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Citation for published version:
Figueroa-O'Farrill, J & Gharamti, M 2014, 'Supersymmetry of hyperbolic monopoles', *Journal of High Energy Physics*, vol. 2014, no. 4, 74. https://doi.org/10.1007/JHEP04(2014)074

Digital Object Identifier (DOI):
10.1007/JHEP04(2014)074

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
Journal of High Energy Physics

SUPERSYMMETRY OF HYPERBOLIC MONOPOLES

JOSÉ FIGUEROA-O'FARRILL AND MOUSTAFA GHARAMTI

Abstract. We investigate what supersymmetry says about the geometry of the moduli space of hyperbolic monopoles. We construct a three-dimensional supersymmetric Yang–Mills–Higgs theory on hyperbolic space whose half-BPS configurations coincide with (complexified) hyperbolic monopoles. We then study the action of the preserved supersymmetry on the collective coordinates and show that demanding closure of the supersymmetry algebra constraints the geometry of the moduli space of hyperbolic monopoles, turning it into a so-called pluricomplex manifold, thus recovering a recent result of Bielawski and Schwachhöfer.

Contents

1. Introduction

BPS monopoles—that is, the solutions of the Bogomol'nyi equation—have been under the microscope by mathematicians and physicists for a long time. This equation and its solutions can be studied on any oriented riemannian 3-manifold, but they are particularly interesting in euclidean and hyperbolic spaces. One inspiring observation about BPS monopoles in these spaces is that they can be viewed as instantons in four-dimensional euclidean space left invariant under the action of a one-parameter subgroup of isometries: translations (resp. rotations) in the case of euclidean (resp. hyperbolic) BPS monopoles. Another way of saying this is that the Bogomol'nyi equation results from the four-dimensional self-duality equation by demanding independence on one of the coordinates.

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To begin with, consider the Bogomol'nyi equation in euclidean space

$$\nabla_A \phi = - \star F_A, \tag{1}$$

where ϕ satisfies some suitable boundary conditions that make the L^2 norm of F_A finite and $\star$ is the Hodge operator of $\mathbb{R}^3$. For a detailed treatment of euclidean monopoles, one can check [1, 2, 3]. The ingredients of the Bogomol'nyi equation can be cast into a geometrical framework, where A can be viewed as a connection on a principal G-bundle P over $\mathbb{R}^3$ and F_A as its curvature. The Higgs field ϕ is a section of the adjoint bundle $\mathrm{ad}P$ over $\mathbb{R}^3$; that is, the associated vector bundle to P corresponding to the adjoint representation of G on its Lie algebra, and ∇_A is the covariant derivative operator induced on $\mathrm{ad}P$. A pair (A, ϕ) satisfying equation (1) is what we call a euclidean monopole. If we now interpret ϕ as being the x_4 component of the connection, then equation (1) becomes the self-duality Yang-Mills equation on $\mathbb{R}^4$

$$F_A = \star F_A, \tag{2}$$

where all the fields are independent of the x_4 coordinate, and the $\star$-operation is now with respect to the flat euclidean metric on $\mathbb{R}^4$.

For the case of hyperbolic monopoles we simply replace the euclidean base space $\mathbb{R}^3$ with hyperbolic space H^3. To construct hyperbolic monopoles from instantons, instead of considering translationally invariant solutions of equation (2) we will, however, look for rotationally invariant solutions [4]. To be specific consider the flat euclidean metric in $\mathbb{R}^4$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \,. \tag{3}$$

If we choose the rotations to be in the (x_1, x_2)-plane and we let r and θ be the polar coordinates in that plane, we have

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 + dx_3^2 + dx_4^2 \\ &= r^2 \left(d\theta^2 + \frac{dr^2 + dx_3^2 + dx_4^2}{r^2} \right) \,. \end{aligned} \tag{4}$$

The rotations now act simply as shifts in the angular variable θ. This coordinate system is valid in the complement $\mathbb{R}^4 \setminus \mathbb{R}^2$ of the $x_1 = x_2 = 0$ plane. Inside the parenthesis we recognise the metric on $S^1 \times H^3$, which is therefore shown to be conformal to $\mathbb{R}^4 \setminus \mathbb{R}^2$.

Now a wonderful fact about the self-duality equation is its conformal invariance: the Hodge $\star$ is conformally invariant acting on middle-dimensional forms in an even-dimensional manifold. This allows us to drop the conformal factor r^2 from the metric without altering the equation. If we now impose the condition that the gauge potential A is S^1 invariant, i.e., rotationally symmetric in the (x_1, x_2)-plane, and if we define A_θ=ϕ, the self-duality equation becomes the Bogomol'nyi equation on H^3. The Bogomol'nyi equation on H^3 is also given by equation (1) but with the $\star$-operation of H^3. The first constructions of a monopole solution on hyperbolic space were first given in [5, 6, 7].

A BPS monopole in hyperbolic space is labelled by a mass $m \in \mathbb{R}^+$ and a charge $k \in \mathbb{Z}^+$ given by

$$\begin{aligned} m &= \lim_{r\to\infty} |\phi(r)| \\ k &= \lim_{r\to\infty} \frac{1}{4\pi m} \int_{H^3} \operatorname{tr}(F_A \wedge \nabla_A \phi) \,, \end{aligned} \tag{5}$$

and it is known [8] that hyperbolic monopoles exist for all values of m and k. In contrast to the euclidean monopoles, m cannot be rescaled to unity in the hyperbolic case, as the value of m affects the monopole solutions [9]. Alternatively, one can normalise the mass to unity, but only at the price of rescaling the hyperbolic metric to one of curvature $-1/m^2$. The rotationally invariant instanton on $\mathbb{R}^4 \setminus \mathbb{R}^2$ corresponding to a hyperbolic monopole of charge k and mass m will extend to a rotationally invariant instanton on all of $\mathbb{R}^4$ if (and only if) $m \in \mathbb{Z}$.

In [10] Manton interpreted low energy dynamics of monopoles as geodesic motion on the moduli space; that is, the space of solutions up to gauge equivalence, and this ushered in an era of much activity in the study of the geometry of the moduli space. For the case of euclidean monopoles, Atiyah and Hitchin showed in [1] that the moduli space has a natural hyperkähler metric and they found the explicit form of the metric for the moduli space of charge 2. Moreover, the metric of the moduli space of well separated monopoles was found in [11], where the monopoles were treated as point particles carrying scalar, electric and magnetic charges.

The hyperbolic case is much less understood. In [4], where Atiyah introduced hyperbolic monopoles, he writes:

> Moreover, by varying the curvature of hyperbolic space and letting it tend to zero, the euclidean case appears as a natural limit of the hyperbolic case. While the details of this limiting procedure are a little delicate, and need much more careful examination than I shall give here, it seems reasonable to conjecture that the moduli of monopoles remains unaltered by passing to the limit.

Atiyah also showed [12] that the moduli space $\mathcal{M}_{k,m}$ of hyperbolic monopoles of charge k and mass m can be identified with the space of rational maps of the form

$$\frac{a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k}{z^k + b_1 z^{k-1} + \cdots + b_k} \quad \text{with } k \geqslant 1, \tag{6}$$

where the polynomials in the numerator and denominator are relatively prime. Since the $a_1, \dots, a_k, b_1, \dots, b_k$ are complex numbers, the moduli space has real dimension $4k$.

Most of the progress in the study of hyperbolic monopoles was focused on finding methods of constructing multimonopole solutions, either by building a hyperbolic version of the Nahm transform [9, 13, 14] or by studying the spectral curves associated with hyperbolic monopoles [15, 16, 17]. Progress on the geometry of the moduli space was hindered by the early realisation [9] that the natural L^2 metric, which in the euclidean case induces upon reduction a hyperkähler metric on the moduli space, does not converge in the case of hyperbolic monopoles, suggesting that the geometry of the moduli space is not in fact riemannian. Nevertheless, Hitchin [18] constructed a family g_m of self-dual Einstein metrics on the moduli space of centered hyperbolic monopoles with mass $m \in \mathbb{Z}$, which in the flat limit $m \to \infty$ recovers the Atiyah–Hitchin metric. It is an interesting open question to relate Hitchin's construction to the physics of hyperbolic monopoles.

The situation has changed dramatically in recent times due to the seminal work of Bielawski and Schwachhöfer, based on earlier work of O. Nash [19]. Nash used a new twistorial construction of $\mathcal{M}_{k,m}$ to show that the complexification of the real geometry of the moduli space of hyperbolic monopoles is similar in some respects to the complexification of a hyperkähler geometry. Building on that work, Bielawski and Schwachhöfer [20] identified the real geometry of the moduli space of hyperbolic monopoles as "pluricomplex geometry", which is equivalent to saying that there is a $\mathbb{C}$-linear hypercomplex structure on the complexification $T_{\mathbb{C}}\mathcal{M}_{k,m}$ of the tangent bundle to the moduli space. Later in [21] Bielawski and Schwachhöfer studied the euclidean limit of the pluricomplex moduli space of hyperbolic monopoles, and showed that in the limit one recovers an enhanced hyperkähler geometry, richer by an additional complex structure.

The fact that BPS monopoles saturate the Bogomol'nyi bound suggests that monopoles are supersymmetric in nature and in this paper we will exhibit this in detail for the case of the hyperbolic monopoles. Similar results for the case of euclidean monopoles were obtained in [22, 23, 24] among others. The aim of this paper is thus to show that the pluricomplex nature of the moduli space of hyperbolic monopoles is a natural consequence of supersymmetry. One novel aspect of our construction is that the contraints coming from supersymmetry are imposed by demanding the closure of the supersymmetry algebra and not the invariance of the effective action for the moduli, which does not exist due to the lack of convergence of the L^2 metric. This is reminiscent of the results of Stelle and Van Proeyen [25] on Wess–Zumino models without an action functional, in which the geometry is relaxed from Kähler to complex flat. In fact, morally one could say that pluricomplex is to hyperkähler what complex flat is to Kähler.

The paper is organised as follows. In Section 2 we construct a supersymmetric Yang–Mills–Higgs theory in hyperbolic space by starting with supersymmetric Yang–Mills theory on Minkowski spacetime, euclideanising to a supersymmetric Yang–Mills theory on $\mathbb{R}^4$, reducing to $\mathbb{R}^3$ and deforming to a supersymmetric theory on H^3. In Section 3 we show that the hyperbolic monopoles coincide with the configurations which preserve precisely one half of the supersymmetry. We also start the analysis of the moduli space by studying the linearisation of the Bogomol'nyi equation and identifying the bosonic and fermionic zero modes and how the unbroken supersymmetry relates them. A possibly surprising result is the fact that supersymmetry suggests a small modification of the Gauss law constraint, which depends explicitly on the hyperbolic curvature. Finally in Section 4 we linearise the unbroken supersymmetry and demanding the on-shell closure of the supersymmetry algebra will yield the conditions satisfied by the geometry of the moduli space. The paper ends with an appendix on the Frölicher–Nijenhuis bracket of two endomorphisms.

2. Hyperbolic supersymmetric Yang–Mills–Higgs

The purpose of this section is to describe a construction of supersymmetric theories in hyperbolic space by the following procedure: start with supersymmetric Yang–Mills in Minkowski spacetime, euclideanise à la van Nieuwenhuizen–Waldron [26], reduce to $\mathbb{R}^3$ and deform to a theory on H^3. The euclideanisation will require complexifying the fields in the theory.

2.1. **Off-shell supersymmetry in euclidean 4-space.**

The first step has been done in [26], except that we expect that auxiliary fields should play an important rôle and thus must promote the theory to one with off-shell closure of supersymmetry (up to possibly gauge transformations).

