curves E and F. Put $\overline{D} = f(D)$. Then $\overline{S}$ is a smooth cubic surface and $\overline{D} \equiv -K_{\overline{S}}$. None of the curves $D_1, \ldots, D_n$ are contracted by the morphism f and

$$\sum_{i=1}^n a_i f(D_i) = \overline{D} \equiv -K_{\overline{S}}.$$

Therefore, we have $a_i \leqslant 1$ for each $i = 1, \ldots, n$ by Lemma 5.2.2. Because $a, b \leqslant 1$, the second case in Remark 1.1.3 implies that the linear system $\mathcal{L}$ associated with the cylinder $S \setminus \mathrm{Supp}(D)$ has a base point, say P. By Theorem 5.1.1 for every effective $\mathbb{R}$-divisor B on S such that $K_S + B$ is pseudo-effective and $\mathrm{Supp}(B) \subset \mathrm{Supp}(D)$, the log pair (S, B) is not log canonical at P . In particular, (S, D) is not log canonical at the point P.

We claim that P belongs to neither E nor F. Indeed, if $P \in E$, then

$$1 > 1 - a = (-K_S + aE) \cdot E = D \cdot E \geqslant \mathrm{mult}_P(D) > 1$$

by Lemma 5.2.1. This shows that $P \notin E$. Similarly, we see that $P \notin F$. Therefore, the birational morphism f is an isomorphism in a neighborhood of the point P. In particular, the log pair $(\overline{S}, \overline{D})$ is not log canonical at $f(P)$.

Let $\overline{T}$ be the unique divisor in $|-K_{\overline{S}}|$ that is singular at $f(P)$. Denote by T its proper transform on the surface S. Since $\overline{D} \equiv -K_{\overline{S}}$ and $(\overline{S}, \overline{D})$ is not log canonical at the point $f(P)$, it follows from Theorem 5.1.2 that $(\overline{S}, \overline{T})$ is not log canonical at $f(P)$ and $\mathrm{Supp}(\overline{T}) \subset \mathrm{Supp}(\overline{D})$. Hence, $\mathrm{Supp}(T) \subset \mathrm{Supp}(D)$.

For every non-negative real number μ, put $D_\mu = (1+\mu)D - \mu T$ and $\overline{D}_\mu = (1+\mu)\overline{D} - \mu\overline{T}$. Since $-K_{\overline{S}} \cdot \overline{T} = K_{\overline{S}}^2 = 3$, the divisor T consists of at most 3 irreducible components. Therefore, $D \neq T$ because the divisor D has at least 9 components by (1.1.2). Put

$$\nu = \sup\Big\{\mu \in \mathbb{R}_{\geqslant 0} \ \Big|\ D_\mu \text{ is effective}\Big\}.$$

Then $\mathrm{Supp}(T) \not\subset \mathrm{Supp}(D_\nu)$ and $\mathrm{Supp}(\overline{T}) \not\subset \mathrm{Supp}(\overline{D}_\nu)$. In particular, we have $\nu > 0$ since $\mathrm{Supp}(T) \subset \mathrm{Supp}(D)$.

We have $\overline{D}_\mu \equiv \overline{D} \equiv \overline{T} \equiv -K_{\overline{S}}$ for each real number μ. This implies that

$$D_\mu \equiv -K_S + a_\mu E + b_\mu F$$

for some real numbers a_μ and b_μ. From $-K_S + E + F \equiv f^*(\overline{D}_\mu) = (1+\mu)f^*(\overline{D}) - \mu f^*(\overline{T})$ and $a_0 = a$, $b_0 = b$ we obtain

$$\begin{cases} a_\mu = \big(\mathrm{mult}_{f(E)}(\overline{T}) - \mathrm{mult}_{f(E)}(\overline{D})\big)\,\mu + a \\ b_\mu = \big(\mathrm{mult}_{f(F)}(\overline{T}) - \mathrm{mult}_{f(F)}(\overline{D})\big)\,\mu + b. \end{cases}$$

Suppose that $a_\nu \geqslant 0$ and $b_\nu \geqslant 0$. Then $K_S + D_\nu$ is pseudo-effective, and hence the log pair (S, D_ν) is not log canonical at the point P by Theorem 5.1.1. Then $(\overline{S}, \overline{D}_\nu)$ is not log canonical at $f(P)$. Since $\mathrm{Supp}(\overline{T}) \not\subset \mathrm{Supp}(\overline{D}_\nu)$, this contradicts Theorem 5.1.2.

Suppose that either $a_\nu < 0$ or $b_\nu < 0$. Since $a_0 = a > 0$ and $b_0 = b > 0$, there is a real number $\lambda \in (0, \nu)$ such that either $a_\lambda = 0$, $b_\lambda \geqslant 0$ or $a_\lambda \geqslant 0$, $b_\lambda = 0$. Without loss of generality we may assume that $a_\lambda = 0$. Since $\lambda < \nu$, $\mathrm{Supp}(T) \subset \mathrm{Supp}(D_\lambda) = \mathrm{Supp}(D)$. Therefore, $S \setminus \mathrm{Supp}(D_\lambda) = S \setminus \mathrm{Supp}(D)$ is a cylinder. However, this contradicts either Theorem 1.2.3 or Theorem 5.2.3 since

$$D_\lambda \equiv -K_S + b_\lambda F.$$

□

6. Cylinders in del Pezzo surfaces of small degrees

6.1. **Del Pezzo surface of degree 2 without a cuspidal anticanonical divisor.** A smooth quartic plane curve can have at most twenty four inflection points. There may be two kinds of inflection points on a smooth quartic curve. One is a point at which its tangent line intersects the quartic with multiplicity 3, and the other with multiplicity 4. The former is called an ordinary inflection point and the latter a hyperinflection point. These inflection points can be spotted with the Hessian curve of the given quartic curve. The Hessian curve intersects the quartic curve transversally at ordinary inflection points and meets the quartic curve at hyperinflection points with multiplicity 2. Since the degree of the Hessian curve is 6, we have

$$\text{the number of ordinary inflection points } + 2 \times \text{ the number of hyperinflection points } = 24.$$

Therefore, a smooth quartic plane curve has exactly twelve hyperinflection points if it contains no ordinary inflection point.

A smooth del Pezzo surface of degree 2 is a double cover of $\mathbb{P}^2$ ramified along a smooth plane quartic curve. An effective anticanonical divisor on a smooth del Pezzo surface of degree 2 is given by the pull-back of a line on $\mathbb{P}^2$ via the double covering map. An effective anticanonical divisor that is a cuspidal rational curve is given exactly by the pull-back of the tangent line at an ordinary inflection point. The pull-back of the tangent line at a hyperinflection point is an effective anticanonical divisor that is a tacnodal curve, i.e., two (-1)-curves intersecting at a single point tangentially. Consequently, a smooth del Pezzo surface of degree 2 contains twelve effective anticanonical divisors that are tacnodal curves if its anticanonical linear system contains no cuspidal rational curve. Each of the twelve tacnodal curves consists of two distinct (-1)-curves intersecting at a single point tangentially. These twenty four (-1)-curves are distinct.

In fact, there are exactly two quartic plane curves without any ordinary inflection point ([8], [24]). One is the Fermat quartic, i.e., the curve defined by

$$x^4 + y^4 + z^4 = 0,$$

and the other is the curve defined by

$$x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + z^2x^2) = 0.$$

As explained above, these have exactly twelve hyperinflection points. The del Pezzo surfaces of degree 2 corresponding these two quartic curves are the only del Pezzo surfaces of degree 2 whose anticanonical linear systems contain no cuspidal rational curves.

6.2. **Cylinders in del Pezzo surfaces of degree 2.** In order to prove Theorem 2.2.4, let S be a smooth del Pezzo surface of degree 2 and let H be an ample $\mathbb{R}$-divisor on S. Let μ and r be the Fujita invariant and the Fujita rank of H. Denote by Δ the Fujita face of H. Let $\phi : S \to Z$ be the contraction given by Δ.

