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A note on the Lasserre hierarchy for different formulations of the maximum independent set problem

M. F. Anjos^{a,d}, Y. Emine^{a,b,c}, A. Lodi^{a,b,c}, Z. Sun^e

^a*GERAD, Montréal (QC), Canada*

^b*Department of Mathematics and Industrial Engineering, Polytechnique Montréal (QC), Canada*

^c*Canada Excellence Research Chair, Polytechnique Montréal (QC), Canada*

^d*School of Mathematics, University of Edinburgh, Edinburgh, United Kingdom*

^e*Msunhealth Technology Corp. Ltd*

Abstract

In this note, we consider several polynomial optimization formulations of the maximum independent set problem and the use of the Lasserre hierarchy with these different formulations. We demonstrate using computational experiments that the choice of formulation may have a significant impact on the resulting bounds. We also provide theoretical justifications for the observed behavior.

Keywords: Maximum Independent Set, Polynomial Optimization, Dual Bounds, Lasserre Hierarchy

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1. Introduction

Polynomial optimization and its close connections with semidefinite and conic optimization have attracted a lot of attention in recent years [1]. It is well known that semidefinite optimization has had a tremendous impact on combinatorial optimization, particularly with the groundbreaking results of

Email addresses: anjos@stanfordalumni.org (M. F. Anjos),
youssef.emine@gerad.ca (Y. Emine), andrea.lodi@polymtl.ca (A. Lodi),
sunzhao@msunsoft.com (Z. Sun)

¹A. Lodi, CERC, Polytechnique Montréal, C.P. 6079, Succ. Centre-ville Montréal (QC) Canada H3C 3A7, andrea.lodi@polymtl.ca

Lovász and Schrijver [2] and Goemans and Williamson [3]. This motivates the study of the application of polynomial optimization to combinatorial optimization problems.

Moreover, combinatorial problems can typically be formulated in different ways, and it is known that different formulations of the same combinatorial problem may lead to different semidefinite relaxations and hence to different global bounds; the example of the maximum cut problem is explored from this perspective in [4].

Finally, various approaches have been proposed to construct hierarchies of semidefinite relaxations for (binary) combinatorial optimization problems (and applied to the stable set polytope). Lovász and Schrijver [2] used a sequence of lift-and-project operations to construct their hierarchy, while Lasserre's work [5] starts with a polynomial formulation and progressively refines it by providing another hierarchy. De Klerk and Pasechnik [6] used copositive programming to construct another hierarchy and yet another one follows by the Reformulation-Linearization Technique approach by Sherali and Adams [7]. All these hierarchies have in common the property of converging to the optimal solution in a finite number of steps and a comparison among Sherali-Adams, Lovász-Schrijver and Lasserre hierarchies was carried out by Laurent [8].

In this note, we focus on the Lasserre approach and how it is impacted by using different polynomial formulations of the same problem. Specifically, we consider several polynomial optimization formulations of the maximum independent set problem and the use of the Lasserre hierarchy with these different formulations. We demonstrate using computational experiments that the choice of formulation may have a significant impact on the resulting bounds. We also provide theoretical justifications for the observed behavior.

A specific comparison among hierarchies for the maximum independent set problem has been considered by Gvozdenović and Laurent [9]. They proved that the Lasserre's is tighter than the Lovász-Schrijver's, and tighter than De Klerk and Pasechnik's as well. However, all the comparisons in the literature used the Lasserre's hierarchy of one polynomial formulation. Our results show that this particular formulation is the best among a set of polynomial formulations, thus confirming the interest of the analysis in [9].

The paper is organized as follows. In Section 2, we give some preliminaries about polynomial optimization, while Section 3 introduces the maximum independent set problem and computationally motivates the interest of looking at different polynomial formulations for the problem. Section 4 provides the

theoretical content of the paper and in Section 5 we draw some concluding remarks.