The euclidean supersymmetric Yang–Mills action in $\mathbb{R}^4$ is obtained by integrating the lagrangian density

$$\mathcal{L}^{(4)} = -\operatorname{Tr}\chi_R^\dagger \not{D}\psi_L - \tfrac{1}{4}\operatorname{Tr}F^2\ , \tag{7}$$

where Tr denotes an ad-invariant inner product on the Lie algebra $\mathfrak{g}$ of the gauge group G, and where the subscripts L, R denote the projections

$$\psi_L = \tfrac{1}{2}(\mathbb{I}+\gamma^5)\psi \qquad \text{and} \qquad \chi_R^\dagger = \tfrac{1}{2}\chi^\dagger(\mathbb{I}-\gamma^5)\ , \tag{8}$$

where $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$, where $\gamma^\mu\gamma^\nu = \gamma^{\mu\nu} + \delta^{\mu\nu}$. This means that that $(\gamma^5)^2 = 1$. We can raise and lower indices with impunity, since the metric is $\delta_{\mu\nu}$. The action defined by $\mathcal{L}^{(4)}$ is invariant under gauge transformations, which infinitesimally take the form

$$\delta_\Lambda\psi_L = [\Lambda,\psi_L] \qquad \delta_\Lambda\chi_R^\dagger = [\Lambda,\chi_R^\dagger] \qquad \text{and} \qquad \delta_\Lambda A_\mu = -D_\mu\Lambda = -\partial_\mu\Lambda + [\Lambda, A_\mu]\ , \tag{9}$$

with $\Lambda \in C^\infty(\mathbb{R}^4;\mathfrak{g})$. Furthermore it is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\varepsilon\psi_L &= \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L \\ \delta_\varepsilon\chi_R^\dagger &= -\tfrac{1}{2}\varepsilon_R^\dagger\gamma^{\mu\nu}F_{\mu\nu} \\ \delta_\varepsilon A_\mu &= -\varepsilon_R^\dagger\gamma_\mu\psi_L + \chi_R^\dagger\gamma_\mu\varepsilon_L\ , \end{aligned} \tag{10}$$

where ε_L and $\varepsilon_R^\dagger$ are constant spinor parameters of the indicated chirality. Since ε_L and $\varepsilon_R^\dagger$ are independent, we actually have two supersymmetry variations, which we will denote δ_L and δ_R and leave the parameter unspecified when there is no danger of confusion. In this notation we have

$$\begin{aligned} \delta_L\psi_L &= \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L & \delta_R\psi_L &= 0 \\ \delta_L\chi_R^\dagger &= 0 & \delta_R\chi_R^\dagger &= -\tfrac{1}{2}\varepsilon_R^\dagger\gamma^{\mu\nu}F_{\mu\nu} \\ \delta_L A_\mu &= \chi_R^\dagger\gamma_\mu\varepsilon_L & \delta_R A_\mu &= -\varepsilon_R^\dagger\gamma_\mu\psi_L\ . \end{aligned} \tag{11}$$

Notice that if δ_L' is defined as δ_L but with a different supersymmetry parameter, say ε_L', then on the gauge field $[\delta_L,\delta_L']A_\mu = 0$, and similarly $[\delta_R,\delta_R']A_\mu = 0$. On the fermion, however, this will not be true off-shell and it is for that reason that we will introduce an auxiliary field. Indeed, one finds

$$[\delta_L,\delta_L']\psi_L = \delta_L(\tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L') - \delta_L'(\tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L)\ . \tag{12}$$

Using that

$$\delta_L F_{\mu\nu} = D_\mu\delta_L A_\nu - D_\nu\delta_L A_\mu = D_\mu(\chi_R^\dagger\gamma_\nu\varepsilon_L) - D_\nu(\chi_R^\dagger\gamma_\mu\varepsilon_L)\ , \tag{13}$$

whence

$$[\delta_L,\delta_L']\psi_L = D_\mu\chi_R^\dagger\gamma_\nu\varepsilon_L\gamma^{\mu\nu}\varepsilon_L' - D_\mu\chi_R^\dagger\gamma_\nu\varepsilon_L'\gamma^{\mu\nu}\varepsilon_L\ , \tag{14}$$

where we have used that $\gamma_\mu^\dagger = \gamma_\mu$ and also that $(\chi^\dagger\psi)^\dagger = +\psi^\dagger\chi$ for anticommuting spinors. (One might think that the $+$ sign violates the sign rule, but it does not because ψ and $\psi^\dagger$ are independent fields, etc.)

In order to further manipulate the right-hand side of $[\delta_L,\delta_L']\psi_L$ we must make use of a Fierz identity. The basic Fierz identity in $\mathbb{R}^4$ for anticommuting spinors is given by

$$\psi\chi^\dagger = -\tfrac{1}{4}\chi^\dagger\psi\mathbb{I} - \tfrac{1}{4}\chi^\dagger\gamma^5\psi\gamma_5 - \tfrac{1}{4}\chi^\dagger\gamma^\mu\psi\gamma_\mu + \tfrac{1}{4}\chi^\dagger\gamma^\mu\gamma^5\psi\gamma_\mu\gamma_5 + \tfrac{1}{8}\chi^\dagger\gamma^{\mu\nu}\psi\gamma_{\mu\nu}\ . \tag{15}$$

Two special cases will play a rôle in what follows:

$$\psi_L\chi_R^\dagger = -\tfrac{1}{2}\chi_R^\dagger\gamma^\mu\psi_L\gamma_\mu P_R\ , \tag{16}$$

and

$$\psi_R\chi_R^\dagger = -\tfrac{1}{2}\chi_R^\dagger\psi_R P_R - \tfrac{1}{8}\chi_R^\dagger\gamma^{\mu\nu}\psi_R\gamma_{\mu\nu}\ , \tag{17}$$

where $P_R = \frac{1}{2}(\mathbb{I}-\gamma_5)$. Of course, for commuting spinors, we simply flip all signs in the right-hand side.

Using the Fierz formula (16), we may rewrite

$$[\delta_L,\delta_L']\psi_L = -\tfrac{1}{2}D_\mu\chi_R^\dagger\gamma^\sigma\varepsilon_L'\gamma^{\mu\nu}\gamma_\sigma\gamma_\nu\varepsilon_L + \tfrac{1}{2}D_\mu\chi_R^\dagger\gamma^\sigma\varepsilon_L\gamma^{\mu\nu}\gamma_\sigma\gamma_\nu\varepsilon_L'\ . \tag{18}$$

Using that $\gamma^{\mu\nu}\gamma_\sigma\gamma_\nu = -\gamma_{\mu\sigma} - 3\delta_{\mu\sigma}$, we rewrite

$$[\delta_L,\delta_L']\psi_L = \tfrac{3}{2}\chi_R^\dagger\overleftarrow{\not{D}}\varepsilon_L'\varepsilon_L - \tfrac{3}{2}\chi_R^\dagger\overleftarrow{\not{D}}\varepsilon_L\varepsilon_L' + \tfrac{1}{2}D_\mu\chi_R^\dagger\gamma_\nu\varepsilon_L'\gamma^{\mu\nu}\varepsilon_L - \tfrac{1}{2}D_\mu\chi_R^\dagger\gamma_\nu\varepsilon_L\gamma^{\mu\nu}\varepsilon_L'\ . \tag{19}$$

Comparing with equation (14) we see that

$$\mu\chi_R^\dagger\gamma_\nu\varepsilon_L\gamma^{\mu\nu}\varepsilon_L' - D_\mu\chi_R^\dagger\gamma_\nu\varepsilon_L'\gamma^{\mu\nu}\varepsilon_L = \chi_R^\dagger\overleftarrow{\not{D}}\varepsilon_L'\varepsilon_L - \chi_R^\dagger\overleftarrow{\not{D}}\varepsilon_L\varepsilon_L'\ , \tag{20}$$

whence, in summary,

$$[\delta_L,\delta_L']\psi_L = \chi_R^\dagger\overleftarrow{\not{D}}\varepsilon_L'\varepsilon_L - \chi_R^\dagger\overleftarrow{\not{D}}\varepsilon_L\varepsilon_L'\ , \tag{21}$$

which vanishes for all $\varepsilon_L, \varepsilon_L'$ if and only if $\chi_R^\dagger \overleftarrow{\not{D}} = 0$, which is the field equation for $\chi_R^\dagger$. This suggests introducing an auxiliary field, historically denoted by D (and who are we to challenge tradition?!), and modifying the supersymmetry variation of ψ_L by a term proportional to D, namely

$$\delta_L \psi_L = D\varepsilon_L + \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L \ . \tag{22}$$

Now, we see that

$$[\delta_L, \delta_L']\psi_L = (\delta_L D - \chi_R^\dagger \overleftarrow{\not{D}}\varepsilon_L)\varepsilon_L' - (\delta_L' D - \chi_R^\dagger \overleftarrow{\not{D}}\varepsilon_L')\varepsilon_L \ , \tag{23}$$

whence we deduce that if we set

$$\delta_L D = \chi_R^\dagger \overleftarrow{\not{D}}\varepsilon_L = D_\mu \chi_R^\dagger \gamma^\mu \varepsilon_L \tag{24}$$

then $[\delta_L, \delta_L']\psi_L = 0$. But now we have to check that $[\delta_L, \delta_L']D = 0$ as well:

$$\begin{aligned}
[\delta_L, \delta_L']D &= \delta_L(D_\mu \chi_R^\dagger \gamma^\mu \varepsilon_L') - \delta_L'(D_\mu \chi_R^\dagger \gamma^\mu \varepsilon_L) \\
&= [\delta_L A_\mu, \chi_R^\dagger]\gamma^\mu \varepsilon_L' - [\delta_L' A_\mu, \chi_R^\dagger]\gamma^\mu \varepsilon_L \\
&= 2[\chi_R^\dagger \gamma_\mu \varepsilon_L, \chi_R^\dagger \gamma^\mu \varepsilon_L'] \ ,
\end{aligned} \tag{25}$$

where we have used that $\delta_L \chi_R^\dagger = 0$. We now use the Fierz identity (16) and (in matrix notation) rewrite

$$\begin{aligned}
[\delta_L, \delta_L']D &= 2\chi_R^\dagger \gamma_\mu \varepsilon_L \chi_R^\dagger \gamma^\mu \varepsilon_L' - 2\chi_R^\dagger \gamma_\mu \varepsilon_L' \chi_R^\dagger \gamma^\mu \varepsilon_L \\
&= -\chi_R^\dagger \gamma_\mu \gamma_\nu \gamma^\mu \varepsilon_L' \chi_R^\dagger \gamma^\nu \varepsilon_L + \chi_R^\dagger \gamma_\mu \gamma_\nu \gamma^\mu \varepsilon_L \chi_R^\dagger \gamma^\nu \varepsilon_L' \\
&= 2\chi_R^\dagger \gamma_\nu \varepsilon_L' \chi_R^\dagger \gamma^\nu \varepsilon_L - 2\chi_R^\dagger \gamma_\nu \varepsilon_L \chi_R^\dagger \gamma^\nu \varepsilon_L' \\
&= 2[\chi_R^\dagger \gamma_\mu \varepsilon_L', \chi_R^\dagger \gamma^\mu \varepsilon_L] \ ,
\end{aligned} \tag{26}$$

which is to be compared with equation (25), from where we see that indeed $[\delta_L, \delta_L']D = 0$.

In a similar way we work out $\delta_R D$ by the requirement that $[\delta_R, \delta_R']\chi_R^\dagger = 0$. Let α be a number to be determined and let

$$\delta_R \chi_R^\dagger = \alpha D \varepsilon_R^\dagger - \tfrac{1}{2}\varepsilon_R^\dagger \gamma^{\mu\nu}F_{\mu\nu} \ . \tag{27}$$

Then

$$\begin{aligned}
[\delta_R, \delta_R']\chi_R^\dagger &= \delta_R\left(\alpha D \varepsilon_R'^\dagger - \tfrac{1}{2}\varepsilon_R'^\dagger \gamma^{\mu\nu}F_{\mu\nu}\right) - \delta_R'\left(\alpha D \varepsilon_R^\dagger - \tfrac{1}{2}\varepsilon_R^\dagger \gamma^{\mu\nu}F_{\mu\nu}\right) \\
&= \alpha \delta_R D \varepsilon_R'^\dagger + \varepsilon_R^\dagger \gamma_\nu D_\mu \psi_L \varepsilon_R'^\dagger \gamma^{\mu\nu} - (\varepsilon_R \leftrightarrow \varepsilon_R') \ .
\end{aligned} \tag{28}$$

We use the Fierz identity (16)

$$D_\mu \psi_L \varepsilon_R'^\dagger = -\tfrac{1}{2}\varepsilon_R'^\dagger \gamma^\sigma D_\mu \psi_L \gamma_\sigma P_R \tag{29}$$

to rewrite

$$[\delta_R, \delta_R']\chi_R^\dagger = \alpha \delta_R D \varepsilon_R'^\dagger - \tfrac{1}{2}\varepsilon_R'^\dagger \gamma^\sigma D_\mu \psi_L \varepsilon_R^\dagger \gamma_\nu \gamma_\sigma \gamma^{\mu\nu} - (\varepsilon_R \leftrightarrow \varepsilon_R') \ . \tag{30}$$

We now use that $\gamma_\nu \gamma_\sigma \gamma^{\mu\nu} = -\gamma_{\mu\sigma} + 3\delta_{\mu\sigma}$ to rewrite the above equation as

$$[\delta_R, \delta_R']\chi_R^\dagger = \alpha \delta_R D \varepsilon_R'^\dagger + \tfrac{1}{2}\varepsilon_R'^\dagger \gamma_\sigma D_\mu \psi_L \varepsilon_R^\dagger \gamma^{\mu\sigma} - \tfrac{3}{2}\varepsilon_R'^\dagger \not{D}\psi_L \varepsilon_R^\dagger - (\varepsilon_R \leftrightarrow \varepsilon_R') \ . \tag{31}$$

Comparing with equation (28), we see that

$$\varepsilon_R'^\dagger \gamma_\sigma D_\mu \psi_L \varepsilon_R^\dagger \gamma^{\mu\sigma} - (\varepsilon_R \leftrightarrow \varepsilon_R') = \varepsilon_R'^\dagger \not{D}\psi_L \varepsilon_R^\dagger - (\varepsilon_R \leftrightarrow \varepsilon_R') \ , \tag{32}$$

whence finally

$$[\delta_R, \delta_R']\chi_R^\dagger = \left(\alpha \delta_R D + \varepsilon_R^\dagger \not{D}\psi_L\right)\varepsilon_R'^\dagger - (\varepsilon_R \leftrightarrow \varepsilon_R') \ , \tag{33}$$

which vanishes provided that

$$\delta_R D = -\frac{1}{\alpha}\varepsilon_R^\dagger \overleftarrow{\not{D}}\psi_L \ . \tag{34}$$

As before, one checks that $[\delta_R, \delta_R']D = 0$.

