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Modelling of Weibull Distributions in Brittle Solids Using 2-Dimensional Peridynamics

L. D. Jones [a,b], L. J. Vandeperre [a,c], T.A. Haynes[a,b], M. R. Wenman[a,b]*

[a]*Department of Materials,* [b]*Centre for Nuclear Engineering, and*
[c]*Centre for Advanced Structural Ceramics,*
Imperial College London, Exhibition Road, London, SW7 2AZ

Abstract

Peridynamics is a continuum mechanics modelling method, which offers advantages over traditional continuum methods when modelling brittle fracture. Brittle fracture typically follows a Weibull fracture distribution, but this behaviour is not well represented in bond-based peridynamics using a single valued bond failure stretch. In order to recreate specific Weibull-type behaviour in bond-based peridynamics, consideration must be given to scaling the distribution to account for the size of peridynamics bonds. Care must also be taken to avoid (wherever possible) non-physical crack arrest, caused by the variations in fracture toughness in the model, distorting the distributions. In this work a method for recreating a variety of Weibull distributions is outlined, based on applying Weibull-type bond behaviour only to surface bonds, including a transition zone across one horizon. The method is shown to be insensitive to variations in mesh refinement.

Keywords: Peridynamics; Brittle Fracture; Non- Local Modelling; Weibull

1. Introduction

For many structural applications, ceramic materials have desirable properties. Their use is especially common in components designed for high temperature environments, where their low thermal expansion, high melting point, and high strength at elevated temperatures are extremely valuable. Structural ceramics are, however, generally brittle. This brittleness means the strength of a ceramic component is limited by the size of flaws and defects within it.

* Corresponding author.
E-mail address: m.wenman@imperial.ac.uk

Since the size distribution of flaws is variable between samples, large differences in strength can occur in components

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made of the same material. Predicting failure in ceramics is therefore an exercise in statistics and reliability analysis. The most widely accepted statistical distribution on which to base this analysis is the Weibull distribution [1], an abstraction based on weakest link theory. As the theory goes, the weakest link in the chain (meaning the largest flaw in the component) will be responsible for fracture. Larger components, then, or longer chains in the Weibull analogy, will typically be weaker, since they are more likely to contain a large flaw.
Depending on the exact material and loading configuration of the component, the effective size may be measured differently. In some brittle materials, most notably glasses, surface area is the dominant factor in determining strength according to the size of the object. Relatively large flaws exist primarily in the surface, while the bulk is near perfect. In materials where the surface is not a source of unusually large defects, the statistical variation of fracture strength is dependent on the effective volume of the component. In all cases, only the surface or volume that is under load is considered. The calculation of these effective surfaces and volumes is non-trivial in three dimensions [2]–[9].

Quantifying the size dependence of strength in ceramics is of particular importance for lab testing of materials. Often structural components are too large and/or expensive to test in laboratories. For this reason, tests to evaluate mechanical reliability of ceramic materials are performed on smaller components. Data from these tests may then be scaled to the relevant size using the Weibull effective volumes and surfaces.

Continuum-scale modelling is dominated by the finite element method (FE), but brittle fracture, where numerous cracks are present, may be more appropriately modelled by techniques such as peridynamics [10]–[14], an emerging continuum-mechanics modelling method in which material is represented by a network of material points connected to each other by overlapping 1D bonds. Since the governing equations of peridynamics are integrals rather than partial differentials, it is a method well-suited to modelling the nucleation and growth of discontinuities such as cracks.

Peridynamics can be formulated as either state-based or bond-based. State-based peridynamics determines the behaviour of a material point based on that of its surrounding material points, and is a common method to use, especially where the effects of creep are important [15], [16]. Bond-based peridynamics uses pairwise force relations between all material points that are within a cut-off radius of the central material point. This radius is referred to as the horizon, δ, and is often controlled using the horizon ratio, m, the ratio of horizon size to material point spacing. Peridynamic bonds can be readily implemented in a FE code using truss elements [17]–[19].

Fracture in bond-based peridynamics is driven by the failure of individual bonds when they reach a given critical stretch s_0. This s_0 value is typically applied uniformly across a model [17], [20], [21]. Although this method has been shown to be sufficient in many cases, giving representative fracture patterns, it is well known that real brittle materials are best described by fracture strength distributions, and it is therefore incumbent on the peridynamics community to investigate the effects of such distributions on peridynamics. There is a scarcity of such information in the literature, and it is noteworthy that even in 1D, recreating a given fracture distribution in peridynamics is non-trivial [22].

The two critical factors in implementing a fracture distribution in peridynamics, as discussed in [22], are heterogenization and size scaling. In order to have the crack growth (in 1D cracking should be instantaneous) controlled by stress state rather than the randomness of bond strengths, it is necessary to ensure that bonds that occupy the same space have largely the same strength. If this isn't true, the model takes on a smeared, homogenous characteristic, and displays non-brittle fracture behaviour more reminiscent of a composite.

In one dimension the problem of size scaling is relatively easily dealt with. Weibull expressed his original theory in terms of a 1D chain, so implementing the same theory in 1D, when $m = 1$, i.e. overlapping bonds, was trivial, made more complex only by the non-locality inherent to peridynamics. Implementing the same idea in 2D is more complex again. The problem has changed from using 1D bonds as part of a 1D model representing (in quite a rudimentary fashion) a 3D object, to now using 1D bonds as part of a 2D model, representing a 3D object. There are now more ways to express the "size" of the model. Where previously the ratio of model size to bond size was a 1D:1D ratio, the 1D size of a bond must now be compared to some measure of the size of a 2D object. In order to determine the nature of this measure, we must look to the behaviour of real materials.
A useable peridynamics representation of a Weibull distribution, such as the one outlined in this paper, would allow

for accurate modelling of fracture in cases where obtaining a large enough sample of physical experimental data is impractical. Weibull distributions are commonly used in fracture mechanics, but the basic framework outlined in this work could be modified to use other distributions.

2. Weibull in Peridynamics: Heterogenisation

In 1D peridynamics, one of the most important issues was avoiding the homogenising effect of having bonds of different strengths overlapping each other [22]. In order to create a heterogeneous random material, bond strengths were defined by the material points they are connected to. Randomly generated values between 0 and 1 were assigned to each material point. These values could be used as probability of failure values to calculate a failure stress using a given Weibull distribution. Each bond compared the two material points it connected to and took on the failure stress that was most different to the mean failure stress of the distribution. This method was used in order to maintain the same average failure stress in the bonds, while also introducing strong and weak areas of the model.

