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Leibniz Gauge Theories and Infinity Structures

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Abstract: We formulate gauge theories based on Leibniz(-Loday) algebras and uncover their underlying mathematical structure. Various special cases have been developed in the context of gauged supergravity and exceptional field theory. These are based on 'tensor hierarchies', which describe towers of p-form gauge fields transforming under non-abelian gauge symmetries and which have been constructed up to low levels. Here we define 'infinity-enhanced Leibniz algebras' that guarantee the existence of consistent tensor hierarchies to arbitrary level. We contrast these algebras with strongly homotopy Lie algebras (L_∞ algebras), which can be used to define topological field theories for which all curvatures vanish. Any infinity-enhanced Leibniz algebra carries an associated L_∞ algebra, which we discuss.

1. Introduction

In this paper we construct the general gauge theory of Leibniz-Loday algebras [1–6], which are algebraic structures generalizing the notion of Lie algebras. These structures have appeared in the context of duality covariant formulations of gauged supergravity [7–10] and of string/M-theory [11–24]. Such gauge theories and their associated tensor hierarchies (towers of p-form gauge fields transforming under non-abelian gauge symmetries) have so far been constructed on a case-by-case basis up to the level needed in a given number of dimensions. Our goal is to develop gauge theories based on Leibniz(-Loday) algebras in all generality and to axiomatize the underlying mathematical structure that guarantees consistency of the tensor hierarchies up to arbitrary levels. This gauge theory construction has a certain degree of universality in that it is based on an algebraic structure encoding the most general bilinear 'product' defining transformations whose closure is governed by the same product, thereby generalizing the adjoint action of a Lie algebra.

We begin by discussing this notion of universality as a way of introducing Leibniz algebras. There is a definite sense in which the most general algebraic structures defining

(infinitesimal) symmetries are Lie algebras; indeed, we usually take the notion of continuous symmetries and Lie algebras to be synonymous. Let us briefly recall a 'proof' of this lore: suppose we are given infinitesimal variations $\delta_\lambda \phi^i$ that leave an action $I[\phi^i]$ invariant, i.e.,

$$0 = \delta_\lambda I[\phi^i] = \int \frac{\delta I}{\delta \phi^i} \delta_\lambda \phi^i, \tag{1.1}$$

where ϕ^i collectively denotes all fields. We can now act with another symmetry variation and antisymmetrize, which yields

$$0 = (\delta_{\lambda_1}\delta_{\lambda_2} - \delta_{\lambda_2}\delta_{\lambda_1})I[\phi^i] = \int \left(2 \frac{\delta^2 I}{\delta \phi^i \delta \phi^j} \delta_{\lambda_{[1}}\phi^i \delta_{\lambda_{2]}}\phi^j + \frac{\delta I}{\delta \phi^i} [\delta_{\lambda_1}, \delta_{\lambda_2}]\phi^i\right). \tag{1.2}$$

Since the second variational derivative is symmetric, the first term vanishes and we infer

$$0 = \int \frac{\delta I}{\delta \phi^i} [\delta_{\lambda_1}, \delta_{\lambda_2}]\phi^i. \tag{1.3}$$

But this means that $[\delta_{\lambda_1}, \delta_{\lambda_2}]\phi^i$ is also an invariance. Therefore, symmetries 'close', so that we can write

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]\phi^i = \delta_{[\lambda_1, \lambda_2]}\phi^i, \tag{1.4}$$

where we now take the λ to parameterize all invariances of $I[\phi^i]$ and $[\cdot, \cdot]$ on the right-hand side to be defined by this relation. Since the left-hand side of (1.4) is just a commutator, the Jacobi identity $[[\delta_{\lambda_1}, \delta_{\lambda_2}], \delta_{\lambda_3}]$+cycl. $= 0$ is identically satisifed. It follows that the antisymmetric bracket $[\cdot, \cdot]$ on the right-hand side of (1.4) also satisfies the Jacobi identity and hence defines a Lie algebra.

The above proof has several loopholes. For instance, the bracket (1.4) could be field-dependent or closure could hold only 'on-shell', i.e., modulo trivial equations-of-motion symmetries. There is a well-developed machinery in (quantum) field theory to deal with such issues, the BV formalism [25] (which in turn is related to L_∞ algebras that in this paper will play a role in a slightly different context). Here, however, we are concerned with another, more algebraic loophole: the existence of 'trivial symmetry parameters' whose action on fields vanishes, so that the Jacobi identity does not need to hold exactly for the bracket $[\cdot, \cdot]$, as long as its 'Jacobiator' lies in the space of trivial parameters. We want to ask: what is the most general bilinear algebraic operation (product) defined on a vector space that gives rise to consistent symmetry variations that close according to the same product? We will now argue that such algebraic structures are Leibniz algebras: they are defined by a bilinear operation $\circ$, satisfying the identity

$$x \circ (y \circ z) - y \circ (x \circ z) = (x \circ y) \circ z. \tag{1.5}$$

Given such a bilinear operation, we can define variations

$$\delta_x y \equiv \mathcal{L}_x y \equiv x \circ y, \tag{1.6}$$

where we introduced the notation $\mathcal{L}_x$ to be used below. Now, the Leibniz relation (1.5) is nothing more or less than the requirement that these are consistent symmetry transformations whose closure is governed by $\circ$:

$$
\begin{aligned}
[\mathcal{L}_x, \mathcal{L}_y]z &\equiv \mathcal{L}_x(\mathcal{L}_y z) - \mathcal{L}_y(\mathcal{L}_x z) \\
&= x \circ (y \circ z) - y \circ (x \circ z) \\
&= (x \circ y) \circ z \\
&= \mathcal{L}_{x \circ y} z.
\end{aligned}
\tag{1.7}
$$

In this sense, Leibniz algebras are the answer to our question above. In particular, we do not need to assume that the product $\circ$ is antisymmetric. If the product is antisymmetric, the Leibniz relation (1.5) coincides with the Jacobi identity, and hence a Lie algebra is a special case of a Leibniz algebra. If the product is not antisymmetric, it carries a non-vanishing symmetric part denoted by $\{\cdot, \cdot\}$. Symmetrizing (1.7) in x, y we infer $\mathcal{L}_{\{x,y\}}z = 0$ for any z, which implies that there is a space of trivial parameters, in which the symmetric part takes values. We will see that the antisymmetric part denoted by $[\cdot, \cdot]$ does not satisfy the Jacobi identity, but its Jacobiator yields a trivial parameter.

The reader may wonder what the significance of Leibniz algebras is, given that the symmetric part $\{\cdot, \cdot\}$, which encodes the deviation from a Lie algebra, acts trivially. Indeed, we will see that the space of trivial parameters forms an ideal of the antisymmetric bracket, hence we could pass to the quotient algebra by modding out the trivial parameters, for which the resulting bracket is antisymmetric and does satisfy the Jacobi identity. In this sense, it is indeed sufficient to work with Lie algebras. So why should we bother with Leibniz algebras? The reason is the same as for gauge symmetries in general. Gauge invariances encode redundancies of the formulation, and hence in principle can be disposed of by working on the 'space of gauge invariant functions' or, alternatively, by 'fixing a gauge'. But the fact of the matter is that a redundant formulation is often greatly beneficial. Typically, a gauge theory formulation is necessary in order to render Lorentz invariance and locality manifest. Similarly, the Leibniz algebras arising in gauged supergravity and exceptional field theory are necessary in order to render duality symmetries manifest. The price to pay is then a yet higher level of redundancy, in which one not only has equivalences between certain field configurations but also equivalences between equivalences, etc., leading to the notion of 'higher gauge theories'. (See [26] for a recent introduction to higher gauge theories.)

The higher gauge theory structures manifest themselves in the form of 'tensor hierarchies', which arise when one attempts to mimic the construction of Yang-Mills theory by introducing one-form gauge fields taking values in the Leibniz algebra. Since the associated antisymmetric bracket $[\cdot, \cdot]$ does not obey the Jacobi identity, however, one cannot define a gauge covariant field strength as in Yang-Mills theory. This can be resolved by introducing two-form potentials taking values in the space of trivial parameters and coupling it to the naive field strength. This, in turn, requires the introduction of three-form potentials in order to define a covariant field strength for the two-forms, indicating a pattern that potentially continues indefinitely. Thus, the seemingly minor relaxation of Lie algebra structures given by Leibniz algebras has profound consequence for the associated gauge theories, leading to a rich structure of higher-form symmetries. In the case of exceptional field theory, this gives a rationale for the presence of higher-form gauge fields in M-theory.

A core feature of the tensor hierarchy construction is that familiar relations from Yang-Mills theory, as closure of gauge transformations for the one-form connection or

covariance of its naive field strength, only hold 'up to higher-form gauge transformations'. The resulting structure closely resembles that of strongly homotopy Lie algebras (L_∞ algebras), in which the standard Lie algebra relations only hold 'up to homotopy', i.e., up to higher brackets that in turn satisfy higher Jacobi identities [27–30]. Indeed, the relation with L_∞ algebras has already been elaborated in a number of publications, see [31–35]. More generally, in the mathematics literature it is well established that many algebraic structures or operations have 'infinity' versions, in which the standard relations only hold 'up to homotopy'. (See, for instance, [36] and [37] for a pedagogical introduction.) Our goal here is to identify the 'infinity structure' that underlies tensor hierarchies. While L_∞ algebras and tensor hierarchies are closely related, it turns out that by themselves the former do not provide a proper axiomatization of the latter. Further structures are needed, beyond the graded antisymmetric brackets of L_∞ algebras, in order to define the most general tensor hierarchies. A first step towards the mathematical characterization of such structures was taken by Strobl, who introduced 'enhanced Leibniz algebras' [38,39], which extend a Leibniz algebra by an additional vector space, together with a new algebraic operation satisfying suitable compatibility conditions with the Leibniz product. This structure is sufficient in order to define tensor hierarchies that end with two-forms. Here we go beyond this by identifying the mathematical structure that can be used to define tensor hierarchies up to arbitrary degrees, which we term 'infinity enhanced Leibniz algebras'. In this we rely heavily on the results obtained in [22,40].

In the following we briefly display some of our core technical results. As a first step, the vector space of the Leibniz algebra is extended to a chain complex $X = \bigoplus_{n=0}^{\infty} X_n$ with a degree-(-1) differential $\mathfrak{D}$ (satisfying $\mathfrak{D}^2 = 0$), of which the Leibniz algebra forms the degree-zero subspace X_0. While the Leibniz product is only defined on X_0, we postulate a degree-$(+1)$ graded symmetric map $\bullet : X \otimes X \to X$, satisfying suitable (compatibility-)conditions with the differential $\mathfrak{D}$ and the Leibniz product, as for instance $\{x, y\} = \frac{1}{2}\mathfrak{D}(x \bullet y)$ for $x, y \in X_0$. Defining the operator $\iota_x u := x \bullet u$ for $x \in X_0$ and u arbitrary one can then define a Lie derivative with respect to $\lambda \in X_0$ in analogy to Cartan's 'magic formula' for the Lie derivative acting on differential forms,

$$\mathcal{L}_\lambda \equiv \iota_\lambda \mathfrak{D} + \mathfrak{D} \iota_\lambda. \tag{1.8}$$

This Lie derivative satisfies all familiar relations as a consequence of the general axioms we formulate. One can then define a gauge theory for a set of p-form gauge fields of arbitrary rank, each form taking values in X_{p-1}. The one-form gauge field taking values in X_0 plays a distinguished role. The resulting formulas can be written very efficiently in terms of formal sums for the remaining gauge fields $\mathcal{A} := \sum_{p=2}^{\infty} A_p$, the curvatures $\mathcal{F} := \sum_{p=2}^{\infty} \mathcal{F}_p$ and the Chern-Simons-type forms $\Omega := \sum_{p=2}^{\infty} \Omega_p$, defined by

$$\Omega_n(A) = \frac{(-1)^n}{(n-1)!} (\iota_A)^{n-2} \big[dA - \frac{1}{n} A \circ A \big]. \tag{1.9}$$

We will show that the curvatures defined by

$$\mathcal{F} = \sum_{N=0}^{\infty} \frac{(-\iota_\mathcal{A})^N}{(N+1)!} \Big[(D + \mathfrak{D})\mathcal{A} + (N+1)\Omega \Big], \tag{1.10}$$

where $D = d - \mathcal{L}_{A_1}$ is the covariant derivative, satisfy the Bianchi identity

$$D\mathcal{F} + \frac{1}{2} \mathcal{F} \bullet \mathcal{F} = \mathfrak{D}\mathcal{F}. \tag{1.11}$$

Writing it out in terms of differential forms, the Bianchi identity takes a hierarchical form that relates the covariant exterior derivative of the p-form field strength $\mathcal{F}_p$ to the differential $\mathfrak{D}$ of the $(p + 1)$-form field strength $\mathcal{F}_{p+1}$, c.f. (4.38) below. The gauge covariance of the lowest field strength $\mathcal{F}_2$ then implies by induction gauge covariance of all field strengths.

The rest of this paper is organized as follows. In Sect. 2 we discuss general results on Leibniz algebras to set the stage for the construction of gauge theories. In order to keep the paper self-contained and accessible we then present a step-by-step construction of the associated tensor hierarchy up to some low form-degree. In Sect. 3 we use various observations made along the way, generalized further to arbitrary degrees, in order to motivate the general axioms of 'infinity-enhanced Leibniz algebras' that will be used in Sect. 4 to construct exact tensor hierarchies. These are not restricted to finite degrees, and we prove the consistency of the tensor hierarchy to all orders. This construction is then contrasted in Sect. 5 with topological field theories based on L_∞ algebras, which are consistent, without the need to introduce infinity-enhanced Leibniz algebras, by virtue of all field strengths being zero. We conclude in Sect. 6 with a brief summary and outlook, while the Appendices include some technical details needed for the proof of the Bianchi identity, as well as a discussion of L_∞ algebras associated to Leibniz algebras.

Note added in proof:
During the review stage of this article there have been further developments in the understanding of infinity-enhanced Leibniz algebras [41,42]. In particular, this gives an improved motivation for the axioms of infinity-enhanced Leibniz algebras as being obtained through a derived construction from a differential graded Lie algebra and a subsequent truncation to spaces of non-negative degree. This is reflected in the new Sect. 3.2 of this paper.

2. Generalities on Leibniz Gauge Theories

In this section we develop Leibniz algebras and discuss the first few steps needed in order to define their associated gauge theories. Specifically, this requires an extension of the original vector space on which the Leibniz algebra is defined by a 'space of trivial parameters' together with a new algebraic operation. Eventually, this construction will be extended to a graded sum of vector spaces with a differential (chain complex) and a bilinear graded symmetric operation. The results of this section will motivate the general axioms to be presented in the next section.

2.1. Leibniz algebras. As outlined in the introduction, a Leibniz (or Loday) algebra is a vector space V equipped with a 'product' or 2-bracket $\circ$ satisfying for $x, y, z \in V$ the Leibniz identity (1.5), which we here rewrite as

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z). \tag{2.1}$$

This form makes it clear that the symmetry variations defined by (1.6), i.e., $\delta_x y = \mathcal{L}_x y = x \circ y$, act according to the Leibniz rule on the product $\circ$, hence explaining the name 'Leibniz algebra'. (Sometimes this is referred to as 'left Leibniz algebra'. One could also introduce a 'right Leibniz algebra', where a vector acts from the right.) Similarly,

it follows that the product is covariant under these transformations:

$$
\begin{aligned}
\delta_x(y \circ z) &\equiv \delta_x y \circ z + y \circ \delta_x z \\
&= (x \circ y) \circ z + y \circ (x \circ z) \\
&= x \circ (y \circ z) \\
&= \mathcal{L}_x(y \circ z).
\end{aligned}
\tag{2.2}
$$

Conversely, demanding that the product $\circ$ defines a symmetry operation that is covariant with respect to itself uniquely leads to the notion of a Leibniz algebra.