2. Preliminaries

Polynomial optimization is NP-hard in general, and the Lasserre hierarchy has great theoretical and practical appeal because it provides a sequence of tractable relaxations whose optimal objective values converge to the global optimum. We briefly review the construction of the Lasserre hierarchy (in the dual form). For more details about Lasserre hierarchy, see e.g. [1, 5, 10, 11].

Given polynomials f, g_1, \dots, g_m , we consider the following general polynomial optimization problem:

$$f_{min} = \min f(x) : \text{s.t. } g_j(x) \geq 0, \forall j = 1, \dots, m. \quad (1)$$

Let $\{x^\alpha\}_{\alpha \in \mathbb{N}^n}$ be a canonical basis for $\mathbb{R}[x]$. Given $y = \{y_\alpha\} \in \mathbb{R}^{\mathbb{N}^n}$, we define $L_y : \mathbb{R}[x] \rightarrow \mathbb{R}$ as the linear functional which maps a polynomial function $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha$ (f_α are the coefficients of f in the canonical basis $\{x^\alpha\}_{\alpha \in \mathbb{N}^n}$)

to the real value $L_y(f) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha y_\alpha$

The *moment matrix* $M_d(y)$ is the matrix of $\mathbb{R}^{\mathbb{N}_d^n \times \mathbb{N}_d^n}$ such that its entries are:

$$M_d(y)(\alpha, \beta) = y_{\alpha+\beta}, \forall \alpha, \beta \in \mathbb{N}_d^n.$$

Given a polynomial function $\theta(x) = \sum_{\gamma \in \mathbb{N}^n} \theta_\gamma x^\gamma$, the *localizing matrix* $M_d(\theta \star y) \in \mathbb{R}^{\mathbb{N}_d^n \times \mathbb{N}_d^n}$ is the matrix of $\mathbb{R}^{\mathbb{N}_d^n \times \mathbb{N}_d^n}$ such that its entries are:

$$M_d(\theta \star y)(\alpha, \beta) = \sum_{\gamma \in \mathbb{N}^n} \theta_\gamma y_{\alpha+\beta+\gamma}, \forall \alpha, \beta \in \mathbb{N}_d^n.$$

Let the degree of g_j is $2v_j$ or $2v_j - 1$. Then, for problem (1), the Lasserre relaxation of order d provides a lower bound ρ_d for f_{min} :

$$\rho_d = \min L_y(f) : \text{s.t. } M_d(y) \succeq 0, M_{d-v_j}(g_j \star y) \succeq 0, \forall j = 1, \dots, m, \quad (2)$$

$$L_y(1) = 1.$$

The sequence of Lasserre relaxations of increasing order $d = 1, 2, 3, \dots$ forms the Lasserre hierarchy.

3. Maximum independent set

Given a graph $G = (V, E)$, the maximum independent set problem consists of determining the maximum cardinality of any subset of vertices such that no two vertices in that subset are connected by an edge of G . We consider four different formulations of this problem using quadratic polynomials. The formulations are:

$$\rho = \max \sum_{i \in V} x_i$$

$$\begin{aligned} \text{s.t.} \quad & x_i x_j = 0, & \forall (i, j) \in E, & \quad x_i^2 - x_i = 0, & \forall i \in V. & \quad (3) \\ \text{or s.t.} \quad & x_i x_j \leq 0, & \forall (i, j) \in E, & \quad x_i^2 - x_i = 0, & \forall i \in V. & \quad (4) \\ \text{or s.t.} \quad & x_i + x_j \leq 1, & \forall (i, j) \in E, & \quad x_i^2 - x_i = 0, & \forall i \in V. & \quad (5) \\ \text{or s.t.} \quad & x_i x_j = 0, & \forall (i, j) \in E, & \quad 0 \leq x_i \leq 1, & \forall i \in V. & \quad (6) \end{aligned}$$

Let us compute the upper bounds arising from the Lasserre relaxation (2) of order $d = 1$ for each of the above four formulations. The bounds are reported in Table 1, where C_n denotes the cycle with n vertices and K_4 denotes the complete graph with 4 vertices.