In 2D peridynamics, material points were still assigned random probability of failure values. Since bonds do not always exactly overlap material points the failure strain of bonds was determined by a neighbour analysis of the material points all around the bond. A material point is within the set for a given bond if it is within 1 bond-length of both ends of the bond (see Fig. 1). On failure, the bond has more influence in the direction perpendicular to itself, with the furthest points being in line with the centre of the bond. These areas are the ones where a large defect would cause the largest stress concentration on the bond, so it follows that these material points should have an effect on the failure strain of the bond. This scanning arrangement causes a single low-value material point to have interesting, and quite useful effects. All bonds in the immediate area (i.e. overlapping) take on the same low failure strain, which should mean a clean break once that critical strain is reached.

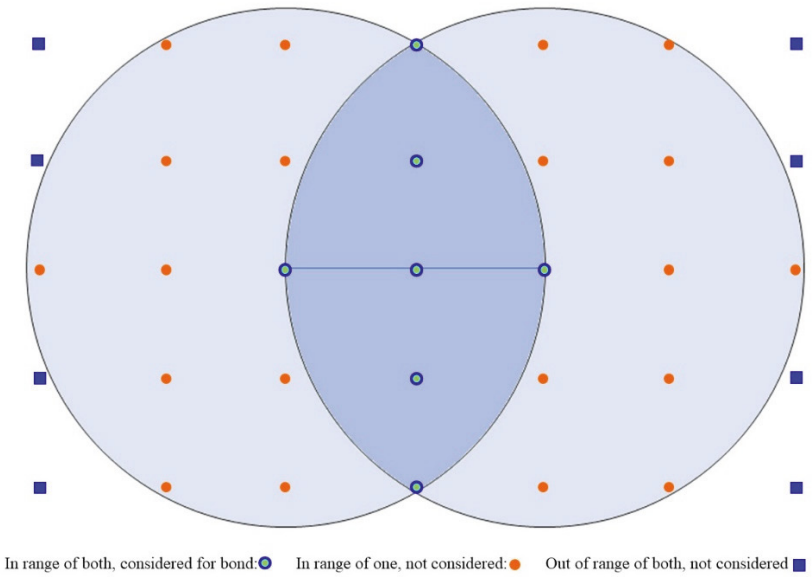

Fig. 1 A schematic of the region of material points used to determine the failure strain of a bond with length equal to twice the material point separation. The circles have radii equal to the length of the bond, represented by a blue line. The light blue region represents the region within the scanning region of one material point, and the dark region is in range of both. The randomly generated probability of failure values of any material points in the overlapping are used to determine the failure strain of the bond.

2.1. Weibull in Peridynamics: Size Scaling

The Weibull distribution is a continuous probability distribution, commonly used in assessment of reliability, life data and failure times. It has no mechanistic basis and is instead used to statistically fit to data. A two-parameter

Weibull distribution (as used in this work) defined by a characteristic value (in this case strain), ε_0, at which point probability of failure is ~63%, and a modulus, β, which defines the steepness of the curve.

Weibull-type behaviour is driven by random variation in the size and orientation of micro-scale flaws in brittle materials. Since explicitly recreating these cracks would require an impractically fine mesh, in this model this variation is represented by a random variation in the strengths of the material points. In order to recreate a particular Weibull distribution in a peridynamics model, it is necessary to adjust the distribution to account for the difference in size between the material points and the model overall. In 1D, size scaling can be derived from the original Weibull chain of links metaphor.

Indeed, if the probability of failure, $P_{f,link}$, at an applied strain, ε, for a single link can be calculated using ε_0, the characteristic strain at which $P_{f,link}$ = ~63%, and β, the Weibull modulus, which describes the spread of possible failure forces:

$$P_{f,link} = 1 - exp\left(-\left[\frac{\varepsilon}{\varepsilon_{0,link}}\right]^{\beta}\right) \tag{1}$$

The probability of failure of a chain consisting of N links is then given by:

$$P_{f,chain} = 1 - exp\left(-N\left[\frac{\varepsilon}{\varepsilon_{0,link}}\right]^{\beta}\right) = 1 - exp\left(-\left[\frac{\varepsilon}{\varepsilon_{0,chain}}\right]^{\beta}\right) \tag{2}$$

From (2), it follows that the characteristic strain for the chain can be obtained from that of the links from:

$$\varepsilon_{0,chain} = \left(\frac{1}{N}\right)^{\frac{1}{\beta}} \varepsilon_{0,link} \tag{3}$$

It is clear that the longer chains (i.e. 1D peridynamics bodies) have a lower characteristic strain than the links (bonds) that make them up. The Weibull modulus is invariant with change of size of the body, and is a material property [1]. In terms of the characteristic strains of a measured sample ($\varepsilon_{0,Sample}$) and of peridynamic bonds ($\varepsilon_{0,Bond}$) this translates as:

$$\varepsilon_{0,Sample} = \left(\frac{1}{N}\right)^{\frac{1}{\beta}} \varepsilon_{0,Bond} \tag{4}$$

or, more usefully:

$$\varepsilon_{0,Bond} = N^{\frac{1}{\beta}} \varepsilon_{0,Sample} \tag{5}$$

Where N is the number of times the Weibull distribution is sampled in the body. Using the heterogenization method outlined in [22], this is the number of material points in the body.
In real ceramic specimens, there are strength dependencies on effective surface area and effective volume. It follows, therefore, that the 2D peridynamic models of such materials would have strength dependencies on their edge length and area or in 3D their surface area and volume. In order to ascertain how the peridynamics models would best describe real materials (either as surface dominant or as volume dominant) two models were constructed. In the first, the

Weibull critical strain was modified according to the edge length (in material points) of the specimen, doubled to account for the two free surfaces of the model. In the second, the Weibull critical strain was modified according to the total number of material points in the model, as an analogue for the volume of the specimen. Examples of the models produced using these scaling methods can be seen in Fig. 2, and the distributions can be seen in Fig. 3.

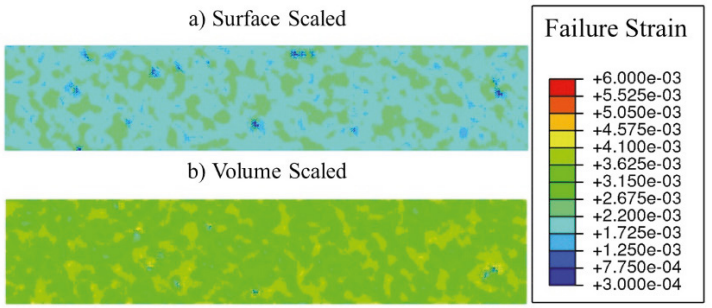

Fig. 2 Failure maps based on using surface scaling (a) and volume scaling (b). Based on a 200 x 40 material point mesh, and an intended distribution of $\beta = 6$ and $\varepsilon_0 = 8.3\text{x}10^{-4}$.