We can now derive some further consequences from the Leibniz relations, in particular from the closure relation (1.7),

$$
[\mathcal{L}_x, \mathcal{L}_y]z = \mathcal{L}_{x \circ y} z.
\tag{2.3}
$$

Defining

$$
\begin{aligned}
\{x, y\} &\equiv \tfrac{1}{2}(x \circ y + y \circ x), \\
[x, y] &\equiv \tfrac{1}{2}(x \circ y - y \circ x),
\end{aligned}
\tag{2.4}
$$

and symmetrizing (2.3) in x, y we have

$$
[\mathcal{L}_x, \mathcal{L}_y]z = \mathcal{L}_{[x,y]} z,
\tag{2.5}
$$

and

$$
\mathcal{L}_{\{x,y\}} z = 0 \quad \forall x, y.
\tag{2.6}
$$

Thus, the antisymmetric part defines the 'structure constants' of the more conventional (antisymmetric) gauge algebra, but we will see shortly that it does not satisfy the Jacobi identity. Indeed, as discussed in the introduction, we infer from (2.6) that in general there is a notion of 'trivial gauge parameters', given by the symmetric part, so that it is sufficient that the 'Jacobiator' is trivial in this sense. We can now prove that the 'Jacobiator' of the bracket [,] is trivial in that

$$
\mathrm{Jac}(x_1, x_2, x_3) \equiv 3[[x_{[1}, x_2], x_{3]}] = \{x_{[1} \circ x_2, x_{3]}\}.
\tag{2.7}
$$

For the proof we suppress the total antisymmetrization in 1, 2, 3. We then need to establish:

$$
6[x_1 \circ x_2, x_3] - 2\{x_1 \circ x_2, x_3\} = 0,
\tag{2.8}
$$

where we multiplied by 2 for convenience. This relation is verified by writing out the brackets, using total antisymmetry and the Leibniz identity (2.1) in the last step:

$$
\begin{aligned}
6[x_1 \circ x_2, x_3] - 2\{x_1 \circ x_2, x_3\} &= 3\,(x_1 \circ x_2) \circ x_3 - 3\,x_3 \circ (x_1 \circ x_2) \\
&\quad - (x_1 \circ x_2) \circ x_3 - x_3 \circ (x_1 \circ x_2) \\
&= 2\,(x_1 \circ x_2) \circ x_3 - 4\,x_3 \circ (x_1 \circ x_2) \\
&= 2\,(x_1 \circ x_2) \circ x_3 + 2\,x_2 \circ (x_1 \circ x_3) - 2\,x_1 \circ (x_2 \circ x_3) \\
&= 0.
\end{aligned}
\tag{2.9}
$$

It should be emphasized that the above structure is only non-trivial iff the symmetric pairing { , } takes values in a proper subspace of V, for otherwise we had with (2.6) that

$\forall x : \mathcal{L}_x z = 0$, i.e., that the product is trivial. If $\{,\} = 0$, we have a Lie algebra. More generally, the above structures define an L_∞ algebra with '2-bracket' $\ell_2(x, y) = [x, y]$. Provided the space of trivial gauge parameters forms an ideal, this follows directly from Theorem 2 in [45]. In order to prove that the trivial parameters form an ideal we have to show that the bracket of an arbitrary vector z with $\{x, y\}$ is again trivial, i.e., writable in terms of $\{,\}$. To this end, we use that the covariance property (2.2) implies the covariance of the symmetric pairing:

$$z \circ \{x, y\} = \{z \circ x, y\} + \{x, z \circ y\}. \tag{2.10}$$

Since this also equals

$$z \circ \{x, y\} = [z, \{x, y\}] + \{z, \{x, y\}\}, \tag{2.11}$$

we have

$$[z, \{x, y\}] = \{z \circ x, y\} + \{x, z \circ y\} - \{z, \{x, y\}\}. \tag{2.12}$$

This completes the proof that the bracket of a trivial element with an arbitrary vector z is itself trivial and hence that the space of trivial vectors forms an ideal. Therefore, as mentioned in the introduction, we could pass to the quotient algebra in which one identifies two vectors that differ by a 'trivial' vector, which then defines a Lie algebra. In applications, however, this can typically not be done in a duality covariant manner.

It will next turn out to be convenient to parameterize the space of trivial parameters more explicitly, so that the symmetric part of the product can be viewed as the image of a linear nilpotent operator of another algebraic operation. Specifically, we introduce a vector space U and a linear operator $\mathfrak{D} : U \to V$, so that

$$\{x, y\} = \tfrac{1}{2}\mathfrak{D}(x \bullet y), \tag{2.13}$$

where $\bullet$ is a symmetric bilinear map $V \otimes V \to U$, and the factor of $\frac{1}{2}$ is for convenience. This relation is motivated by 'infinity' structures such as L_∞ algebras, where a nilpotent differential on a chain complex governs the homotopy versions of algebraic relations, and also will turn out to be necessary in order to define tensor hierarchies explicitly. One can assume (2.13) without loss of generality. For instance, if $\{,\}$ lives in a subspace of V, we can take U to be isomorphic to this subspace and $\mathfrak{D}$ the inclusion map that views an element of U as an element of V. However, $\mathfrak{D}$ can be more general, and in particular have a non-trivial kernel. In examples, $\mathfrak{D}$ typically emerges naturally as a non-trivial operator.

Let us spell out some further assumptions on the space U and then derive some consequences of (2.13). First, (2.6) in combination with (2.13) implies $\mathcal{L}_{\mathfrak{D}(x \bullet y)} z = \mathfrak{D}(x \bullet y) \circ z = 0$ for all $x, y \in V$. We will assume that the space U has been chosen so as to precisely encode the trivial parameters in that

$$\forall u \in U : \quad \mathfrak{D}u \circ x = \mathcal{L}_{\mathfrak{D}u} x = 0. \tag{2.14}$$

Immediate corollaries are

$$\forall x \in V\, u \in U : \quad \{x, \mathfrak{D}u\} = \{\mathfrak{D}u, x\} = \tfrac{1}{2}x \circ \mathfrak{D}u, \tag{2.15}$$

and therefore with (2.13)

$$\forall x \in V\, u \in U : \quad \mathfrak{D}(x \bullet \mathfrak{D}u) = x \circ \mathfrak{D}u. \tag{2.16}$$

This means that the Leibniz product of an arbitrary vector with any trivial ($\mathfrak{D}$ exact) vector is itself $\mathfrak{D}$ exact and hence trivial. Another consequence is derived by setting $x = \mathfrak{D}v$ in (2.16),

$$\mathfrak{D}(\mathfrak{D}v \bullet \mathfrak{D}u) \;=\; \mathfrak{D}v \circ \mathfrak{D}u \;=\; 0. \tag{2.17}$$

Put differently, the $\bullet$ product of two $\mathfrak{D}$ exact elements takes values in the kernel of $\mathfrak{D}$:

$$\mathfrak{D}v \bullet \mathfrak{D}u \;\in\; \mathrm{Ker}(\mathfrak{D}). \tag{2.18}$$

In the remainder of this subsection we make the assumption that the kernel of $\mathfrak{D}$ is trivial in order to exemplify the resulting structures in the simplest possible setting and to connect to the 'enhanced Leibniz algebras' discussed recently in [39]. Put differently, we assume a structure given by the 2-term chain complex

$$U \;\xrightarrow{\mathfrak{D}}\; V, \tag{2.19}$$

so that $\mathfrak{D}u = 0$ implies $u = 0$. Although somewhat degenerate, this setup already allows us to exhibit some features that later will recur in the general context.

Our first goal is to prove that the bilinear operation $\bullet$ is covariant w.r.t. a natural action of the Leibniz algebra on $u \in U$ given by

$$\mathcal{L}_x u \;\equiv\; x \bullet \mathfrak{D}u. \tag{2.20}$$

Thus, we want to prove that

$$\delta_z(x \bullet y) \;\equiv\; (z \circ x) \bullet y + x \bullet (z \circ y) \;=\; z \bullet \mathfrak{D}(x \bullet y). \tag{2.21}$$

To this end we employ the covariance of the Leibniz product $\circ$ w.r.t. its own action, as expressed in (2.2), (2.10), to compute

$$\delta_z(\mathfrak{D}(x \bullet y)) \;=\; 2\,\delta_z\{x, y\} \;=\; 2\,z \circ \{x, y\} \;=\; z \circ \mathfrak{D}(x \bullet y) \;=\; \mathfrak{D}(z \bullet \mathfrak{D}(x \bullet y)), \tag{2.22}$$

where we used (2.16) in the last step. Since, by definition of variations, the left-hand side equals $\mathfrak{D}(\delta_z(x \bullet y))$, we have established:

$$\mathfrak{D}\big(\delta_z(x \bullet y) - z \bullet \mathfrak{D}(x \bullet y)\big) \;=\; 0. \tag{2.23}$$

Since we assumed the kernel of $\mathfrak{D}$ to be trivial, (2.21) follows, as we wanted to prove.

We can now prove that the action (2.20) of the Leibniz algebra on U closes according to the Leibniz product. We first note the general fact that the following combination lives in the kernel of $\mathfrak{D}$:

$$x \bullet (y \circ \mathfrak{D}a) - y \bullet (x \circ \mathfrak{D}a) - (x \circ y) \bullet \mathfrak{D}a \;\in\; \mathrm{Ker}(\mathfrak{D}). \tag{2.24}$$

This is verified by acting with $\mathfrak{D}$, using the defining relation (2.13) and writing out the Leibniz products:

$$
\begin{aligned}
2\big(\{x, \, & y \circ \mathfrak{D}a\} - \{y, \, x \circ \mathfrak{D}a\} - \{x \circ y, \, \mathfrak{D}a\}\big) \\
&= x \circ (y \circ \mathfrak{D}a) + (y \circ \mathfrak{D}a) \circ x - y \circ (x \circ \mathfrak{D}a) - (x \circ \mathfrak{D}a) \circ y \\
&\quad - (x \circ y) \circ \mathfrak{D}a - \mathfrak{D}a \circ (x \circ y) \\
&= (y \circ \mathfrak{D}a) \circ x - (x \circ \mathfrak{D}a) \circ y - \mathfrak{D}a \circ (x \circ y)
\end{aligned}
$$

$$\begin{aligned}
&= (y \circ \mathfrak{D}a) \circ x - (x \circ \mathfrak{D}a) \circ y \\
&= 0,
\end{aligned}$$
(2.25)

where we used the Leibniz algebra relations (2.1) and the properties of trivial parameters (2.14). This completes the proof of (2.24). Since we assume that the kernel of $\mathfrak{D}$ is trivial, it follows that the expression to the left of (2.24) vanishes. Closure of (2.20) then follows:

$$\begin{aligned}
[\mathcal{L}_{z_1}, \mathcal{L}_{z_2}]u &= z_1 \bullet \mathfrak{D}(z_2 \bullet \mathfrak{D}u) - z_1 \bullet \mathfrak{D}(z_2 \bullet \mathfrak{D}u) \\
&= z_1 \bullet (z_2 \circ \mathfrak{D}u) - z_2 \bullet (z_1 \circ \mathfrak{D}u) \\
&= (z_1 \circ z_2) \bullet \mathfrak{D}u \\
&= \mathcal{L}_{z_1 \circ z_2} u,
\end{aligned}$$
(2.26)

where we used (2.16) in the second line.

2.2. Generalization to non-trivial kernel. We will now relax some of the assumptions above. First, we allow $\mathfrak{D}$ to have a non-trivial kernel. This implies that the chain complex (2.19) has to be extended by an additional space and a new differential $\mathfrak{D}$ whose image parameterizes the kernel of the previous differential. Adopting a notation for vector spaces labelled by their degree (w.r.t. the grading of the chain complex to be developed shortly), we consider the complex

$$\cdots \quad \to \quad X_2 \quad \xrightarrow{\mathfrak{D}_2} \quad X_1 \quad \xrightarrow{\mathfrak{D}_1} \quad X_0 \,,$$
(2.27)

where we inserted, as a subscript, the space on which $\mathfrak{D}$ acts. The graded vector space, together with the linear maps $\mathfrak{D}$, forms a chain complex, which means that $\mathfrak{D}^2 = 0$ or, more precisely, $\mathfrak{D}_i \circ \mathfrak{D}_{i+1} = 0$.[1] Thus, there is a notion of *homology*: the quotient space of $\mathfrak{D}$-closed elements modulo $\mathfrak{D}$-exact elements. In the remainder of this section we will assume this homology to be trivial, so that any $\mathfrak{D}$-closed element is $\mathfrak{D}$-exact. This allows us to derive relations needed for the construction of tensor hierarchies, the beginning of which will be discussed in the next subsection. In Sects. 3 and 4 below we will then take these relations to be imposed axiomatically, so that the homology need not be trivial.

Let us now develop some relations involving elements of the new space X_2. From (2.18) we infer that the symmetric pairing of two $\mathfrak{D}_1$ exact elements takes values in the kernel of $\mathfrak{D}_1$. Thus, by the assumption of trivial homology, the result is $\mathfrak{D}$-exact — with respect to the new $\mathfrak{D}_2$. In analogy to (2.13) we then introduce a new bilinear operation $\bullet$ to write

$$\forall a, b \in X_1 : \quad \mathfrak{D}_1 a \bullet \mathfrak{D}_1 b \equiv -\mathfrak{D}_2(a \bullet \mathfrak{D}_1 b) \equiv -\mathfrak{D}_2(b \bullet \mathfrak{D}_1 a),$$
(2.28)

where the sign is for later convenience. Moreover, we have introduced on the r.h.s. maps $\bullet : X_1 \otimes X_0 \to X_2$ of intrinsic degree +1. The equality of both forms on the r.h.s. follows from the l.h.s. being symmetric in a, b, so that we can assume that the r.h.s. is also symmetric. Put differently, we can assume that the antisymmetric part is $\mathfrak{D}$ exact and write

$$a \bullet \mathfrak{D}_1 b - b \bullet \mathfrak{D}_1 a = \mathfrak{D}_3(a \bullet b),$$
(2.29)

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[1] As customary, we also denote the composition of maps by $\circ$. It should always be clear from the context whether we mean the Leibniz product or composition.

with $a \bullet b \in X_3$ and a new differential $\mathfrak{D}_3 : X_3 \to X_2$. More generally, we can anticipate the existence of a bilinear operation $\bullet$ of intrinsic degree 1 that is *graded symmetric*, i.e.,

$$\forall A, B \in X : \quad A \bullet B = (-1)^{|A||B|} B \bullet A, \quad A \bullet B \in X_{|A||B|+1}. \quad (2.30)$$

Indeed, according to the grading for $a, b \in X_1$ the product needs to be antisymmetric, in agreement with the implicit definition (2.29).

Our next goal is to define a generalization of the Leibniz action (2.20) that is valid on the entire chain complex. Specifically, we define a generalized Lie derivative via 'Cartan's magic formula'

$$\mathcal{L}_z a \equiv z \bullet \mathfrak{D}a + \mathfrak{D}(z \bullet a), \quad (2.31)$$

for $z \in X_0$ and $a \in X_i$, $i > 0$. The complete analogy to Cartan's formula for Lie derivatives of differential forms can be made manifest by introducing the map

$$\iota_z : X_i \to X_{i+1}, \quad \iota_z(a) \equiv z \bullet a, \quad (2.32)$$

for $z \in X_0$, because then (2.31) can be written as

$$\mathcal{L}_z = \iota_z \mathfrak{D} + \mathfrak{D} \iota_z. \quad (2.33)$$

We will next try to establish standard relations for Lie derivatives, which in turn requires imposing further relations between $\mathfrak{D}$ and $\bullet$. We first show that an element of the original Leibniz algebra that is $\mathfrak{D}$-exact acts trivially according to the Cartan formula — as it should be in view of the interpretation of X_1 as the 'space of trivial parameters'. To this end we set $z = \mathfrak{D}b$ and compute with (2.31)

$$\mathcal{L}_{\mathfrak{D}b} a = \mathfrak{D}b \bullet \mathfrak{D}a + \mathfrak{D}(\mathfrak{D}b \bullet a) = -\mathfrak{D}(a \bullet \mathfrak{D}b) + \mathfrak{D}(\mathfrak{D}b \bullet a) = 0, \quad (2.34)$$

using (2.28) and the graded commutativity of $\bullet$. (The sign choice in (2.28) was made such that the trivial parameters of the action given by the Cartan formula are $\mathfrak{D}$-exact.) Another direct consequence of the definition (2.31) is that Lie derivatives commute with $\mathfrak{D}$:

$$[\mathfrak{D}, \mathcal{L}_x] = \mathfrak{D}(\iota_x \mathfrak{D} + \mathfrak{D} \iota_x) - (\iota_x \mathfrak{D} + \mathfrak{D} \iota_x)\mathfrak{D} = 0. \quad (2.35)$$

Put differently, $\mathfrak{D}$ is a covariant operation.