Table 1: Bounds from the Lasserre relaxation of order $d = 1$ for different formulations

Graph	Optimal bound	Bound from (3)	Bound from (4)	Bound from (5)	Bound from (6)
C_3	1	1	1	1.5	3
C_4	2	2	2	2	4
K_4	1	1	1	2	4
C_5	2	2.236	2.236	2.5	5
C_6	3	3	3	3	6
C_7	3	3.318	3.318	3.5	7
Petersen graph	4	4	4	5	10

We observe that the bounds obtained using (3) and (4) are always equal, and are the best for all of these graphs. On the other hand, the bounds from (6) are consistently the weakest; indeed the bound obtained using (6) is

always equal to the trivial upper bound $|V|$, as we prove formally in Proposition 4.1 below. Finally, the bounds from (5) are equal or moderately weaker than those from (3) and (4).

Although these results are only for 7 small graphs, they clearly show that the choice of formulation dramatically impacts the quality of the resulting bound. The remainder of this note is concerned with providing some theoretical justification for the results in Table 1.

4. Independent Set Formulations and Lasserre Relaxations

4.1. Notation

Let us consider the following polynomials of $\mathbb{R}^{|V|}$

$$\begin{aligned} g_{ij}^+(x) &= x_i x_j, \quad g_{ij}^-(x) = -x_i x_j, \quad l_{ij}(x) = 1 - x_i - x_j, \quad \forall (i, j) \in E \\ h_i^+(x) &= x_i^2 - x_i, \quad h_i^-(x) = -x_i^2 + x_i, \quad q_i^+(x) = x_i, \quad q_i^-(x) = 1 - x_i, \quad \forall i \in V \\ f(x) &= \sum_{i \in V} x_i. \end{aligned}$$

Let $(e_i)_{i \in V}$ be the canonical basis of $\mathbb{R}^{|V|}$. For a fixed degree d , the Lasserre relaxations of order d of the above formulations are:

$$\begin{aligned} \rho_{d,3} &= \max L_y(f) & \rho_{d,4} &= \max L_y(f) \\ \text{s.t. } & M_d(y) \succeq 0, & \text{s.t. } & M_d(y) \succeq 0, \\ M_{d-1}(g_{ij}^+ \star y) &= 0, \quad \forall (i, j) \in E, & M_{d-1}(g_{ij}^- \star y) &\succeq 0, \quad \forall (i, j) \in E, \\ M_{d-1}(h_i^+ \star y) &= 0, \quad \forall i \in V, & M_{d-1}(h_i^+ \star y) &= 0, \quad \forall i \in V, \\ & L_y(1) = 1. & & L_y(1) = 1. \end{aligned}$$

$$\begin{aligned} \rho_{d,5} &= \max L_y(f) & \rho_{d,6} &= \max L_y(f) \\ \text{s.t. } & M_d(y) \succeq 0, & \text{s.t. } & M_d(y) \succeq 0, \\ M_{d-1}(l_{ij} \star y) &\succeq 0, \quad \forall (i, j) \in E, & M_{d-1}(g_{ij}^+ \star y) &= 0, \quad \forall (i, j) \in E, \\ M_{d-1}(h_i^+ \star y) &= 0, \quad \forall i \in V, & M_{d-1}(q_i^+ \star y) &\succeq 0, \quad \forall i \in V, \\ & L_y(1) = 1. & M_{d-1}(q_i^- \star y) &\succeq 0, \quad \forall i \in V, \\ & & & L_y(1) = 1. \end{aligned}$$

4.2. Value of $\rho_{1,6}$ for every graph G

Our first result is the proof that the optimal value of the Lasserre relaxation of order $d = 1$ using formulation (6) is equal to $|V|$ for every graph G .