We fix α by closing the supersymmetry algebra on the gauge field: we expect that it should close to a translation up to a gauge transformation. Indeed,

$$\begin{aligned}[\delta_L, \delta_R]A_\mu &= \delta_L(-\varepsilon_R^\dagger\gamma_\mu\psi_L) - \delta_R(\chi_R^\dagger\gamma_\mu\varepsilon_L)\\ &= -\varepsilon_R^\dagger\gamma_\mu\left(D + \tfrac{1}{2}\gamma^{\nu\rho}F_{\nu\rho}\right)\varepsilon_L - \varepsilon_R^\dagger\left(\alpha D - \tfrac{1}{2}\gamma^{\nu\rho}F_{\nu\rho}\right)\gamma_\mu\varepsilon_L\\ &= -(1+\alpha)\varepsilon_R^\dagger\gamma_\mu\varepsilon_L D - \tfrac{1}{2}\varepsilon_R^\dagger\left(\gamma_\mu\gamma^{\nu\rho} - \gamma^{\nu\rho}\gamma_\mu\right)\varepsilon_L F_{\nu\rho} \ ,\end{aligned} \tag{35}$$

whence we see that $\alpha = -1$ and using that $[\gamma_\mu, \gamma^{\nu\rho}] = 2\delta_\mu^\nu\gamma^\rho - 2\delta_\mu^\rho\gamma^\nu$, we rewrite

$$\begin{aligned}[\delta_L, \delta_R]A_\mu &= 2\varepsilon_R^\dagger\gamma^\rho\varepsilon_L F_{\rho\mu}\\ &= 2\varepsilon_R^\dagger\gamma^\rho\varepsilon_L(\partial_\rho A_\mu - \partial_\mu A_\rho + [A_\rho, A_\mu])\\ &= \xi^\rho\partial_\rho A_\mu - D_\mu\Lambda \ ,\end{aligned} \tag{36}$$

where $\xi^\rho = 2\varepsilon_R^\dagger\gamma^\rho\varepsilon_L$ and $\Lambda = \xi^\rho A_\rho$.

In a similar way, one shows that the algebra closes as expected also on ψ_L, $\chi_R^\dagger$ and D. Indeed, on ψ_L one has

$$\begin{aligned}[\delta_L, \delta_R]\psi_L &= -\delta_R(D\varepsilon_L + \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L)\\ &= -\varepsilon_R^\dagger\not{D}\psi_L\varepsilon_L - \gamma^{\nu\mu}\varepsilon_R^\dagger\gamma_\nu D_\mu\psi_L\varepsilon_L\\ &= -\gamma^\nu\gamma^\mu\varepsilon_L\varepsilon_R^\dagger\gamma_\nu D_\mu\psi_L \ ,\end{aligned} \tag{37}$$

which upon using the Fierz identity (15) for $\varepsilon_L\varepsilon_R^\dagger$ becomes

$$[\delta_L, \delta_R]\psi_L = \tfrac{1}{2}\varepsilon_R^\dagger\gamma^\rho\varepsilon_L\gamma^\nu\gamma^\mu\gamma_\rho\gamma_\nu D_\mu\psi_L \ . \tag{38}$$

Now, we use that $\gamma^\nu\gamma_{\mu\rho}\gamma_\nu = 0$ in four dimensions in order to rewrite this as

$$[\delta_L, \delta_R]\psi_L = 2\varepsilon_R^\dagger\gamma^\mu\varepsilon_L D_\mu\psi_L = \xi^\mu\partial_\mu\psi_L + [\Lambda, \psi_L] \ , \tag{39}$$

as expected. The calculation for $[\delta_L, \delta_R]\chi_R^\dagger$ is similar. Finally, we check closure on D:

$$\begin{aligned}[\delta_L, \delta_R]D &= \delta_L(\varepsilon_R^\dagger\not{D}\psi_L) - \delta_R(\chi_R^\dagger\overleftarrow{\not{D}}\varepsilon_L)\\ &= \varepsilon_R^\dagger\gamma_\mu[\chi_R^\dagger\gamma^\mu\varepsilon_L, \psi_L] + \varepsilon_R^\dagger\not{D}(D\varepsilon_L + \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L)\\ &\quad + (\varepsilon_R^\dagger D + \tfrac{1}{2}\varepsilon_R^\dagger\gamma^{\mu\nu}F_{\mu\nu})\overleftarrow{\not{D}}\varepsilon_L + [\varepsilon_R^\dagger\gamma_\mu\psi_L, \chi_R^\dagger]\gamma^\mu\varepsilon_L\\ &= \varepsilon_R^\dagger\gamma^\rho(D_\rho D + \tfrac{1}{2}\gamma^{\mu\nu}D_\rho F_{\mu\nu})\varepsilon_L + \varepsilon_R^\dagger(D_\rho D + \tfrac{1}{2}\varepsilon_R^\dagger\gamma^{\mu\nu}D_\rho F_{\mu\nu})\gamma^\rho\varepsilon_L\\ &= 2\varepsilon_R^\dagger\gamma^\rho D_\rho D\varepsilon_L + \tfrac{1}{2}\varepsilon_R^\dagger(\gamma^\rho\gamma^{\mu\nu} + \gamma^{\mu\nu}\gamma^\rho)D_\rho F_{\mu\nu}\varepsilon_L \ .\end{aligned} \tag{40}$$

Using that $\gamma^\rho\gamma^{\mu\nu} + \gamma^{\mu\nu}\gamma^\rho = 2\gamma^{\rho\mu\nu}$ and the Bianchi identity $D_{[\rho}F_{\mu\nu]} = 0$, we conclude that

$$[\delta_L, \delta_R]D = 2\varepsilon_R^\dagger\gamma^\rho D_\rho D\varepsilon_L = \xi^\rho\partial_\rho D + [\Lambda, D] \ , \tag{41}$$

as desired.

In summary, the following supersymmetry transformations

$$\begin{aligned} \delta_L A_\mu &= \chi_R^\dagger\gamma_\mu\varepsilon_L & \delta_R A_\mu &= -\varepsilon_R^\dagger\gamma_\mu\psi_L\\ \delta_L\psi_L &= D\varepsilon_L + \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L & \delta_R\psi_L &= 0\\ \delta_L\chi_R^\dagger &= 0 & \delta_R\chi_R^\dagger &= -\varepsilon_R^\dagger D - \tfrac{1}{2}\varepsilon_R^\dagger\gamma^{\mu\nu}F_{\mu\nu}\\ \delta_L D &= \chi_R^\dagger\overleftarrow{\not{D}}\varepsilon_L & \delta_R D &= \varepsilon_R^\dagger\not{D}\psi_L \end{aligned} \tag{42}$$

obey

$$[\delta_L, \delta_L'] = 0 \qquad [\delta_R, \delta_R'] = 0 \qquad \text{whereas} \qquad [\delta_L, \delta_R] = \mathcal{L}_\xi + \delta_\Lambda^{\text{gauge}} \;, \tag{43}$$

where $\xi^\mu = 2\varepsilon_R^\dagger \gamma^\mu \varepsilon_L$ and $\Lambda = \xi^\mu A_\mu$.

The action given by the lagrangian (7) is not invariant under the supersymmetry transformations in (42) unless we also add a term depending on the auxiliary field. Indeed, the invariant action is given by

$$\mathcal{L}^{(4)} = -\operatorname{Tr} \chi_R^\dagger \not{D} \psi_L - \tfrac{1}{4} \operatorname{Tr} F^2 - \tfrac{1}{2} \operatorname{Tr} D^2 \;. \tag{44}$$

It should be remarked that the euclideanisation has in fact complexified the fields in the original Yang–Mills theory. Indeed, the spinor representation in euclidean signature is not of real type, as it is in lorentzian signature and the supersymmetry transformations further force the bosonic fields to be complex as well.

We may promote this action to an arbitrary riemannian 4-manifold simply by covariantising the derivatives, so that D_μ now also contains the spin connection. Doing so and taking ε_L and $\varepsilon_R^\dagger$ to be spinor fields, we find that

$$\delta_L \mathcal{L}^{(4)} = -\nabla_\mu \operatorname{Tr} \chi_R^\dagger \gamma_\nu \varepsilon_L (D g^{\mu\nu} + F^{\mu\nu}) - \tfrac{1}{2} \operatorname{Tr} \chi_R^\dagger \gamma^\rho \gamma^{\mu\nu} F_{\mu\nu} \nabla_\rho \varepsilon_L \;, \tag{45}$$

and

$$\delta_R \mathcal{L}^{(4)} = \tfrac{1}{2} \nabla_\rho \operatorname{Tr} F_{\mu\nu} \varepsilon_R^\dagger \gamma^{\mu\nu\rho} \psi_L - \tfrac{1}{2} \operatorname{Tr} \nabla_\rho \varepsilon_R^\dagger \gamma^{\mu\nu} \gamma^\rho F_{\mu\nu} \psi_L \;, \tag{46}$$

from where we see that if ε_L and $\varepsilon_R^\dagger$ are not parallel, the action is not invariant. This will be remedied for the dimensionally reduced action in three dimensions by adding further terms in the action provided that ε_L and $\varepsilon_R^\dagger$ are Killing spinors.

2.2. **Reduction to euclidean 3-space.** The spin group in four dimensions is $\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. The spin group in three dimensions is $\mathrm{Spin}(3)$ and embeds in $\mathrm{Spin}(4)$ as the diagonal $\mathrm{Spin}(3)$ in $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$. Therefore in three dimensions there is no distinction between L and R spinors. We reduce to three dimensions along the fourth coordinate, whence we assume that $\partial_4 = 0$ on all fields and parameters.

We take the following explicit realisation for the four-dimensional gamma matrices:

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma^j \\ i\sigma^j & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \qquad \text{and hence} \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \;. \tag{47}$$

This means that we can take $\psi_L = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ and $\chi_R^\dagger = \begin{pmatrix} 0 & \chi^\dagger \end{pmatrix}$. The basic Fierz identity for anticommuting spinors in three dimensions is

$$\psi\chi^\dagger = -\tfrac{1}{2}\chi^\dagger\psi - \tfrac{1}{2}\chi^\dagger\sigma^j\psi\sigma^j \;. \tag{48}$$

The gauge field decomposes as $A_\mu \rightsquigarrow (A_i, \phi)$. The supersymmetry parameters ε_L and $\varepsilon_R^\dagger$ also decompose as ψ_L and $\chi_R^\dagger$ do: $\varepsilon_L = \begin{pmatrix} \epsilon_L \\ 0 \end{pmatrix}$ and $\varepsilon_R^\dagger = \begin{pmatrix} 0 & \epsilon_R^\dagger \end{pmatrix}$. In terms of the three-dimensional quantities we have the following supersymmetry transformations:

$$\begin{aligned}
\delta_L A_i &= i\chi^\dagger\sigma_i\epsilon_L & \delta_R A_i &= -i\epsilon_R^\dagger\sigma_i\psi \\
\delta_L \phi &= \chi^\dagger\epsilon_L & \delta_R \phi &= -\epsilon_R^\dagger\psi \\
\delta_L \chi^\dagger &= 0 & \delta_R \chi^\dagger &= -D\epsilon_R^\dagger - \tfrac{i}{2}\varepsilon_{ijk}F^{ij}\epsilon_R^\dagger\sigma^k - i\epsilon_R^\dagger\sigma^i D_i\phi \\
\delta_L D &= i\chi^\dagger \overleftarrow{\not{D}} \epsilon_L + [\phi, \chi^\dagger\epsilon_L] & \delta_R D &= i\epsilon_R^\dagger \not{D}\psi + \epsilon_R^\dagger[\phi, \psi] \\
\delta_L \psi &= D\epsilon_L + \tfrac{i}{2}\varepsilon_{ijk}F^{ij}\sigma^k\epsilon_L - iD_i\phi\sigma^i\epsilon_L & \delta_R \psi &= 0 \;,
\end{aligned} \tag{49}$$

where now

$$[\delta_L, \delta_L'] = 0 = [\delta_R, \delta_R'] \qquad \text{and} \qquad [\delta_L, \delta_R] = \mathcal{L}_\xi + \delta_\Lambda^{\text{gauge}} , \tag{50}$$

with $\xi^i = 2i\epsilon_R^\dagger \sigma^i \epsilon_L$ and $\Lambda = \xi^i A_i + 2\epsilon_R^\dagger \epsilon_L \phi$.