If it is assumed that fracture will occur at the surface (a reasonable assumption in many applications) the effect on fracture behaviour of the strength inside the bulk of the 2D model should ideally be negligible. This was achieved by applying the Weibull distribution only to the bonds at or near the surface and using a single value failure criterion for the remaining bulk. The value chosen for the bulk failure criterion was somewhat arbitrary. It is necessary that the bulk bond strength was high enough that fracture initiation does not occur in the bulk, but low enough that should failure occur in the Weibull region, the progress of a crack is not hindered by encountering too strong bonds in the bulk. The value chosen for this work was the strain equivalent to 99.9% probability of failure, based on the Weibull distribution for the entire part (i.e. not the distribution for the material points).

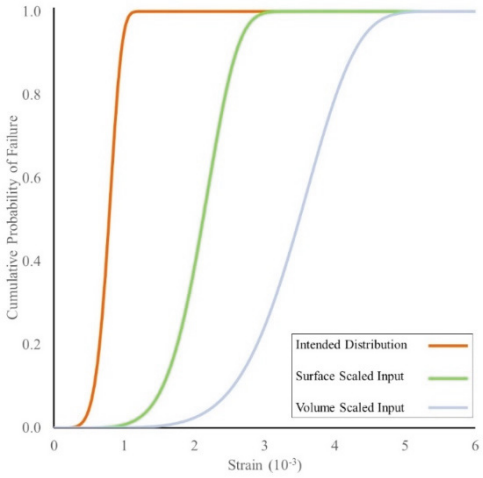

Fig. 3 The distributions used to generate bond strengths for surface and volume scaled bonds, relative to the intended distribution. Based on a 200 x 40 material point mesh, and an intended distribution of $\beta = 6$ and $\varepsilon_0 = 8.3x10^{-4}$

3. Peridynamics in Abaqus

This work was completed in Abaqus, a commercial software designed for FE simulations, rather than a peridynamics-

specific code. This necessitated a conversion of some of the terminology typically seen in peridynamics into concepts more compatible with Abaqus. There is a naming mismatch between Abaqus and peridynamics, in that bonds are represented by truss elements, and material points are represented by nodes, to which a mass must be assigned separately in Abaqus. The exact way these elements are used is outlined below, but it should be noted that where the word 'truss' is used, the reference is to Abaqus, and where the word 'bond' is used, the reference is to the peridynamics mathematical framework.

3.1. Elastic Behaviour

Peridynamics is defined by the force, *F,* response of a bond under a stretch, *s*. This relationship is controlled by *c*, the micro-modulus.

$$F = cs \tag{6}$$

The stretch, *s*, is defined as

$$s = \frac{l_{deformed} - l_{undeformed}}{l_{undeformed}} \tag{7}$$

Where *l* is the length of a bond.

We follow the definition for 2D micromodulus given by Le and Bobaru [23]:

$$c_{2D} = \frac{12E_{material}}{\pi\tau(1+\upsilon)} \tag{8}$$

Where c_{2D} is the micro-modulus in two dimensions, *E* is the elastic modulus of the material, υ is Poisson's ratio, and τ is the plate thickness.

Here, peridynamic bonds were represented by one dimensional truss elements, (type T2D2T) with linear stress response to force defined by the force, *F*, exerted on them and A_{truss}, a fictional cross-section assigned to the trusses in keeping with the Abaqus framework [18], [24].

$$\sigma_{truss} = \frac{F}{A_{truss}} \tag{9}$$

The value of A_{truss} is given as the product of plate thickness, τ_{plate}, and node spacing, *d*:

$$A_{truss} = \tau_{plate} * d \tag{10}$$

The elastic modulus input to Abaqus is the elastic modulus of these trusses, E_T. The strain of an individual truss (or the stretch of a bond) is equal to

$$\varepsilon_{truss} = \frac{\sigma_{truss}}{E_{truss}} = s_{truss} \tag{11}$$

Substituting (9) into (11) gives:

$$F = E_{truss} A\varepsilon \tag{12}$$

And comparing with (6) yields:

$$E_{truss} = \frac{c}{A_{truss}} \tag{13}$$

A correction factor, λ for converting macroscopic material elastic modulus to truss elastic modulus can therefore be defined by

$$E_{truss} \equiv \lambda E_{material} \tag{14}$$

Such that:

$$\lambda = \frac{12}{\pi\tau\delta^3(2+\upsilon)} \tag{15}$$

where δ is the horizon size.

A significant limitation of bond-based peridynamics is that certain material constants (namely Poisson's ratio) are restricted to certain values by the assumption that PD forces between two material points must be equal in magnitude and opposite in direction [10]. In this work in two dimensions, the value is fixed at ⅓, and is most appropriate to a plane stress condition.

At the edges of peridynamics meshes, elastic properties are distorted by the lack of a full horizon's worth of material points for bonds to connect to. It is therefore necessary to alter the properties of such bonds to recreate the bulk elastic properties in edge regions. Le and Bobaru [23] proposed an adjustment factor, Ω_{ij}, based on the ratio of the volume of a complete horizon, V_{max}, and the volumes associated with the material points, i and j at either end of a given bond.

$$\Omega_{ij} \equiv \frac{2V_{max}}{V_i + V_j} \tag{16}$$

In this work, the volumes in (16) are represented by the number, N, of Abaqus trusses connected to each node, such that:

$$\Omega_{ij} \approx \frac{2N_{max}}{N_i + N_j} \tag{17}$$

Where N_{max} is the maximum number of bonds connected to any material point, i.e. those in the bulk of the material. Even using this correction, the displacement under force does not match perfectly between edge and bulk regions, but the error is small enough for this to be a satisfactory approach. The external load in this work is strain controlled, and the failure criterion is defined in terms of strain, so this inaccuracy is not an issue in this work.

3.2. Bond Failure

Bonds were considered broken above a critical strain, after which their stiffness was changed to an arbitrarily low value, close to zero. This was achieved using the ductile damage model in Abaqus, so as to make use of an implicit time-step, ensuring only a small number of bonds are ever fractured in a given time-step. The stress-strain response is displayed graphically in Fig. 4.