Proposition 4.1. *For every graph G , $\rho_{1,6} = |V|$.*

Proof. For every feasible solution $\{y_\alpha\}$, we have $0 \leq y_{e_i} \leq 1$, $\forall i \in V$, therefore

$$\sum_{i \in V} y_{e_i} = L_y(f) \leq |V|.$$

To show attainment, consider $y^* = \{y_\alpha^*\}$ such that:

$$\begin{cases} y_0^* = 1, \\ y_{e_i}^* = 1, & \forall i \in V, \\ y_{2e_i}^* = |V| + 1, & \forall i \in V, \\ y_{e_i+e_j}^* = 0, & \forall (i, j) \in V^2, \quad i \neq j. \end{cases}$$

It is straightforward to check that y^* is a feasible solution of the Lasserre relaxation of order $d = 1$ for formulation (6), and that this solution achieves the objective value $|V|$. \square

4.3. Relationship between $\rho_{d,3}$ and $\rho_{d,4}$

The next proposition shows that the set of feasible solutions of the d order Lasserre relaxation for formulation (3) is a subset of the set of feasible solutions the relaxation with the same order for formulation (4). Moreover, for $d \geq 2$, the two feasible sets are equal, and hence so are the bounds.

Proposition 4.2. *For every graph G and order $d \geq 1$, $\rho_{d,3} \leq \rho_{d,4}$. Moreover, if $d \geq 2$, then $\rho_{d,3} = \rho_{d,4}$.*

Proof. The first claim follows from the observation that $M_{d-1}(g_{ij}^+ \star y) = 0$ implies $M_{d-1}(g_{ij}^- \star y) \succeq 0$.

To prove the second claim, let $y = \{y_\alpha\}$ be a feasible solution of the Lasserre hierarchy of order d for formulation (4). We know that for every $(\alpha, \beta) \in \mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_{d-1}^{|V|}$

$$\begin{aligned} M_{d-1}(g_{ij}^- \star y)(\alpha, \beta) &= -y_{\alpha+\beta+e_i+e_j}, & \forall (i, j) \in E, \\ M_{d-1}(h_i^+ \star y)(\alpha, \beta) &= y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i} = 0, & \forall i \in V. \end{aligned}$$

then for every $\alpha \in \mathbb{N}_{d-2}^{|V|}$ and $k \in V$,

$$\begin{aligned}
M_{d-1}(g_{ij}^- \star y)(\alpha, \alpha) &= -y_{2\alpha+e_i+e_j} \\
&= -y_{2\alpha+2e_i+e_j} \\
&= -y_{2\alpha+2e_i+2e_j} \\
&= -M_d(y)(\alpha + e_i + e_j, \alpha + e_i + e_j).
\end{aligned}$$

and

$$\begin{aligned}
\det [M_{d-1}(g_{ij}^- \star y)_{\{\alpha, \alpha+e_k\}, \{\alpha, \alpha+e_k\}}] &= \begin{vmatrix} -y_{2\alpha+e_i+e_j} & -y_{2\alpha+e_k+e_i+e_j} \\ -y_{2\alpha+e_k+e_i+e_j} & -y_{2\alpha+2e_k+e_i+e_j} \end{vmatrix} \\
&= \begin{vmatrix} -y_{2\alpha+e_i+e_j} & -y_{2\alpha+e_k+e_i+e_j} \\ -y_{2\alpha+e_k+e_i+e_j} & -y_{2\alpha+e_k+e_i+e_j} \end{vmatrix} \\
&= y_{2\alpha+e_i+e_j}y_{2\alpha+e_k+e_i+e_j} - y_{2\alpha+e_k+e_i+e_j}^2.
\end{aligned}$$

Since $M_{d-1}(g_{ij}^- \star y)$ and $M_d(y)$ are semi-definite positive matrices then

$$\begin{cases} M_{d-1}(g_{ij}^- \star y)(\alpha, \alpha) \geq 0, \\ M_d(y)(\alpha + e_i + e_j, \alpha + e_i + e_j) \geq 0, \\ \det [M_{d-1}(g_{ij}^- \star y)_{\{\alpha, \alpha+e_k\}, \{\alpha, \alpha+e_k\}}] \geq 0. \end{cases}$$