We first consider ample $\mathbb{R}$-divisors of type $B(r)$. Let $E_1, \ldots, E_r$ be the r disjoint (-1)-curves that generate the face Δ. We may then write

$$K_S + \mu H \equiv \sum_{i=1}^{r} a_i E_i \tag{6.2.1}$$

for some positive real numbers $a_1, \ldots, a_r$ (see (2.1.3)).

Theorem 6.2.2. *If the ample $\mathbb{R}$-divisor H is of type $B(r)$ with $3 \leqslant r \leqslant 7$, then S contains an H-polar cylinder.*

Proof. The proof is divided into two cases. One is the case when S has a cuspidal rational curve in $|-K_S|$, and the other is the case when it does not.

Case 1. The surface S has no cuspidal rational curve in $|-K_S|$.

In this case, as mentioned in the previous subsection, S has exactly twelve pairs of (-1)-curves $\{C_i, C_i'\}$, $i = 1, \ldots, 12$ such that each $C_i + C_i'$ is a tacnodal anticanonical divisor.

Choose one tacnodal anticanonical divisor, say $C_1 + C_1'$. Since we have more than seven tacnodal anticanonical divisors, we may assume that

- none of E_i are C_1 or C_1';
- if $r = 6$, then neither $\phi(C_1)$ nor $\phi(C_1')$ is a (-1)-curve.

Let m be the number of the curves E_i's intersecting C_1 and m' be the number of the curves E_i's intersecting C_1'. We may assume that $E_1, \ldots, E_m$ intersect C_1 and that $E_{m+1}, \ldots, E_r$ meet C_1'. Furthermore, we may assume that $m \geqslant m'$. Note that $m + m' = r$. Furthermore, by the assumption above, $2 \leqslant m \leqslant 5$. Let $\overline{C}_1$ and $\overline{C}_1'$ be the images C_1 and C_1' by ϕ, respectively. The curve $\overline{C}_1$ is an $(m-1)$-curve on the del Pezzo surface Z. Since $m \geqslant 2$, the complete linear system $|\overline{C}_1|$ induces a birational morphism ψ of Z into $\mathbb{P}^m$. Furthermore, its image is isomorphic to

$$\begin{cases} \mathbb{P}^2 \text{ for } m=2; \\ \mathbb{P}^1\times\mathbb{P}^1 \cong \text{a smooth quadratic surface} \subset \mathbb{P}^3 \text{ for } m=3; \\ \mathbb{F}_1 \cong \text{ a smooth rational normal scroll of degree 3 in } \mathbb{P}^4 \text{ for } m=4; \\ \mathbb{P}^2 \cong \text{ a Veronese surface in } \mathbb{P}^5 \text{ for } m=5. \end{cases}$$

Put $\sigma=\psi\circ\phi$.

For $m=2$ and 5, we have $(7-r)$ disjoint (-1)-curves on Z that are contracted by ψ. These curves do not intersect $\overline{C}_1$ but they meet $\overline{C}_1'$ since $\overline{C}_1+\overline{C}_1'$ is an anticanonical divisor of Z. Let $F_1,\ldots,F_{7-r}$ be the pull-backs of these $(7-r)$ disjoint (-1)-curves by ϕ. Then the curve C_1 intersects exactly m curves $E_1,\ldots E_m$ and the curve C_1' meets the other m' curves $E_{m+1},\ldots E_r$ and all the $(7-r)$ curves $F_1,\ldots,F_{7-r}$.

For $m=2$, $\sigma(C_1)$ is a line and $\sigma(C_1')$ is a conic in $\mathbb{P}^2$. For $m=5$, $\sigma(C_1)$ is a conic and $\sigma(C_1')$ is a line in $\mathbb{P}^2$. They intersect tangentially at a single point. Therefore,

$$-K_S\equiv(1-a\varepsilon)C_1+(1+b\varepsilon)C_1'-a\varepsilon\sum_{i=1}^{m}E_i+b\varepsilon\sum_{i=m+1}^{r}E_i+b\varepsilon\sum_{i=1}^{7-r}F_i,$$

and hence

$$\mu H\equiv(1-a\varepsilon)C_1+(1+b\varepsilon)C_1'+\sum_{i=1}^{m}(a_i-a\varepsilon)E_i+\sum_{i=m+1}^{r}(a_i+b\varepsilon)E_i+b\varepsilon\sum_{i=1}^{7-r}F_i,$$

where $a=2$, $b=1$ if $m=2$ and $a=1$, $b=2$ if $m=5$. For a sufficiently small positive real number ε, H yields a cylinder because

$$S\setminus(C_1\cup C_1'\cup E_1\cup\ldots\cup E_r\cup F_1\cup\ldots\cup F_{7-r})\cong\mathbb{P}^2\setminus(\sigma(C_1)\cup\sigma(C_1')).$$

For $m=3$ and 4, we have $(6-r)$ disjoint (-1)-curves on Z that are contracted by ψ. These curves do not intersect $\overline{C}_1$ but they meet $\overline{C}_1'$ since $\overline{C}_1+\overline{C}_1'$ is an anticanonical divisor of Z. Again we let $F_1,\ldots,F_{6-r}$ be the pull-backs of these $(6-r)$ disjoint (-1)-curves by ϕ. Then the curve C_1 intersects exactly m curves $E_1,\ldots E_m$ and the curve C_1' meets the other m' curves $E_{m+1},\ldots E_r$ and all the $(6-r)$ curves $F_1,\ldots,F_{6-r}$.

For $m=3$, $\sigma(C_1)$ is an irreducible curve of bidegree $(1,1)$ in $\mathbb{P}^1\times\mathbb{P}^1$ since its self-intersection number is 2. The irreducible curve $\sigma(C_1')$ is also of bidegree $(1,1)$ because $\sigma(C_1)+\sigma(C_1')$ is an anticanonical divisor of $\mathbb{P}^1\times\mathbb{P}^1$. They intersect tangentially at a single point Q. Let L_1 and L_2 be the curves of bidegrees $(1,0)$ and $(0,1)$, respectively, passing through Q. Then,

$$-K_S\equiv(1-2\varepsilon)C_1+(1+\varepsilon)C_1'+\varepsilon(\tilde{L}_1+\tilde{L}_2)-2\varepsilon\sum_{i=1}^{3}E_i+\varepsilon\sum_{i=4}^{r}E_i+\varepsilon\sum_{i=1}^{6-r}F_i,$$

where $\tilde{L}_1$ and $\tilde{L}_2$ are the proper transforms of L_1 and L_2 by σ, respectively. Therefore

$$\mu H\equiv(1-2\varepsilon)C_1+(1+\varepsilon)C_1'+\varepsilon(\tilde{L}_1+\tilde{L}_2)+\sum_{i=1}^{3}(a_i-2\varepsilon)E_i+\sum_{i=4}^{r}(a_i+\varepsilon)E_i+\varepsilon\sum_{i=1}^{6-r}F_i.$$

For a sufficiently small positive real number ε, we obtain an H-polar cylinder because

$$S\setminus(C_1\cup C_1'\cup\tilde{L}_1\cup\tilde{L}_2\cup E_1\cup\ldots\cup E_r\cup F_1\cup\ldots\cup F_{6-r})\cong\mathbb{P}^1\times\mathbb{P}^1\setminus(\sigma(C_1)\cup\sigma(C_1')\cup L_1\cup L_2)$$

(see Example 4.1.7).

For $m=4$, $\sigma(C_1)$ is a 3-curve in $\mathbb{F}_1$. The irreducible curve $\sigma(C_1')$ is a 1-curve intersecting $\sigma(C_1)$ at a single point Q tangentially. Let M be the 0-curve passing through the point Q.