Let us now address the crucial question whether the generalized Lie derivatives (2.33) form an algebra. This can be easily seen to be the case if and only if the Lie derivative is 'covariant' w.r.t its own action. Since, as just established, $\mathfrak{D}$ is covariant, we expect that the Lie derivatives close if we assume that the operations $\bullet$ are defined so as to be covariant. More precisely, we demand that the operation (2.32) transforms covariantly under the generalized Lie derivatives (2.31) in the sense that

$$\delta_x(\iota_y(A)) \equiv \iota_{x \circ y}(A) + \iota_y(\mathcal{L}_x A) \equiv \mathcal{L}_x(\iota_y(A)). \quad (2.36)$$

The last equality is the statement of covariance. This can be rewritten as

$$\mathcal{L}_x \iota_y - \iota_y \mathcal{L}_x = \iota_{x \circ y}. \quad (2.37)$$

Even shorter, and together with (2.35), we thus have

$$[\mathcal{L}_x, \mathfrak{D}] = 0, \quad [\mathcal{L}_x, \iota_y] = \iota_{x \circ y}. \quad (2.38)$$

This is sufficient in order to prove closure of the algebra of generalized Lie derivatives:

$$
\begin{aligned}
[\mathcal{L}_x, \mathcal{L}_y] \\
= \mathcal{L}_x(\iota_y\mathfrak{D} + \mathfrak{D}\iota_y) - (\iota_y\mathfrak{D} + \mathfrak{D}\iota_y)\mathcal{L}_x \\
= (\mathcal{L}_x\iota_y - \iota_y\mathcal{L}_x)\mathfrak{D} + \mathfrak{D}(\mathcal{L}_x\iota_y - \iota_y\mathcal{L}_x) \\
= \iota_{x\circ y}\mathfrak{D} + \mathfrak{D}\iota_{x\circ y} \\
= \mathcal{L}_{x\circ y}.
\end{aligned}
\tag{2.39}
$$

We recall that Cartan's formula and hence the above proof only hold when acting on objects in X_i, $i > 0$. Of course, for elements in X_0 closure follows from the Leibniz algebra properties.

2.3. Tensor hierarchy at low levels. We will now turn to the formulation of gauge theories based on algebraic structures satisfying the relations discussed in the previous subsection, exhibiting the first few steps in the construction of a tensor hierarchy. In this one tries to mimic the construction of Yang-Mills theory: one introduces one-forms $A = A_\mu \, dx^\mu$, with x^μ the coordinates of the base 'spacetime' manifold, but taking values in a Leibniz algebra instead of a Lie algebra. Following the standard textbook treatment of gauge theories, we next aim to define covariant derivatives and field strengths. We can define covariant derivatives for any fields in X by

$$
D_\mu = \partial_\mu - \mathcal{L}_{A_\mu},
\tag{2.40}
$$

with the universal form (2.31) of the generalized Lie derivative. It is a quick computation using the closure relation (2.39) to verify that the covariant derivative transforms covariantly, i.e., according to the same Lie derivative (2.31). (For an explicit display of this proof see eq. (139) in [43].) Moreover, since the $\bullet$ operation is covariant under these Lie derivatives we immediately have the Leibniz rule:

$$
D_\mu(a \bullet b) = D_\mu a \bullet b + a \bullet D_\mu b,
\tag{2.41}
$$

for arbitrary $a, b \in X$.

For the gauge potential A_μ, we postulate gauge transformations w.r.t. to a Leibniz-algebra valued gauge parameter $\lambda \in X_0$:

$$
\delta_\lambda A_\mu = D_\mu\lambda \equiv \partial_\mu\lambda - A_\mu \circ \lambda.
\tag{2.42}
$$

The important difference to Yang-Mills theory originates from the fact that in general these gauge transformations do not close by themselves. Using the Leibniz algebra relations and (2.13) we compute for the commutator of (2.42):

$$
\begin{aligned}
[\delta_{\lambda_1}, \delta_{\lambda_2}]A_\mu &= -2\,\partial_\mu\lambda_{[1} \circ \lambda_{2]} + 2\,(A_\mu \circ \lambda_{[1}) \circ \lambda_{2]} \\
&= -\partial_\mu\lambda_{[1} \circ \lambda_{2]} - \lambda_{[1} \circ \partial_\mu\lambda_{2]} + 2\{\lambda_{[1}, \partial_\mu\lambda_{2]}\} \\
&\quad + A_\mu \circ (\lambda_{[1} \circ \lambda_{2]}) - 2\{\lambda_{[1}, A_\mu \circ \lambda_{2]}\} \\
&= -D_\mu(\lambda_{[1} \circ \lambda_{2]}) + 2\{\lambda_{[1}, D_\mu\lambda_{2]}\} \\
&= D_\mu[\lambda_2, \lambda_1] + \mathfrak{D}(\lambda_{[1} \bullet D_\mu\lambda_{2]}).
\end{aligned}
\tag{2.43}
$$

The first term on the right-hand side takes the form of $\delta_{12}A_\mu$, with $\lambda_{12} = [\lambda_2, \lambda_1]$, but the second term is inconsistent with closure. The notation above suggests already the resolution: one postulates a new gauge symmetry with a parameter $\lambda_\mu \in X_1$:

$$
\delta_\lambda A_\mu = D_\mu\lambda - \mathfrak{D}\lambda_\mu.
\tag{2.44}
$$

We then have closure according to $[\delta_{\lambda_1}, \delta_{\lambda_2}]A_\mu = D_\mu\lambda_{12} - \mathfrak{D}\lambda_{12\mu}$, where

$$\lambda_{12} = [\lambda_2, \lambda_1], \qquad \lambda_{12\mu} = \lambda_{[2} \bullet D_\mu\lambda_{1]}. \tag{2.45}$$

We now turn to the definition of a non-abelian field strength for A_μ, starting from the ansatz

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu], \tag{2.46}$$

where further (2-form) terms will be added to achieve gauge covariance. The need for this modification is most efficiently shown by computing the general variation of the field strength and demanding covariance. Under a general variation δA_μ we compute

$$\begin{aligned}
\delta F_{\mu\nu} &= 2\,\partial_{[\mu}(\delta A_{\nu]}) - 2[A_{[\mu}, \delta A_{\nu]}] \\
&= 2(\partial_{[\mu}\delta A_{\nu]} - A_{[\mu} \circ \delta A_{\nu]} + \{A_{[\mu}, \delta A_{\nu]}\}) \\
&= 2\,D_{[\mu}\delta A_{\nu]} + \mathfrak{D}(A_{[\mu} \bullet \delta A_{\nu]}),
\end{aligned} \tag{2.47}$$

where we used (2.13). Thus, we have succeeded to write $\delta F_{\mu\nu}$ in terms of the covariant derivative of δA_μ only up to $\mathfrak{D}$-exact terms. This is now remedied by introducing a 2-form $B_{\mu\nu} \in X_1$ and completing the definition of the field strength as

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] + \mathfrak{D}B_{\mu\nu}. \tag{2.48}$$

It follows with (2.47) that the general variation under $\delta A_\mu, \delta B_{\mu\nu}$ takes the form

$$\delta\mathcal{F}_{\mu\nu} = 2\,D_{[\mu}\,\delta A_{\nu]} + \mathfrak{D}(\Delta B_{\mu\nu}), \tag{2.49}$$

where we defined the 'covariant variations'

$$\Delta B_{\mu\nu} \equiv \delta B_{\mu\nu} + A_{[\mu} \bullet \delta A_{\nu]}. \tag{2.50}$$

We will see that these covariant variations of higher forms recur in all covariant formulas.

We can now determine the gauge transformations of the 2-forms so that the field strength transforms covariantly. To this end we use that with (2.39) we have for the commutator of covariant derivatives:

$$[D_\mu, D_\nu] = -\mathcal{L}_{F_{\mu\nu}} \equiv -\mathcal{L}_{\mathcal{F}_{\mu\nu}}. \tag{2.51}$$

Note that, due to (2.34), in this formula it is immaterial whether the field strength on the right-hand side contains the 2-form term in (2.48) or not. Covariance of $\mathcal{F}_{\mu\nu}$ under (2.42) now follows, provided we postulate the following gauge transformations for $B_{\mu\nu}$, written in terms of (2.50),

$$\Delta_\lambda B_{\mu\nu} = 2\,D_{[\mu}\lambda_{\nu]} + \mathcal{F}_{\mu\nu} \bullet \lambda. \tag{2.52}$$

Here we also introduced a new gauge parameter $\lambda_\mu \in X_1$, for which $B_{\mu\nu}$ is the gauge field. Indeed, with (2.51) and (2.49) we then compute

$$\begin{aligned}
\delta_\lambda \mathcal{F}_{\mu\nu} &= [D_\mu, D_\nu]\lambda + \mathfrak{D}(\Delta_\lambda B_{\mu\nu}) \\
&= \lambda \circ \mathcal{F}_{\mu\nu} - 2\{\mathcal{F}_{\mu\nu}, \lambda\} + \mathfrak{D}(\Delta_\lambda B_{\mu\nu}) \\
&= \lambda \circ \mathcal{F}_{\mu\nu} + \mathfrak{D}(\Delta_\lambda B_{\mu\nu} - \mathcal{F}_{\mu\nu} \bullet \lambda) \\
&= \mathcal{L}_\lambda \mathcal{F}_{\mu\nu}.
\end{aligned} \tag{2.53}$$

Next, we have to prove invariance under the new shift transformation w.r.t. $\lambda_\mu \in X_1$. With (2.44) we compute

$$
\begin{aligned}
\delta_\lambda \mathcal{F}_{\mu\nu} &= -2\, D_{[\mu} \mathfrak{D} \lambda_{\nu]} + 2\, \mathfrak{D} (D_{[\mu} \lambda_{\nu]}) \\
&= -2\, \partial_{[\mu} \mathfrak{D} \lambda_{\nu]} + 2\, A_{[\mu} \circ \mathfrak{D} \lambda_{\nu]} + 2\, \mathfrak{D} \big(\partial_{[\mu} \lambda_{\nu]} - A_{[\mu} \bullet \mathfrak{D} \lambda_{\nu]} \big) - \mathfrak{D} (A_{[\mu} \bullet \lambda_{\nu]}) \\
&= 2\, A_{[\mu} \circ \mathfrak{D} \lambda_{\nu]} - 4 \{ A_{[\mu}, \mathfrak{D} \lambda_{\nu]} \} \\
&= -2\, \mathfrak{D} \lambda_{[\nu} \circ A_{\mu]} \\
&= 0,
\end{aligned}
\tag{2.54}
$$

using $\mathfrak{D}^2 = 0$ and, in the last step, (2.14).

Having established the covariance (and invariance) properties of $\mathcal{F}_{\mu\nu}$, we next ask whether there is a covariant 3-form field strength for the 2-form. This 3-form curvature emerges naturally upon inspecting the possible Bianchi identities for $\mathcal{F}_{\mu\nu}$. In contrast to the Bianchi identity in Yang-Mills theory based on Lie algebras, the covariant curl of $\mathcal{F}_{\mu\nu}$ in general is not zero but only $\mathfrak{D}$-exact, thereby introducing a 3-form that is covariant (again, up to $\mathfrak{D}$-exact terms). Specifically, we have the following generalized Bianchi identity,

$$
3\, D_{[\mu} \mathcal{F}_{\nu\rho]} \;=\; \mathfrak{D} H_{\mu\nu\rho},
\tag{2.55}
$$

where

$$
\begin{aligned}
H_{\mu\nu\rho} &= 3 \Big(\partial_{[\mu} B_{\nu\rho]} - A_{[\mu} \bullet \mathfrak{D} B_{\nu\rho]} - \mathfrak{D}(A_{[\mu} \bullet B_{\nu\rho]}) - A_{[\mu} \bullet \partial_\nu A_{\rho]} + \tfrac{1}{3} A_{[\mu} \bullet A_\nu \circ A_{\rho]} \Big) \\
&= 3 \left(D_{[\mu} B_{\nu\rho]} - \Omega_{\mu\nu\rho}(A) \right),
\end{aligned}
\tag{2.56}
$$

and we introduced the Chern-Simons three-form

$$
\Omega_{\mu\nu\rho}(A) \;\equiv\; A_{[\mu} \bullet \partial_\nu A_{\rho]} - \tfrac{1}{3} A_{[\mu} \bullet (A_\nu \circ A_{\rho]}).
\tag{2.57}
$$

Note that $H_{\mu\nu\rho}$ is determined by (2.55) only up to contributions that are $\mathfrak{D}$-closed and hence $\mathfrak{D}$-exact. In (2.56) we added a $\mathfrak{D}$-exact term in order to build the full covariant derivative. Moreover, we should not expect $H_{\mu\nu\rho}$ to be fully gauge covariant, but only up to $\mathfrak{D}$-exact contributions. These, in turn, can be fixed by introducing a 3-form gauge potential. The proof of the Bianchi identity (2.55) proceeds by a straightforwards computation, using repeatedly (2.16) and performing similar calculations as above.[2]

In order to further develop the general pattern of tensor hierarchies, we close this section by completing the definition of the 3-form curvature by introducing a 3-form potential taking values in X_2. To this end it is again convenient to inspect the general variation of $H_{\mu\nu\rho}$. First, under an arbitrary variation δA_μ, we compute for the Chern-Simons term[3]

$$
\delta\, \Omega_{\mu\nu\rho}(A) \;=\; -D_\mu(A_\nu \bullet \delta A_\rho) + \delta A_\mu \bullet \mathcal{F}_{\nu\rho} - \delta A_\mu \bullet \mathfrak{D} B_{\nu\rho} - \tfrac{1}{3} \mathfrak{D}(A_\mu \bullet (A_\nu \bullet \delta A_\rho)),
\tag{2.59}
$$

---

[2] In particular, one has to use the perhaps somewhat surprising relations

$$
3\, A_{[\mu} \circ (A_\nu \circ A_{\rho]}) \;=\; \mathfrak{D}(A_{[\mu} \bullet (A_\nu \circ A_{\rho]})), \qquad A_\mu \circ \mathfrak{D} B_{\nu\rho} \;=\; \mathfrak{D}(A_\mu \bullet \mathfrak{D} B_{\nu\rho}).
\tag{2.58}
$$

[3] Here and in the following total antisymmetrization of form indices is understood.

where we used the covariance relation (2.36). The general variation of the three-form curvature is then given by

$$\delta H_{\mu\nu\rho} = 3\Big(D_\mu \delta B_{\nu\rho} - \delta\Omega_{\mu\nu\rho} - \delta A_\mu \bullet \mathfrak{D} B_{\nu\rho} - \mathfrak{D}(\delta A_\mu \bullet B_{\nu\rho})\Big)$$
$$= 3\Big(D_\mu \Delta B_{\nu\rho} - \delta A_\mu \bullet \mathcal{F}_{\nu\rho} - \mathfrak{D}\big(\delta A_\mu \bullet B_{\nu\rho} - \tfrac{1}{3} A_\mu \bullet (A_\nu \bullet \delta A_\rho)\big)\Big). \quad (2.60)$$