The reduction of the action (44) to three dimensions is

$$\mathcal{L}^{(3)} = -i \operatorname{Tr} \chi^\dagger \not{D} \psi - \operatorname{Tr} \chi^\dagger [\phi, \psi] - \tfrac{1}{4} \operatorname{Tr} F^2 - \tfrac{1}{2} \operatorname{Tr} |D\phi|^2 - \tfrac{1}{2} \operatorname{Tr} D^2 , \tag{51}$$

where $\not{D} = \sigma^i D_i$, $F^2 = F_{ij}F^{ij}$ and $|D\phi|^2 = D_i\phi D^i\phi$. It can again be suitably covariantised to define it on a riemannian 3-manifold. Its variation under supersymmetry can be read off from equations (45) and (46). Doing so, one finds

$$\delta_L \mathcal{L}^{(3)} = -i\nabla_i \operatorname{Tr} \chi^\dagger \left(\sigma^i D + \sigma_j F^{ij} - iD^i\phi\right) \epsilon_L + \operatorname{Tr} \chi^\dagger \sigma^i \sigma^\ell \left(\tfrac{1}{2}\varepsilon_{jk\ell} F^{jk} - D_\ell \phi\right) \nabla_i \epsilon_L \tag{52}$$

and

$$\delta_R \mathcal{L}^{(3)} = \nabla_i \operatorname{Tr} \varepsilon^{ijk} \epsilon_R^\dagger \left(-\tfrac{1}{2} F_{jk} + iD_j\phi\sigma_k\right) \psi + \operatorname{Tr} \nabla_i \epsilon_R^\dagger \left(\tfrac{1}{2}\varepsilon_{jk\ell}F^{jk} + D_\ell\phi\right) \sigma^\ell \sigma^i \psi . \tag{53}$$

2.3. **Deforming to curved space.** We now wish to improve the action $\mathcal{L}^{(3)}$ and the supersymmetry transformations of the fermions and the auxiliary field in order for the new $\mathcal{L}^{(3)}$ to transform into a total derivative when the spinor parameters are not necessarily parallel. Instead we will take them to be Killing: $\nabla_i \epsilon_L = \lambda_L \sigma_i \epsilon_L$ and $\nabla_i \epsilon_R^\dagger = \lambda_R \epsilon_R^\dagger \sigma_i$ for some (either real or imaginary) constants λ_L and λ_R. We add terms

$$\mathcal{L}^{(3)} \rightsquigarrow \mathcal{L}^{(3)} + \alpha_1 \operatorname{Tr} \chi^\dagger \psi + \tfrac{1}{2}\alpha_2 \operatorname{Tr} \phi^2 + \alpha_3 \operatorname{Tr} \phi D + \tfrac{1}{2}\alpha_4 \operatorname{Tr} D^2 \tag{54}$$

to the lagrangian and also

$$\begin{aligned} \delta_L \psi &\rightsquigarrow \delta_L \psi + \beta_1 \phi \epsilon_L & \qquad \delta_R \chi^\dagger &\rightsquigarrow \delta_R \chi^\dagger - \beta_3 \epsilon_R^\dagger \phi \\ \delta_L D &\rightsquigarrow \delta_L D + \beta_2 \chi^\dagger \epsilon_L & \qquad \delta_R D &\rightsquigarrow \delta_R D + \beta_4 \epsilon_R^\dagger \psi , \end{aligned} \tag{55}$$

for some constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ to be determined.

We start by computing $\delta_L \mathcal{L}^{(3)}$. Using equation (52), we arrive at (henceforth dropping Tr from the notation)

$$\begin{aligned} \delta_L \mathcal{L}^{(3)} = \nabla_i X_L^i &- \lambda_L (\tfrac{1}{2}\varepsilon_{jk\ell}F^{jk} - D_\ell\phi)\chi^\dagger \sigma^\ell \epsilon_L - i\beta_1 \chi^\dagger \not{D}(\phi\epsilon_L) - \beta_2 D \chi^\dagger \epsilon_L \\ &+ \alpha_1 \chi^\dagger \left((D + \beta_1\phi)\epsilon_L + i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} - D_k\phi)\sigma^k \epsilon_L\right) + \alpha_2 \phi \chi^\dagger \epsilon_L + \alpha_3 D \chi^\dagger \epsilon_L \\ &+ \alpha_3 \phi \left(i\chi^\dagger \overleftarrow{\not{D}} \epsilon_L + \beta_2 \chi^\dagger \epsilon_L\right) + \alpha_4 D \left(i\chi^\dagger \overleftarrow{\not{D}} \epsilon_L + [\phi, \chi^\dagger \epsilon_L] + \beta_2 \chi^\dagger \epsilon_L\right) , \end{aligned} \tag{56}$$

where $X_L^i = -i\chi^\dagger \left(\sigma^i D + \sigma_j F^{ij} - iD^i\phi\right) \epsilon_L$, and where we have used that $\sigma^i \sigma_j \sigma_i = -\sigma_j$.

The $\chi^\dagger F$ terms vanish provided that $\alpha_1 = -i\lambda_L$, which also takes care of the $\chi^\dagger D_i \phi$ terms. The $\chi^\dagger D A_i$ terms impose $\alpha_4 = 0$, whereas the $\chi^\dagger \phi A_i$ terms become a total derivative $\nabla_i Y_L^i$, with $Y_L^i = -i\beta_1 \phi \chi^\dagger \sigma^i \epsilon_L$, provided that $\alpha_3 = -\beta_1$. The $\chi^\dagger D$ terms vanish if $\beta_2 = -(\beta_1 + i\lambda_L)$ and the $\chi^\dagger \phi$ terms vanish provided that $\alpha_2 = -\beta_1^2$.

In summary,

$$\mathcal{L}^{(3)} := -i\chi^\dagger \not{D}\psi - \chi^\dagger[\phi, \psi] - i\lambda_L \chi^\dagger \psi - \tfrac{1}{4}F^2 - \tfrac{1}{2}|D\phi|^2 - \tfrac{1}{2}(D + \beta_1\phi)^2 \tag{57}$$

transforms as

$$\delta_L \mathcal{L}^{(3)} = \nabla_i \left(-i\chi^\dagger \left(\sigma^i (D + \beta_1 \phi) + \sigma_j F^{ij} - iD^i\phi\right) \epsilon_L\right) , \tag{58}$$

under

$$\begin{aligned}
\delta_L A_i &= i\chi^\dagger \sigma_i \epsilon_L\\
\delta_L \phi &= \chi^\dagger \epsilon_L\\
\delta_L \chi^\dagger &= 0\\
\delta_L \psi &= (D + \beta_1\phi)\epsilon_L + \tfrac{i}{2}\varepsilon_{ijk}F^{ij}\sigma^k\epsilon_L - iD_i\phi\sigma^i\epsilon_L\\
\delta_L D &= i\chi^\dagger \overleftarrow{\not{D}}\epsilon_L + [\phi, \chi^\dagger\epsilon_L] - (\beta_1 + i\lambda_L)\chi^\dagger\epsilon_L ,
\end{aligned} \tag{59}$$

with $\nabla_i\epsilon_L = \lambda_L\sigma_i\epsilon_L$.

Notice that the action depends on λ_L, hence once the action is fixed, the sign of the Killing constant in the Killing spinor equation is also fixed.

Next we compute $\delta_R\mathcal{L}^{(3)}$ and use equation (53) to find

$$\begin{aligned}
\delta_R\mathcal{L}^{(3)} = \nabla_i X^i_R - \lambda_R(\tfrac{1}{2}\varepsilon_{jk\ell} + D_\ell\phi)\epsilon_R^\dagger\sigma^\ell\psi + i\beta_3\phi\epsilon_R^\dagger\not{D}\psi - \beta_4 D\epsilon_R^\dagger\psi + \beta_1^2\phi\epsilon_R^\dagger\psi\\
+ i\lambda_L\Big((D+\beta_3\phi)\epsilon_R^\dagger\psi + i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} + D_k\phi)\epsilon_R^\dagger\sigma^k\psi\Big)\\
+ \beta_1 D\epsilon_R^\dagger\psi - \beta_1\phi(i\epsilon_R^\dagger\not{D}\psi + \beta_4\epsilon_R^\dagger\psi) ,
\end{aligned} \tag{60}$$

where we have again used $\sigma^i\sigma_j\sigma_i = -\sigma_j$ and where $X^i_R = \varepsilon^{ijk}\epsilon_R^\dagger\left(-\tfrac{1}{2}F_{jk} + iD_j\phi\sigma_k\right)\psi$.

The $F\psi$ terms vanish provided that $\lambda_R = -\lambda_L$, and this also takes care of the $D_i\phi\psi$ terms. Notice that this means that the vector field $\xi^i = 2i\epsilon_R^\dagger\sigma^i\epsilon_L$ is a Killing vector, and not merely conformal Killing. Indeed,

$$\begin{aligned}
\nabla_i\xi_j &= 2i\lambda_R\epsilon_R^\dagger\sigma_i\sigma_j\epsilon_L + 2i\lambda_L\epsilon_R^\dagger\sigma_j\sigma_i\epsilon_L\\
&= -2i\lambda_L\epsilon_R^\dagger(\sigma_i\sigma_j - \sigma_j\sigma_i)\epsilon_L\\
&= -2i\lambda_L\varepsilon_{ijk}\xi^k ,
\end{aligned} \tag{61}$$

whence $\nabla_i\xi_j + \nabla_j\xi_i = 0$.

The $A_i\phi\psi$ terms vanish provided that $\beta_3 = \beta_1$, whereas the vanishing of the $D\psi$ terms set $\beta_4 = \beta_1 + i\lambda_L$, which also takes care of the $\phi\psi$ terms.

In summary, and letting $\lambda_L = -\lambda_R = \lambda$,

$$\mathcal{L}^{(3)} := -i\chi^\dagger\not{D}\psi - \chi^\dagger[\phi,\psi] - i\lambda\chi^\dagger\psi - \tfrac{1}{4}F^2 - \tfrac{1}{2}|D\phi|^2 - \tfrac{1}{2}(D+\beta_1\phi)^2 \tag{62}$$

transforms as

$$\delta_R\mathcal{L}^{(3)} = \nabla_i\left(\varepsilon^{ijk}\epsilon_R^\dagger\left(-\tfrac{1}{2}F_{jk} + iD_j\phi\sigma_k\right)\psi\right) , \tag{63}$$

under

$$\begin{aligned}
\delta_R A_i &= -i\epsilon_R^\dagger\sigma_i\psi\\
\delta_R\phi &= -\epsilon_R^\dagger\psi\\
\delta_R\chi^\dagger &= -(D+\beta_1\phi)\epsilon_R^\dagger - i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} + D_k\phi)\epsilon_R^\dagger\sigma^k\\
\delta_R\psi &= 0\\
\delta_R D &= i\epsilon_R^\dagger\not{D}\psi + \epsilon_R^\dagger[\phi,\psi] + (\beta_1 + i\lambda)\epsilon_R^\dagger\psi ,
\end{aligned} \tag{64}$$

with $\nabla_i\epsilon_L = \lambda\sigma_i\epsilon_L$ and $\nabla_i\epsilon_R^\dagger = -\lambda\epsilon_R^\dagger\sigma_i$.