Example 4.1.9 shows that

$$-K_S \equiv (1-\varepsilon)C_1 + (1+\varepsilon)C_1' + \varepsilon\tilde{M} - \varepsilon\sum_{i=1}^{4} E_i + \varepsilon\sum_{i=5}^{r} E_i + \varepsilon\sum_{i=1}^{6-r} F_i$$

and

$$\mu H \equiv (1-\varepsilon)C_1 + (1+\varepsilon)C_1' + \varepsilon\tilde{M} + \sum_{i=1}^{4}(a_i-\varepsilon)E_i + \sum_{i=5}^{r}(a_i+\varepsilon)E_i + \varepsilon\sum_{i=1}^{6-r} F_i,$$

where $\tilde{M}$ is the proper transform of M. We see also from Example 4.1.9 that H defines a cylinder with a sufficiently small positive real number ε.

Case 2. The surface S possesses a cuspidal rational curve C in $|-K_S|$.

Let P be the point at which the curve C has the cusp. Each E_i intersects the curve C at a single smooth point. This cuspidal curve C plays a key role in constructing H-polar cylinders case by case, according to r.

Subcase 1. $r = 3$.

In this subcase, the surface S has five 0-curves $F_1, \ldots, F_5$ such that

- they pass through P;
- they do not meet each other outside P;
- they are disjoint from the curves E_1, E_2 and E_3.

(For the better understanding of this construction, see Example 4.1.13 after contracting the (-1)-curves E_1, E_2, E_3 to a smooth del Pezzo surface of degree 5.)

Let $\pi : \tilde{S} \to S$ be the blow up at the point P and let E be the exceptional curve of π. Then $\tilde{S}$ is a weak del Pezzo surface of degree 1 and it has exactly one (-2)-curve, the proper transform $\tilde{C}$ of C. Denote the proper transforms on $\tilde{S}$ of the curves E_1, E_2, E_3, $F_1, \ldots, F_5$ by $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$, $\tilde{F}_1, \ldots, \tilde{F}_5$. Since these (-1)-curves are disjoint, they give us a contraction $\psi : \tilde{S} \to \mathbb{P}^2$.

Since $\psi(E)$ is a conic and $\psi(\tilde{C})$ is a line on $\mathbb{P}^2$, we immediately see that

$$-K_{\tilde{S}} \equiv (1-2\varepsilon)\tilde{C} + (1+\varepsilon)E + \varepsilon\sum_{i=1}^{5}\tilde{F}_i - 2\varepsilon(\tilde{E}_1+\tilde{E}_2+\tilde{E}_3),$$

and hence

$$-K_S \equiv (1-2\varepsilon)C + \varepsilon\sum_{i=1}^{5} F_i - 2\varepsilon(E_1+E_2+E_3).$$

Therefore,

$$\mu H \equiv (1-2\varepsilon)C + \varepsilon\sum_{i=1}^{5} F_i + (a_1-2\varepsilon)E_1 + (a_2-2\varepsilon)E_2 + (a_3-2\varepsilon)E_3.$$

For a sufficiently small positive real number ε, this is an H-polar cylinder because

$$S \setminus (C \cup F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 \cup E_1 \cup E_2 \cup E_3) \cong \mathbb{P}^2 \setminus (\psi(E) \cup \psi(\tilde{C})).$$

Note that the conic $\psi(E)$ and the line $\psi(\tilde{C})$ meet tangentially.

Subcase 2. $r = 4$.

In this subcase, the surface S has three 0-curves F_1, F_2, F_3 such that

- they pass through P;
- they do not intersect each other outside P;
- they are disjoint from the curves E_1, E_2, E_3, E_4.

In addition, it has two 1-curves G_1, G_2 such that

- they intersect C only at P;
- they do not meet each other outside P;
- they are disjoint from the curves E_1, E_2, E_3, E_4.

(See Example 4.1.11 after contracting the (-1)-curves E_1, E_2, E_3, E_4 to a smooth del Pezzo surface of degree 6.)

Let $\pi : \tilde{S} \to S$ be the blow up at the point P and let E be the exceptional curve of π. Then $\tilde{S}$ is a weak del Pezzo surface of degree 1 and it has exactly one (-2)-curve, the proper transform $\tilde{C}$ of C. Denote the proper transforms on $\tilde{S}$ of the curves $E_1, \ldots, E_4$, F_1, F_2, F_3, G_1, G_2 by $\tilde{E}_1, \ldots, \tilde{E}_4$, $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_3$, $\tilde{G}_1$, $\tilde{G}_2$. Contracting the seven (-1)-curves $\tilde{E}_1, \ldots, \tilde{E}_4$, $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_3$, we obtain a birational morphism $\psi : \tilde{S} \to \mathbb{P}^1 \times \mathbb{P}^1$. Since $\psi(E)$ and $\psi(\tilde{C})$ are curves of bidegree $(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\psi(\tilde{G}_1)$, $\psi(\tilde{G}_2)$ are curves of bidegrees $(1, 0)$ and $(0, 1)$, respectively, on $\mathbb{P}^1 \times \mathbb{P}^1$,

$$-K_{\tilde{S}} \equiv (1-2\varepsilon)\tilde{C} + (1+\varepsilon)E + \varepsilon(\tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3) + \varepsilon(\tilde{G}_1 + \tilde{G}_2) - 2\varepsilon \sum_{i=1}^{4} \tilde{E}_i,$$

and hence

$$-K_S \equiv (1-2\varepsilon)C + \varepsilon(F_1 + F_2 + F_3) + \varepsilon(G_1 + G_2) - 2\varepsilon \sum_{i=1}^{4} E_i.$$

Therefore,

$$\mu H \equiv (1-2\varepsilon)C + \varepsilon(F_1 + F_2 + F_3) + \varepsilon(G_1 + G_2) + \sum_{i=1}^{4} (a_i - 2\varepsilon)E_i.$$

For a sufficiently small positive real number ε, this defines an H-polar cylinder because

$$S \setminus (C \cup F_1 \cup F_2 \cup F_3 \cup G_1 \cup G_2 \cup E_1 \cup E_2 \cup E_3 \cup E_4) \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus (\psi(E) \cup \psi(\tilde{C}) \cup \psi(\tilde{G}_1) \cup \psi(\tilde{G}_2)).$$

Note that the curves $\psi(E)$ and $\psi(\tilde{C})$ meet tangentially at one point and the curves $\psi(\tilde{G}_1)$, $\psi(\tilde{G}_2)$ pass through this point (see Example 4.1.7).

Subcase 3. $r = 5$.

There are two 0-curves L_1, L_2 such that

- they pass through P;
- they do not intersect each other outside P;
- they are disjoint from the curves $E_1, \ldots, E_5$.

In addition, there is a unique 1-curve T that meets C only at the point P and that does not intersect any of E_i's (see Example 4.1.10).

By contracting the (-1)-curves $E_1, \ldots, E_5$, we immediately see from Example 4.1.10 that

$$-K_S \equiv (1-\varepsilon)C + \varepsilon L_1 + \varepsilon L_2 + \varepsilon T - \varepsilon \sum_{i=1}^{5} E_i,$$

and hence

$$\mu H \equiv (1-\varepsilon)C + \varepsilon L_1 + \varepsilon L_2 + \varepsilon T + \sum_{i=1}^{5} (a_i - \varepsilon)E_i.$$

Example 4.1.10 shows that for a sufficiently small positive real number ε, this defines an H-polar cylinder.

Subcase 4. $r = 6$ and $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$.

There are exactly two 0-curves F_1, F_2 passing through the point P and not intersecting any of E_i's. (By contracting $E_1, \ldots, E_6$ into the surface $\mathbb{P}^1 \times \mathbb{P}^1$, we can easily detect such 0-curves.)