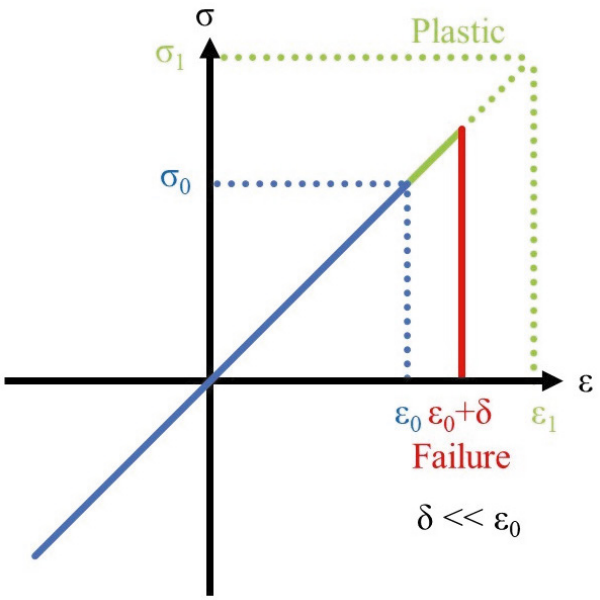

Fig. 4 A schematic of the stress-strain graph of the ductile damage model

In order to define this stress-strain response in Abaqus, four values are needed:

- σ_0: The stress at which failure occurs
- ε_1 & σ_1: Arbitrary stress and strain values > σ_0 and ε_0, but with the same ratio. This ensures any plastic strain that does accrue has the same elastic modulus as the elastic strain
- δ: an arbitrarily small value (we used 1.0 x 10^{-15}) defining the amount of plastic damage that is required to totally damage the bond. The size of this value is greatly exaggerated in Fig. 4

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- δ: an arbitrarily small value (we used 1.0 x 10^{-15}) defining the amount of plastic damage that is required to totally damage the bond. The size of this value is greatly exaggerated in Figure 1

These values are input to Abaqus using the *ELASTIC and *PLASTIC keywords, and using dependencies upon field variables allows for variation from bond to bond of these values. When in compression, a user-defined field subroutine sets the fracture strain to 1.0 x 10^{10}, ensuring that fracture does not occur in compression.

3.3. Inertial Response

Traditionally in peridynamics, bonds are massless, but Abaqus requires that some mass be assigned to them. In this formulation 1% of mass is assigned to bonds, and 99% to nodes (material points) using the *MASS keyword. Determining the mass to assign to each material point is a trivial matter of determining 99% of the mass of the component and dividing by the number of material points, which is all performed by the meshing pre-processor.
The mass assigned to the bonds is slightly more complex, in that the Abaqus input is the density of the truss elements. This requires noting the cross-section of the trusses, A_{truss}, the total length, Σl_i, of all trusses and the mass, m, attributed

to the total object.

$$\rho_{truss} \approx \frac{0.01 * m}{\sum_{i=1}^{n}(l_i) * A_{truss}} \tag{18}$$

Assigning cross-section and volume to the Abaqus truss elements is not an attempt to represent the thickness and volume of the component. It is simply a requirement of Abaqus that these properties exist. It should also be noted that the material properties in this work are arbitrary. No particular material is being recreated, so the important factor is just that response to deformation is uniform.

3.4. Simulation setup

The intention of this work is to develop a method of applying Weibull distributions in peridynamic simulations of any geometry, loading regime, or indeed code. Isolating the influence of the Weibull distribution on the peridynamics simulations required a simple loading regime. In order to keep strain uniform across the entire sample, and therefore control the effective area and length, a tensile test (see Fig. 5) was used. Typically, a 4-point flexural test is used when testing the fracture properties of brittle materials. This is due to practical concerns around the application of force in a tensile test, which do not apply in this simulation, so a tensile test was deemed appropriate. In order to prevent the slight inaccuracies near surfaces from dominating the Weibull behaviour, the bonds near the application of the strain were prohibited from failing, and the static boundary condition was applied to one whole horizon of material points. This shortening was accounted for when calculating the length of the model for Weibull scaling. For all simulations the specimen was 10 mm long, and 2 mm wide.

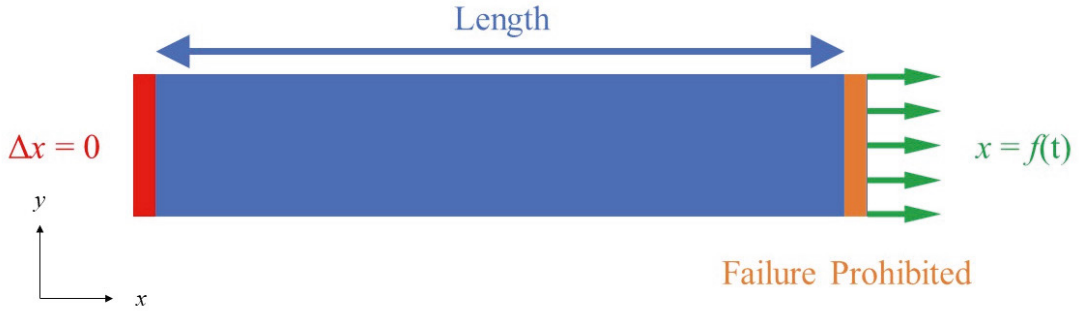

Fig. 5 Setup of simulations to evaluate Weibull behaviour in peridynamics. A strain-controlled tensile test was used.

A sample 30 models, each using a different random seed to assign material point and therefore bond strengths, can be used to fit a Weibull distribution. Different random seeds were used between samples, since simulations with the same random seed would have similar bond strength distributions, even where different Weibull parameters were used. In order to plot these output data points alongside the intended distributions, probability of failure values may be assigned to each result by ranking results by failure strain, and assigning $Pf = (i-0.5)/N$ where i is the result's rank, and N is the total number of results in the sample. In order to estimate the Weibull modulus, a graph is plotted where:

$$x_i = \ln \varepsilon_i \tag{19}$$

And

$$y_i = \ln\left(\ln\left(1 - P_{f,i}\right)\right) \tag{20}$$

The gradient of the resultant line is the Weibull modulus of the dataset.
In order to test that changes of setup did not affect the ability to reproduce Weibull distributions, some parameters were varied. Table 1 shows the default values for parameters, which were used in any case where it is not explicitly mentioned otherwise.

Table 1 Default variables for testing of Weibull distributions in peridynamics.

Parameter	Symbol	Default Value
Nodal Spacing	d	0.05 (mm)
Weibull Modulus	β	6
Characteristic Strain	ε_0	8.3 x 10^{-4}
Horizon Ratio	m	3

The default surface-scaled and volume scaled characteristic strengths were $2.2\text{x}10^{-3}$ and $3.7\text{x}10^{-3}$ respectively, and the default number of bonds was 111,036.

Modelling Weibull distributions in a tensile test is, in itself, not very useful, however the goal of this work is to create a method for applying Weibull distributions that could then be used in more complex and interesting loading schemes. The method would be considered a success if it:

1. reproduces a given Weibull distribution with reasonable accuracy;
2. was insensitive to changes in setup (i.e. different nodal spacings, different horizon ratios);
3. was sensitive to changes in Weibull parameters;
4. did not display any unphysical behaviour.