The first two terms on the right-hand side are covariant, but there is also a non-covariant but $\mathfrak{D}$-exact term. Again, this can be remedied by introducing $C_{\mu\nu\rho} \in X_2$ and defining

$$\mathcal{H}_{\mu\nu\rho} \equiv H_{\mu\nu\rho} + \mathfrak{D} C_{\mu\nu\rho}. \quad (2.61)$$

The general variation can then be written as

$$\delta\mathcal{H}_{\mu\nu\rho} = 3 D_\mu \Delta B_{\nu\rho} - 3 \delta A_\mu \bullet \mathcal{F}_{\nu\rho} + \mathfrak{D}\Delta C_{\mu\nu\rho}, \quad (2.62)$$

with the covariant variation of the 3-form

$$\Delta C_{\mu\nu\rho} \equiv \delta C_{\mu\nu\rho} - 3 \delta A_\mu \bullet B_{\nu\rho} + A_\mu \bullet (A_\nu \bullet \delta A_\rho). \quad (2.63)$$

With the relations established so far it is now a direct computation to verify gauge covariance of the field strength under

$$\Delta C_{\mu\nu\rho} = 3 D_\mu \Sigma_{\nu\rho} + 3 \mathcal{F}_{\mu\nu} \bullet \lambda_\rho + \lambda \bullet \mathcal{H}_{\mu\nu\rho} - \mathfrak{D} \Xi_{\mu\nu\rho},$$
$$\Delta B_{\mu\nu} = 2 D_\mu \lambda_\nu + \lambda \bullet \mathcal{F}_{\mu\nu} - \mathfrak{D}\Sigma_{\mu\nu}, \quad (2.64)$$
$$\delta A_\mu = D_\mu \lambda - \mathfrak{D}\lambda_\mu,$$

where we introduced higher shift gauge parameters $\Sigma_{\mu\nu} \in X_2$, $\Xi_{\mu\nu\rho} \in X_3$. We can use this to quickly verify that there are trivial parameters of the following form

$$\lambda = \mathfrak{D}\chi,$$
$$\lambda_\mu = \mathfrak{D}\chi_\mu + D_\mu\chi,$$
$$\Sigma_{\mu\nu} = \mathfrak{D}\chi_{\mu\nu} + 2 D_\mu\chi_\nu - \chi \bullet \mathcal{F}_{\mu\nu}, \quad (2.65)$$
$$\Xi_{\mu\nu\rho} = \mathfrak{D}\chi_{\mu\nu\rho} + 3 D_\mu\chi_{\nu\rho} - 3 \mathcal{F}_{\mu\nu} \bullet \chi_\rho + \chi \bullet \mathcal{H}_{\mu\nu\rho}.$$

For this one uses (2.41) and, for the final relation, (2.29).

It should now be fairly clear how the pattern continues: at each level (form degree) one can construct consistent gauge transformations, covariant curvatures, etc., that have the familiar properties up to $\mathfrak{D}$-exact contributions that, in turn, can be fixed by introducing forms of one higher degree. The exact (or closed-form) formulation of the complete tensor hierarchy will be developed in the next two sections.

3. Infinity Enhanced Leibniz Algebra

In the previous section we have seen how the step-by-step construction of the tensor hierarchy proceeds in parallel to the introduction of spaces of higher degree X_n, as well as differentials $\mathfrak{D}$ and graded symmetric maps $\bullet$. In this section we will give a set of axioms, involving the Leibniz product and the higher structures, that define what we call an infinity enhanced Leibniz algebra. We will then show that such an algebraic structure is sufficient in order to construct a tensor hierarchy to all orders. Before listing the axioms, we will show how they can be motivated from the properties of the original Leibniz algebra. This will hint to a close relation with differential graded Lie algebras, that will be discussed as well.

3.1. Motivation. As discussed in the previous sections, the Leibniz product provides a natural notion of symmetry transformations (in the following often referred to as Lie derivative):

$$\delta_x y \equiv \mathcal{L}_x y := x \circ y, \tag{3.1}$$

that closes and is covariant, *i.e.*,

$$[\mathcal{L}_x, \mathcal{L}_y] z = \mathcal{L}_{[x,y]} z, \quad \mathcal{L}_x (y \circ z) = (\mathcal{L}_x y) \circ z + y \circ (\mathcal{L}_x z). \tag{3.2}$$

The trivial action of the symmetric pairing, $\mathcal{L}_{\{x,y\}} z = 0$, prompts us to introduce the space X_1 and the first bullet operator $\bullet : X_0 \otimes X_0 \to X_1$ as

$$x \circ y + y \circ x = \mathfrak{D}(x \bullet y), \tag{3.3}$$

where the right hand side can be viewed as the definition of $\bullet$. Since $\mathcal{L}_{\mathfrak{D}(x \bullet y)} z = 0$ for any $x, y \in X_0$, one is led to associate triviality of the Lie derivative with $\mathfrak{D}$-exactness, and thus postulate

$$\mathfrak{D}u \circ x = 0, \quad \forall u \in X_1, \ x \in X_0. \tag{3.4}$$

The Lie derivative is at the core of constructing the gauge theory, in that it defines covariant derivatives and gauge variations. Since higher form gauge fields are valued in spaces X_n with $n > 0$, it is necessary to extend the definition of the Lie derivative to spaces of arbitrary degree in a way that preserves closure and covariance. In order to determine the form of $\mathcal{L}_x$ acting on elements of higher degree, we notice that

$$(x \circ y) \bullet z + (x \circ z) \bullet y - x \bullet (y \circ z + z \circ y) \tag{3.5}$$

is $\mathfrak{D}$-closed, thanks to the Leibniz property of $\circ$. Imposing it to be $\mathfrak{D}$-exact amounts to define the higher bullet $\bullet : X_0 \otimes X_1 \to X_2$ by

$$(x \circ y) \bullet z + (x \circ z) \bullet y - x \bullet (y \circ z + z \circ y) = \mathfrak{D}(x \bullet (y \bullet z)). \tag{3.6}$$

This is motivated by the assumption that everything should be writable only in terms of $\mathfrak{D}$ and $\bullet$, and by manifest symmetry in $y \leftrightarrow z$. By defining

$$\mathcal{L}_x a := x \bullet \mathfrak{D}a + \mathfrak{D}(x \bullet a), \quad a \in X_n, n > 0, \tag{3.7}$$

this is equivalent to $y \bullet z$ being covariant under $\mathcal{L}_x$. Totally symmetrizing the relation (3.6), and recalling that $\bullet$ is symmetric for degree-zero objects, we infer

$$\mathfrak{D}\big[x \bullet (y \bullet z) + y \bullet (z \bullet x) + z \bullet (x \bullet y) \big] = 0, \tag{3.8}$$

so the expression in parenthesis is $\mathfrak{D}$-closed, and we impose it to be $\mathfrak{D}$-exact. Since there is nothing writable in terms of $\bullet$ yielding a degree +3 object from (x, y, z), this amounts to postulating

$$x \bullet (y \bullet z) + y \bullet (z \bullet x) + z \bullet (x \bullet y) = 0. \tag{3.9}$$

By using the notation $\iota_x = x \bullet$, we notice that (3.9) can be rewritten in the form

$$- \iota_x (y \bullet z) = (\iota_x y) \bullet z + y \bullet (\iota_x z), \quad x, y, z \in X_0 \tag{3.10}$$

viewed as a (twisted) Leibniz property of the operator ι_x. This suggests the graded extension

$$- \iota_x(a \bullet b) = (\iota_x a) \bullet b + (-1)^{|a|} a \bullet (\iota_x b), \quad x \in X_0, \quad a, b \in X \quad (3.11)$$

to the whole space. Given (3.11), one can act repeatedly with ι_{x_k} to prove by induction

$$(-1)^{|\mathbf{X}_n|+1} \mathbf{X}_n \bullet (a \bullet b) = (\mathbf{X}_n \bullet a) \bullet b + (-1)^{|a||b|} (\mathbf{X}_n \bullet b) \bullet a, \quad \forall \, a, b \in X \quad (3.12)$$

with $\mathbf{X}_n := x_1 \bullet (x_2 \bullet (...(x_{n-1} \bullet x_n)))$ an element of degree $n-1$ generated by nested products of degree zero elements x_i, thereby suggesting the general relation

$$(-1)^{|a|+1} a \bullet (b \bullet c) = (a \bullet b) \bullet c + (-1)^{|b||c|} (a \bullet c) \bullet b. \quad (3.13)$$

Coming back to the properties of the $\mathfrak{D}$ operator, we see from (3.4) that $\mathfrak{D}u \bullet \mathfrak{D}v$ is $\mathfrak{D}$-closed for $u, v \in X_1$, since $\mathfrak{D}(\mathfrak{D}u \bullet \mathfrak{D}v) = \mathfrak{D}u \circ \mathfrak{D}v + \mathfrak{D}v \circ \mathfrak{D}u$. Postulating this is $\mathfrak{D}$-exact amounts to requiring there exists a degree 2 element, depending on u and v, whose $\mathfrak{D}$-boundary is $\mathfrak{D}u \bullet \mathfrak{D}v$; we can therefore define $-u \bullet \mathfrak{D}v$ to be this element, so that

$$\mathfrak{D}u \bullet \mathfrak{D}v = -\mathfrak{D}(u \bullet \mathfrak{D}v) = -\mathfrak{D}(v \bullet \mathfrak{D}u), \quad (3.14)$$

where the sign has been chosen such that $\mathcal{L}_{\mathfrak{D}u} v = 0$, maintaining triviality of the Lie derivative along $\mathfrak{D}$-exact elements. Antisymmetrizing the above relation one finds that $u \bullet \mathfrak{D}v - v \bullet \mathfrak{D}u$ is $\mathfrak{D}$-closed. Imposing again that it is $\mathfrak{D}$-exact allows us to define yet a higher bullet product $\bullet : X_1 \otimes X_1 \to X_3$ via

$$u \bullet \mathfrak{D}v - v \bullet \mathfrak{D}u = \mathfrak{D}(u \bullet v), \quad (3.15)$$

suggesting the twisted Leibniz property

$$- \mathfrak{D}(a \bullet b) = (\mathfrak{D}a) \bullet b + (-1)^{|a|} a \bullet \mathfrak{D}b, \quad |a|, |b| > 0. \quad (3.16)$$

From the relations (3.11) and (3.16) it is possible to prove covariance of the product $a \bullet b$ under the action of the Lie derivative when neither a nor b have degree zero, as will be shown explicitly later. As we have already mentioned, (3.6) ensures covariance of $y \bullet z$ under the Lie derivative when both y and z have degree zero. On the other hand, when only one argument has degree zero, (3.11) and (3.16) only allow us to determine

$$\begin{aligned}
&\mathcal{L}_x(y \bullet u) + \mathcal{L}_y(x \bullet u) \\
&= x \bullet \mathfrak{D}(y \bullet u) + y \bullet \mathfrak{D}(x \bullet u) + \mathfrak{D}\big[x \bullet (y \bullet u) + y \bullet (x \bullet u)\big] \\
&= x \bullet \mathcal{L}_y u + y \bullet \mathcal{L}_x u - x \bullet (y \bullet \mathfrak{D}u) - y \bullet (x \bullet \mathfrak{D}u) - \mathfrak{D}\big[(x \bullet y) \bullet u\big] \\
&= x \bullet \mathcal{L}_y u + y \bullet \mathcal{L}_x u + \mathfrak{D}(x \bullet y) \bullet u \\
&= \big[(\mathcal{L}_x y) \bullet u + y \bullet \mathcal{L}_x u\big] + \big[(\mathcal{L}_y x) \bullet u + x \bullet \mathcal{L}_y u\big],
\end{aligned} \quad (3.17)$$

showing that one needs to demand

$$\begin{aligned}
&\mathfrak{D}(x_{[1} \bullet (x_{2]} \bullet u)) \\
&= 2\, x_{[2} \bullet \mathfrak{D}(x_{1]} \bullet u) + x_{[2} \bullet (x_{1]} \bullet \mathfrak{D}u) + [x_1, x_2] \bullet u, \quad |x_i| = 0, \quad |u| > 0,
\end{aligned} \quad (3.18)$$

for the product $(x \bullet u)$ to be covariant. Notice that the structures on the right hand side above are completely fixed by degree and symmetry, and the assumption is actually on the relative coefficients.

3.2. Relation with differential graded Lie algebras. Looking at the properties (3.13) and (3.16) one immediately notices the resemblance with the graded Jacobi identity and graded Leibniz rule of differential graded Lie algebras. In fact, one can show that this is precisely the case upon suspension, *i.e.* degree shifting, of the graded vector spaces X_n. To start with, we shall define the degree shifted vector space $\widetilde{X} = \bigoplus_{n=1}^{\infty} \widetilde{X}_n$ and the suspension s by

$$
\begin{aligned}
&s : X_n \to \widetilde{X}_{n+1}, \quad \tilde{a} := sa, \\
&|\tilde{a}| = |a| + 1.
\end{aligned}
\tag{3.19}
$$

The bullet product on X translates, upon suspension, to a graded antisymmetric bracket on $\widetilde{X}$ defined by

$$
[\tilde{a}, \tilde{b}] := (-1)^{|a|+1} s(a \bullet b),
\tag{3.20}
$$

which should not be confused with the antisymmetrization of the Leibniz product. The differential $\mathfrak{D}$ can also be defined on the suspended spaces by $\mathfrak{D}\tilde{a} := s\mathfrak{D}a$. Degree counting shows that the differential, that we still denote by $\mathfrak{D}$, retains degree -1, while the bracket (3.20) has degree zero.

With these definitions one can show that properties (3.13) and (3.16) become the usual graded Jacobi identity for the bracket (3.20) and graded Leibniz compatibility of the differential, namely

$$
\begin{aligned}
&[[\tilde{a}, \tilde{b}], \tilde{c}] + (-1)^{|\tilde{a}|(|\tilde{b}|+|\tilde{c}|)}[[\tilde{b}, \tilde{c}], \tilde{a}] + (-1)^{|\tilde{c}|(|\tilde{a}|+|\tilde{b}|)}[[\tilde{c}, \tilde{a}], \tilde{b}] = 0, \\
&\mathfrak{D}[\tilde{a}, \tilde{b}] = [\mathfrak{D}\tilde{a}, \tilde{b}] + (-1)^{|\tilde{a}|}[\tilde{a}, \mathfrak{D}\tilde{b}], \quad |\tilde{a}|, |\tilde{b}| > 1.
\end{aligned}
\tag{3.21}
$$

Finally, the original Leibniz algebra $(X_0, \circ)$ can be transported to $\widetilde{X}_1$ by

$$
\tilde{x} \circ \tilde{y} := s(x \circ y),
\tag{3.22}
$$

so that the Leibniz property is unchanged. The suspended Leibniz product has intrinsic degree -1, and indeed closes on $\widetilde{X}_1$.

Even though the differential graded Lie algebra (dgLa) structure appearing in (3.21) allows for a more familiar interpretation of the properties (3.13) and (3.16), one should keep in mind that the Leibniz product $\circ$ is an independent algebraic structure. In particular, its compatibility properties with the differential and the dgLa bracket, given by the suspended version of (3.3), (3.4), (3.6) and (3.18), need to be imposed as separate requirements.