Let $\pi : \tilde{S} \to S$ be the blow up at the point P and let E be the exceptional curve of π. Then $\tilde{S}$ is a weak del Pezzo surface of degree 1 and it has exactly one (-2)-curve, the proper transform $\tilde{C}$ of C. Denote the proper transforms on $\tilde{S}$ of the curves $E_1, \ldots, E_6$, F_1, F_2 by $\tilde{E}_1, \ldots, \tilde{E}_6$, $\tilde{F}_1$, $\tilde{F}_2$. Contracting the (-1)-curves $\tilde{E}_1, \ldots, \tilde{E}_6$, $\tilde{F}_1$, $\tilde{F}_2$, we obtain a birational morphism $\psi : \tilde{S} \to \mathbb{P}^2$. Note that $\psi(C)$ and $\psi(E)$ are a conic and a line meeting tangentially on $\mathbb{P}^2$. Therefore,

$$-K_S \equiv (1-\varepsilon)C + 2\varepsilon(F_1+F_2) - \varepsilon\sum_{i=1}^{6} E_i,$$

and hence

$$\mu H \equiv (1-\varepsilon)C + 2\varepsilon(F_1+F_2) + \sum_{i=1}^{6}(a_i-\varepsilon)E_i.$$

For a sufficiently small positive real number ε, this defines an H-polar cylinder since

$$S \setminus (C \cup F_1 \cup F_2 \cup E_1 \cup \ldots \cup E_6) \cong \mathbb{P}^2 \setminus (\psi(\tilde{C}) \cup \psi(E)).$$

Subcase 5. $r = 6$ and $Z \cong \mathbb{F}_1$.

There is a unique 0-curve L passing through the point P and not meeting any of E_i's. In addition, there is a unique 1-curve T that intersects C only at the point P and that does not intersect any of E_i's. (By contracting $E_1, \ldots, E_6$ into the Hirzebruch surface $\mathbb{F}_1$, we can easily detect such curves.)

Example 4.1.8 shows that

$$-K_S \equiv (1-\varepsilon)C + \varepsilon L + 2\varepsilon T - \varepsilon\sum_{i=1}^{6} E_i,$$

and hence

$$\mu H \equiv (1-\varepsilon)C + \varepsilon L + 2\varepsilon T + \sum_{i=1}^{6}(a_i-\varepsilon)E_i.$$

It also shows that for a sufficiently small positive real number ε, this defines an H-polar cylinder since

$$S \setminus (C \cup L \cup T \cup E_1 \cup \ldots \cup E_6) \cong \mathbb{F}_1 \setminus (\phi(C) \cup \phi(L) \cup \phi(T)).$$

Subcase 6. $r = 7$.

By contracting $E_1, \ldots, E_7$ we obtain a birational morphism π of S onto the projective plane $\mathbb{P}^2$. The curve $\pi(C)$ is a cuspidal cubic curve passing through all the points $\pi(E_i)$'s. Let T be the Zariski tangent line to the curve $\pi(C)$ at its cuspidal point. We immediately see that

$$-K_S \equiv (1-\varepsilon)\pi^*(\pi(C)) + 3\varepsilon\pi^*(T) - \sum_{i=1}^{7} E_i \equiv (1-\varepsilon)C + 3\varepsilon\tilde{T} - \varepsilon\sum_{i=1}^{7} E_i,$$

where $\tilde{T}$ is the proper transform of T by π, and hence

$$\mu H \equiv (1-\varepsilon)C + 3\varepsilon\tilde{T} + \sum_{i=1}^{7}(a_i-\varepsilon)E_i.$$

For a sufficiently small positive real number ε, this defines an H-polar cylinder since

$$S \setminus (C \cup \tilde{T} \cup E_1 \cup \ldots \cup E_7) \cong \mathbb{P}^2 \setminus (\pi(C) \cup T)$$

(see Example 4.1.3). □

For the following theorem we assume that $a_1 \geqslant \ldots \geqslant a_r$ in (6.2.1).

Theorem 6.2.3. *Suppose that H is of type $B(2)$. If one of the following conditions holds*

- $2a_2 > 1$;
- $2a_1 + a_2 > 2$,

then S contains an H-polar cylinder.

Proof. There are five (-1)-curves $E_3, \ldots, E_7$ on S such that they, together with E_1 and E_2, define a birational morphism $\sigma : S \to \mathbb{P}^2$. Denote the point $\sigma(E_i)$ by P_i for $i = 1, \ldots, 7$. Let C_1 be the conic that passes through the points $P_3, \ldots, P_7$.

Suppose that the inequality $2a_2 > 1$ is satisfied.

There is a conic C_2 passing through the points P_1, P_2 and meeting the conic C_1 only at a single point. Let T be the tangent line to both the conics C_1 and C_2 at the intersection point of C_1 and C_2. For any real number ε we have $-K_{\mathbb{P}^2} \equiv (1+\varepsilon)\, C_1 + \left(\frac{1}{2} - 2\varepsilon\right) C_2 + 2\varepsilon T$. Hence,

$$\begin{aligned} -K_S &\sim_{\mathbb{Q}} \sigma^*\left(-K_{\mathbb{P}^2}\right) - \sum_{i=1}^{7} E_i \\ &\equiv (1+\varepsilon)\, \tilde{C}_1 + \left(\frac{1}{2} - 2\varepsilon\right) \tilde{C}_2 + 2\varepsilon \tilde{T} - \left(\frac{1}{2} + 2\varepsilon\right)(E_1 + E_2) + \varepsilon \sum_{i=3}^{7} E_i, \end{aligned}$$

where $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{T}$ are the proper transforms of C_1, C_2, and T, respectively. Thus, we have

$$\begin{aligned} H \equiv \frac{1}{\mu} \Bigg\{ &(1+\varepsilon)\, \tilde{C}_1 + \left(\frac{1}{2} - 2\varepsilon\right) \tilde{C}_2 + 2\varepsilon \tilde{T} + \\ &+ \left(a_1 - \frac{1}{2} - 2\varepsilon\right) E_1 + \left(a_2 - \frac{1}{2} - 2\varepsilon\right) E_2 + \varepsilon \sum_{i=3}^{7} E_i. \Bigg\} \end{aligned}$$

Since $a_1 - \frac{1}{2} \geqslant a_2 - \frac{1}{2} > 0$, for a sufficiently small positive real number ε this defines an H-polar cylinder on S.

Suppose that the inequality $2a_1 + a_2 > 2$ is satisfied.

Let L be a line passing through the point P_2 and tangent to the conic C_1. Let C_3 be the conic that intersects C_1 only at the point where C_1 and L meet and that passes through P_1. For any real numbers β and ε we have

$$-K_{\mathbb{P}^2} \equiv (1+2\varepsilon) C_1 + (\beta - \varepsilon) C_3 + (1 - 2\beta - 2\varepsilon) L.$$

Hence,

$$\begin{aligned} -K_S &\sim_{\mathbb{Q}} \sigma^*\left(-K_{\mathbb{P}^2}\right) - \sum_{i=1}^{7} E_i \\ &\equiv (1+2\varepsilon)\, \tilde{C}_1 + (\beta - \varepsilon)\, \tilde{C}_3 + (1 - 2\beta - 2\varepsilon)\, \tilde{L} + (\beta - \varepsilon - 1)\, E_1 - 2\,(\beta + \varepsilon)\, E_2 + 2\varepsilon \sum_{i=3}^{7} E_i, \end{aligned}$$

where $\tilde{C}_3$, $\tilde{L}$ are the proper transforms of C_3, L, respectively. Thus, we have

$$\begin{aligned} H \equiv \frac{1}{\mu} \Bigg\{ &(1+2\varepsilon)\, \tilde{C}_1 + (\beta - \varepsilon)\, \tilde{C}_3 + (1 - 2\beta - 2\varepsilon)\, \tilde{L} + \\ &+ (a_1 + \beta - \varepsilon - 1)\, E_1 + (a_2 - 2\beta - 2\varepsilon)\, E_2 + 2\varepsilon \sum_{i=3}^{7} E_i \Bigg\}. \end{aligned}$$