4. Results and Discussion

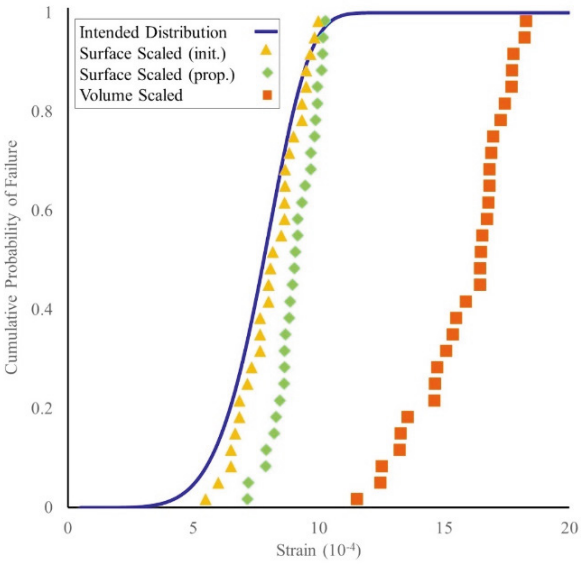

Fig. 6 The results of using entirely surface-scaled or entirely volume-scaled bond strength. The intended Weibull parameters are $\varepsilon_0 = 8.3x10^{-4}$and $\beta = 6$. The surface scaled method gave values of $\varepsilon_0 = 9.38\ x10^{-4}$ and $\beta = 12.46$. The volume scaled method gave $\beta = 8.65$ and $\varepsilon_0 = 17\ x10^{-4}$. The distribution of surface-scaled crack initiation strains had parameters: $\varepsilon_0 = 8.7\ x10^{-4}$ and $\beta = 7.94$.

Fig. 6 shows resulting distributions for both surface and volume scaled methods and show that scaling for the surface area of the model, while imperfect, is by far the better solution. The Weibull modulus is inaccurate, but this issue pales into comparison with the 104% error in the characteristic strain of the volume-scaled method. It is clear the surface-scaled method, while far from perfect, is a much closer fit than the volume-scaled method. Weibull behaviour, modelled in peridynamics in this way, captures surface dominated processes far more appropriately than volume dominated processes. While ideally the model would be flexible enough to model materials with either bulk-initiated

or surface-initiated cracks, there are plenty of applications for a model of this type.
One reason that the Weibull modulus is higher than expected in the surface-scaled method is that low strength models, in particular, are stronger than expected. This arises from crack arrest, when there is a large difference between the lowest strength bonds, where failure is initiated, and stronger bonds in the bulk. This resulted in behaviour where global strain could increase massively (often >50%, and up to ~100%) between the initiation of a crack, and the crack propagating through the model. When plotting the strains at which the cracks initiate in the surface-scaled method, a very good fit with the intended distribution can be seen. This is more evidence that scaling according to surface size is the correct method, but the application of Weibull-type variation to all bonds causes distortion of the distribution.

A more quantitative analysis of this crack arrest is shown in section 4.2. While temporary crack arrest does occur in some real materials, notably in low-temperature tested ferritic steels, [25] here this behaviour is an artefact of the attempt to control fracture initiation using Weibull distributions and is not desirable in this model of truly brittle solids.

4.1. Surface-Only Randomisation

The results of the edge-only method were better than the surface-scaled method, although still not perfect (See Fig. 7). What separates this method from the blanket application of a surface-scaled Weibull distribution is that the overestimation of strength is consistent across the distribution, meaning the shape of the distribution is roughly maintained ($\beta_{Result} = 8.16$, $\beta_{Intended} = 6$) even if at a slightly higher ε_0 value ($\varepsilon_{0,Result} = 9.28$, $\varepsilon_{0,Intended} = 6$) than desired. The improvement in accuracy relative to the all surface-scaled case can largely be attributed to a decrease in crack arrest. A quantitative comparison can be found in section 4.2.

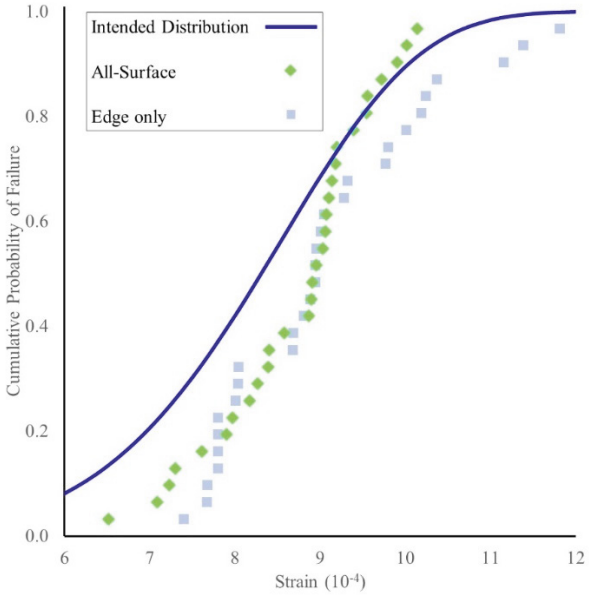

Fig. 7 The results of using the edge-only Weibull method. The intended Weibull parameters are $\varepsilon_0 = 8.3\ x10^{-4}$ and $\beta = 6$. The surface scaled method gave values of $\varepsilon_0 = 9.38\ x10^{-4}$ and $\beta = 12.46$. The edge-only method gave values of $\varepsilon_0 = 9.28x10^{-4}$ and $\beta = 10.79$.

The edge-only method largely proved sensitive to changes in input Weibull modulus (see
Table 2). There is an unwanted increase in fracture strains uniformly, at high, medium and low probability of failure values. This manifests in the output Weibull parameters as an increase in ε_0 and β. The "S-curve" has essentially been translated along the x-axis to higher strains, but without changing shape. In order to represent this in a Weibull distribution, the Weibull modulus increases. There also seems to be a lower limit cut-off around $\beta = 7.5$, where the method no longer recreates the Weibull curve faithfully. When using broader Weibull distributions (i.e. a lower β value) the overestimation of ε_0 increases. Since there is a wider range of possible values, and therefore a greater degree of randomness, it makes sense that errors in failure strain would be larger at lower β values. That the error is seemingly

systematic though, suggests that it is not just an expression of increased randomness. Between the input of Weibull parameters and the eventual calculation of the simulated ones, there are a number of steps. Any one of these could be introducing this small (in moderate β-value cases) error.

It is worth noting that the relatively small sample size of this work implies a certain degree of uncertainty in the Weibull parameters, even excluding any effects of this application of peridynamics improperly reproducing Weibull-type behaviour. This uncertainty is difficult to quantify in the peridynamics case, but work on uncertainty in Weibull distributions in general is extensive. For a sample size of 30, using i-0.5/N as a P_f estimator, Bergman [26] shows that the standard deviation of 0.186β can be expected. This variation could potentially describe the variation of the β_{input} = 6 case away from the roughly linear relationship between β_{input} and β_{output}. In section 3.2, however, it is shown that this is an effect of the particular method of using peridynamics to reproduce a Weibull distribution.