One can show that the supersymmetry algebra closes as follows:

$$[\delta_L, \delta_L'] = 0 = [\delta_R, \delta_R'] \qquad \text{and} \qquad [\delta_L, \delta_R] = \mathcal{L}_\xi + \delta_\Lambda^{\text{gauge}} + \delta_\varpi^R , \tag{65}$$

for $\xi^i = 2i\epsilon_R^\dagger\sigma^i\epsilon_L$ and $\Lambda = \xi^i A_i + 2\epsilon_R^\dagger\epsilon_L\phi$, and where δ_ϖ^R is an R-symmetry transformation with $\varpi = -4\lambda\epsilon_R^\dagger\epsilon_L$, where

$$\delta_\varpi^R\psi = i\varpi\psi \qquad \text{and} \qquad \delta_\varpi^R\chi^\dagger = -i\varpi\chi^\dagger . \tag{66}$$

Indeed, it's induced from four-dimensions, where it is generated by γ^5. Notice that ϖ is actually constant, so that this is indeed a rigid R-symmetry transformation. Similarly, it is worth remarking that $\mathcal{L}_\xi$ now means the spinorial Lie derivative [27] on the spinor fields, which in our case becomes

$$\mathcal{L}_\xi\psi = \xi^i\nabla_i\psi + \lambda\xi^i\sigma_i\psi \qquad \text{and} \qquad \mathcal{L}_\xi\chi^\dagger = \xi^i\nabla_i\chi^\dagger - \lambda\xi^i\chi^\dagger\sigma_i \,. \tag{67}$$

One can check that this is indeed the expression which follows by evaluating the definition $\mathcal{L}_\xi = \nabla_\xi + \rho(A_\xi)$, with A_ξ the skew-symmetric endomorphism of the tangent bundle defined by $A_\xi(X) = -\nabla_X\xi$ and where ρ is the spin representation.

The parameter β_1 remains free and can be set to zero if so desired. This is equivalent to the field redefinition $D \rightsquigarrow D+\beta_1\phi$. Doing so, we have that the action with lagrangian

$$\mathcal{L}^{(3)} = -i\chi^\dagger \not{D}\psi - \chi^\dagger[\phi,\psi] - i\lambda\chi^\dagger\psi - \tfrac{1}{4}F^2 - \tfrac{1}{2}|D\phi|^2 - \tfrac{1}{2}D^2 \tag{68}$$

transforms as

$$\delta_L\mathcal{L}^{(3)} = \nabla_i\left(-i\chi^\dagger\left(\sigma^i D + \sigma_j F^{ij} - iD^i\phi\right)\epsilon_L\right) \tag{69}$$

$$\delta_R\mathcal{L}^{(3)} = \nabla_i\left(\varepsilon^{ijk}\epsilon_R^\dagger\left(-\tfrac{1}{2}F_{jk} + iD_j\phi\sigma_k\right)\psi\right) \tag{70}$$

under

$$\begin{aligned}
\delta_L A_i &= i\chi^\dagger\sigma_i\epsilon_L & \delta_R A_i &= -i\epsilon_R^\dagger\sigma_i\psi\\
\delta_L\phi &= \chi^\dagger\epsilon_L & \delta_R\phi &= -\epsilon_R^\dagger\psi\\
\delta_L\chi^\dagger &= 0 & \delta_R\chi^\dagger &= -D\epsilon_R^\dagger - i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} + D_k\phi)\epsilon_R^\dagger\sigma^k\\
\delta_L\psi &= D\epsilon_L + i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} - D_k\phi)\sigma^k\epsilon_L & \delta_R\psi &= 0\\
\delta_L D &= i\chi^\dagger\overleftarrow{\not{D}}\epsilon_L + [\phi,\chi^\dagger]\epsilon_L - i\lambda\chi^\dagger\epsilon_L\,, & \delta_R D &= i\epsilon_R^\dagger\not{D}\psi + \epsilon_R^\dagger[\phi,\psi] + i\lambda\epsilon_R^\dagger\psi\,,
\end{aligned} \tag{71}$$

with $\nabla_i\epsilon_L = \lambda\sigma_i\epsilon_L$ and $\nabla_i\epsilon_R^\dagger = -\lambda\epsilon_R^\dagger\sigma_i$.

2.4. **Some remarks.** The first remark is that there is only a mass term for the fermions, yet none for the scalar. (This is a choice.) The choice of λ is dictated by the geometry up to a sign, but that sign is immaterial since λ appears in the action.

Secondly, it seems that the action is not "exact" in that $\mathcal{L}^{(3)}\epsilon_R^\dagger\epsilon_L \neq \delta_L\delta_R\Xi$ for any reasonable Ξ.

Thirdly, we remark that this theory agrees morally with one of the theories in Family A in [28]. In fact, if we eliminate the auxiliary field, then it agrees with the theory described by equation (3.10) in that paper, denoted $N = 2$ in $d = 3$.

Finally, let us comment on the geometry of the manifolds admitting Killing spinors. The integrability condition for solutions of the Killing spinor equation $\nabla_i\epsilon_L = \lambda\sigma_i\epsilon_L$ says that the metric is Einstein. The vanishing of the Weyl tensor in three dimensions implies that the Riemann curvature tensor of a Einstein three-dimensional riemannian manifold can be written purely in terms of the scalar curvature and the metric; in other words, it has constant sectional curvature, where the value of the scalar curvature is related to the Killing constant λ by $R = -24\lambda^2$ in our conventions. Therefore the existence of Killing spinors with real λ forces the manifold to be hyperbolic, whereas for imaginary λ it would be spherical. In the simply-connected case, we have three-dimensional hyperbolic space and the 3-sphere, respectively, which admit the maximum number of such Killing spinors, with either sign of the Killing constant.

3. Moduli space of BPS configurations

In this section we start the analysis of the geometry of the moduli space of BPS configurations. The first observation, which is crucial for this approach to the problem, is that the BPS configurations are precisely the BPS monopoles with $D = 0$. More precisely, bosonic configurations for which $\delta_L \psi = 0$ are precisely those obeying $D = 0$ and $D_k \phi = \frac{1}{2}\varepsilon_{ijk}F^{ij}$, for which the δ_L supersymmetries with parameter ϵ_L obeying $\nabla_i \epsilon_L = \lambda \sigma_i \epsilon_L$ are preserved. This is easy to see by writing

$$\delta_L \psi = (D + i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} - D_k\phi)\sigma^k)\epsilon_L \tag{72}$$

and noticing that the determinant of $D + i(\frac{1}{2}\varepsilon_{ijk}F^{ij} - D_k\phi)\sigma^k$ is zero if and only if $D = 0$ and $\frac{1}{2}\varepsilon_{ijk}F^{ij} - D_k\phi = 0$. Similarly, the bosonic configurations with $D_k\phi = -\frac{1}{2}\varepsilon_{ijk}F^{ij}$ and $D = 0$ are precisely the ones which preserve the δ_R supersymmetries with parameter $\epsilon_R^\dagger$ obeying $\nabla_i \epsilon_R^\dagger = -\lambda \epsilon_R^\dagger \sigma_i$. It is the these latter bosonic BPS configurations whose moduli space $\mathcal{M}$ we will study in the rest of this paper. The moduli space $\mathcal{M}$ is defined as the quotient $\mathcal{P}/\mathcal{G}$ of the space $\mathcal{P}$ of solutions of the Bogomol'nyi equation

$$D_i\phi + \varepsilon_{ijk}F^{jk} = 0 \tag{73}$$

by the action of the group $\mathcal{G}$ of gauge transformations:

$$A \mapsto gAg^{-1} - dgg^{-1} \qquad \text{and} \qquad \phi \mapsto g\phi g^{-1}\,, \tag{74}$$

where $g : H^3 \to G$ is a smooth function. We mention once again that the euclidean theory has complex fields, so that strictly speaking the half-BPS states actually correspond to complexified hyperbolic monopoles with $D = 0$.

3.1. **Zero modes.** Consider a one-parameter family $A_i(s), \phi(s)$ of bosonic BPS configurations, where s is a formal parameter. This means that for all s, $A_i(s)$ and $\phi(s)$ obey the Bogomol'nyi equation

$$D_i(s)\phi(s) + \varepsilon_{ijk}F^{jk}(s) = 0\,. \tag{75}$$

Differentiating with respect to s at $s = 0$, we find

$$D_i(0)\dot\phi - [\phi(0), \dot A_i] + \varepsilon_{ijk}D^j(0)\dot A^k = 0\,, \tag{76}$$

where $\dot A_i = \frac{\partial A_i}{\partial s}\big|_{s=0}$, $\dot\phi = \frac{\partial \phi}{\partial s}\big|_{s=0}$ and $D_i(0) = \partial_i + [A_i(0), -]$. Equation (76) is the linearisation at $(A_i(0), \phi(0))$ of the Bogomol'nyi equation and solutions of that equation will be termed *bosonic zero modes*.

One way to generate bosonic zero modes is to consider the tangent vector to the orbit of a one-parameter subgroup of the group of gauge transformations. The subspace of such zero modes is the tangent space to the gauge orbit of $(A_i(0), \phi(0))$. The true tangent space to the moduli space can be identified with a suitable complement of that subspace. A choice of such a complement is essentially a choice of connection on the principal $\mathcal{G}$-bundle $\mathcal{P} \to \mathcal{M}$. In the absence of a natural riemannian metric on $\mathcal{P}$, we will employ supersymmetry to define this connection.

Supersymmetry relates the bosonic zero modes to *fermionic zero modes* $\dot\psi$ which are solutions of the (already linear) field equations for ψ at $(A_i(0), \phi(0))$:

$$\not{D}(0)\dot\psi - i[\phi(0), \dot\psi] + \lambda\dot\psi = 0\,. \tag{77}$$

Let η, ζ be Killing spinors on hyperbolic space satisfying

$$\nabla_i \eta = \lambda\sigma_i\eta \qquad \text{and} \qquad \nabla_i\zeta^\dagger = -\lambda\zeta^\dagger\sigma_i\,. \tag{78}$$

Of course, hyperbolic space has the maximal number of either class of such Killing spinors.

Let $(\dot{A}_i, \dot{\phi})$ satisfy the linearised Bogomol'nyi equation (76) and let

$$\dot{\psi} = i\dot{A}_i\sigma^i\eta - \dot{\phi}\eta \; . \tag{79}$$

We claim that $\dot{\psi}$ so defined is a fermionic zero mode provided that $(\dot{A}_i, \dot{\phi})$ obey in addition the *generalised Gauss law*

$$D^i(0)\dot{A}_i + [\phi(0), \dot{\phi}] + 4i\lambda\dot{\phi} = 0 \; . \tag{80}$$

Indeed, with the tacit evaluation at $s = 0$,

$$\begin{aligned}
&\not{D}\left(i\dot{A}_i\sigma^i\eta - \dot{\phi}\eta\right) + i\left[\left(i\dot{A}_i\sigma^i\eta - \dot{\phi}\eta\right), \phi\right] + \lambda\left(i\dot{A}_i\sigma^i\eta - \dot{\phi}\eta\right)\\
&= iD_j\dot{A}_i\sigma^j\sigma^i\eta + i\dot{A}_i\sigma^j\sigma^i\nabla_j\eta - D_i\dot{\phi}\sigma^i\eta - \dot{\phi}\not{\nabla}\eta - [\dot{A}_i, \phi]\sigma^i\eta - i[\dot{\phi}, \phi]\eta + i\lambda\dot{A}_i\sigma^i\eta - \lambda\dot{\phi}\eta\\
&= iD^i\dot{A}_i\eta - \varepsilon^{ijk}D_i\dot{A}_j\sigma_k\eta - D_i\dot{\phi}\sigma^i\eta - 4\lambda\dot{\phi}\eta - [\dot{A}_i, \phi]\sigma^i\eta - i[\dot{\phi}, \phi]\eta \; ,
\end{aligned}$$

where we have used that $\sigma^j\sigma_i\sigma_j = -\sigma_i$ and that $\not{\nabla}\eta = 3\lambda\eta$. We can rewrite the resulting expression as follows

$$\left(iD^i\dot{A}_i - i[\dot{\phi}, \phi] - 4\lambda\dot{\phi}\right)\eta - \left(\varepsilon^{ijk}D_i\dot{A}_j + D^k\dot{\phi} + [\dot{A}^k, \phi]\right)\sigma_k\eta \; , \tag{81}$$

which contains two kinds of terms: those which are proportional to $\sigma_k\eta$ vanish because of the linearised Bogomol'nyi equation (76), whereas the ones proportional to η cancel if and only if the generalised Gauss law (80) is satisfied.

One might be surprised by the last term in the generalised Gauss law as this is absent in the case of euclidean monopoles. And indeed, we see that in the flat space limit $\lambda \to 0$ this term disappears. The Gauss law is a gauge-fixing condition, or more geometrically, it is an Ehresmann connection on the principal gauge bundle $\mathcal{P} \to \mathcal{M}$ over the moduli space; that is, a $\mathcal{G}$-invariant complement to the tangent space to the gauge orbit through every point of $\mathcal{P}$. It is not hard to see that condition (80) is $\mathcal{G}$-invariant and that it provides a complement to the gauge orbits. However it is not, as in the case of euclidean monopoles, the perpendicular complement to the tangent space to the gauge orbits relative to a $\mathcal{G}$-invariant metric on $\mathcal{P}$.