By putting $\beta = \frac{3}{2}\varepsilon + 1 - a_1$ with a sufficiently small positive real number ε, we are able to obtain an H-polar cylinder on S. □

From now on, we suppose that the ample $\mathbb{R}$-divisor H is of type $C(7)$ with length ℓ. Then the morphism $\phi : S \to Z$ is a conic bundle, i.e., $Z = \mathbb{P}^1$. We may write

$$K_S + \mu H \equiv aB + \sum_{i=1}^{\ell} a_i E_i$$

where a and a_i are positive real numbers, B is an irreducible fiber of ϕ, and E_i's are disjoint (-1)-curves in fibers of ϕ. There exist $(6-\ell)$ disjoint (-1)-curves $\hat{E}_1, \ldots, \hat{E}_{6-\ell}$ such that they are in fibers of ϕ and they generate the face Δ together with B and E_i's. Let $\phi_1 : S \to W$ be the birational morphism obtained by contracting the disjoint (-1)-curves $E_1, \ldots, E_\ell$, $\hat{E}_1, \ldots, \hat{E}_{6-\ell}$. Let $\hat{E}'_j$ be the (-1)-curve such that $\hat{E}_j + \hat{E}'_j$ is a fiber of ϕ.

Lemma 6.2.4. *Suppose that the surface S has no cuspidal rational curve in $|-K_S|$. If the ample $\mathbb{R}$-divisor H is of type $C(7)$ with length $3 \leqslant \ell \leqslant 6$, then S contains an H-polar cylinder.*

Proof. The surface S has exactly twelve pairs of (-1)-curves $\{C_i, C'_i\}$, $i = 1, \ldots, 12$ such that each $C_i + C'_i$ is a tacnodal anticanonical divisor. Choose one tacnodal anticanonical divisor, say $C_1 + C'_1$. Since we have more than ten tacnodal anticanonical divisors, we may assume that

- none of $E_1, \ldots, E_\ell$ are C_1 or C'_1;
- none of $\hat{E}_1, \hat{E}'_1 \ldots, \hat{E}_{6-\ell}, \hat{E}'_{6-\ell}$ are C_1 or C'_1;
- neither $\phi_1(C_1)$ nor $\phi_1(C'_1)$ is a (-1)-curve on W.

Each of E_i and $\hat{E}_j$ intersects exclusively either C_1 or C'_1 once. Note that if $\hat{E}_i$ intersects C_1, then $\hat{E}'_i$ meets C'_1 and if $\hat{E}_i$ intersects C'_1, then $\hat{E}'_i$ meets C_1. Let m_1 (resp. m'_1) be the number of E_i with $E_i \cdot C_1 = 1$ (resp. $E_i \cdot C'_1 = 1$) and let m_0 (resp. m'_0) be the number of $\hat{E}_j$ with $\hat{E}_j \cdot C_1 = 1$ (resp. $\hat{E}_j \cdot C'_1 = 1$). Put $m = m_0 + m_1$ and $m' = m'_0 + m'_1$. We may assume that

- $m \geqslant m'$;
- if $m = m'$, then $m_1 \geqslant m'_1$;
- $E_1, \ldots, E_{m_1}$ intersect C_1 and $E_{m_1+1}, \ldots, E_\ell$ intersect C'_1;
- $\hat{E}_1, \ldots, \hat{E}_{m_0}$ intersect C_1 and $\hat{E}_{m_0+1}, \ldots, \hat{E}_{6-\ell}$ intersect C'_1.

Since $m+m' = 6$ and neither $\phi_1(C_1)$ nor $\phi_1(C'_1)$ is a (-1)-curve on W, we have three possibilities, $(5,1)$, $(4,2)$, $(3,3)$ for (m,m').

Suppose that $(m,m') = (5,1)$ and $W \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $\phi_1(C_1)$ is a 4-curve. We may assume that this is an irreducible curve of bidegree $(1,2)$. Then $\phi_1(C'_1)$ is a curve of bidegree $(1,0)$. The curves $\phi_1(C_1)$ and $\phi_1(C'_1)$ meets tangentially at a single point Q. Let L be the curve of bidegree $(0,1)$ passing through Q. Note that $\phi_1(B)$ is a curve of bidegree $(0,1)$ since $B \cdot C_1 = B \cdot C'_1 = 1$. We immediately see from Example 4.1.6 that for an arbitrary real number ε

$$-K_S+aB \equiv (1-\varepsilon)C_1+(1+\varepsilon)C'_1+(a+2\varepsilon)\tilde{L}-\sum_{i=1}^{m_1} \varepsilon E_i-\sum_{i=1}^{m_0} \varepsilon \hat{E}_i+\sum_{i=m_1+1}^{\ell} \varepsilon E_i+\sum_{i=m_0+1}^{6-\ell} \varepsilon \hat{E}_i, \tag{6.2.5}$$

where $\tilde{L}$ is the proper transform of L by ϕ_1. In a similar way, we obtain

$$-K_S+aB \equiv (1+\varepsilon)C_1+(1-\varepsilon)C'_1+(a-2\varepsilon)\tilde{L}+\sum_{i=1}^{m_1} \varepsilon E_i+\sum_{i=1}^{m_0} \varepsilon \hat{E}_i-\sum_{i=m_1+1}^{\ell} \varepsilon E_i-\sum_{i=m_0+1}^{6-\ell} \varepsilon \hat{E}_i. \tag{6.2.6}$$

Example 4.1.6 also shows that the complements of the supports of the right hand sides of (6.2.5) and (6.2.6) are cylinders for a sufficiently small positive real number ε.

Suppose that $(m,m') = (4,2)$. Then $\phi_1(C_1)$ is a 3-curve, and hence $W \cong \mathbb{F}_1$. There is a unique 0-curve M on S such that $\phi_1(M)$ is the 0-curve passing through the intersection point of $\phi_1(C_1)$ and $\phi_1(C_1')$. From Example 4.1.9 we obtain

$$(6.2.7)\quad -K_S+aB \equiv (1-\varepsilon)C_1+(1+\varepsilon)C_1'+(a+\varepsilon)M-\sum_{i=1}^{m_1}\varepsilon E_i-\sum_{i=1}^{m_0}\varepsilon \hat{E}_i+\sum_{i=m_1+1}^{\ell}\varepsilon E_i+\sum_{i=m_0+1}^{6-\ell}\varepsilon \hat{E}_i$$

for an arbitrary real number ε. We can also obtain

$$(6.2.8)\quad -K_S+aB \equiv (1+\varepsilon)C_1+(1-\varepsilon)C_1'+(a-\varepsilon)M+\sum_{i=1}^{m_1}\varepsilon E_i+\sum_{i=1}^{m_0}\varepsilon \hat{E}_i-\sum_{i=m_1+1}^{\ell}\varepsilon E_i-\sum_{i=m_0+1}^{6-\ell}\varepsilon \hat{E}_i.$$

With a sufficiently small positive real number ε, these two divisors on the right hand sides define cylinders (see Example 4.1.9).