This construction simplifies considerably if one extends the graded vector space $\widetilde{X}$ to non-positive degrees and declares the differential and the bracket to be defined on the whole space, thus making $(\widetilde{X}, [\,,\,], \mathfrak{D})$ into a differential graded Lie algebra. In particular, the newly introduced space $\widetilde{X}_0 = s X_{-1}$ is a genuine Lie algebra $\mathfrak{g}$ and the dgLa bracket gives to all graded spaces $\widetilde{X}_n$ a $\mathfrak{g}-$module structure via

$$
\rho_{\tilde{t}}\, \tilde{a} := [\tilde{t}, \tilde{a}], \quad \tilde{t} \in \widetilde{X}_0\,, \ \tilde{a} \in \widetilde{X}.
\tag{3.23}
$$

The crucial difference of the extended space formulation is that the differential can now act on the Leibniz space $\widetilde{X}_1 = s X_0$, yielding a map

$$
\mathfrak{D} : \widetilde{X}_1 \to \mathfrak{g}
\tag{3.24}
$$

that can be interpreted as an abstract embedding tensor, in the language of gauged supergravity. This in turn allows one to *define* the Leibniz product as a derived one via

$$\tilde{x} \circ \tilde{y} := -[\mathfrak{D}\tilde{x}, \tilde{y}], \tag{3.25}$$

or, before suspension, as $x \circ y := -\mathfrak{D}x \bullet y$. The graded Jacobi identity and compatibility of $\mathfrak{D}$ are sufficient to prove that the product (3.25) does indeed obey the Leibniz property. Moreover, the Lie derivative is also universally defined by

$$\mathcal{L}_{\tilde{x}}\tilde{a} := -[\mathfrak{D}\tilde{x}, \tilde{a}], \quad \tilde{x} \in \widetilde{X}_1 , \ \tilde{a} \in \widetilde{X}. \tag{3.26}$$

Finally, all the compatibility conditions (3.3), (3.4), (3.6) and (3.18) are ensured by the dgLa structure of the entire vector space, requiring only the graded Jacobi identity and compatibility of the differential.

At this point it is evident that the dgLa structure on the unbounded space allows for a more concise construction. However, it should be noted that assuming the existence of the Lie algebra $\mathfrak{g}$ and its action on $\widetilde{X}$ is not needed in order to construct the tensor hierarchy and is a considerable piece of extra data to be given as input. Moreover, so far there is no clear field theoretic interpretation of the spaces in negative degrees. Nevertheless, given any Leibniz algebra $(X_0, \circ)$ it is always possible to define an associated Lie algebra by modding out the symmetric part of the Leibniz product. This yields a Lie algebra $\mathfrak{g}_0 \sim X_0/\mathfrak{D}(X_1)$ that, however, almost never coincides with the Lie algebra $\mathfrak{g} = s X_{-1}$, but is rather a subalgebra of the latter. For instance, in gauged supergravity the underlying Lie algebra $\mathfrak{g}$ is given by the global symmetry algebra of the ungauged phase, while in double and exceptional field theory such an algebra was identified only recently [5,43]. This shows that augmenting the original graded vector space X to negative degrees and extending the action of the differential and the bullet requires a significant amount of extra structure, that is not necessary in order to define a consistent tensor hierarchy. For this reason, in the present paper we shall investigate tensor hierarchies in terms of the most minimal set of algebraic structures that guarantee their consistency. We will consider the graded vector space X concentrated in non-negative degrees, with the Leibniz algebra $(X_0, \circ)$ as a fundamental structure. As discussed in the present section, the consistency relations (3.3), (3.4), (3.6) and (3.18) cannot be derived, in this case, by more fundamental ones, and we shall demand them as additional axioms that will be collected in the next subsection.

3.3. Axioms. We are now ready to provide the list of structures and axioms defining what we name an infinity-enhanced Leibniz algebra. In this section we will prove that the given axioms allow one to define a generalized Lie derivative that acts covariantly on all algebraic structures, and closes on itself modulo trivial transformations.

An infinity-enhanced Leibniz algebra consists of the quadruple $(X, \circ, \mathfrak{D}, \bullet)$. X is an $\mathbb{N}$-graded vector space, where sometimes we single out the degree zero subspace:

$$X = \bigoplus_{n=0}^{\infty} X_n = X_0 \oplus \bar{X}. \tag{3.27}$$

X_0 is endowed with a (left) Leibniz product $\circ : X_0 \otimes X_0 \to X_0$, obeying

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z). \tag{3.28}$$

$\mathfrak{D}$ is a degree -1 differential acting on $\bar{X}$:

$$\dots \longrightarrow X_n \xrightarrow{\mathfrak{D}} X_{n-1}\dots \xrightarrow{\mathfrak{D}} X_1 \xrightarrow{\mathfrak{D}} X_0, \qquad \mathfrak{D}^2 = 0, \qquad (3.29)$$

and $\bullet$ is a graded commutative product of degree $+1$ defined on the whole space X :

$$\bullet : X_i \otimes X_j \to X_{i+j+1}, \quad a \bullet b = (-1)^{|a||b|} b \bullet a. \qquad (3.30)$$

This quadruple defines an infinity-enhanced Leibniz algebra provided

1) $\mathfrak{D}u \circ x = 0, \quad \forall\, u \in X_1, \ x \in X_0,$

2) $\mathfrak{D}(x \bullet y) = x \circ y + y \circ x, \quad \forall\, x, y \in X_0,$

3) $\mathfrak{D}(x \bullet (y \bullet z)) = (x \circ y) \bullet z + (x \circ z) \bullet y - (y \circ z + z \circ y) \bullet x, \quad \forall\, x, y, z \in X_0,$

4) $\mathfrak{D}(x_{[1} \bullet (x_{2]} \bullet u)) = 2\, x_{[2} \bullet \mathfrak{D}(x_{1]} \bullet u) + x_{[2} \bullet (x_{1]} \bullet \mathfrak{D}u)$

$\quad + [x_1, x_2] \bullet u, \quad \forall\, x_1, x_2 \in X_0, u \in \bar{X},$ $\hspace{3cm}$ (3.31)

5) $\mathfrak{D}(u \bullet v) + \mathfrak{D}u \bullet v + (-1)^{|u|} u \bullet \mathfrak{D}v = 0, \quad \forall\, u, v \in \bar{X},$

6) $(-1)^{|a|} a \bullet (b \bullet c) + (a \bullet b) \bullet c + (-1)^{|b||c|}(a \bullet c) \bullet b = 0, \quad \forall\, a, b, c \in X.$

The generalized Lie derivative is defined as

$$\begin{aligned} \mathcal{L}_x y &:= x \circ y, \quad \forall\, x, y \in X_0, \\ \mathcal{L}_x u &:= x \bullet \mathfrak{D}u + \mathfrak{D}(x \bullet u), \quad \forall\, x \in X_0, u \in \bar{X}. \end{aligned} \qquad (3.32)$$

From axioms 1) and 5) it is immediate that $\mathfrak{D}$-exact degree zero elements generate trivial Lie derivatives, *i.e.*

$$\mathcal{L}_{\mathfrak{D}u} a = 0, \quad \forall\, u \in X_1, a \in X. \qquad (3.33)$$

Covariance. As the first statement of covariance, we see that the Lie derivative commutes with the differential $\mathfrak{D}$,

$$\left[\mathcal{L}_x, \mathfrak{D}\right] = 0, \quad \forall\, x \in X_0. \qquad (3.34)$$

This is obvious by construction when the commutator acts on an element u with $|u| > 1$, since then $\mathcal{L}_x = \iota_x \mathfrak{D} + \mathfrak{D}\iota_x$, c.f. (2.35). For $|u| = 1$ one has

$$\left[\mathcal{L}_x, \mathfrak{D}\right] u = x \circ \mathfrak{D}u - \mathfrak{D}(x \bullet \mathfrak{D}u + \mathfrak{D}(x \bullet u)) = x \circ \mathfrak{D}u - \mathfrak{D}(x \bullet \mathfrak{D}u) = 0,$$
$$(3.35)$$

upon using 1) and 2). Covariance of the Leibniz product $\circ$ itself is just a rewriting of its defining property:

$$\mathcal{L}_x(y \circ z) = (\mathcal{L}_x y) \circ z + y \circ (\mathcal{L}_x z). \qquad (3.36)$$

Covariance of the bullet product, *i.e.*

$$\mathcal{L}_x(a \bullet b) = (\mathcal{L}_x a) \bullet b + a \bullet (\mathcal{L}_x b), \quad \forall\, x \in X_0, \ a, b \in X, \qquad (3.37)$$

has to be proved in different steps, depending on the degrees of a and b . For a and b in X_0 one has

$$\mathcal{L}_x(a \bullet b) = x \bullet \mathfrak{D}(a \bullet b) + \mathfrak{D}(x \bullet (a \bullet b)) = x \bullet (a \circ b + b \circ a) + \mathfrak{D}(x \bullet (a \bullet b))$$
$$(3.38)$$
$$= (x \circ a) \bullet b + (x \circ b) \bullet a = (\mathcal{L}_x a) \bullet b + a \bullet (\mathcal{L}_x b)$$

thanks to 3). In the case $a \in X_0$ and $b \in \bar{X}$ we have to use properties 4), 5) and 6) (in the proof we rename $a = y$ to make clear that it has zero degree):

$$
\begin{aligned}
\mathcal{L}_x(y \bullet b) &= x \bullet \mathfrak{D}(y \bullet b) + \mathfrak{D}(x \bullet (y \bullet b)) \\
&= x \bullet \mathfrak{D}(y \bullet b) + \tfrac{1}{2}\Big[\mathfrak{D}(x \bullet (y \bullet b)) + \mathfrak{D}(y \bullet (x \bullet b))\Big] \\
&\quad + \tfrac{1}{2}\Big[\mathfrak{D}(x \bullet (y \bullet b)) - \mathfrak{D}(y \bullet (x \bullet b))\Big] \\
&= -\tfrac{1}{2}\mathfrak{D}((x \bullet y) \bullet b) + y \bullet \mathfrak{D}(x \bullet b) + \tfrac{1}{2}\,y \bullet (x \bullet \mathfrak{D}b) - \tfrac{1}{2}\,x \bullet (y \bullet \mathfrak{D}b) + [x,y] \bullet b \\
&= \{x,y\} \bullet b - \tfrac{1}{2}\,(x \bullet y) \bullet \mathfrak{D}b + y \bullet \mathcal{L}_x b \\
&\quad - \tfrac{1}{2}\,y \bullet (x \bullet \mathfrak{D}b) \\
&\quad - \tfrac{1}{2}\,x \bullet (y \bullet \mathfrak{D}b) + [x,y] \bullet b \\
&= (x \circ y) \bullet b + y \bullet \mathcal{L}_x b.
\end{aligned}
$$
(3.39)

For both a and b in $\bar{X}$ one uses repeatedly properties 5) and 6) to get

$$
\begin{aligned}
\mathcal{L}_x(a \bullet b) &= x \bullet \mathfrak{D}(a \bullet b) + \mathfrak{D}(x \bullet (a \bullet b)) \\
&= -x \bullet [\mathfrak{D}a \bullet b + (-1)^{|a|}a \bullet \mathfrak{D}b] - \mathfrak{D}[(x \bullet a) \bullet b + (-1)^{|a|}a \bullet (x \bullet b)] \\
&= (x \bullet \mathfrak{D}a) \bullet b - (-1)^{|a|}\mathfrak{D}a \bullet (x \bullet b) + (-1)^{|a|}(x \bullet a) \bullet \mathfrak{D}b + a \bullet (x \bullet \mathfrak{D}b) \\
&\quad - \mathfrak{D}[(x \bullet a) \bullet b + (-1)^{|a|}a \bullet (x \bullet b)] \\
&= (\mathcal{L}_x a) \bullet b - \mathfrak{D}(x \bullet a) \bullet b - (-1)^{|a|}\mathfrak{D}a \bullet (x \bullet b) + (-1)^{|a|}(x \bullet a) \bullet \mathfrak{D}b \\
&\quad + a \bullet (\mathcal{L}_x b) - a \bullet \mathfrak{D}(x \bullet b) \\
&\quad - \mathfrak{D}[(x \bullet a) \bullet b + (-1)^{|a|}a \bullet (x \bullet b)] \\
&= (\mathcal{L}_x a) \bullet b + a \bullet (\mathcal{L}_x b),
\end{aligned}
$$
(3.40)

thus proving covariance (3.37) for arbitrary elements $a, b \in X$.

Closure. Since the Lie derivative will be used to define symmetry variations and covariant derivatives, we have to show that it closes under commutation, namely

$$
\big[\mathcal{L}_x, \mathcal{L}_y\big]a = \mathcal{L}_{[x,y]}\,a. \tag{3.41}
$$

When acting on a degree zero element, this is ensured by the Leibniz property of $\circ$, as displayed in (1.7), which yields (3.41) together with the triviality property $\mathcal{L}_{\{x,y\}} = 0$. In order to prove closure on higher degree elements we need the covariance properties (3.34) and (3.37) that have just been proven:

$$
\begin{aligned}
\mathcal{L}_x \mathcal{L}_y a &= \mathcal{L}_x\Big[y \bullet \mathfrak{D}a + \mathfrak{D}(y \bullet a)\Big] \\
&= (x \circ y) \bullet \mathfrak{D}a + y \bullet \mathfrak{D}(\mathcal{L}_x a) + \mathfrak{D}\Big[(x \circ y) \bullet a + y \bullet (\mathcal{L}_x a)\Big] \\
&= \mathcal{L}_{x \circ y} a + \mathcal{L}_y \mathcal{L}_x a.
\end{aligned}
\tag{3.42}
$$

We close this section by giving a brief summary of where the algebraic structures axiomatized here appear in physical models. For instance, in gauged supergravity (see e.g. [7]) the vector fields take values in some representation R_1 of the Lie algebra $\mathfrak{g}$

of global symmetries of the ungauged phase. The Leibniz space X_0 is then identified with the representation space R_1, and the so called embedding tensor map $\vartheta : R_1 \to \mathfrak{g}$ allows one to define the Leibniz product as $x \circ y = \rho_{\vartheta(x)} y$ for $x, y \in R_1$. The first bullet product $\bullet$ is then defined as a projection from the tensor product of two R_1 representations to the $R_2 \equiv X_1$ representation carried by the two-forms: $\bullet : R_1 \otimes R_1 \to R_2$, and so on. In double field theory [43] and exceptional field theory [5,22,43] the Leibniz product is given in terms of generalized Lie derivatives, and both the bullet product $\bullet$ and differential $\mathfrak{D}$ are given by algebraic and/or differential operators acting on the internal space. Similarly, for a 3D gauge theory based on an infinite-dimensional Leibniz algebra see [44].

4. Exact Tensor Hierarchy

The main goal of this section is to show that the infinity-enhanced Leibniz algebra, defined in the previous section by the set of axioms (3.31), allows us to construct the tensor hierarchy to all orders. In particular, we will show that it is possible to define gauge covariant curvatures $\mathcal{F}_{p+1}$ for p-form gauge fields A_p of arbitrary degree. Consistency of the tensor hierarchy is established by showing that the curvatures obey a set of Bianchi identities. Indirectly, this establishes gauge covariance.

4.1. General strategy. We start by briefly outlining the general strategy: When constructing the gauge theory step by step, as in Sect. 2.3, one starts from the one-form[4] $A_1 = A_\mu \, dx^\mu$, taking values in the Leibniz algebra X_0, and postulates the gauge transformation[5] $\delta A_1 = D\lambda_0 - \mathfrak{D}\lambda_1$, where the covariant derivative is defined as

$$D = dx^\mu \, D_\mu := dx^\mu \, (\partial_\mu - \mathcal{L}_{A_\mu}). \tag{4.1}$$

The naive Yang-Mills curvature $F_2 = dA_1 - \frac{1}{2} A_1 \circ A_1$ transforms covariantly only modulo a $\mathfrak{D}$-exact term: $\delta F_2 = \mathcal{L}_{\lambda_0} F_2 + \mathfrak{D}(...)$, forcing us to introduce an X_1-valued two-form A_2 whose gauge transformation can be adjusted to make the full two-form curvature $\mathcal{F}_2 = F_2 + \mathfrak{D}A_2$ covariant, *i.e.* $\delta\mathcal{F}_2 = \mathcal{L}_{\lambda_0}\mathcal{F}_2$. At this point, the most efficient way to determine the three-form curvature is to take the covariant curl of $\mathcal{F}_2$, yielding the Bianchi identity $D\mathcal{F}_2 = \mathfrak{D}\mathcal{F}_3$ with

$$
\begin{aligned}
F_3 &= DA_2 + \Omega_3, \\
\Omega_3 &= -\frac{1}{2} A_1 \bullet dA_1 + \frac{1}{6} A_1 \bullet (A_1 \circ A_1).
\end{aligned}
\tag{4.2}
$$

As for the lower order, the curvature F_3 is covariant only modulo $\mathfrak{D}$-exact terms: $\delta F_3 = \mathcal{L}_{\lambda_0} F_3 + \mathfrak{D}(...)$. Again, this can be cured by introducing an X_2-valued gauge potential A_3 and fixing its gauge transformation so that $\mathcal{F}_3 = F_3 + \mathfrak{D}A_3$ transforms covariantly. The procedure continues in the same way up to the top form, for a given spacetime dimension, but it becomes very cumbersome quite quickly, as can be appreciated from the first steps carried out explicitly in Sect. 2.3.