Which implies that

$$\begin{cases} y_{2\alpha+e_i+e_j} = 0, \\ -y_{2\alpha+e_k+e_i+e_j}^2 \geq 0. \end{cases} \quad \text{and so} \quad \begin{cases} y_{2\alpha+e_i+e_j} = 0, \\ y_{2\alpha+2e_k+e_i+e_j}^2 = 0. \end{cases}$$

This proves that $M_{d-1}(g_{ij}^- \star y)(\alpha, \alpha) = 0$ for every $\alpha \in \mathbb{N}_{d-1}^{|V|}$. Therefore $M_{d-1}(g_{ij}^- \star y)$ is semi-definite positive matrix with zero on the diagonal. It is the zero matrix and this proves that y is also a feasible solution of the level d of the Lasserre hierarchy for formulation (3). \square

4.4. Relationship between $\rho_{d,4}$ and $\rho_{d,5}$

The next result is that the feasible set of the Lasserre relaxation of order d using the formulation (4) is a subset of the feasible set of the relaxation of the same order for formulation (5). Hence, the bound $\rho_{d,5}$ is always dominated by the bound $\rho_{d,4}$.

Proposition 4.3. *For every graph G and order $d \geq 1$, $\rho_{d,4} \leq \rho_{d,5}$.*

Proof. Let $y = \{y_\alpha\}$ be a feasible solution of the relaxation of (4) of order d . We know that for every $(\alpha, \beta) \in \mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_{d-1}^{|V|}$:

$$\begin{aligned} M_{d-1}(g_{ij}^- \star y)(\alpha, \beta) &= -y_{\alpha+\beta+e_i+e_j} & \forall (i, j) \in E, \\ M_{d-1}(h_i^+ \star y)(\alpha, \beta) &= y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i} & \forall i \in V, \\ M_{d-1}(l_{ij} \star y)(\alpha, \beta) &= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+\beta+e_j} & \forall (i, j) \in E. \end{aligned}$$

$$\text{Let } A \in \mathbb{R}^{\mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_d^{|V|}} \text{ such that: } A(\alpha, \gamma) = \begin{cases} 1 & \text{if } \gamma = \alpha, \\ -1 & \text{if } \gamma = \alpha + e_i, \\ -1 & \text{if } \gamma = \alpha + e_j, \\ 0 & \text{otherwise.} \end{cases} \text{ For every}$$

$$(\alpha, \beta) \in \mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_{d-1}^{|V|}:$$

$$\begin{aligned} [AM_d(y)A^T](\alpha, \beta) &= \sum_{\gamma \in \mathbb{N}_d^{|V|}} \sum_{\delta \in \mathbb{N}_d^{|V|}} A(\alpha, \gamma) M_d(y)(\gamma, \delta) A^T(\delta, \beta) \\ &= \sum_{\gamma \in \mathbb{N}_d^{|V|}} \sum_{\delta \in \mathbb{N}_d^{|V|}} A(\alpha, \gamma) M_d(y)(\gamma, \delta) A(\beta, \delta) \\ &= \sum_{\gamma \in \mathbb{N}_d^{|V|}} \sum_{\delta \in \mathbb{N}_d^{|V|}} A(\alpha, \gamma) y_{\gamma+\delta} A(\beta, \delta) \\ &= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+\beta+e_j} \\ &\quad - y_{\alpha+e_i+\beta} + y_{\alpha+e_i+\beta+e_j} + y_{\alpha+e_i+\beta+e_j} \\ &\quad - y_{\alpha+e_j+\beta} + y_{\alpha+e_j+\beta+e_i} + y_{\alpha+e_j+\beta+e_j} \\ &= \underbrace{y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+\beta+e_j}}_{=M_{d-1}(l_{ij} \star y)(\alpha, \beta)} + \underbrace{y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i}}_{=M_{d-1}(h_i^+ \star y)(\alpha, \beta)} \\ &\quad + \underbrace{y_{\alpha+\beta+2e_j} - y_{\alpha+\beta+e_j}}_{=M_{d-1}(h_j^+ \star y)(\alpha, \beta)} + 2 \underbrace{y_{\alpha+\beta+e_i+e_j}}_{=-M_{d-1}(g_{ij}^- \star y)(\alpha, \beta)} \end{aligned}$$

which implies that

$$M_{d-1}(l_{ij} \star y) = AM_d(y)A^T + 2M_{d-1}(g_{ij}^- \star y) - M_{d-1}(h_j^+ \star y) - M_{d-1}(h_i^+ \star y).$$