Suppose that $(m, m') = (3, 3)$. Then $\phi_1(C_1)$ and $\phi_1(C_1')$ are 2-curves, and hence $W \cong \mathbb{P}^1\times\mathbb{P}^1$. Moreover, $\phi_1(C_1)$ and $\phi_1(C_1')$ are irreducible curves of bidegree $(1, 1)$ intersecting tangentially at a single point Q. Let L_1 and L_2 be the curves of bidegrees $(1, 0)$ and $(0, 1)$, respectively, passing through Q. The curve $\phi_1(B)$ is a curve of bidegree $(1, 0)$ or $(0, 1)$. Without loss of generality, we may assume that $\phi_1(B)$ is a curve of bidegree $(1, 0)$. Then for an arbitrary real number ε
(6.2.9)

$$-K_S+aB \equiv (1-2\varepsilon)C_1+(1+\varepsilon)C_1'+(a+\varepsilon)\tilde{L}_1+\varepsilon\tilde{L}_2-\sum_{i=1}^{m_1}2\varepsilon E_i-\sum_{i=1}^{m_0}2\varepsilon \hat{E}_i+\sum_{i=m_1+1}^{\ell}\varepsilon E_i+\sum_{i=m_0+1}^{6-\ell}\varepsilon \hat{E}_i,$$

where $\tilde{L}_1$ and $\tilde{L}_2$ are the proper transforms of L_1 and L_2 by ϕ_1. In particular, the complement of the support of the divisor on the right hand side is a cylinder for a sufficiently small positive real number ε (see Example 4.1.7).

Now we construct H-polar cylinders case by case, according to the length ℓ.

Case 1. $\ell = 6$ and $W \cong \mathbb{P}^1 \times \mathbb{P}^1$.

If $m = 5$, then we use (6.2.5) to obtain

$$\mu H \equiv (1-\varepsilon)C_1 + (1+\varepsilon)C_1' + (a+2\varepsilon)\tilde{L} + (a_6+\varepsilon)E_6 + \sum_{i=1}^{5}(a_i-\varepsilon)E_i.$$

If $m = 3$, then we apply (6.2.9) to yield

$$\mu H \equiv (1-2\varepsilon)C_1 + (1+\varepsilon)C_1' + (a+\varepsilon)\tilde{L}_1 + \varepsilon\tilde{L}_2 + \sum_{i=1}^{3}(a_i-2\varepsilon)E_i + \sum_{i=4}^{6}(a_i+\varepsilon)E_i.$$

These show that S has an H-polar cylinder.

Case 2. $\ell = 6$ and $W \cong \mathbb{F}^1$.

Suppose that $m = 5$. Then $\phi_1(C_1)$ is a 4-curve and $\phi_1(C_1')$ is a 0-curve. They meet tangentially at a single point. There is a (-1)-curve E on S such that $\phi_1(E)$ is the negative section of $W \cong \mathbb{F}^1$. The curve $\phi_1(B)$ is equivalent to $\phi_1(C_1')$. Therefore,

$$-K_S + aB \equiv (1-\varepsilon)C_1 + (1+a+2\varepsilon)C_1' + 2\varepsilon E + (a+2\varepsilon)E_6 - \varepsilon\sum_{i=1}^{5}E_i,$$

and

$$\mu H \equiv (1-\varepsilon)C_1 + (1+a+2\varepsilon)C_1' + 2\varepsilon E + (a_6+a+2\varepsilon)E_6 + \sum_{i=1}^{5}(a_i-\varepsilon)E_i.$$

Suppose that $m = 4$. Then we apply (6.2.7) to obtain

$$\mu H \equiv (1-\varepsilon)C_1 + (1+\varepsilon)C_1' + (a+\varepsilon)M + \sum_{i=1}^{4}(a_i - \varepsilon)E_i + \sum_{i=4}^{6}(a_i + \varepsilon)E_i.$$

The divisors on the right hand sides produce H-polar cylinders on S.

From now, we consider the cases where $\ell < 6$. By contracting $\hat{E}_1'$ instead of $\hat{E}_1$, if necessary, we may always assume that $W \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Case 3. $\ell = 5$.

If $(m_1, m_0) = (5, 0)$, we apply (6.2.5) to yield

$$\mu H \equiv (1-\varepsilon)C_1 + (1+\varepsilon)C_1' + (a+2\varepsilon)\tilde{L} + \varepsilon\hat{E}_1 + \sum_{i=1}^{5}(a_i - \varepsilon)E_i.$$

If $(m_1, m_0) = (4, 1)$, then we use (6.2.6) to obtain

$$\mu H \equiv (1+\varepsilon)C_1 + (1-\varepsilon)C_1' + (a-2\varepsilon)\tilde{L} + \varepsilon\hat{E}_1 + (a_5 - \varepsilon)E_5 + \sum_{i=1}^{4}(a_i + \varepsilon)E_i.$$

If $(m_1, m_0) = (3, 0)$, then (6.2.9) shows

$$\mu H \equiv (1-2\varepsilon)C_1 + (1+\varepsilon)C_1' + (a+\varepsilon)\tilde{L}_1 + \varepsilon\tilde{L}_2 + \varepsilon\hat{E}_1 + \sum_{i=1}^{3}(a_i - 2\varepsilon)E_i + \sum_{i=4}^{5}(a_i + \varepsilon)E_i.$$

In each case, the divisor on the right hand side produces an H-polar cylinder on S with a sufficiently small positive real number ε.

Case 4. $\ell = 4$.

Suppose that $(m_1, m_0) = (4, 1)$. When we obtain the birational morphism ϕ_1, we contract $\hat{E}_1'$ instead of $\hat{E}_1$. Then this new contraction maps S onto $\mathbb{F}_1$. The curve C_1 meets $E_1, \dots, E_4$ and the curve C_1' intersects $\hat{E}_1'$ and $\hat{E}_2$. We apply (6.2.7) to this new set-up. Then we obtain

$$\mu H \equiv (1-\varepsilon)C_1 + (1+\varepsilon)C_1' + (a+\varepsilon)M + \varepsilon\hat{E}_1' + \varepsilon\hat{E}_2 + \sum_{i=1}^{4}(a_i - \varepsilon)E_i.$$

If $(m_1, m_0) = (3, 2)$, then we apply (6.2.6) to yield

$$\mu H \equiv (1+\varepsilon)C_1 + (1-\varepsilon)C_1' + (a-2\varepsilon)\tilde{L} + (a_4 - \varepsilon)E_4 + \sum_{i=1}^{2}\varepsilon\hat{E}_i + \sum_{i=1}^{3}(a_i + \varepsilon)E_i.$$

If $(m_1, m_0) = (3, 0)$, then use (6.2.9), and obtain

$$\mu H \equiv (1-2\varepsilon)C_1 + (1+\varepsilon)C_1' + (a+\varepsilon)\tilde{L}_1 + \varepsilon\tilde{L}_2 + (a_4 + \varepsilon)E_4 + \sum_{i=1}^{3}(a_i - 2\varepsilon)E_i + \sum_{i=1}^{2}\varepsilon\hat{E}_i.$$

If $(m_1, m_0) = (2, 1)$, then we contract $\hat{E}_2'$ instead of $\hat{E}_2$ when we obtain the birational morphism ϕ_1. This new contraction sends S to $\mathbb{F}_1$. The curve C_1 meets $E_1, E_2, \hat{E}_1, \hat{E}_2'$ and the curve C_1' intersects E_2 and E_4. Apply (6.2.8) to the new contraction, and we obtain

$$\mu H \equiv (1+\varepsilon)C_1 + (1-\varepsilon)C_1' + (a-\varepsilon)M + \varepsilon\hat{E}_1 + \varepsilon\hat{E}_2' + \sum_{i=1}^{2}(a_i + \varepsilon)E_i + \sum_{i=3}^{4}(a_i - \varepsilon)E_i.$$

These four equivalences show that S has an H-polar cylinder if $\ell = 4$.

Case 5. $\ell = 3$.

If $(m_1, m_0) = (3, 2)$, then we contract $\hat{E}_1'$ and $\hat{E}_2'$ instead of $\hat{E}_1$ and $\hat{E}_2$ when we obtain the birational morphism ϕ_1. Then new contraction maps S to $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, it is enough to consider the case $(m_1, m_0) = (3, 0)$ below.