Conversely, if $\dot{\psi}$ obeys equation (77), then

$$\dot{A}_i = -i\zeta^\dagger\sigma_i\dot{\psi} \qquad \text{and} \qquad \dot{\phi} = -\zeta^\dagger\dot{\psi} \tag{82}$$

obey the linearised Bogomol'nyi equation (76) and the generalised Gauss law (80). Indeed, and again with the tacit evaluation at $s = 0$,

$$\begin{aligned}
D_i&\left(-\zeta^\dagger\dot{\psi}\right) + \varepsilon_{ijk}D^j\left(-i\zeta^\dagger\sigma^k\dot{\psi}\right) - \left[\phi, \left(-i\zeta^\dagger\sigma_i\dot{\psi}\right)\right]\\
&= -\nabla_i\zeta^\dagger\dot{\psi} - \zeta^\dagger D_i\dot{\psi} - i\varepsilon_{ijk}\nabla^j\zeta^\dagger\sigma^k\dot{\psi} - i\varepsilon_{ijk}\zeta^\dagger\sigma^k D^j\dot{\psi} + i\zeta^\dagger\sigma_i[\phi, \dot{\psi}]\\
&= \lambda\zeta^\dagger\sigma_i\dot{\psi} - \zeta^\dagger D_i\dot{\psi} + i\zeta^\dagger[\phi, \dot{\psi}] + i\lambda\varepsilon_{ijk}\zeta^\dagger\sigma^{jk}\dot{\psi} - i\varepsilon_{ijk}\zeta^\dagger\sigma^k D^j\dot{\psi} \; .
\end{aligned}$$

We now use that $\varepsilon_{ijk}\sigma^{jk} = 2i\sigma_i$ and that $i[\phi, \dot{\psi}] = \not{D}\dot{\psi} + \lambda\dot{\psi}$ to arrive at

$$\begin{aligned}
D_i\left(-\zeta^\dagger\dot{\psi}\right) + \varepsilon_{ijk}D^j\left(-i\zeta^\dagger\sigma^k\dot{\psi}\right) - \left[\phi, \left(-i\zeta^\dagger\sigma_i\dot{\psi}\right)\right]&\\
= -\zeta^\dagger D_i\dot{\psi} - i\varepsilon_{ijk}\zeta^\dagger\sigma^k D^j\dot{\psi} + \zeta^\dagger\sigma_i\not{D}\dot{\psi} \; ,&
\end{aligned}$$

which is seen to vanish after using that $\sigma_i\sigma_j = g_{ij} + i\varepsilon_{ijk}\sigma^k$ to expand $\sigma_i\not{D}\dot{\psi}$.

3.2. **A four-dimensional formalism.** It is convenient for calculations to introduce a four-dimensional language. This amounts to working on the four-dimensional manifold $H^3 \times S^1$, but where the fields are invariant under translations in S^1. The relevant Clifford algebra is now generated by $\Gamma_\mu = (\Gamma_i, \Gamma_4)$ given by

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \Gamma_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{83}$$

which satisfy $\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\delta_{\mu\nu}\mathbb{I}$. Let $\zeta_R = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}$ and $\eta_R = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$, which obey the Killing spinor equations

$$\nabla_i \eta_R = -i\lambda\Gamma_i\Gamma_4\eta_R \qquad \text{and} \qquad \nabla_i\zeta_R^\dagger = -i\lambda\zeta_R^\dagger\Gamma_4\Gamma_i\ , \tag{84}$$

and in addition $\nabla_4\eta_R = 0$ and $\nabla_4\zeta_R^\dagger = 0$. The zero modes are now $\dot\Psi_L = \begin{pmatrix} \dot\psi \\ 0 \end{pmatrix}$ and $\dot A_\mu = (\dot A_i, \dot\phi)$ and the relations (79) and (82) between them can now be rewritten respectively as

$$\dot\Psi_L = i\dot A_\mu\Gamma^\mu\eta_R \qquad \text{and} \qquad \dot A_\mu = -i\zeta_R^\dagger\Gamma_\mu\dot\Psi_L\ . \tag{85}$$

It is perhaps pertinent to remark that these equations are not meant to be understood as mutual inverse relations; that is, substituting the first equation for $\dot A_\mu$ in the second equation does *not* lead to an identity and neither does substituting the second equation for $\dot\Psi_L$ into the first. What these relations do mean is that given a bosonic zero mode $\dot A_\mu$ and a Killing spinor η on H^3, the RHS of the second of the above equations defines a fermionic zero mode; and that, conversely, given a fermionic zero mode $\dot\Psi_L$ and a Killing spinor ζ on H^3, the RHS of the first of the above equations defines a bosonic zero mode.

More formally, let us define the vector spaces

$$K^\pm = \{\xi_R | \nabla_i\xi_R = \mp i\lambda\Gamma_i\Gamma_4\xi_R \quad \text{and} \quad \nabla_4\xi_R = 0\}\ . \tag{86}$$

$K^\pm$ is a two-dimensional complex vector space isomorphic to the vector space of Killing spinor fields on H^3 with the stated sign of the Killing constant; that is,

$$K^\pm \cong \{\xi | \nabla_i\xi = \pm\lambda\sigma_i\xi\}\ . \tag{87}$$

Then letting Z_0 and Z_1 stand for the vector spaces of (complexified) bosonic and fermionic zero modes, respectively, we have exhibited real bilinear maps

$$\begin{aligned} K^+ \times Z_0 &\to Z_1 \\ (\eta_R, \dot A_\mu) &\mapsto i\dot A_\mu\Gamma^\mu\eta_R \end{aligned} \qquad \text{and} \qquad \begin{aligned} K^- \times Z_1 &\to Z_0 \\ (\zeta_R, \dot\Psi_L) &\mapsto -i\zeta_R^\dagger\Gamma_\mu\dot\Psi_L\ . \end{aligned} \tag{88}$$

We may compose the maps to arrive at

$$\begin{aligned} K^+ \times K^- \times Z_0 &\to Z_0 \\ (\eta_R, \zeta_R, \dot A_\mu) &\mapsto \zeta_R^\dagger\eta_R\dot A_\mu + \zeta_R^\dagger\Gamma_\mu{}^\nu\eta_R\dot A_\nu \end{aligned} \tag{89}$$

and

$$\begin{aligned} K^+ \times K^- \times Z_1 &\to Z_1 \\ (\eta_R, \zeta_R, \dot\Psi_L) &\mapsto 2\zeta_R^\dagger\eta_R\dot\Psi_L\ , \end{aligned} \tag{90}$$

where in deriving these identities we have used the Fierz identity (17) for commuting spinors.

If we fix ζ_R and η_R such that $\zeta_R^\dagger\eta_R = \frac{1}{2}$, which we can always do, then the composite map in equation (90) is the identity, which implies that the maps in equation (88) are invertible. In particular, this implies that the vector spaces Z_0 and Z_1 of (complexified) bosonic and fermionic zero modes, respectively, are isomorphic. This is the hyperbolic analogue of the result of Zumino [29] for euclidean monopoles. That result can be rederived without using supersymmetry via the calculation of the index of the Dirac operator in the presence of a monopole. For hyperbolic monopoles this calculation has not been performed, to our knowledge, but it is conceivable that it may be possible using the generalisation of the Callias index theorem [30] in [31].

We end this section by recording that in four-dimensional language the fermionic zero modes are defined by the equation

$$\not{D}\dot{\Psi}_L = -i\lambda\Gamma_4\dot{\Psi}_L \ , \tag{91}$$

whereas those defining the bosonic zero modes are

$$D_{[\mu}\dot{A}_{\nu]} = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}D^\rho\dot{A}^\sigma \qquad \text{and} \qquad D^\mu\dot{A}_\mu = -4i\lambda\dot{A}_4 \ . \tag{92}$$

The first equation is simply the statement that the $\mathfrak{g}$-valued 2-form $D_{[\mu}\dot{A}_{\nu]}$ is antiself-dual.

3.3. **Complex structures.** We start by defining some natural endomorphisms of the complexified tangent bundle of $H^3 \times S^1$ which can be built out of the Killing spinors.

Let us choose a complex basis $\eta_{R\alpha}$ and $\zeta_{R\beta}$, for $\alpha, \beta = 1, 2$, for the vector spaces K^+ and K^- of Killing spinors, respectively, which satisfies in addition the normalisation condition $\zeta^\dagger_{R\alpha}\eta_{R\beta} = \delta_{\alpha\beta}$. Let $A_{\alpha\beta}$ be the endomorphism of $T_{\mathbb{C}}(H^3 \times S^1)$ defined by

$$A_{\alpha\beta\,\mu}{}^\nu = -i\zeta^\dagger_{R\alpha}\Gamma_\mu{}^\nu\eta_{R\beta} \ . \tag{93}$$

Then one can show that the linear combinations

$$I = A_{11} \qquad J = \tfrac{1}{2}(A_{12} + A_{21}) \qquad K = -\frac{i}{2}(A_{12} - A_{21}) \tag{94}$$

satisfy the quaternion algebra

$$I^2 = J^2 = -\mathbb{I} \qquad IJ = -JI = K \ . \tag{95}$$

More invariantly, if $\eta_R \in K^+$ and $\zeta_R \in K^-$, let

$$E_\mu{}^\nu = -i\zeta^\dagger_R\Gamma_\mu{}^\nu\eta_R \tag{96}$$

denote the corresponding endomorphism of $T_{\mathbb{C}}(H^3 \times S^1)$. It follows from the fact that η_R, ζ_R have negative chirality, i.e., $\Gamma_{1234}\eta_R = -\eta_R$ and similarly for ζ_R, that $E_{\mu\nu}$ is self-dual:

$$\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}E^{\rho\sigma} = E_{\mu\nu} \ , \tag{97}$$

and also that

$$E_\mu{}^\rho E_\rho{}^\nu = -(\zeta^\dagger_R\eta_R)^2\delta_\mu{}^\nu \ . \tag{98}$$

The proof of this expression follows from the Fierz identity (17) and tedious use of the Clifford relations. Hence if we choose η_R and ζ_R such that $\zeta^\dagger_R\eta_R = 1$, then the endomorphism E is a (complex-linear) almost complex structure on $T_{\mathbb{C}}(H^3 \times S^1)$.

In addition, from from the fact that η_R, ζ_R are Killing spinors it also follows that

$$\nabla_4 E_{\mu\nu} = 0, \qquad \nabla_i E_{4j} = 2i\lambda E_{ij} \qquad \nabla_i E_{jk} = -2i\lambda\left(\delta_{ij}E_{4k} - \delta_{ik}E_{4j}\right) \ . \tag{99}$$

Indeed, the first equation follows from the fact that $\nabla_4\zeta_R = 0 = \nabla_4\eta_R$. The second equation follows from the following calculation:

$$\begin{aligned}
\nabla_i E_{4j} &= \nabla_i \left(-i\zeta_R^\dagger \Gamma_4 \Gamma_j \eta_R\right) \\
&= -i\left(-i\lambda\zeta_R^\dagger \Gamma_4\Gamma_i\right)\Gamma_4\Gamma_j\eta_R - i\zeta_R^\dagger\Gamma_4\Gamma_j\left(-i\lambda\Gamma_i\Gamma_4\eta_R\right) \\
&= -\lambda\zeta_R^\dagger\Gamma_4\Gamma_i\Gamma_4\Gamma_j\eta_R - \lambda\zeta_R^\dagger\Gamma_4\Gamma_j\Gamma_i\Gamma_4\eta_R \\
&= \lambda\zeta_R^\dagger\left(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i\right)\eta_R \\
&= 2\lambda\zeta_R^\dagger\Gamma_{ij}\eta_R \\
&= 2i\lambda E_{ij}\ ,
\end{aligned} \tag{100}$$

where we have used the Clifford relations and the fact that $\nabla_i\zeta_R^\dagger = -i\lambda\zeta_R^\dagger\Gamma_4\Gamma_i$.