---

[4] From now on, having to deal with forms of arbitrary degree, we use intrinsic differential form notation with normalization $A_p = \frac{1}{p!} A_{\mu_1...\mu_p} dx^{\mu_1}...dx^{\mu_p}$, and different forms are only named by their degree, so that $A_{2\,\mu\nu} = B_{\mu\nu}$ of Sect. 2.3 and so on. Moreover, we use a slightly different normalization for the Ω_3 Chern-Simons form compared to Sect. 2.3, in order to avoid cluttering formulas with factorial coefficients.

[5] As shown in Sect. 2.3 the $\mathfrak{D}$-exact part of the gauge transformation is required by closure.

From the examples at low form degree it is clear that at every order the Bianchi identities alone fix the next order curvature up to a $\mathfrak{D}$-exact term. We aim thus at finding curvatures $\mathcal{F}_p = DA_{p-1} + \mathfrak{D}A_p + \cdots$ that obey Bianchi identities among themselves for arbitrary form degrees.

In order to do so, we shall first focus on the Chern-Simons-like terms such as the Ω_3 in (4.2). Such composite p-forms, entirely built out of the one-form A_1, appear at every order in a brute-force calculation of the curvatures: $\mathcal{F}_p = DA_{p-1} + \mathfrak{D}A_p + \Omega_p(A_1) + \cdots$. Perhaps counter-intuitively, it is the term $\Omega_p(A_1)$, rather than DA_{p-1}, that helps us to determine the all-order structure of $\mathcal{F}_p$ since, as we shall prove in Appendix A, these pseudo-Chern-Simons forms already obey Bianchi identities. We thus define the pseudo Chern-Simons (CS) n-form by

$$\Omega_n(A) = \frac{(-1)^n}{(n-1)!}(\iota_A)^{n-2}\big[dA - \tfrac{1}{n}A \circ A\big], \quad |\Omega_n| = n - 2, \tag{4.3}$$

where $A \equiv A_1$ and we have introduced the operator

$$\iota_A x := A \bullet x. \tag{4.4}$$

The form index n is related to the powers of A as $\Omega_n \sim A^{n-2}dA + A^n$ such that, for instance, one has $\Omega_3 = -\tfrac{1}{2}A \bullet dA + \tfrac{1}{6}A \bullet (A \circ A)$, $\Omega_4 = \tfrac{1}{6}A \bullet (A \bullet dA) - \tfrac{1}{24}A \bullet (A \bullet (A \circ A))$ and so on. It will be convenient to include the pure Yang-Mills curvature F_2 in this family as $\Omega_2 = dA - \tfrac{1}{2}A \circ A \equiv F_2$. Notice that the Ω_n have form degree n and intrinsic degree $n - 2$, making them $\bullet$-commutative, *i.e.*

$$\Omega_k \bullet \Omega_l = \Omega_l \bullet \Omega_k, \quad \forall\, k, l \geqslant 2. \tag{4.5}$$

Moreover, from assumption 6) of (3.31), upon combining internal and form degrees, one can derive

$$-\iota_A(\Omega_k \bullet \Omega_l) = (\iota_A\Omega_k) \bullet \Omega_l + \Omega_k \bullet (\iota_A\Omega_l). \tag{4.6}$$

The crucial property of this family of differential forms is that it closes (quadratically) under the action of the covariant derivative $D = d - \mathcal{L}_A$:

$$D\Omega_n + \tfrac{1}{2}\sum_{k=2}^{n-1}\Omega_k \bullet \Omega_{n+1-k} = \mathfrak{D}\Omega_{n+1}, \quad n \geqslant 2. \tag{4.7}$$

We will prove (4.7) by induction in Appendix A.

4.2. Curvatures and Bianchi identities to all orders. In this section we are going to work with arbitrary p-form gauge potentials A_p, defined as differential p-forms over a spacetime manifold M, taking values in X_{p-1}, namely

$$A_p := \tfrac{1}{p!}A_{\mu_1 \ldots \mu_p}(x)\, dx^{\mu_1} \cdots dx^{\mu_p} \in \Omega^p(M) \otimes X_{p-1}, \tag{4.8}$$

where from now on we shall omit the symbol for wedge products.

We are now ready to introduce curvatures $\mathcal{F}_{p+1} \in \Omega^{p+1}(M) \otimes X_{p-1}$ for arbitrary p-form gauge fields as

$$\mathcal{F}_{p+1} = \sum_{k_i \geqslant 2}^{p} \sum_{N=1}^{\left[\frac{p}{2}\right]} \frac{(-1)^{N-1}}{N!}\iota_{k_1} \ldots \iota_{k_{N-1}} DA_{k_N} + \sum_{k_i \geqslant 2}^{p+1} \sum_{N=1}^{\left[\frac{p+1}{2}\right]} \frac{(-1)^{N-1}}{N!}\iota_{k_1} \ldots \iota_{k_{N-1}}\Big(\mathfrak{D}A_{k_N} + N\,\Omega_{k_N}\Big),$$

$$\tag{4.9}$$

where A_p with $p \geqslant 2$ are the higher form gauge fields,

$$\Omega_p = \frac{(-1)^p}{(p-1)!} \iota_1^{p-2} \left(dA_1 - \frac{1}{n} A_1 \circ A_1 \right), \quad p \geqslant 2 \tag{4.10}$$

are the pseudo CS forms just introduced, and we use the shorthand notation

$$\iota_k x := A_k \bullet x. \tag{4.11}$$

The sums $\sum_{k_i \geqslant 2}^n$ run over all possible $\{k_1, ..., k_N\}$ with $k_i \geqslant 2$, constrained by $\sum_{i=1}^N k_i = n$. We will prove that the curvatures defined in (4.9) obey the Bianchi identities

$$D\mathcal{F}_n + \frac{1}{2} \sum_{k=2}^{n-1} \mathcal{F}_k \bullet \mathcal{F}_{n+1-k} = \mathfrak{D}\mathcal{F}_{n+1}, \quad n \geqslant 2, \tag{4.12}$$

that are the benchmark for proving recursively the gauge covariance of the $\mathcal{F}_n$'s. Before giving the proof of (4.12) for arbitrary form degree, we spend a few words to justify the form of (4.9). When constructing the tensor hierarchy step by step, starting from the one-form, one is led to identify the first few gauge covariant curvatures as

$$\begin{aligned}
\mathcal{F}_2 &= dA_1 - \frac{1}{2} A_1 \circ A_1 + \mathfrak{D}A_2 = F_2 + \mathfrak{D}A_2 \equiv \Omega_2 + \mathfrak{D}A_2, \\
\mathcal{F}_3 &= DA_2 + \Omega_3 + \mathfrak{D}A_3, \\
\mathcal{F}_4 &= DA_3 + \Omega_4 - A_2 \bullet \Omega_2 - \frac{1}{2} A_2 \bullet \mathfrak{D}A_2 + \mathfrak{D}A_4,
\end{aligned} \tag{4.13}$$

that immediately suggest to consider

$$\mathcal{F}_n = DA_{n-1} + \Omega_n + \mathfrak{D}A_n + \cdots \tag{4.14}$$

as a starting point for the curvature, which indeed coincides with the $N = 1$ term of (4.9). When considering the DA_{n-1} term, we recall that DA_1 is not defined (the pure Yang-Mills part of $\mathcal{F}_2$ is Ω_2) and we formally set $DA_1 \equiv 0$ when necessary. In order to guess the structure of the extra terms in (4.14) it is useful to push the step by step procedure a bit further to find

$$\begin{aligned}
\mathcal{F}_5 &= DA_4 + \Omega_5 - A_2 \bullet \Omega_3 - A_3 \bullet \Omega_2 - \frac{1}{2} A_2 \bullet DA_2 - \alpha A_2 \bullet \mathfrak{D}A_3 \\
&\quad + (\alpha - 1) A_3 \bullet \mathfrak{D}A_2 + \mathfrak{D}A_5.
\end{aligned} \tag{4.15}$$

The ambiguity, parametrized by $\alpha \in \mathbb{R}$, amounts to a field redefinition $A_5 \to A_5 + A_2 \bullet A_3$, that we fix by choosing the symmetric point $\alpha = \frac{1}{2}$. One can easily see that, starting from $\mathcal{F}_5$, analogous ambiguities show up at every level, due to possible field redefinitions of the $\mathfrak{D}$-exact term $\mathfrak{D}A_n$ in (4.14). We choose the symmetric point for all of them by demanding that the coefficients, as displayed in (4.9), do not depend on the set $\{k_i\}_{i=1}^N$ of form degrees.

In order to formulate the ansatz for $\mathcal{F}_n$ we define a new degree, that we name twist, given by the difference between form degree and internal degree of a given field. We recall that gauge fields A_p have internal degree $p - 1$, the differential $\mathfrak{D}$ has internal degree -1, and the products $\circ$ and $\bullet$ have degree zero and $+1$, respectively. It is thus clear that all gauge fields have twist $+1$, while all curvatures have twist $+2$ and the operator ι_k has twist zero. Therefore, the only way to make a twist $+2$ object from a higher gauge

field A_p with $p \geqslant 2$ is to act with an arbitrary number of ι_{k_i} on the building blocks $\mathfrak{D}A_p$ and DA_p .[6] As for the vector A_1 , a basis of twist $+2$ forms can be constructed by acting with an arbitrary number of ι_{k_i} on the building block[7] Ω_p , finally leading to the ansatz

$$\mathcal{F}_n = DA_{n-1} + \Omega_n + \mathfrak{D}A_n + \sum_{k_i \geqslant 2}^{n} \sum_{N=2}^{\left[\frac{n}{2}\right]} \iota_{k_1}...\iota_{k_{N-1}} \left(\alpha_N \, DA_{k_N-1} + \beta_N \, \mathfrak{D}A_{k_N} + \gamma_N \, \Omega_{k_N} \right),$$

(4.16)

where the leading terms can be included as the $N = 1$ part of the sum for the initial values $\alpha_1 = \beta_1 = \gamma_1 = 1$. Despite the usual point of view of seeing DA_{n-1} as the leading term of the field strength $\mathcal{F}_n$, it is more convenient to split it as

$$\mathcal{F}_n = \Omega_n + \Delta \mathcal{F}_n$$

(4.17)

in order to prove the Bianchi identities, since the pseudo CS forms already obey (4.7), yielding

$$D\mathcal{F}_n = D\Omega_n + D\Delta\mathcal{F}_n = -\frac{1}{2}\sum_{k=2}^{n-1} \Omega_k \bullet \Omega_{n+1-k} + \mathfrak{D}\Omega_{n+1} + D\Delta\mathcal{F}_n$$

$$= -\frac{1}{2}\sum_{k=2}^{n-1} \mathcal{F}_k \bullet \mathcal{F}_{n+1-k} + \mathfrak{D}\mathcal{F}_{n+1}$$

(4.18)

$$+ \sum_{k=2}^{n-1} \left[\Omega_k \bullet \Delta\mathcal{F}_{n+1-k} + \tfrac{1}{2}\Delta\mathcal{F}_k \bullet \Delta\mathcal{F}_{n+1-k} \right] + D\Delta\mathcal{F}_n - \mathfrak{D}\Delta\mathcal{F}_{n+1}.$$

Proving the Bianchi identity (4.12) thus amounts to showing that

$$\mathfrak{D}\Delta\mathcal{F}_{n+1} - D\Delta\mathcal{F}_n = \sum_{k=2}^{n-1} \left[\Omega_k \bullet \Delta\mathcal{F}_{n+1-k} + \tfrac{1}{2}\Delta\mathcal{F}_k \bullet \Delta\mathcal{F}_{n+1-k} \right],$$

(4.19)

where

$$\Delta\mathcal{F}_n = f_n^A + f_n^\Omega, \quad \text{with}$$

$$f_n^A := \sum_{k_i \geqslant 2}^{n-1} \sum_{N \geqslant 1} \alpha_N \, \iota_{k_1}...\iota_{k_{N-1}} DA_{k_N} + \sum_{k_i \geqslant 2}^{n} \sum_{N \geqslant 1} \beta_N \, \iota_{k_1}...\iota_{k_{N-1}} \mathfrak{D}A_{k_N},$$

(4.20)

$$f_n^\Omega := \sum_{k_i \geqslant 2}^{n} \sum_{N \geqslant 2} \gamma_N \, \iota_{k_1}...\iota_{k_{N-1}} \Omega_{k_N}.$$

---

[6] In principle one can consider dA_p and $\mathcal{L}_{A_1}A_p$ separately, but there is no point in breaking the covariant derivative.

[7] As before, one could consider dA_1 and $A_1 \circ A_1$ separately, but there is no advantage in breaking up $F_2 = \Omega_2$.

Let us first focus on the term $Df_n^A - \mathfrak{D} f_{n+1}^A$, for which we have to treat objects of the form (at fixed N)

$$\sum_{k_i \geqslant 2} D[\iota_{k_1}...\iota_{k_{N-1}} B_{k_N}], \qquad \sum_{k_i \geqslant 2} \mathfrak{D}[\iota_{k_1}...\iota_{k_{N-1}} B_{k_N}], \qquad (4.21)$$

where B_k is a k-form of twist $+2$. By using the graded Leibniz rule of the covariant derivative one has

$$D[\iota_{k_1}...\iota_{k_{N-1}} B_{k_N}] = \sum_{l=0}^{N-2} (-1)^{k_1+\cdots+k_l} \iota_{k_1}...\iota_{k_l} [DA_{k_{l+1}} \bullet (\iota_{k_{l+2}}...\iota_{k_{N-1}} B_{k_N})] \\ + (-1)^{k_1+\cdots+k_{N-1}} \iota_{k_1} \cdots \iota_{k_{N-1}} DB_{k_N}. \qquad (4.22)$$

The term in square brackets can be further manipulated by recursively using the identity

$$(-1)^k \iota_k (f \bullet g) = (\iota_k f) \bullet g + f \bullet (\iota_k g), \qquad (4.23)$$

that is axiom 6) of (3.31) for f and g with even twist, to get

$$\sum_{k_i \geqslant 2} \sum_{l=0}^{N-2} (-1)^{k_1+\cdots+k_l} \iota_{k_1} \cdots \iota_{k_l} [DA_{k_{l+1}} \bullet (\iota_{k_{l+2}}...\iota_{k_{N-1}} B_{k_N})] \\ = \sum_{k_i \geqslant 2} \sum_{l=0}^{N-2} \sum_{m=0}^{l} \binom{l}{m} (\iota_{k_1}...\iota_{k_m} DA_{k_{m+1}}) \bullet (\iota_{k_{m+2}}...\iota_{k_{N-1}} B_{k_N}) \\ = \sum_{k_i \geqslant 2} \sum_{m=0}^{N-2} \Big[\sum_{l=m}^{N-2} \binom{l}{m} \Big] (\iota_{k_1}...\iota_{k_m} DA_{k_{m+1}}) \bullet (\iota_{k_{m+2}}...\iota_{k_{N-1}} B_{k_N}) \\ = \sum_{k_i \geqslant 2} \sum_{m=0}^{N-2} \binom{N-1}{m+1} (\iota_{k_1}...\iota_{k_m} DA_{k_{m+1}}) \bullet (\iota_{k_{m+2}}...\iota_{k_{N-1}} B_{k_N}), \qquad (4.24)$$

finally giving

$$\sum_{k_i \geqslant 2} D[\iota_{k_1}...\iota_{k_{N-1}} B_{k_N}] = \sum_{k_i \geqslant 2} \Big\{ \sum_{m=0}^{N-2} \binom{N-1}{m+1} (\iota_{k_1}...\iota_{k_m} DA_{k_{m+1}}) \bullet (\iota_{k_{m+2}}...\iota_{k_{N-1}} B_{k_N}) \\ + (-1)^{k_1+\cdots+k_{N-1}} \iota_{k_1}...\iota_{k_{N-1}} DB_{k_N} \Big\}. \qquad (4.25)$$