If $(m_1, m_0) = (2, 3)$, then apply (6.2.6) and we obtain

$$\mu H \equiv (1+\varepsilon)C_1 + (1-\varepsilon)C_1' + (a-2\varepsilon)\tilde{L} + (a_3-\varepsilon)E_3 + \sum_{i=1}^{2}(a_i+\varepsilon)E_i + \sum_{i=1}^{3}\varepsilon\hat{E}_i.$$

If $(m_1, m_0) = (3, 0)$, then we use (6.2.9) to get

$$\mu H \equiv (1-2\varepsilon)C_1 + (1+\varepsilon)C_1' + (a+\varepsilon)\tilde{L}_1 + \varepsilon\tilde{L}_2 + \sum_{i=1}^{3}(a_i-2\varepsilon)E_i + \sum_{i=1}^{3}\varepsilon\hat{E}_i.$$

Suppose that $(m_1, m_0) = (2, 1)$. Then we contract $\hat{E}_2'$ and $\hat{E}_3'$ instead of $\hat{E}_2$ and $\hat{E}_3$. This new contraction reduces this case to the case where $(m_1, m_0) = (2, 3)$ above.

Consequently, these two equivalences verify that S has an H-polar cylinder. □

Theorem 6.2.10. *If the ample $\mathbb{R}$-divisor H is of type $C(7)$ with length $3 \leqslant \ell \leqslant 6$, then S contains an H-polar cylinder.*

Proof. Due to Lemma 6.2.4, we may assume that there exists a cuspidal rational curve C in $|-K_S|$. Let P be the point at which the curve C has the cusp. Each E_i intersects the curve C at a single smooth point.

We construct H-polar cylinders case by case, according to the length ℓ.

Case 1. $\ell = 6$ and $W \cong \mathbb{P}^1 \times \mathbb{P}^1$.

There are two 0-curves F_1, F_2 passing through the point P and not meeting any of E_i's. Note that the curve B must intersect one of the 0-curves F_1, F_2. We may assume that it intersects F_1. Then $B \equiv F_2$. We see that

$$-K_S \equiv (1-\varepsilon)C + 2\varepsilon F_1 + (a+2\varepsilon)F_2 - aB - \varepsilon\sum_{i=1}^{6}E_i,$$

and hence

$$\mu H \equiv (1-\varepsilon)C + 2\varepsilon F_1 + (a+2\varepsilon)F_2 + \sum_{i=1}^{6}(a_i-\varepsilon)E_i.$$

Let $\pi : \tilde{S} \to S$ be the blow up at the point P and let E be the exceptional curve of π. Then $\tilde{S}$ is a weak del Pezzo surface of degree 1 and it has exactly one (-2)-curve, the proper transform $\tilde{C}$ of C. Denote the proper transforms on $\tilde{S}$ of the curves $E_1, \ldots, E_6$, F_1, F_2 by $\tilde{E}_1, \ldots, \tilde{E}_6$, $\tilde{F}_1$, $\tilde{F}_2$. Contracting the (-1)-curves $\tilde{E}_1, \ldots, \tilde{E}_6$, $\tilde{F}_1$, $\tilde{F}_2$, we obtain a birational morphism $\psi : \tilde{S} \to \mathbb{P}^2$.

For a sufficiently small positive real number ε, the divisor above defines an H-polar cylinder since

$$S \setminus (C \cup F_1 \cup F_2 \cup E_1 \cup \ldots \cup E_6) \cong \mathbb{P}^2 \setminus (\psi(\tilde{C}) \cup \psi(E)),$$

where $\psi(\tilde{C})$ and $\psi(E)$ are a conic and a line meeting tangentially at a single point on $\mathbb{P}^2$.

Case 2. $\ell = 6$ and $W \cong \mathbb{F}_1$.

In this case, we have only one 0-curve F passing through the point P and not intersecting any of E_i's. Instead, we consider the Zariski tangent line M to C at the point P. This is a 1-curve.

Note that the curve $\phi_1(B)$ is a 0-curve and it intersects the unique (-1)-curve on $\mathbb{F}_1$. We have

$$-K_S \equiv (1-\varepsilon)\,C + 2\varepsilon M + (a+\varepsilon)F - aB - \varepsilon\sum_{i=1}^{6} E_i,$$

and hence

$$\mu H \equiv (1-\varepsilon)\,C + 2\varepsilon M + (a+\varepsilon)F + \sum_{i=1}^{6}(a_i-\varepsilon)\,E_i.$$

For a sufficiently small positive real number ε, this defines an H-polar cylinder (see Example 4.1.8).

Case 3. $\ell = 5$.

There are two 0-curves L_1, L_2 such that

- they pass through P;
- they do not meet each other outside P;
- they are disjoint from the curves $E_1, \ldots, E_5$.

In addition, there is a unique 1-curve T that intersects C only at P and that does not meet any of E_i's (see Example 4.1.10). Note that the curve B is a 0-curve. It may be assumed to intersect L_1 but not L_2. Then

$$-K_S \equiv (1-\varepsilon)C + \varepsilon L_1 + (a+\varepsilon)L_2 + \varepsilon T - aB - \varepsilon\sum_{i=1}^{5} E_i,$$

and hence

$$\mu H \equiv (1-\varepsilon)C + \varepsilon L_1 + (a+\varepsilon)L_2 + \varepsilon T + \sum_{i=1}^{5}(a_i-\varepsilon)E_i.$$

For a sufficiently small positive real number ε, this defines an H-polar cylinder (see Example 4.1.10).

Case 4. $\ell = 4$.

The surface S has three 0-curves F_1, F_2, F_3 such that

- they pass through P;
- they do not intersect each other outside P;
- they are disjoint from the curves E_1, E_2, E_3, E_4.

In addition, it has two 1-curves G_1, G_2 such that

- they meet C only at P;
- they do not intersect each other outside P;
- they are disjoint from the curves E_1, E_2, E_3, E_4

(see Example 4.1.11). The curve B is a 0-curve. We may assume that it intersects F_1, F_2 but not F_3. Then

$$-K_S \equiv (1-2\varepsilon)C + \varepsilon(F_1+F_2) + (a+\varepsilon)F_3 + \varepsilon(G_1+G_2) - aB - 2\varepsilon\sum_{i=1}^{4} E_i.$$

Therefore,

$$\mu H \equiv (1-2\varepsilon)C + \varepsilon(F_1+F_2) + (a+\varepsilon)F_3 + \varepsilon(G_1+G_2) + \sum_{i=1}^{4}(a_i-2\varepsilon)E_i.$$

For a sufficiently small positive real number ε, this defines an H-polar cylinder as shown in Example 4.1.11.

Case 5. $\ell = 3$.

In this case, the surface S has five 0-curves $F_1, \ldots, F_5$ such that

- they pass through P;
- they do not intersect each other outside P;
- they are disjoint from the curves E_1, E_2, E_3

(see Example 4.1.13). The curve B is a 0-curve. We may assume that it meets F_1, F_2, F_3, F_4 but not F_5. Then

$$-K_S \equiv (1-2\varepsilon)C + \varepsilon\sum_{i=1}^{4} F_i + (a+\varepsilon)F_5 - aB - 2\varepsilon(E_1+E_2+E_3).$$

Therefore,

$$\mu H \equiv (1-2\varepsilon)C + \varepsilon\sum_{i=1}^{4} F_i + (a+\varepsilon)F_5 + (a_1-2\varepsilon)E_1 + (a_2-2\varepsilon)E_2 + (a_3-2\varepsilon)E_3.$$

For a sufficiently small positive real number ε, this defines an H-polar cylinder. □

Theorem 6.2.11. *If H is of type $C(7)$ with $a > \frac{10}{3}$, then S contains an H-polar cylinder.*

Proof. Put $\phi_1(E_i) = P_i$ for $i = 1, \ldots, 6$.