Table 2 The effect on Weibull parameters output based on varying Weibull modulus input for the edge-only models.

Input β	**Output β**	**Input ε_0 (10^{-4})**	**Output ε_0 (10^{-4})**
6.0	10.79	8.30	9.30
7.5	8.38	8.30	9.14
9.0	10.06	8.30	9.00
12.0	13.59	8.30	8.97
1.0 x10^5	N/A	8.30	8.30

The source of the overestimation error in characteristic strain does not appear to be the scaling. To investigate this the bond strengths in Abaqus were output and analysed before running the simulation, to find the lowest value, and build a Weibull distribution out of these "presumed" failure strains. Even in cases where the parameters of this distribution do not exactly match the input (because of the slightly inexact nature of samples of this size) there is often a significant difference between the ε_0 of the bond distribution, and the final output distribution based on fracture of models. This can only mean that something in the model is preventing these weaker bonds from failing at the correct strain, or that these fracture events are not always causing a catastrophic failure of the model as they should.

Testing of the bonds and observation of the models during fracture suggests that initiation of cracks is in some cases inhibited by either the stronger bonds around it, or the bulk single strength bonds. A more aggressive strategy with respect to the failure strain value for the bulk bonds (i.e. reducing it, at the risk of allowing some failure to initiate in the bulk) could reduce the overestimation of ε_0 and its dependency on Weibull modulus. A single test with an arbitrarily high Weibull modulus yielded a perfect reproduction of the characteristic strain. Although it is not known at what Weibull modulus value this error becomes negligible, it is likely to be far higher than the modulus of any engineering material. The effects of mesh refinement and horizon size on this are discussed in sections **Errore. L'origine riferimento non è stata trovata.** and 4.4 respectively.

The overestimation of the characteristic strain is small, relative to the characteristic strain itself. The sensitivity to Weibull modulus is also small in the Weibull moduli of interest. In some applications, this error would make this method unsuitable. Given that there is otherwise no other method of modelling brittle fracture according to a Weibull distribution, reproducing the slope of the curve is seen as more desirable, with the characteristic strain being less critical, as long as the error remains reasonable.

4.2. Crack Arrest

When recreating Weibull distributions in 1D, the most difficult issue to overcome was that failure of a single bond (analogous to crack initiation) did not always cause failure of the entire model.

In 2D, crack arrest was defined by the increase in global applied strain that occurred between the initiation of a through-crack and its eventual propagation. Figure 8 shows this value plotted against applied strain at initial failure for 3 cases: all surface-scaled and edge-only, with $\beta = 6$; and edge-only with $\beta = 7.5$. In all three, it is clear that there is a relationship between strain at crack initiation and the additional strain needed to propagate cracks and therefore to crack arrest. In low strain at initiation cases, the additional strain to propagate cracks is larger and hence crack arrest is a larger effect.

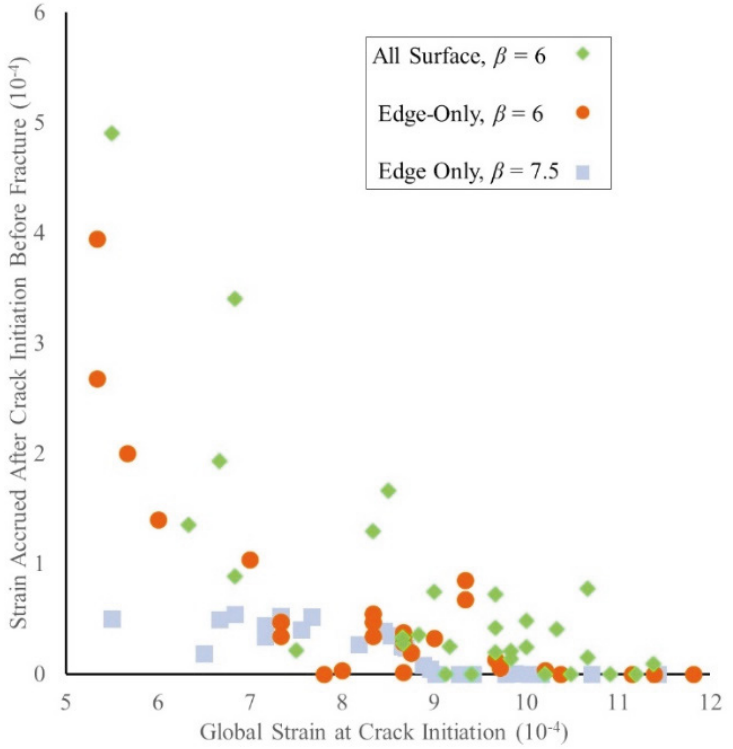

Figure 8 The additional strain to propagate cracks as a proxy for crack arrest profiles of the all surface-scaled method, and the edge-only method with β = 6.0 and β = 7.5

In the all surface-scaled case, crack arrest is an issue in the vast majority of models and increases rapidly with decreasing initiation strain. The edge-only model, with $\beta = 6$, displays a broadly similar shape, although with less crack arrest generally. It is notable that in this case, crack arrest seems to disappear at initiation strains above 1×10^{-3}, which does not occur in the all-surface case. In the all-surface case initiation generally occurs earlier because there are more opportunities for low failure strains. In the case of the edge-only model with $\beta = 7.5$ there is some similarity in shape (in that crack arrest is broadly a greater issue at low initiation strains, which occur often in low-modulus cases) the crack arrest values never reach the same scale as in the other cases.

The effect of crack arrest in the all surface-scaled method is actually underestimated by measuring it in this way, though. Take for example, Fig 9. A crack (arrowed) begins to initiate at low strain, but is prevented from growing further by strong bonds in front of the crack tip. A second through-crack at a different location does not appear until much later, at much higher strain. By the method used for generating Figure 8, this is recorded as a crack initiating at

1.09E-3 and almost immediately propagating. when in fact, crack arrest caused the failure strain of this model to increase by 118%.

One possible advantage of using the all surface-scaled method is the presence of small, dispersed areas of damage, throughout the body, which could be interpreted as realistic micro-cracks. Microcracks play an important role in brittle fracture, even when they do not initiate a fracture event in isolation. Crack paths can be diverted by the stress fields around micro-cracks and nucleation may occur as a result of the confluence of stress fields around several micro-cracks. Implementing fracture strength distributions in the bulk of objects, to give rise to these micro-cracks without also improperly distorting the Weibull properties, would be a worthy goal of any future work on this subject. The distortion and branching of the crack paths seen in Fig 9 do not fit with the expected crack path (i.e. a straight line, perpendicular to the force) in a simple tensile test such as this one. One cause of this branching is the variation of fracture strain in the bulk of the sample. This highlights an important reason to use Weibull methods at the edge-only, since they are designed to model crack initiation, not crack growth.