In an almost identical way, by using 5) of (3.31), one proves that

$$-\sum_{k_i \geqslant 2} \mathfrak{D}[\iota_{k_1}...\iota_{k_{N-1}} B_{k_N}] = \sum_{k_i \geqslant 2} \Big\{ \sum_{m=0}^{N-2} \binom{N-1}{m+1} (\iota_{k_1}...\iota_{k_m} \mathfrak{D}A_{k_{m+1}}) \bullet (\iota_{k_{m+2}}...\iota_{k_{N-1}} B_{k_N}) \\ - (-1)^{k_1+\cdots+k_{N-1}} \iota_{k_1}...\iota_{k_{N-1}} \mathfrak{D}B_{k_N} \Big\}. \qquad (4.26)$$

Using the identities (4.25) and (4.26) on $Df_n^A - \mathfrak{D}f_{n+1}^A$ one obtains

$$
\begin{aligned}
Df_n^A - \mathfrak{D}f_{n+1}^A = &\sum_{k_i \geqslant 2} \sum_{N \geqslant 1}^{n} (\beta_N - \alpha_N)(-1)^{k_1 + \cdots + k_{N-1}} \iota_{k_1} \ldots \iota_{k_{N-1}} D\mathfrak{D}A_{k_N} \\
&+ \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n} \sum_{m=0}^{N-2} \Big[\beta_N \binom{N-1}{m+1} \\
&+ \alpha_N \binom{N-1}{N-1-m} \Big] (\iota_{k_1} \ldots \iota_{k_m} DA_{k_{m+1}}) \bullet (\iota_{k_{m+2}} \ldots \iota_{k_{N-1}} \mathfrak{D}A_{k_N}) \\
&+ \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n-1} \alpha_N \sum_{m=0}^{N-2} \binom{N-1}{m+1} (\iota_{k_1} \ldots \iota_{k_m} DA_{k_{m+1}}) \bullet (\iota_{k_{m+2}} \ldots \iota_{k_{N-1}} DA_{k_N}) \\
&+ \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n+1} \beta_N \sum_{m=0}^{N-2} \binom{N-1}{m+1} (\iota_{k_1} \ldots \iota_{k_m} \mathfrak{D}A_{k_{m+1}}) \bullet (\iota_{k_{m+2}} \ldots \iota_{k_{N-1}} \mathfrak{D}A_{k_N}) \\
&- \sum_{k_i \geqslant 2} \sum_{N \geqslant 1}^{n-1} \alpha_N (-1)^{k_1 + \cdots + k_{N-1}} \iota_{k_1} \ldots \iota_{k_{N-1}} \mathcal{L}_{\Omega_2} A_{k_N},
\end{aligned}
\tag{4.27}
$$

where we have used $D^2 = -\mathcal{L}_{\Omega_2}$. By looking at the right hand side of (4.19) one sees that there are no terms containing $D\mathfrak{D}A_k$, hence fixing $\beta_N = \alpha_N$ from the first term of (4.27). Furthermore, by looking at the diagonal terms of the form $(\iota^m DA) \bullet (\iota^{N-2-m} DA)$ and $(\iota^m \mathfrak{D}A) \bullet (\iota^{N-2-m} \mathfrak{D}A)$, one notices that they are manifestly symmetric (under the $\sum_{k_i}$) upon the exchange $m \to N-2-m$, thereby projecting the corresponding coefficients to their manifest symmetric part:

$$
\binom{N-1}{m+1} \to \frac{1}{2}\left[\binom{N-1}{m+1} + \binom{N-1}{N-1-m}\right] = \frac{1}{2}\left[\binom{N-1}{m+1} + \binom{N-1}{m}\right] = \frac{1}{2}\binom{N}{m+1}.
$$

After setting $\beta_N = \alpha_N$ (4.27) becomes

$$
\begin{aligned}
&Df_n^A - \mathfrak{D}f_{n+1}^A \\
&= \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n} \sum_{m=0}^{N-2} \alpha_N \binom{N}{m+1} (\iota_{k_1} \ldots \iota_{k_m} DA_{k_{m+1}}) \bullet (\iota_{k_{m+2}} \ldots \iota_{k_{N-1}} \mathfrak{D}A_{k_N}) \\
&+ \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n-1} \alpha_N \sum_{m=0}^{N-2} \frac{1}{2}\binom{N}{m+1} (\iota_{k_1} \ldots \iota_{k_m} DA_{k_{m+1}}) \bullet (\iota_{k_{m+2}} \ldots \iota_{k_{N-1}} DA_{k_N}) \\
&+ \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n+1} \alpha_N \sum_{m=0}^{N-2} \frac{1}{2}\binom{N}{m+1} (\iota_{k_1} \ldots \iota_{k_m} \mathfrak{D}A_{k_{m+1}}) \bullet (\iota_{k_{m+2}} \ldots \iota_{k_{N-1}} \mathfrak{D}A_{k_N}) \\
&- \sum_{k_i \geqslant 2} \sum_{N \geqslant 1}^{n-1} \alpha_N (-1)^{k_1 + \cdots + k_{N-1}} \iota_{k_1} \ldots \iota_{k_{N-1}} \mathcal{L}_{\Omega_2} A_{k_N},
\end{aligned}
\tag{4.28}
$$

that has to be compared to the Ω-independent part of the r.h.s. of (4.19):

$$
\frac{1}{2} \sum_{k=2}^{n-1} f_k^A \bullet f_{n+1-k}^A
$$

$$
= \sum_{k_i \geqslant 2} \sum_{N \geqslant 2} \sum_{m=0}^{N-2} \alpha_{m+1} \alpha_{N-m-1} \left(\iota_{k_1} ... \iota_{k_m} D A_{k_{m+1}} \right) \bullet \left(\iota_{k_{m+2}} ... \iota_{k_{N-1}} \mathfrak{D} A_{k_N} \right)
$$

$$
+ \frac{1}{2} \sum_{k_i \geqslant 2} \sum_{N \geqslant 2} \sum_{m=0}^{N-2} \alpha_{m+1} \alpha_{N-m-1} \left(\iota_{k_1} ... \iota_{k_m} D A_{k_{m+1}} \right) \bullet \left(\iota_{k_{m+2}} ... \iota_{k_{N-1}} D A_{k_N} \right)
$$

$$
+ \frac{1}{2} \sum_{k_i \geqslant 2} \sum_{N \geqslant 2} \sum_{m=0}^{N-2} \alpha_{m+1} \alpha_{N-m-1} \left(\iota_{k_1} ... \iota_{k_m} \mathfrak{D} A_{k_{m+1}} \right) \bullet \left(\iota_{k_{m+2}} ... \iota_{k_{N-1}} \mathfrak{D} A_{k_N} \right),
$$

(4.29)

thus requiring $\alpha_N \binom{N}{m+1} = -\alpha_{m+1} \alpha_{N-m-1}$. By setting $c_N := N! \alpha_N$ the requirement reads $c_N = -c_{m+1} c_{N-m-1}$, thus fixing $c_N = (-1)^{N-1}$ and

$$
\alpha_N = \beta_N = \frac{(-1)^{N-1}}{N!}.
$$

(4.30)

From (4.19) one is left to prove

$$
\mathfrak{D} f_{n+1}^\Omega - D f_n^\Omega + \sum_{k_i \geqslant 2} \sum_{N \geqslant 1} \alpha_N (-1)^{k_1 + \cdots + k_{N-1}} \iota_{k_1} ... \iota_{k_{N-1}} \mathcal{L}_{\Omega_2} A_{k_N}
$$

$$
= \sum_{k=0}^{n-1} \left[(\Omega_k + f_k^\Omega) \bullet f_{n+1-k}^A + (\Omega_k + \tfrac{1}{2} f_k^\Omega) \bullet f_{n+1-k}^\Omega \right].
$$

(4.31)

The term $D f_n^\Omega$ is no different from $D f_n^A$ and obeys the same identity (4.25):

$$
D f_n^\Omega = \sum_{k_i \geqslant 2} \sum_{N \geqslant 2} \gamma_N \, D[i_{k_1} ... i_{k_{N-1}} \Omega_{k_N}]
$$

$$
= \sum_{k_i \geqslant 2} \sum_{N \geqslant 2} \gamma_N \sum_{m=0}^{N-2} \binom{N-1}{m+1} \left(\iota_{k_1} ... \iota_{k_m} D A_{k_{m+1}} \right) \bullet \left(\iota_{k_{m+2}} ... \iota_{k_{N-1}} \Omega_{k_N} \right)
$$

$$
+ \sum_{k_i \geqslant 2} \sum_{N \geqslant 2} \gamma_N (-1)^{k_1 + \cdots + k_{N-1}} \iota_{k_1} ... \iota_{k_{N-1}} D \Omega_{k_N},
$$

(4.32)

while one has to be more careful with $\mathfrak{D} f_{n+1}^{\Omega}$, since the identity (4.26) does not apply immediately, given that $\mathfrak{D}\Omega_2$ does not exist. We have instead

$$
\begin{aligned}
-\mathfrak{D} f_{n+1}^{\Omega} &= -\sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n+1} \gamma_N \, \mathfrak{D}[\iota_{k_1}...\iota_{k_{N-1}}\Omega_{k_N}] \\
&= \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n+1} \gamma_N \Big\{ \sum_{l=0}^{N-3}(-1)^{k_1+\cdots+k_l}\iota_{k_1}...\iota_{k_l}\big[\mathfrak{D}A_{k_{l+1}} \bullet (\iota_{k_{l+2}}...\iota_{k_{N-1}}\Omega_{k_N})\big] \\
&\quad - (-1)^{k_1+\cdots+k_{N-2}}\iota_{k_1}...\iota_{k_{N-2}}\mathfrak{D}(A_{k_{N-1}} \bullet \Omega_{k_N}) \Big\}.
\end{aligned}
\tag{4.33}
$$

At this point one has to treat the $k_N = 2$ term in the sum separately. Using in particular $\mathfrak{D}(A_{k_{N-1}} \bullet \Omega_2) = \mathcal{L}_{\Omega_2}A_{k_{N-1}} - \mathfrak{D}A_{k_{N-1}} \bullet \Omega_2$ we get

$$
\begin{aligned}
\mathfrak{D} f_{n+1}^{\Omega} &= -\sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n+1} \gamma_N \sum_{m=0}^{N-2}\binom{N-1}{m+1}(\iota_{k_1}...\iota_{k_m}\mathfrak{D}A_{k_{m+1}}) \bullet (\iota_{k_{m+2}}...\iota_{k_{N-1}}\Omega_{k_N}) \\
&\quad + \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n} \gamma_N \, (-1)^{k_1+\cdots+k_{N-1}}\iota_{k_1}...\iota_{k_{N-1}}\mathfrak{D}\Omega_{k_N+1} \\
&\quad + \sum_{k_i \geqslant 2} \sum_{N \geqslant 1}^{n-1} \gamma_{N+1} \, (-1)^{k_1+\cdots+k_{N-1}}\iota_{k_1}...\iota_{k_{N-1}}\mathcal{L}_{\Omega_2}A_{k_N}.
\end{aligned}
\tag{4.34}
$$

This finally yields

$$
\begin{aligned}
Df_n^{\Omega} &- \mathfrak{D} f_{n+1}^{\Omega} \\
&= \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n+1} \gamma_N \sum_{m=0}^{N-2}\binom{N-1}{m+1}\big[\iota_{k_1}...\iota_{k_m}(DA_{k_{m+1}-1} + \mathfrak{D}A_{k_{m+1}})\big] \bullet (\iota_{k_{m+2}}...\iota_{k_{N-1}}\Omega_{k_N}) \\
&\quad + \sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n} \gamma_N \, (-1)^{k_1+\cdots+k_{N-1}}\iota_{k_1}...\iota_{k_{N-1}}(D\Omega_{k_N} - \mathfrak{D}\Omega_{k_N+1}) \\
&\quad - \sum_{k_i \geqslant 2} \sum_{N \geqslant 1}^{n-1} \gamma_{N+1} \, (-1)^{k_1+\cdots+k_{N-1}}\iota_{k_1}...\iota_{k_{N-1}}\mathcal{L}_{\Omega_2}A_{k_N}
\end{aligned}
\tag{4.35}
$$

and, demanding that the $\mathcal{L}_{\Omega_2}$ terms cancel, fixes

$$
\gamma_N = -\alpha_{N-1} = \frac{(-1)^{N-1}}{(N-1)!}.
\tag{4.36}
$$

With this value of γ_N it is easy to see that the first line of (4.35) matches the $(\Omega + f^\Omega) \bullet f^A$ term in (4.31), while the second line is rewritten by using the identity (4.7):

$$\sum_{k_i \geqslant 2} \sum_{N \geqslant 2}^{n} \gamma_N \, (-1)^{k_1 + \cdots + k_{N-1}} \iota_{k_1} \ldots \iota_{k_{N-1}} (\mathfrak{D}\Omega_{k_N+1} - D\Omega_{k_N})$$

$$= \tfrac{1}{2} \sum_{k_i \geqslant 2} \sum_{N \geqslant 3}^{n+1} \gamma_{N-1} \, (-1)^{k_1 + \cdots + k_{N-2}} \iota_{k_1} \ldots \iota_{k_{N-2}} (\Omega_{k_{N-1}} \bullet \Omega_{k_N}) \qquad (4.37)$$

$$= \tfrac{1}{2} \sum_{k_i \geqslant 2} \sum_{N \geqslant 3}^{n+1} \gamma_{N-1} \sum_{l=0}^{N-2} \binom{N-2}{l} (\iota_{k_1} \ldots \iota_{k_l} \Omega_{k_{l+1}}) \bullet (\iota_{k_{l+2}} \ldots \iota_{k_{N-1}} \Omega_{k_N}),$$

where we used (4.24) in the last step. With $\gamma_N = \frac{(-1)^{N-1}}{(N-1)!}$ it is again easy to see that the above expression takes care of the final $(\Omega + \tfrac{1}{2} f^\Omega) \bullet f^\Omega$ term of (4.31), finishing the proof of the Bianchi identity (4.12).

4.3. Gauge covariance. Here our goal is to use the Bianchi identities

$$D\mathcal{F}_n + \tfrac{1}{2} \sum_{k=2}^{n-1} \mathcal{F}_k \bullet \mathcal{F}_{n+1-k} = \mathfrak{D}\mathcal{F}_{n+1} \qquad (4.38)$$

to prove gauge covariance of the curvatures by induction. It is only at this point that we demand the differential $\mathfrak{D}$ to have trivial cohomology. Although it is necessary for this indirect proof, explicit computations for low degrees show that almost certainly this assumption is not actually needed, but we leave the necessary explicit formulation of the gauge transformations to future work.