Suppose that $W \cong \mathbb{P}^1 \times \mathbb{P}^1$. We may assume that $\phi_1(B)$ is a curve of bidegree $(0, 1)$. Let C be the curve of bidegree $(1, 0)$ passing through the point P_1. Let F_i be the curve of bidegree $(0, 1)$ passing through the point P_i for $i = 1, \ldots, 6$. We have

$$-K_{\mathbb{P}^1\times\mathbb{P}^1} \equiv 2C + \frac{1}{3}\sum_{i=1}^{6} F_i.$$

We then obtain

$$-K_S \equiv 2\tilde{C} + \frac{1}{3}\tilde{F}_1 + \frac{4}{3}E_1 + \frac{1}{3}\sum_{i=2}^{6}\tilde{F}_i - \frac{2}{3}\sum_{i=2}^{6} E_i,$$

where $\tilde{C}$ and $\tilde{F}_i$'s are the proper transforms of C and F_i's by ϕ_1, respectively. Since $B \equiv \tilde{F}_i + E_i$ for each i, we have

$$-K_S + aB \equiv 2\tilde{C} + \frac{1}{3}\tilde{F}_1 + \frac{4}{3}E_1 + \left(\frac{a}{5}+\frac{1}{3}\right)\sum_{i=2}^{6}\tilde{F}_i + \left(\frac{a}{5}-\frac{2}{3}\right)\sum_{i=2}^{6} E_i.$$

Since $\frac{a}{5} - \frac{2}{3} > 0$, Example 4.1.4 shows that S has an H-polar cylinder.

We now suppose that $W \cong \mathbb{F}_1$. Let C be the negative section of $\mathbb{F}_1$. Note that C cannot pass through any of the points P_i. Take the fiber F_i of the $\mathbb{P}^1$-bundle morphism of $\mathbb{F}_1$ to $\mathbb{P}^1$ that passes through the point P_i for each i. We have

$$-K_{\mathbb{F}_1} \equiv 2C + \frac{1}{2}\sum_{i=1}^{6} F_i.$$

We then obtain

$$-K_S \equiv 2\tilde{C} + \frac{1}{2}\sum_{i=1}^{6}(\tilde{F}_i - E_i),$$

where $\tilde{C}$ and $\tilde{F}_i$'s are the proper transforms of C and F_i by ϕ_1, respectively. Since $B \equiv \tilde{F}_i + E_i$ for each i, we have

$$-K_S + aB \equiv 2\tilde{C} + \left(\frac{a}{6}+\frac{1}{2}\right)\sum_{i=1}^{6}\tilde{F}_i + \left(\frac{a}{6}-\frac{1}{2}\right)\sum_{i=1}^{6} E_i.$$

Since $a > 3$, Example 4.1.1 verifies that S has an H-polar cylinder. □

Theorems 6.2.2, 6.2.3, 6.2.10 and 6.2.11 imply (1), (2), (3), and (4) in Theorem 2.2.4, respectively.

6.3. **Cylinders in del Pezzo surfaces of degree** 1. In order to prove Theorem 2.2.5, let S be a smooth del Pezzo surface of degree 1 and let H be an ample $\mathbb{R}$-divisor on S. We use the same notations as those at the beginning of Subsection 6.2.

Again we first consider ample $\mathbb{R}$-divisors of type $B(r)$. Let $E_1, \ldots, E_r$ be the r disjoint (-1)-curves that generate the face Δ. We may then write

$$K_S + \mu H \equiv \sum_{i=1}^{r} a_i E_i,$$

for some positive real numbers $a_1, \ldots, a_r$. We may assume that $a_1 \geqslant \ldots \geqslant a_r$.

Proposition 6.3.1. *Suppose that $r \geqslant 3$, $2a_1 + 2a_2 + a_3 > 4$, the contraction ϕ is a birational, and $Z \not\cong \mathbb{P}^1 \times \mathbb{P}^1$. Then S contains an H-polar cylinder.*

Proof. Note that Z is a smooth del Pezzo surface of degree $r+1$. Moreover, by our assumption $Z \not\cong \mathbb{P}^1 \times \mathbb{P}^1$. Thus, either $Z = \mathbb{P}^2$ or Z is a blow up of $\mathbb{P}^2$ at $(8-r)$ points in general position. For both the cases, let $\psi\colon Z \to \mathbb{P}^2$ be the blow up. If $8 - r > 0$, denote the proper transforms of these ψ-exceptional curves on S by $E_{r+1}, \ldots, E_8$. Put $P_i = \sigma(E_i)$ and $\sigma = \psi \circ \phi$.

Let C be the conic in $\mathbb{P}^2$ passing through the points $P_4, \ldots, P_8$. Let L be a line passing through the point P_3 and tangent to the conic C and let Q be the intersection point of the line L and the conic C. For $i = 1, 2$ let C_i be the conic passing through the point P_i and intersecting C only at the point Q.

The open subset $U = \mathbb{P}^2 \setminus (L \cup C \cup C_1 \cup C_2)$ is a cylinder.

We claim that the cylinder $U' := \sigma^{-1}(U) \simeq U$ is H-polar. Indeed, for a real number ε we have

$$-K_{\mathbb{P}^2} \equiv (1+3\varepsilon)C + (\alpha_1 - \varepsilon)\, C_1 + (\alpha_2 - \varepsilon)\, C_2 + (\alpha_3 - 2\varepsilon)\, L,$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$ and $2\alpha_1 + 2\alpha_2 + \alpha_3 = 1$. Hence,

$$\begin{aligned} -K_S &\sim \sigma^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{8} E_i \\ &\equiv (1+3\varepsilon)\tilde{C} + (\alpha_1 - \varepsilon)\, \tilde{C}_1 + (\alpha_2 - \varepsilon)\, \tilde{C}_2 + (\alpha_3 - 2\varepsilon)\, \tilde{L} + \\ &+ (\alpha_1 - \varepsilon - 1)\, E_1 + (\alpha_2 - \varepsilon - 1)\, E_2 + (\alpha_3 - 2\varepsilon - 1)\, E_3 + 3\varepsilon \sum_{i=4}^{8} E_i, \end{aligned}$$

where $\tilde{L}$, $\tilde{C}$ and $\tilde{C}_j$ are the proper transforms of the line L and the conics C, C_j, respectively. We then obtain

$$\begin{aligned} \mu H &\equiv (1+3\varepsilon)\tilde{C} + (\alpha_1 - \varepsilon)\, \tilde{C}_1 + (\alpha_2 - \varepsilon)\, \tilde{C}_2 + (\alpha_3 - 2\varepsilon)\, \tilde{L} + \\ &+ (\alpha_1 + a_1 - \varepsilon - 1)\, E_1 + (\alpha_2 + a_2 - \varepsilon - 1)\, E_2 + (\alpha_3 + a_3 - 2\varepsilon - 1)\, E_3 + \sum_{i=4}^{8} (3\varepsilon + a_i) E_i, \end{aligned}$$

where $a_i = 0$ if $i > r$. Put $\alpha_1 = \frac{3}{2}\varepsilon + 1 - a_1$, $\alpha_2 = \frac{3}{2}\varepsilon + 1 - a_2$ and $\alpha_3 = 2a_1 + 2a_2 - 6\varepsilon - 3$. Since $2a_1 + 2a_2 + a_3 > 4$, for a sufficiently small positive real number ε, all the coefficients in the divisor above are positive. This proves our claim. □

We now suppose that the morphism $\phi : S \to Z$ is a conic bundle, i.e., $Z = \mathbb{P}^1$. We may write

$$K_S + \mu H \equiv aB + \sum_{i=1}^{7} a_i E_i$$

where B is an irreducible fiber of ϕ, E_i's are disjoint (-1)-curves in fibers of ϕ, a is a positive real number, and a_i's are non-negative real numbers.

Proposition 6.3.2. *If H is of type $C(8)$ with $a > \frac{30}{7}$, then S has H-polar cylinders.*

Proof. The proof of Theorem 6.2.11 works almost verbatim for this case. □

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