Since $M_d(y) \succeq 0$ then $AM_d(y)A^T \succeq 0$. Moreover

$$\begin{cases} M_{d-1}(g_{ij}^- \star y) \succeq 0, \\ M_{d-1}(h_j^+ \star y) = M_{d-1}(h_i^+ \star y) = 0. \end{cases}$$

Then $M_{d-1}(l_{ij} \star y) \succeq 0$ for all $(i, j) \in E$, and thus y is a feasible solution of the d order Lasserre relaxation for formulation (5). \square

4.5. Relationship between $\rho_{d,3}$ and $\rho_{d,6}$

The next result is that the feasible set of the Lasserre relaxation of order d using (3) is a subset of the feasible set of the relaxation of the same order for formulation (6). Hence, the bound $\rho_{d,6}$ is always dominated by the bound $\rho_{d,3}$.

Proposition 4.4. *For every graph G and order $d \geq 1$, $\rho_{d,3} \leq \rho_{d,6}$.*

Proof. Let $y = \{y_\alpha\}$ be a feasible solution of the relaxation of (3) of order d . We know that for every $(\alpha, \beta) \in \mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_{d-1}^{|V|}$:

$$\begin{aligned} M_{d-1}(g_{ij}^- \star y)(\alpha, \beta) &= -y_{\alpha+\beta+e_i+e_j} & \forall (i, j) \in E, \\ M_{d-1}(h_i^+ \star y)(\alpha, \beta) &= y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i} & \forall i \in V, \\ M_{d-1}(q_i^+ \star y)(\alpha, \beta) &= y_{\alpha+\beta+e_i} & \forall i \in V, \\ M_{d-1}(q_i^- \star y)(\alpha, \beta) &= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} & \forall i \in V, \end{aligned}$$

Let $A \in \mathbb{R}^{\mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_{d-1}^{|V|}}$ such that: $A(\alpha, \gamma) = \begin{cases} 1 & \text{if } \gamma = \alpha + e_i, \\ 0 & \text{otherwise.} \end{cases}$ For every $(\alpha, \beta) \in \mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_{d-1}^{|V|}$:

$$\begin{aligned} [AM_d(y)A^T](\alpha, \beta) &= \sum_{\gamma \in \mathbb{N}_d^{|V|}} \sum_{\delta \in \mathbb{N}_d^{|V|}} A(\alpha, \gamma) M_d(y)(\gamma, \delta) A^T(\delta, \beta) \\ &= \sum_{\gamma \in \mathbb{N}_d^{|V|}} \sum_{\delta \in \mathbb{N}_d^{|V|}} A(\alpha, \gamma) M_d(y)(\gamma, \delta) A(\beta, \delta) \\ &= \sum_{\gamma \in \mathbb{N}_d^{|V|}} \sum_{\delta \in \mathbb{N}_d^{|V|}} A(\alpha, \gamma) y_{\gamma+\delta} A(\beta, \delta) \\ &= y_{\alpha+e_i+\beta+e_i} \\ &= y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i} + y_{\alpha+\beta+e_i} \\ &= M_{d-1}(h_i^+ \star y)(\alpha, \beta) + M_{d-1}(q_i^+ \star y)(\alpha, \beta) \end{aligned}$$

which implies that

$$M_{d-1}(q_i^+ \star y) = AM_d(y)A^T - M_{d-1}(h_i^+ \star y).$$

Since $M_d(y) \succeq 0$ then $AM_d(y)A^T \succeq 0$. Moreover $M_{d-1}(h_i^+ \star y) = 0$. Then $M_{d-1}(q_i^+ \star y) \succeq 0$ for all $i \in V$. On the other hand, let $B \in \mathbb{R}^{\mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_d^{|V|}}$ such