The third and final equation follows from a similar calculation:

$$\begin{aligned}
\nabla_i E_{jk} &= \nabla_i\left(-i\zeta_R^\dagger\Gamma_{jk}\eta_R\right) \\
&= -i\left(-i\lambda\zeta_R^\dagger\Gamma_4\Gamma_i\right)\Gamma_{jk}\eta_R - i\zeta_R^\dagger\Gamma_{jk}\left(-i\lambda\Gamma_i\Gamma_4\eta_R\right) \\
&= -\lambda\zeta_R^\dagger\Gamma_4\Gamma_i\Gamma_{jk}\eta_R - \lambda\zeta_R^\dagger\Gamma_{jk}\Gamma_i\Gamma_4\eta_R \\
&= -\lambda\zeta_R^\dagger\Gamma_4\left(\Gamma_i\Gamma_{jk} - \Gamma_{jk}\Gamma_i\right)\eta_R\ .
\end{aligned} \tag{101}$$

We now use the following consequences of the Clifford relations:

$$\Gamma_i\Gamma_{jk} = \Gamma_{ijk} + \delta_{ij}\Gamma_k - \delta_{ik}\Gamma_j \qquad \text{and} \qquad \Gamma_{jk}\Gamma_i = \Gamma_{jki} + \delta_{ik}\Gamma_j - \delta_{ij}\Gamma_k \tag{102}$$

whence

$$\Gamma_i\Gamma_{jk} - \Gamma_{jk}\Gamma_i = 2\delta_{ij}\Gamma_k - 2\delta_{ik}\Gamma_j\ , \tag{103}$$

and hence

$$\begin{aligned}
\nabla_i E_{jk} &= -\lambda\zeta_R^\dagger\Gamma_4\left(2\delta_{ij}\Gamma_k - 2\delta_{ik}\Gamma_j\right)\eta_R \\
&= -2\lambda\delta_{ij}\zeta_R^\dagger\Gamma_4\Gamma_k\eta_R + 2\lambda\delta_{ik}\zeta_R^\dagger\Gamma_4\Gamma_j\eta_R \\
&= -2i\lambda\left(\delta_{ij}E_{4k} - \delta_{ik}E_{4j}\right)\ .
\end{aligned} \tag{104}$$

Now we show that the endomorphisms $E_\mu{}^\nu$ act naturally on the bosonic zero modes $\dot{A}_\mu$. In other words, we show that if $\dot{A}_\mu$ obeys the linearised Bogomol'nyi equation (76) and the generalised Gauss law (80), then so does its image $\dot{B}_\mu := E_\mu{}^\nu\dot{A}_\nu$ under such an endomorphism.

We start with the generalised Gauss law (80). By definition,

$$\begin{aligned}
D^\mu\dot{B}_\mu &= D^\mu\left(E_\mu{}^\nu\dot{A}_\nu\right) \\
&= \nabla^\mu E_\mu{}^\nu\dot{A}_\nu + E^{\mu\nu}D_\mu A_\nu \\
&= \nabla^i E_i{}^\nu\dot{A}_\nu + E^{\mu\nu}D_{[\mu}A_{\nu]} \\
&= -4i\lambda E_4{}^j\dot{A}_j \\
&= -4i\lambda\dot{B}_4\ ,
\end{aligned} \tag{105}$$

where we have used equation (99) and the fact that, since $E^{\mu\nu}$ is selfdual and $D_{[\mu}A_{\nu]}$ antiselfdual, their inner product vanishes. Thus we see that $\dot{B}_\mu$ obeys the generalised Gauss law (80).

Next we show that $\dot{B}_\mu$ obeys the linearised Bogomol'nyi equation (76), which says that $D_{[\mu}\dot{B}_{\nu]}$ is antiselfdual, or equivalently, that

$$D_i\dot{B}_4 + \varepsilon_{ijk}D_j\dot{B}_k = 0\ . \tag{106}$$

Using equations (97) and (99), we calculate the first term in the left-hand side:

$$\begin{aligned} D_i \dot{B}_4 &= D_i \left(E_{4j} \dot{A}_j \right) \\ &= \nabla_i E_{4j} \dot{A}_j + E_{4j} D_i \dot{A}_j \\ &= 2i\lambda E_{ij} \dot{A}_j + E_{4j} D_i \dot{A}_j \\ &= -2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j + E_{4j} D_i \dot{A}_j \ , \end{aligned} \tag{107}$$

and then also the second term:

$$\begin{aligned} \varepsilon_{ijk} D_j \dot{B}_k &= \varepsilon_{ijk} D_j \left(E_{kl} \dot{A}_l + E_{k4} \dot{A}_4 \right) \\ &= \varepsilon_{ijk} \left(\nabla_j E_{kl} \dot{A}_l - \nabla_j E_{4k} \dot{A}_4 + E_{kl} D_j \dot{A}_l + E_{k4} D_j \dot{A}_4 \right) \\ &= \varepsilon_{ijk} \left(2i\lambda E_{4k} \dot{A}_j - 2i\lambda \varepsilon_{jkl} E_{4l} \dot{A}_4 + E_{kl} D_j \dot{A}_l - E_{4k} D_j \dot{A}_4 \right) \\ &= 2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j + 4i\lambda E_{4i} \dot{A}_4 - \varepsilon_{ijk} \varepsilon_{klm} E_{4m} D_j \dot{A}_l - E_{4k} \varepsilon_{ijk} D_j \dot{A}_4 \\ &= 2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j - E_{4j} D_j \dot{A}_i - E_{4k} \varepsilon_{ijk} D_j \dot{A}_4 \\ &= 2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j - E_{4j} D_j \dot{A}_i - E_{4k} (D_i \dot{A}_k - D_k \dot{A}_i) \\ &= 2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j - E_{4k} D_i \dot{A}_k \ , \end{aligned} \tag{108}$$

where we have used that $\dot{A}_\mu$ obeys the linearised Bogomol'nyi equation (76) and the generalised Gauss law (80). Finally, we notice that the sum of the two terms vanish.

In summary, we have shown that the vector $E_\mu{}^\nu \dot{A}_\nu$ is tangent to the moduli space. Since there is a quaternion algebra in the span of the endomorphisms $E_\mu{}^\nu$, we see that the complexified tangent space to the moduli space is a quaternionic vector space. Indeed, if we let $\dot{A}_{a\mu}$ denote a complex frame for the complexified tangent space to $\mathcal{M}$ at (A, ϕ), then we may define endomorphisms $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ of the tangent space at that point by

$$\mathcal{I}_a{}^b \dot{A}_{b\mu} = I_\mu{}^\nu \dot{A}_{a\nu} \qquad \mathcal{J}_a{}^b \dot{A}_{b\mu} = J_\mu{}^\nu \dot{A}_{a\nu} \qquad \mathcal{K}_a{}^b \dot{A}_{b\mu} = K_\mu{}^\nu \dot{A}_{a\nu} \ . \tag{109}$$

Letting the point (A, ϕ) vary we obtain a field of endomorphisms of $T_{\mathbb{C}}\mathcal{M}$ which we also call $\mathcal{I}, \mathcal{J}, \mathcal{K}$. It is evident that just like I, J, K generate a quaternion algebra, so do $\mathcal{I}, \mathcal{J}, \mathcal{K}$.

It is worth emphasising that $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are *complex linear* endomorphisms of $T_{\mathbb{C}}\mathcal{M}$; that is, they commute with the complex structure introduced when we complexified the tangent bundle of $\mathcal{M}$. That complex structure is unrelated to $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$. In fact, what we have is an action of the quaternions, say, on the right and an action of the complex numbers on the left, whence an action of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}(2, \mathbb{C})$.

4. Geometry of the moduli space

In order to probe the geometry of the moduli space $\mathcal{M}$ of hyperbolic monopoles, we will consider the multiplet corresponding to a one-dimensional sigma model, except that we do not have an action for this model. In other words, we consider maps $X : \mathbb{R} \to \mathcal{M}$, $t \mapsto X(t)$, and the associated fermions θ which are sections of $\Pi X^* T_{\mathbb{C}}\mathcal{M}$: the (oddified) pullback by X of the complexified tangent bundle of $\mathcal{M}$. In this section we will first linearise the supersymmetry transformations and in this way arrive at an expression for the supersymmetry transformations of the bosonic moduli. We will then derive the supersymmetry transformations of the fermionic moduli by demanding closure of the one-dimensional $N = 4$ supersymmetry algebra. This will also reveal the geometry of the moduli space to be that of a pluricomplex manifold.

4.1. **Linearising the supersymmetry transformations.** In this section we will derive the supersymmetry transformations for the bosonic zero modes by linearising the supersymmetry transformations preserved by the monopoles.

The δ_R supersymmetry transformations preserved by hyperbolic monopole configurations are given by equation (71). On the gauge field, and in four-dimensional language, it can be written as

$$\delta_\epsilon A_\mu = -i\epsilon_R^\dagger \Gamma_\mu \Psi_L , \tag{110}$$

which is already linear, hence at the level of the zero modes becomes

$$\delta_\epsilon \dot{A}_\mu = -i\epsilon_R^\dagger \Gamma_\mu \dot{\Psi}_L . \tag{111}$$

Choose a basis $\dot{\Psi}_{La}$ for the space Z_1 of fermionic zero modes. This defines a basis $\dot{A}_{a\mu}$ for the space Z_0 of complexified bosonic zero modes via the second map in equation (88): namely,

$$\dot{A}_{a\mu} := -i\zeta_R^\dagger \Gamma_\mu \dot{\Psi}_{La} , \tag{112}$$

where $\zeta_R \in K^-$ is a fixed Killing spinor. From equation (90) we may invert this to write $\dot{\Psi}_{La} = i\dot{A}_{a\mu}\Gamma^\mu \eta_R$ for some $\eta_R \in K^+$ such that $\zeta_R^\dagger \eta_R = \frac{1}{2}$.

We now expand the general bosonic zero mode $\dot{A}_\mu = \dot{A}_{a\mu} X^a$ as a linear combination of the basis $\dot{A}_{a\mu}$ and similarly for the general fermionic zero mode $\dot{\Psi}_L = \dot{\Psi}_{La}\theta^a$. Inserting this in equation (111), we obtain

$$\delta_\epsilon \dot{A}_\mu = \dot{A}_{a\mu}\delta_\epsilon X^a = \dot{A}_{a\nu}\epsilon_R^\dagger \Gamma_\mu \Gamma^\nu \eta_R \theta^a = \dot{A}_{a\mu}\epsilon_R^\dagger \eta_R \theta^a + \epsilon_R^\dagger \Gamma_\mu{}^\nu \eta_R \dot{A}_{a\nu}\theta^a . \tag{113}$$

The term $\epsilon_R^\dagger \Gamma_\mu{}^\nu \eta_R$ is a linear combination of the almost complex structures $I_\mu{}^\nu$, $J_\mu{}^\nu$ and $K_\mu{}^\nu$:

$$\epsilon_R^\dagger \Gamma_\mu{}^\nu \eta_R = \varepsilon_1 I_\mu{}^\nu + \varepsilon_2 J_\mu{}^\nu + \varepsilon_3 K_\mu{}^\nu , \tag{114}$$

whence

$$\dot{A}_{a\mu}\delta_\epsilon X^a = \left(\varepsilon^1 I_\mu{}^\nu + \varepsilon^2 J_\mu{}^\nu + \varepsilon^3 K_\mu{}^\nu\right)\dot{A}_{a\nu}\theta^b + \epsilon_R^\dagger \eta_R \dot{A}_{a\mu}\theta^a . \tag{115}$$

From equation (109), we may write the action of these complex structures on $\dot{A}_{a\nu}$ in terms of the almost complex structures $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ on $T_{\mathbb{C}}\mathcal{M}$. The end result is that

$$\dot{A}_{a\mu}\delta_\epsilon X^a = \left(\varepsilon^1 \mathcal{I}_b{}^a + \varepsilon^2 \mathcal{J}_b{}^a + \varepsilon^3 \mathcal{K}_b{}^a + \varepsilon^4 \mathbb{I}_b{}^a\right)\dot{A}_{a\mu}\theta^b , \tag{116}$$

where we have defined $\varepsilon^4 = \epsilon_R^\dagger \eta_R$. We remark that the $\varepsilon^{1,2,3,4}$ are Grassmann odd since so is ϵ_R. Since the $\dot{A}_{a\mu}$ are linearly independent, equation (116) is equivalent to

$$\delta_\epsilon X^a = \left(\varepsilon^1 \mathcal{I}_b{}^a + \varepsilon^2 \mathcal{J}_b{}^a + \varepsilon^3 \mathcal{K}_b{}^a + \varepsilon^4 \mathbb{I}_b{}^a\right)\theta^b , \tag{117}$$

which defines the supersymmetry transformations for the bosonic moduli X^a.

It should be possible to derive the supersymmetry transformations for the fermionic moduli θ^a from the gauge theory as well, but we have been unable to do this and instead we will derive them by demanding the closure of the supersymmetry algebra.

4.2. **Closure of the moduli space supersymmetry algebra.** We shall now constrain the geometry of the moduli space by demanding closure of the supersymmetry algebra. In contrast with the case of euclidean monopoles, where the geometry of the moduli is constrained by demanding the invariance under supersymmetry of the effective action for the zero modes, the lack of convergence of the L^2 metric means that we cannot write down an action for the zero modes. It is the closure of the supersymmetry on the zero modes which will give us geometrical information.