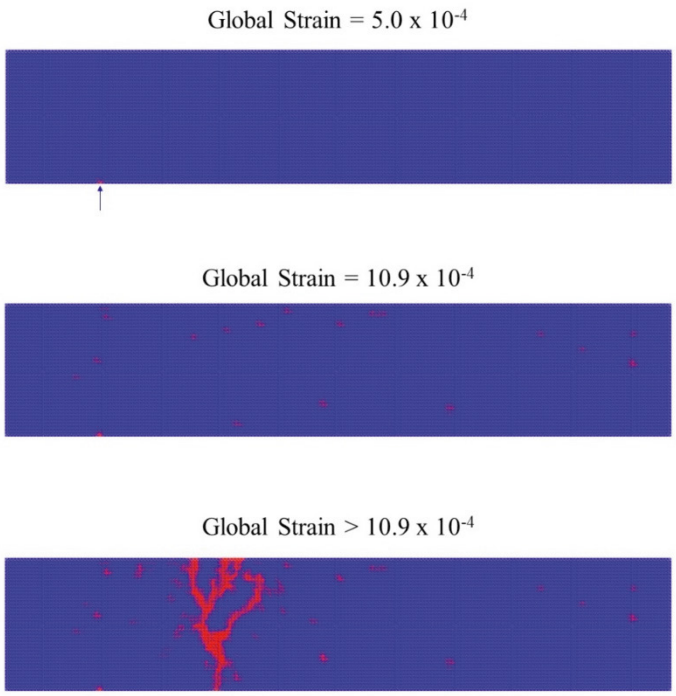

Fig 9 Using the all surface-scaled method, cracks can initiate, but then be stopped before they reach critical size. This can lead to much higher strain at failure.

Nodal Spacing (mm)	Output β	Output ε_0 (10^{-4})
0.1250	8.08	9.76
0.1000	8.60	9.50
0.0625	7.63	9.79
0.0500	8.24	9.30
0.0400	9.18	9.60

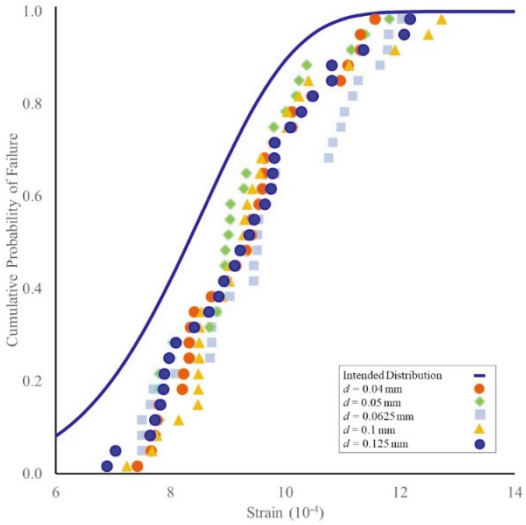

Fig. 10 The results of varying material point spacing from 0.125mm to 0.04 mm. The intended Weibull parameters are $\varepsilon_0 = 8.3x10^{-4}$ and $\beta = 6$.

4.3. Mesh Refinement Tests

In order to "stress-test" the model's sensitivity to varying mesh refinement, a value of $\beta = 6$ was used for 3 different material point spacings. Since errors are larger in lower β cases, using this lower bound value would exaggerate any issues, and make them more apparent. On examination of the Weibull parameters from these three meshes, there appears to be some sensitivity of ε_0 and β to mesh refinement with no obvious pattern (see Fig. 10). However, upon plotting the distribution graphically, it becomes clear that the five meshes are reproducing very similar Weibull distributions (See Fig. 10). The reason for the variation in ε_0 is the relatively small sample size, and it is also an expression of the relatively poor fit of the data to the Weibull curve. For each Weibull distribution, R^2 values were taken as compared to the reported Weibull parameters. For $\beta = 6$, these values were typically below 0.9, but for $\beta \geq 7.5$ the R^2 value was never below 0.96. This is further evidence that this model does not respond correctly when taken to lower β values, but even with these errors appears to be relatively insensitive to varying mesh refinement.

Since crack arrest was identified as a source of the distortion in Weibull parameters, a crack initiation profile was plotted for each mesh (see Fig. 11). The profiles of the three finest meshes are qualitatively similar, and can all be split into three ranges based on strain at crack initiation ($\varepsilon < 7 \times 10^{-4}$, $7 \times 10^{-4} > \varepsilon > 10 \times 10^{-4}$ and $\varepsilon > 10 \times 10^{-4}$) where the crack arrest is, respectively, intolerable, and exponentially increasing with decreasing initiation strain; tolerable; and largely near zero. This is further evidence that the behaviour which distorts Weibull parameters is not sensitive to mesh refinement. The two coarsest meshes display occasional deviance from this pattern, suggesting that very coarse meshes may start to degrade some part of the method. There appears to be convergence below around 0.0625 mm (320 edge material points, 5120 total material points) and more certainly below 0.05 mm (400 edge material points, 8000 total material points). Given the other deleterious effects of decreasing mesh refinement, this seems unlikely to be a limiting factor on any future simulations using this method.

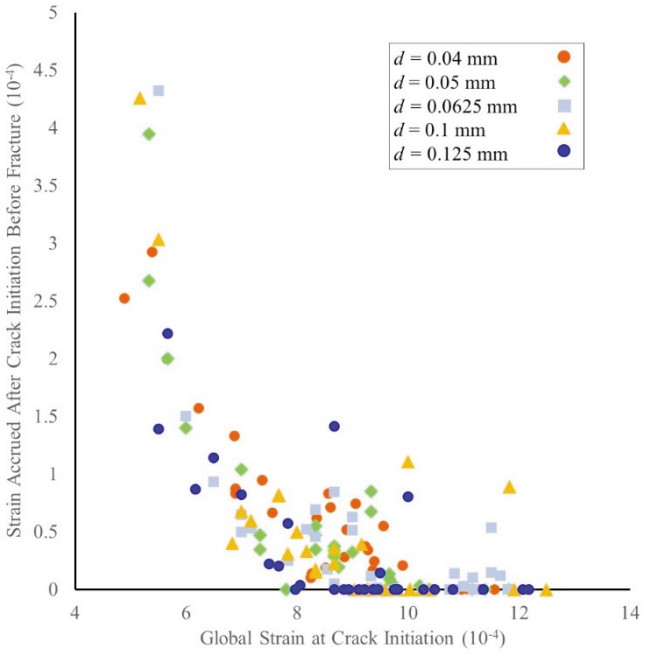

Fig. 11 The additional strain to propagate cracks as a proxy for crack arrest profiles of five different mesh refinements.