$$\text{that: } B(\alpha, \gamma) = \begin{cases} 1 & \text{if } \gamma = \alpha, \\ -1 & \text{if } \gamma = \alpha + e_i, \\ 0 & \text{otherwise.} \end{cases} \text{ For every } (\alpha, \beta) \in \mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_d^{|V|}:$$

$$\begin{aligned} [BM_d(y)B^T](\alpha, \beta) &= \sum_{\gamma \in \mathbb{N}_d^{|V|}} \sum_{\delta \in \mathbb{N}_d^{|V|}} B(\alpha, \gamma)M_d(y)(\gamma, \delta)B^T(\delta, \beta) \\ &= \sum_{\gamma \in \mathbb{N}_d^{|V|}} \sum_{\delta \in \mathbb{N}_d^{|V|}} B(\alpha, \gamma)M_d(y)(\gamma, \delta)B(\beta, \delta) \\ &= \sum_{\gamma \in \mathbb{N}_d^{|V|}} \sum_{\delta \in \mathbb{N}_d^{|V|}} B(\alpha, \gamma)y_{\gamma+\delta}B(\beta, \delta) \\ &= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+e_i+\beta} + y_{\alpha+e_i+\beta+e_i} \\ &= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+\beta+e_i} + y_{\alpha+\beta+2e_i} \\ &= M_{d-1}(q_i^- \star y)(\alpha, \beta) + M_{d-1}(h_i^+ \star y)(\alpha, \beta). \end{aligned}$$

which implies that

$$M_{d-1}(q_i^- \star y) = BM_d(y)B^T - M_{d-1}(h_i^+ \star y).$$

Since $M_d(y) \succeq 0$ then $BM_d(y)B^T \succeq 0$. Moreover $M_{d-1}(h_i^+ \star y) = 0$. Then $M_{d-1}(q_i^- \star y) \succeq 0$ for all $i \in V$. Since $M_{d-1}(g_{ij}^- \star y) = 0$, y is a feasible solution of the d order Lasserre relaxation for formulation (6). \square

4.6. Relationships with the linear programming relaxations of maximum independent set

In this section, we establish the relationship among some of the formulations discussed above and two famous linear programming (LP) relaxations for the maximum independent set problem. More precisely, we consider two LP formulations and we refer to them as "weak" and "strong". The weak formulation is that with constraints

$$x_i + x_j \leq 1 \quad \forall (i, j) \in E \tag{7}$$

and nonnegativity, whereas the strong formulation replaces constraints (7) with constraints

$$\sum_{i \in C} x_i \leq 1 \quad \forall C \in \mathcal{C}, \tag{8}$$

where \mathcal{C} is the set of all maximum cliques in G .

Proposition 4.5. *For every graph $G = (V, E)$, the optimal value of the order 1 of the Lasserre hierarchy for formulation (5) is equal to the value of the LP relaxation of the weak formulation of the independent set problem.*

Proof. Let $y = \{y_\alpha\}_\alpha$ be a feasible solution of the order 1 of the Lasserre hierarchy of formulation (5), then

$$\begin{cases} y_0 = 1, \\ y_{e_i} = y_{2e_i}, & \forall i \in V, \\ y_{e_i} + y_{e_j} \leq 1, & \forall (i, j) \in E. \end{cases} \implies \begin{cases} y_0 = 1, \\ 0 \leq y_{e_i} \leq 1, & \forall i \in V, \\ y_{e_i} + y_{e_j} \leq 1, & \forall (i, j) \in E, \end{cases}$$

and $(y_{e_i})_{i \in V}$ is a feasible solution of the LP relaxation of the weak formulation of the independent set problem. Conversely, let $(x_i)_{i \in V}$ be a feasible solution of the LP relaxation of the weak formulation of the independent