To this end let us define odd derivations δ_A, $A = 1, \ldots, 4$, by $\delta_\epsilon X^a = \varepsilon^A \delta_A X^a$; that is,

$$\delta_A X^a = \theta^b \mathcal{E}_{Ab}{}^a , \tag{118}$$

where $\mathcal{E}_A = (\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathbb{I})$, or completely explicitly,

$$\delta_1 X^a = \theta^b \mathcal{I}_b{}^a \qquad \delta_2 X^a = \theta^b \mathcal{J}_b{}^a \qquad \delta_3 X^a = \theta^b \mathcal{K}_b{}^a \qquad \delta_4 X^a = \theta^a \ . \tag{119}$$

Hyperbolic monopoles are half-BPS, whence they preserve 4 of the 8 supercharges of the supersymmetric Yang–Mills theory and this means that the supersymmetry on the zero modes should close on the one-dimensional $N = 4$ supersymmetry algebra:

$$\delta_A \delta_B + \delta_B \delta_A = 2i \delta_{AB} \frac{d}{dt} \ , \tag{120}$$

where t parametrises the curves $X(t), \theta(t)$. We shall denote the action of $\frac{d}{dt}$ by a prime.

Imposing this on X^a will determine the supersymmetry transformations of the fermionic moduli θ^a. For example,

$$\delta_4^2 X^a = i X'^a \implies \delta_4 \theta^a = i X'^a \ , \tag{121}$$

and also

$$\delta_1^2 X^a = i X'^a \implies \delta_1 \theta^a = -i X'^b \mathcal{I}_b{}^a - \theta^b \theta^d \partial_c \mathcal{I}_b{}^e \mathcal{I}_d{}^c \mathcal{I}_e{}^a \ , \tag{122}$$

and similarly for δ_2 and δ_3 by replacing $\mathcal{I}$ by $\mathcal{J}$ and $\mathcal{K}$, respectively. Next we impose $\delta_4 \delta_i X^a = -\delta_i \delta_4 X^a$ for $i = 1, 2, 3$. For example,

$$0 = \delta_1 \delta_4 X^a + \delta_4 \delta_1 X^a = \theta^d \theta^b \left(\partial_d \mathcal{I}_b{}^a + \partial_c \mathcal{I}_b{}^e \mathcal{I}_d{}^c \mathcal{I}_e{}^a \right) \ , \tag{123}$$

and similarly for $\mathcal{J}$ and $\mathcal{K}$. This allows to rewrite in a slightly simpler way the supersymmetry transformations for the θ^a:

$$\begin{aligned} \delta_1 \theta^a &= -i X'^b \mathcal{I}_b{}^a + \theta^b \theta^c \partial_c \mathcal{I}_b{}^a \\ \delta_2 \theta^a &= -i X'^b \mathcal{J}_b{}^a + \theta^b \theta^c \partial_c \mathcal{J}_b{}^a \\ \delta_3 \theta^a &= -i X'^b \mathcal{K}_b{}^a + \theta^b \theta^c \partial_c \mathcal{K}_b{}^a \\ \delta_4 \theta^a &= i X'^a \ , \end{aligned} \tag{124}$$

together with the conditions

$$\begin{aligned} \partial_{[b} \mathcal{I}_{c]}{}^a - \partial_d \mathcal{I}_{[b}{}^e \mathcal{I}_{c]}{}^d \mathcal{I}_e{}^a &= 0 \\ \partial_{[b} \mathcal{J}_{c]}{}^a - \partial_d \mathcal{J}_{[b}{}^e \mathcal{J}_{c]}{}^d \mathcal{J}_e{}^a &= 0 \\ \partial_{[b} \mathcal{K}_{c]}{}^a - \partial_d \mathcal{K}_{[b}{}^e \mathcal{K}_{c]}{}^d \mathcal{K}_e{}^a &= 0 \ . \end{aligned} \tag{125}$$

Multiplying each equation by the corresponding almost complex structure $\mathcal{E}_a{}^f$, we obtain the equivalent conditions

$$\begin{aligned} \partial_{[b} \mathcal{I}_{c]}{}^a \mathcal{I}_a{}^f + \partial_d \mathcal{I}_{[b}{}^f \mathcal{I}_{c]}{}^d &= 0 \\ \partial_{[b} \mathcal{J}_{c]}{}^a \mathcal{J}_a{}^f + \partial_d \mathcal{J}_{[b}{}^f \mathcal{J}_{c]}{}^d &= 0 \\ \partial_{[b} \mathcal{K}_{c]}{}^a \mathcal{K}_a{}^f + \partial_d \mathcal{K}_{[b}{}^f \mathcal{K}_{c]}{}^d &= 0 \end{aligned} \tag{126}$$

Comparing with equation (136) in Appendix A, we see that these conditions are precisely the vanishing of the following Frölicher–Nijenhuis brackets $[\mathcal{I}, \mathcal{I}] = 0$, $[\mathcal{J}, \mathcal{J}] = 0$ and $[\mathcal{K}, \mathcal{K}] = 0$, which are precisely the vanishing of the Nijenhuis tensors of the corresponding almost complex structures. In other words, $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ are (integrable) complex structures.

Finally we consider the relations imposed by $\delta_i \delta_j X^a = -\delta_j \delta_i X^a$, for $i, j = 1, 2, 3$ but $i \neq j$. For example,

$$\begin{aligned} 0 &= \delta_1 \delta_2 X^a + \delta_2 \delta_1 X^a \\ &= \theta^f \theta^d \left(\mathcal{I}_d{}^c \partial_c \mathcal{J}_f{}^a + \mathcal{J}_d{}^c \partial_c \mathcal{I}_f{}^a - \partial_d \mathcal{I}_f{}^b \mathcal{J}_b{}^a - \partial_d \mathcal{J}_f{}^b \mathcal{I}_b{}^a \right) \ , \end{aligned} \tag{127}$$

and similarly for the two pairs $(i,j) = (2,3),(3,1)$. Comparing with equation (134) in Appendix A, we see that these conditions are precisely the vanishing of the following Frölicher–Nijenhuis brackets $[\mathcal{I},\mathcal{J}] = 0$, $[\mathcal{J},\mathcal{K}] = 0$ and $[\mathcal{K},\mathcal{I}] = 0$.

Closure of the algebra on the fermionic moduli imposes no further constraints on the geometry, as we now show. First we consider

$$\delta_4\delta_4\theta^a = \delta_4\delta_4\delta_4 X^a = \delta_4(iX'^a) = i\theta'^a \;, \tag{128}$$

where we have used that δ_4 and $\frac{d}{dt}$ commute on X^a and, being derivations, on any differentiable function of X^a. In particular this implies that $\delta_4^2 = i\frac{d}{dt}$ on any (differentiable) function of X and θ. Now let us consider, for example,

$$\delta_1\delta_4\theta^a = \delta_1\delta_4^2 X^a = \delta_1(iX'^a) = i(\delta_1 X^a)' \;, \tag{129}$$

whereas on the other hand

$$\delta_4\delta_1\theta^a = \delta_4\delta_1\delta_4 X^a = -\delta_4^2\delta_1 X^a = -i(\delta_1 X^a) \;, \tag{130}$$

where we have used that $\delta_4^2 = i\frac{d}{dt}$ on $\delta_1 X^a$. Therefore we see that $\delta_1\delta_4\theta^a + \delta_4\delta_1\theta^a = 0$ and similarly for δ_2 and δ_3. This means that for all $i = 1,2,3$, $\delta_i\delta_4 + \delta_4\delta_i = 0$ on any (differentiable) function of X^a and θ^a. Now consider

$$\delta_1^2\theta^a = \delta_1^2\delta_4 X^a = -\delta_1\delta_4\delta_1 X^a = +\delta_4\delta_1^2 X^a = i\delta_4 X'^a = i\theta'^a \;. \tag{131}$$

Similar calculations show that $\delta_i^2\theta^a = i\theta'^a$ for $i = 1,2,3$, whence $\delta_i^2 = i\frac{d}{dt}$ on any (differentiable) function of X^a and θ^a. Finally, consider

$$\begin{aligned}\delta_1\delta_2\theta^a &= \delta_1\delta_2\delta_4 X^a = -\delta_1\delta_4\delta_2 X^a = +\delta_4\delta_1\delta_2 X^a\\ &= -\delta_4\delta_2\delta_1 X^a = +\delta_2\delta_4\delta_1 X^a = -\delta_2\delta_1\delta_4 X^a = -\delta_2\delta_1\theta^a \;,\end{aligned} \tag{132}$$

and similarly for the other combinations, whence we see that $\delta_i\delta_j\theta^a = -\delta_j\delta_i\theta^a$ for $i \neq j$.

In summary, if $\mathcal{E}$ is any linear combination $\mathcal{E} = \alpha\mathcal{I} + \beta\mathcal{J} + \gamma\mathcal{K}$, then $[\mathcal{E},\mathcal{E}] = 0$ and if in addition, $\alpha^2 + \beta^2 + \gamma^2 = 1$, so that $\mathcal{E}$ is an almost complex structure, the condition $[\mathcal{E},\mathcal{E}] = 0$ says that it is integrable. Hence the complexified tangent bundle to the hyperbolic monopole moduli space has a 2-sphere worth of integrable complex structures which act complex linearly. In other words, $\mathcal{M}$ has a *pluricomplex structure*, a concept introduced in [20], and which we have hereby shown to follow naturally from supersymmetry.

Acknowledgments

It is a pleasure to thank Michael Atiyah, Roger Bielawski and Michael Singer for very stimulating conversations. A preliminary version of the work described here was presented in October 2012 by the first author at the "Cycles, calibrations and nonlinear partial differential equations" conference celebrating the 70th birthday of Blaine Lawson at the Simons Center in Stony Brook, New York, and he would like to thank the organisers, particularly Michael Anderson, for the invitation to speak. This work was supported in part by the grant ST/J000329/1 "Particle Theory at the Tait Institute" from the UK Science and Technology Facilities Council. Finally, we are grateful to an anonymous referee for comments and suggestions which we believe have improved the paper.

Appendix A. The Frölicher–Nijenhuis bracket of endomorphisms

The Frölicher–Nijenhuis bracket defines graded Lie superalgebra structure on the space $\Omega^\bullet(M;TM)$ of vector-valued differential forms on a manifold M. For a modern treatment see [32, Chapter 8]. This bracket extends the Lie bracket of vector fields, thought of as elements of $\Omega^0(M;TM)$. Endomorphisms of TM can be thought of as elements of $\Omega^1(M;TM)$ and the Frölicher–Nijenhuis bracket defines a symmetric bilinear map $[-,-] : \Omega^1(M;TM) \times \Omega^1(M;TM) \to \Omega^2(M;TM)$. Paragraph 8.12 in [32] gives an explicit expression of the Frölicher–Nijenhuis bracket $[K,L]$ of two endomorphisms K, L in terms of the Lie bracket of vector fields: namely,

$$\begin{aligned}[K,L](X,Y) = [KX,LY] - [KY,LX] - L[KX,Y] + L[KY,X]\\ - K[LX,Y] + K[LY,X] + (LK+KL)[X,Y] \ .\end{aligned} \tag{133}$$

Applying this to $X = \partial_a$ and $Y = \partial_b$, we find

$$\begin{aligned}[K,L](\partial_a,\partial_b) &= [K_a{}^c\partial_c, L_b{}^d\partial_d] - [K_b{}^c\partial_c, L_a{}^d\partial_d] - L[K_a{}^c\partial_c,\partial_b]\\ &\quad + L[K_b{}^c\partial_c,\partial_a] - K[L_a{}^c\partial_c,\partial_b] + K[L_b{}^c\partial_c,\partial_a]\\ &= \big(K_a{}^c\partial_c L_b{}^d - L_b{}^c\partial_c K_a{}^d - K_b{}^c\partial_c L_a{}^d + L_a{}^c\partial_c K_b{}^d\\ &\qquad + \partial_b K_a{}^c L_c{}^d - \partial_a K_b{}^c L_c{}^d + \partial_b L_a{}^c K_c{}^d - \partial_a L_b{}^c K_c{}^d\big)\,\partial_d \ .\end{aligned} \tag{134}$$

It is perhaps easier to remember the case $K = L$:

$$\tfrac{1}{2}[K,K](X,Y) = [KX,KY] - K[KX,Y] + K[KY,X] + K^2[X,Y] \ , \tag{135}$$

from which we can recover the general case by the standard polarisation trick. Applying this to $X = \partial_a$ and $Y = \partial_b$, we find

$$\begin{aligned}\tfrac{1}{2}[K,K](\partial_a,\partial_b) &= [K_a{}^c\partial_c, K_b{}^d\partial_d] - K[K_a{}^c\partial_c,\partial_b] + K[K_b{}^c\partial_c,\partial_a]\\ &= \big(K_a{}^c\partial_c K_b{}^d - K_b{}^c\partial_c K_a{}^d - \partial_b K_a{}^c K_c{}^d + \partial_a K_b{}^c K_c{}^d\big)\,\partial_d \ .\end{aligned} \tag{136}$$

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Maxwell and Tait Institutes, School of Mathematics, University of Edinburgh