4.4. Horizon Ratio Sensitivity Testing

The surface-only method's response to changing horizon sizes was tested using horizon ratios ranging from 2-4 (see Fig. 12). The material point spacing was fixed at 0.0625 mm. As in the case of varying mesh refinement, there is some variation in Weibull parameters of the output curves, but graphically they appear similar. In the case of $m = 4$ though, the Weibull modulus is significantly higher than the others. Increasing m does appear to improve the output distribution, producing a much clearer characteristic s-curve shape. It also, however, leads to some concerning errors, the least of which is the increase in β. The "presumed" failure strain values, taken from the lowest failure strain values of the bonds, are much lower than intended, with the increase in m seemingly disrupting the scaling method. This is paired with a significantly increased tendency for crack arrest (in one case strain increased by 136% between crack initiation and propagation), which masks both errors to some degree, since they distort the distribution in opposite directions. Using $m = 2$ produces a more accurate set of Weibull parameters, but does not appear graphically to improve the fit in any meaningful way. In any case, $m = 2$ is a horizon ratio size that is not often considered useful in peridynamics, although it is worth noting that the error in recreating Weibull distributions seems to be related to non-locality, and increasing the level of non-locality exacerbates it.

Fig. 13 shows that the crack arrest behaviour is insensitive to horizon ratio, with the exception of $m = 4$. For m values of < 4, the additional strain to propagate cracks profile is largely similar for all other m values, with some small (generally < 1 x 10^{-4}) crack arrest for crack initiation values above 6 x 10^{-4}, very little for crack initiation above around 1 x 10^{-3}, and a seemingly exponential increase in crack arrest for initiation values below 6 x 10^{-4}. With $m = 4$ however, the crack arrest in the middle (6 x 10^{-4} to 1 x 10^{-3}) is consistently 2 to 3 times larger than the smaller horizon cases. Crack arrest is expected in some form, since this test was done with $\beta = 6$, previously established to be outside of the working range for Weibull modulus. There is an issue in that the method responds inconsistently and unpredictably, dependent on horizon ratio, since aberrant crack arrest values seem to appear more often in high m cases (note the relatively high crack arrest values in the high initiation strain region), using m values other than 3 cannot be recommended on this evidence. This is a limitation to the method, but $m = 3$ is probably the most commonly used

peridynamics setup so there is still significant scope for the use of the method in its current form.

Horizon Ratio	Output β	Output ε_0 (10^{-4})
2	7.38	9.51
$\sqrt{5}$	9.00	9.58
3	7.63	9.79
$\sqrt{10}$	8.64	9.39
4	11.28	9.31

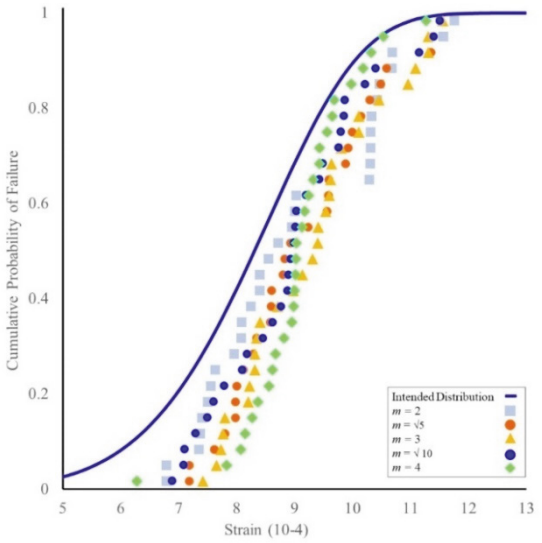

Fig. 12 The results of varying horizon ratio, m. The intended Weibull parameters are $\varepsilon_0 = 8.3\ x10^{-4}$ and $\beta = 6$.

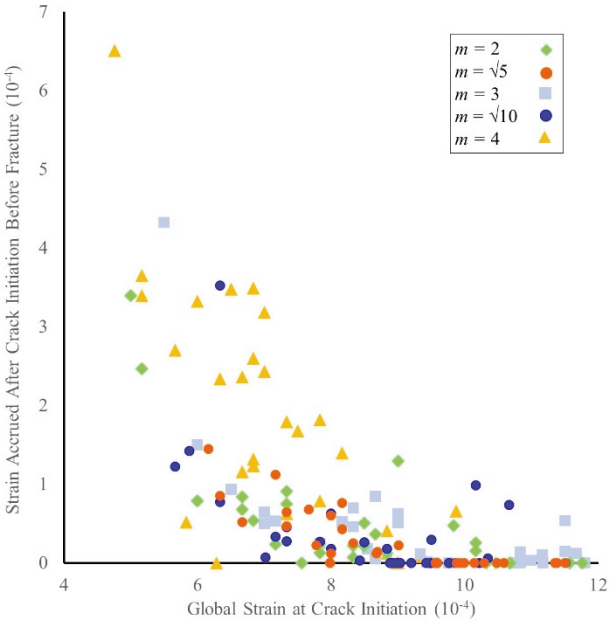

Fig. 13 The additional strain to propagate cracks as a proxy for crack arrest profiles of three different horizon ratios. The intended Weibull parameters are $\varepsilon_0 = 8.3\ x10^{-4}$ and $\beta = 6.0$. The mesh spacing is 0.0625 mm.

5. Conclusions

Weibull distributions were recreated in 2D tensile tests modelled in peridynamics by applying a scaled Weibull distribution only to the material points where fracture events are expected to initiate (i.e. the edges), and applying a single failure criterion to the remaining (bulk) bonds.

The proposed method of modelling Weibull-type fracture behaviour in peridynamics is successful in that:

- It recreated the shape of a given Weibull distribution with reasonable accuracy.
- It responds properly to changes in Weibull modulus.
- It does not change its Weibull output parameters when varying mesh fineness.

There are limitations to the model in that:

- It can only model materials dominated by surface defects, and is less suitable for materials where strength is determined by volume defects.
- There is a slight overestimate in the output Weibull characteristic strain, $\varepsilon_{0,}$ which is inversely proportional to Weibull modulus, β.
- Below $\beta = 7.5$ it does not reliably recreate Weibull behaviour, because very broad Weibull distributions introduce large amounts of crack arrest, disproportionately affecting low failure strain simulations, and distorting the Weibull distribution.
- Increasing horizon ratio, $m > 3$ causes significant error in the model behaviour. Inconsistency in crack arrest behaviour leads to different Weibull distributions for otherwise identical simulations when increasing m to 4.

The proposed method is therefore of considerable utility, given that it allows modelling of most ceramic materials with reasonable accuracy, especially in the form of qualitative comparisons where the only varied parameter is Weibull modulus. This is certainly not presented as a complete method for modelling any fracture distribution in peridynamics, but it does have many practical applications. Another important use of this work may be that it prompts work on modelling such behaviour in peridynamics, a modelling method well-suited to the task.

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