set problem, let $X \in \mathbb{R}^{|V|+1}$: $\begin{cases} X_0 = 1, \\ X_i = x_i, & \forall i \in V, \end{cases}$ $A \in \mathbb{R}^{(|V|+1) \times (|V|+1)}$:

$$\begin{cases} A \text{ diagonal}, \\ A_{0,0} = 0, \\ A_{i,i} = x_i(1 - x_i) & \forall i \in V, \end{cases} \text{ and } y = \{y_\alpha\}_{\alpha \in \mathbb{N}_2^{|V|}} \text{ such that}$$

$$\begin{cases} y_0 = 1, \\ y_{e_i} = y_{2e_i} = x_i, & \forall i \in V, \\ y_{e_i+e_j} = x_i \times x_j, & \forall (i, j) \in V^2 \quad i \neq j. \end{cases}$$

Then,

$$\begin{cases} L_y(1) = y_0 = 1, \\ M_0(l_{ij} \star y) = 1 - y_{e_i} - y_{e_j} = 1 - x_i - x_j \geq 0, & \forall (i, j) \in E, \\ M_0(h_i^+ \star y) = y_{2e_i} - y_{e_i} = 0, & \forall i \in V, \\ M_1(y) = XX^T + A \succeq 0. \end{cases}$$

This proves that y is a feasible solution of the level 1 of the Lasserre hierarchy of formulation (5) with the value equal to $\sum_{i \in V} y_{e_i} = \sum_{i \in V} x_i$. \square

In Theorem 10.4, Conforti et al. [12] proved that $\rho_{1,3}$ is smaller than the value of the LP relaxation of the strong formulation of the independent set problem. In the following proposition, we extend this result to prove a stronger relationship, namely that between the LP relaxation of the strong formulation and the order 1 of the Lasserre hierarchy for formulation (4). Such a result, with a different proof, was given independently by Szegedy [13].

Proposition 4.6 ([13]). *For every graph $G = (V, E)$, the optimal value of the order 1 of the Lasserre hierarchy for formulation (4) is smaller than the value of the LP relaxation of the strong formulation of the independent set problem.*

Proof. Let $y = \{y_\alpha\}_\alpha$ be a feasible solution of the level one of the Lasserre hierarchy of formulation (4), then

$$\begin{cases} y_0 = 1, \\ y_{e_i} = y_{2e_i}, \quad \forall i \in V, \\ y_{e_i+e_j} \leq 0, \quad \forall (i, j) \in E. \end{cases}$$

Let W be a clique of G and $X \in \mathbb{R}^{|V|+1}$ such that :

$$\begin{cases} X_0 = 1, \\ X_i = -1, \quad \text{if } i \in W, \\ X_i = 0, \quad \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} 0 \leq X^T M_1(y) X &= y_0 - 2 \sum_{i \in W} y_{e_i} + \sum_{i \in W} y_{2e_i} + \sum_{\substack{(i,j) \in W^2, \\ i \neq j}} y_{e_i+e_j} \\ &\leq y_0 - 2 \sum_{i \in W} y_{e_i} + \sum_{i \in W} y_{e_i} \\ &\leq 1 - \sum_{i \in W} y_{e_i}. \end{aligned}$$

This proves that $(y_{e_i})_{i \in V}$ is a feasible solution of the LP relaxation of the strong formulation of the independent set problem. \square

5. Summary of Results and Future Research

Using the notation previously defined, we summarize our results as follows: For $d = 1$: $\rho_{1,3} \leq \rho_{1,4} \leq \text{LP}_{\text{strong}} \leq \text{LP}_{\text{weak}} = \rho_{1,5} \leq |V| = \rho_{1,6}$, and for $d \geq 2$: $\rho_{d,3} = \rho_{d,4} \leq \rho_{d,5}$ and $\rho_{d,3} = \rho_{d,4} \leq \rho_{d,6}$. We believe these results give an interesting, initial perspective on evaluating the quality of a formulation not only in terms of its relaxation but also with respect to the Lasserre relaxations originated by it.

In future research, it would be interesting to further study this question for other combinatorial problems and for the other hierarchies in Section 1.

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