As before, the lowest case $n = 2$ is proven directly, which we briefly recall in the present notation: One starts from

$$F_2 = dA_1 - \tfrac{1}{2} A_1 \circ A_1, \qquad (4.39)$$

and postulates the gauge symmetry

$$\delta A_1 = D\lambda_0 - \mathfrak{D}\lambda_1. \qquad (4.40)$$

The general variation of F_2 is given by

$$\delta F_2 = d\delta A_1 - \tfrac{1}{2} (\delta A_1 \circ A_1 + A_1 \circ \delta A_1) = D\delta A_1 + \tfrac{1}{2} \mathfrak{D}(A_1 \bullet \delta A_1), \qquad (4.41)$$

that, upon using (4.40), yields

$$\delta F_2 = -F_2 \circ \lambda_0 + \mathfrak{D}\left(\tfrac{1}{2} A_1 \bullet (D\lambda_0 - \mathfrak{D}\lambda_1) - D\lambda_1 \right)$$

$$= \mathcal{L}_{\lambda_0} F_2 + \mathfrak{D}\left(\tfrac{1}{2} A_1 \bullet (D\lambda_0 - \mathfrak{D}\lambda_1) - D\lambda_1 - F_2 \bullet \lambda_0 \right) \qquad (4.42)$$

$$= \mathcal{L}_{\lambda_0} F_2 + \mathfrak{D}\Delta_2.$$

One now introduces the two-form gauge field A_2 and the full two-form curvature as $\mathcal{F}_2 = F_2 + \mathfrak{D}A_2$, obeying

$$
\begin{aligned}
\delta\mathcal{F}_2 = \delta(F_2 + \mathfrak{D}A_2) &= \mathcal{L}_{\lambda_0}F_2 + \mathfrak{D}(\delta A_2 + \Delta_2) \\
&= \mathcal{L}_{\lambda_0}\mathcal{F}_2 + \mathfrak{D}(\delta A_2 - \mathcal{L}_{\lambda_0}A_2 + \Delta_2)
\end{aligned}
\tag{4.43}
$$

so that, by adjusting

$$
\delta A_2 = \mathcal{L}_{\lambda_0}A_2 - \Delta_2 - \mathfrak{D}\tilde{\lambda}_2
\tag{4.44}
$$

it is possible to achieve $\delta\mathcal{F}_2 = \mathcal{L}_{\lambda_0}\mathcal{F}_2$. Notice that, by using the explicit form of Δ_2 we get the usual transformation

$$
\Delta A_2 := \delta A_2 + \tfrac{1}{2}A_1 \bullet \delta A_1 = D\lambda_1 + \mathcal{F}_2 \bullet \lambda_0 - \mathfrak{D}\lambda_2,
\tag{4.45}
$$

with $\lambda_2 = \tilde{\lambda}_2 - \lambda_0 \bullet A_2$. Suppose next that we have fixed the gauge transformations of gauge fields A_p with $1 \leqslant p \leqslant n$ so that

$$
\delta\mathcal{F}_p = \mathcal{L}_{\lambda_0}\mathcal{F}_p, \quad \text{for } 2 \leqslant p \leqslant n.
\tag{4.46}
$$

We split $\mathcal{F}_{n+1} = F_{n+1} + \mathfrak{D}A_{n+1}$, since F_{n+1} only contains gauge fields A_p with $p \leqslant n$. The Bianchi identity (4.38), thanks to covariance and Leibniz property of the Lie derivative, ensures that

$$
\mathfrak{D}(\delta F_{n+1} - \mathcal{L}_{\lambda_0}F_{n+1}) = 0.
\tag{4.47}
$$

Assuming that the differential $\mathfrak{D}$ has trivial cohomology, we can thus write

$$
\delta F_{n+1} = \mathcal{L}_{\lambda_0}F_{n+1} + \mathfrak{D}\Delta_{n+1},
\tag{4.48}
$$

and

$$
\begin{aligned}
\delta\mathcal{F}_{n+1} &= \mathcal{L}_{\lambda_0}F_{n+1} + \mathfrak{D}\Delta_{n+1} + \mathfrak{D}\delta A_{n+1} \\
&= \mathcal{L}_{\lambda_0}\mathcal{F}_{n+1} + \mathfrak{D}\big(\delta A_{n+1} - \mathcal{L}_{\lambda_0}A_{n+1} + \Delta_{n+1}\big),
\end{aligned}
\tag{4.49}
$$

such that, by fixing

$$
\delta A_{n+1} = \mathcal{L}_{\lambda_0}A_{n+1} - \Delta_{n+1} - \mathfrak{D}\lambda_{n+1},
\tag{4.50}
$$

the covariance of $\mathcal{F}_{n+1}$ is established, and hence proven for all n.

As a final comment, we notice that by taking the formal sum of the higher form potentials, as well as the pseudo CS forms and curvatures:

$$
\mathcal{A} := \sum_{p=2}^{\infty} A_p, \quad \Omega := \sum_{p=2}^{\infty} \Omega_p, \quad \mathcal{F} := \sum_{p=2}^{\infty} \mathcal{F}_p,
\tag{4.51}
$$

it is possible to recast the Bianchi identities, as well as the definition of the curvatures, in the compact form

$$
\begin{aligned}
&D\mathcal{F} + \tfrac{1}{2}\mathcal{F} \bullet \mathcal{F} = \mathfrak{D}\mathcal{F}, \\
&\mathcal{F} = \sum_{N=0}^{\infty} \frac{(-\iota_{\mathcal{A}})^N}{N!}\left[\frac{1}{N+1}(D + \mathfrak{D})\mathcal{A} + \Omega\right].
\end{aligned}
\tag{4.52}
$$

In this paper we have discussed the algebraic structure underlying general tensor hierarchies purely at the kinematical level, determining gauge covariant curvatures of arbitrary rank together with their Bianchi identities. In order to provide dynamics to the above construction,[8] natural candidates are action principles (or pseudo-actions supported by first-order duality relations) of the schematic form $\int \langle \mathcal{F} \wedge, \star \mathcal{M} \mathcal{F} \rangle$, where $\langle \cdot, \cdot \rangle$ denotes a suitable inner product and $\mathcal{M}$ plays the role of generalized metric built out of the scalar fields of the theory.

5. Topological Theories Based on L_∞ Algebras

Our goal in this section is to relate the infinity enhanced Leibniz algebra introduced above to the closely related L_∞ algebras. It is known that for an 'enhanced Leibniz algebra' as defined in [39], consisting of the vector space $X_0 \oplus X_1$, there is an associated Lie 2-algebra, *i.e.* an L_∞ algebra whose highest bracket is the three bracket l_3. Similarly, given the structure of an infinity enhanced Leibniz algebra, one can define L_∞ algebras characterized by a nilpotent operator $l_1 \equiv \mathfrak{D}$, and graded antisymmetric brackets l_n obeying higher Jacobi-like relations. However, we will show that, in contrast to Leibniz algebras and extensions thereof, the L_∞ structure alone is not sufficient to define gauge covariant curvatures for higher form potentials. It is still possible, by means of L_∞ brackets alone, to define field strengths that transform into themselves under gauge transformations. This allows us to define topological tensor hierarchies, whose field equations amount to zero curvature conditions.

5.1. Warm-up. We begin with a warm-up example, trying to construct curvatures for the lowest gauge potentials by means of L_∞ brackets. This will show problems already at the level of the three-form curvature, essentially due to the lack of a proper covariant derivative. We will then proceed to construct a topological tensor hierarchy to all orders in terms of a general L_∞ algebra, not necessarily based on an underlying Leibniz algebra. We consider here a set of differential forms and gauge parameters taking value in an L_∞ algebra, with L_∞ degree assignments $|A_\mu| = |\lambda| = 0$, $|A_{\mu\nu}| = |\lambda_\mu| = 1$ and so on. More specifically, we identify

$$
\begin{aligned}
&\text{'trivial parameters':} && \chi_0, \ \chi_1, \ldots \\
&\text{gauge parameters:} && \lambda_0, \ \lambda_1, \ \lambda_2, \ldots \\
&\text{gauge fields:} && A_1, \ A_2, \ A_3, \ldots \\
&\text{field strengths:} && \mathcal{F}_2, \ \mathcal{F}_3, \ldots,
\end{aligned}
\tag{5.1}
$$

where now the index denotes the form degree and the L_∞ degree can be inferred from the following diagram:

---

[8] *Note added in proof:* during the review stage, dynamical field equations were provided in terms of first-order duality relations in [42], where possible action principles have also been discussed.

$$\cdots \xrightarrow{l_1} \quad X^2_{[0]} \quad \xrightarrow{l_1} \quad X^1_{[0]}(\chi_0) \quad \xrightarrow{l_1} \quad X^0_{[0]}(\lambda_0)$$

$$\downarrow \text{d} \qquad\qquad \downarrow \text{d} \qquad\qquad \downarrow \text{d} \qquad\qquad \downarrow \text{d}$$

$$\cdots \xrightarrow{l_1} X^2_{[1]}(\chi_1) \xrightarrow{l_1} X^1_{[1]}(\lambda_1) \xrightarrow{l_1} X^0_{[1]}(A_1)$$

$$\downarrow \text{d} \qquad\qquad \downarrow \text{d} \qquad\qquad \downarrow \text{d} \qquad\qquad \downarrow \text{d} \tag{5.2}$$

$$\cdots \xrightarrow{l_1} X^2_{[2]}(\lambda_2) \xrightarrow{l_1} X^1_{[2]}(A_2) \xrightarrow{l_1} X^0_{[2]}(\mathcal{F}_2)$$

$$\downarrow \text{d} \qquad\qquad \downarrow \text{d} \qquad\qquad \downarrow \text{d} \qquad\qquad \downarrow \text{d}$$

$$\cdots \xrightarrow{l_1} X^2_{[3]}(A_3) \xrightarrow{l_1} X^1_{[3]}(\mathcal{F}_3) \xrightarrow{l_1} X^0_{[3]}(\text{d}\mathcal{F}_2)$$

Note that the sequence of gauge parameters, fields, field strengths, respectively, runs 'diagonally' through the diagram (from north-east to south-west), in principle indefinitely.

We start again, in parallel with the construction of Sect. 2, by postulating the gauge symmetry for the one-form as

$$\delta A_\mu = \partial_\mu \lambda - l_2(A_\mu, \lambda) - l_1 \lambda_\mu. \tag{5.3}$$

Here l_1 is the nilpotent, degree -1, operator of the L_∞ algebra, that is a differential w.r.t. the l_2 bracket:

$$l_1 l_2(a, b) = l_2(l_1 a, b) + (-1)^{|a|} l_2(a, l_1 b), \tag{5.4}$$

and can be identified with the $\mathfrak{D}$ operator of the Leibniz algebra of the previous sections. The only other L_∞ relation needed at this level is

$$0 = l_1 \, l_3(a, b, c) + l_3(l_1 a, b, c) + (-1)^{|a|} l_3(a, l_1 b, c) + (-1)^{|a|+|b|} l_3(a, b, l_1 c)$$
$$+ l_2(l_2(a, b), c) + (-1)^{|a|(|b|+|c|)} l_2(l_2(b, c), a) + (-1)^{|c|(|a|+|b|)} l_2(l_2(c, a), b), \tag{5.5}$$

recalling that l_1 does not act on degree zero elements. By using (5.4) and (5.5) one still finds that the two-form curvature defined by

$$\mathcal{F}_{\mu\nu} = 2 \, \partial_{[\mu} A_{\nu]} - l_2(A_\mu, A_\nu) + l_1 B_{\mu\nu} \tag{5.6}$$

transforms covariantly: $\delta \mathcal{F}_{\mu\nu} = l_2(\lambda, \mathcal{F}_{\mu\nu})$, provided that the two-form gauge field transforms as

$$\delta B_{\mu\nu} = 2 \, \partial_{[\mu} \lambda_{\nu]} - 2 \, l_2(A_{[\mu}, \lambda_{\nu]}) + l_2(\lambda, B_{\mu\nu}) - l_3(A_\mu, A_\nu, \lambda) - l_1 \, \lambda_{\mu\nu}. \tag{5.7}$$

This apparently suggests to define a "covariant" derivative: $D_\mu x := \partial_\mu x - l_2(A_\mu, x)$ that allows to rewrite

$$\delta A_\mu = D_\mu \lambda - l_1 \lambda_\mu,$$
$$\delta B_{\mu\nu} = 2 \, D_{[\mu} \lambda_{\nu]} + l_2(\lambda, B_{\mu\nu}) - l_3(A_\mu, A_\nu, \lambda) - l_1 \, \lambda_{\mu\nu}, \tag{5.8}$$

but is effectively of little use, since D_μ is *not* a covariant operation in the usual sense. Indeed, given any element x that transforms covariantly in the sense that

$$\delta x = l_2(\lambda, x), \tag{5.9}$$

one has

$$\delta D_\mu x = l_2(\lambda, D_\mu x) + l_1 l_3(A_\mu, x, \lambda) + l_3(A_\mu, l_1 x, \lambda), \tag{5.10}$$

which contains bare gauge fields. At this stage, the most general ansatz for the three-form curvature is

$$\mathcal{H}_{\mu\nu\lambda} = 3\,\partial_{[\mu} B_{\nu\lambda]} + \alpha\, l_2(A_{[\mu}, B_{\nu\lambda]}) + \beta\, l_3(A_\mu, A_\nu, A_\lambda) + l_1\, C_{\mu\nu\lambda}, \tag{5.11}$$

and using (5.8) one finds the variation

$$
\begin{aligned}
\delta\mathcal{H}_{\mu\nu\rho} &= (\alpha + 3)\, l_2(\partial_{[\mu}\lambda, B_{\nu\rho]}) + 3(\beta - 1)\, l_3(\partial_{[\mu}\lambda, A_\nu, A_{\rho]}) \\
&\quad + 2(\alpha + 3)\, l_2(A_{[\mu}, \partial_\nu\lambda_{\rho]}) + \cdots \\
&\quad + l_1\big(\delta C_{\mu\nu\rho} - 3\,\partial_{[\mu}\lambda_{\nu\rho]} + \cdots\big),
\end{aligned} \tag{5.12}
$$

where the omitted terms do not contain derivatives of the gauge parameters. This already fixes $\alpha = -3$ and $\beta = 1$, yielding

$$\mathcal{H}_{\mu\nu\lambda} = 3\, D_{[\mu} B_{\nu\lambda]} + l_3(A_\mu, A_\nu, A_\lambda) + l_1\, C_{\mu\nu\lambda}. \tag{5.13}$$

The l_1-exact terms in $\delta\mathcal{H}_{\mu\nu\rho}$ can be absorbed by setting[9]

$$
\begin{aligned}
\delta C_{\mu\nu\rho} &= 3\, D_{[\mu}\lambda_{\nu\rho]} + l_2(\lambda, C_{\mu\nu\rho}) - 3\, l_2(\lambda_{[\mu}, B_{\nu\rho]}) \\
&\quad - 3\, l_3(\lambda, A_{[\mu}, B_{\nu\rho]}) - 3\, l_3(\lambda_{[\mu}, A_\nu, A_{\rho]}) + l_4(\lambda, A_\mu, A_\nu, A_\rho),
\end{aligned} \tag{5.14}
$$

but one is still left with

$$\delta\mathcal{H}_{\mu\nu\rho} = l_2(\lambda, \mathcal{H}_{\mu\nu\rho}) + 3\, l_2(\lambda_{[\mu}, \mathcal{F}_{\nu\rho]}) + 3\, l_3(\lambda, A_{[\mu}, \mathcal{F}_{\nu\rho]}), \tag{5.15}$$

that is not covariant even in terms of a formal sum of curvatures, due to the bare gauge field in the last term. However, since $\delta\mathcal{H}_{\mu\nu\rho}$ is proportional to itself and $\mathcal{F}_{\mu\nu}$, it still allows for the zero curvature conditions

$$\mathcal{H}_{\mu\nu\rho} = 0, \quad \mathcal{F}_{\mu\nu} = 0, \tag{5.16}$$

to be gauge invariant topological field equations.

After this explicit example for the lowest ranks of the tensor hierarchy, we will now turn to determine the gauge transformations and field strengths for differential forms $A_{\mu_1 \ldots \mu_p}$ of arbitrary degree taking values in an L_∞ algebra, that are consistent as topological field equations of the form $\mathcal{F}_{\mu_1 \ldots \mu_{p+1}} = 0$.

---

[9] To find this one also needs the relation $l_1 l_4(x_1, x_2, x_3, x_4) = 4 l_2(l_3(x_{[1}, x_2, x_3), x_4]) - 6 l_3(l_2(x_{[1}, x_2), x_3, x_4])$ for $|x_i| = 0$.

5.2. Differential forms taking values in L_∞ algebras. In this section we will only assume that the field content consists of a set of differential forms of arbitrary degree, taking values in an L_∞ algebra with multilinear, graded antisymmetric, maps l_n of intrinsic degree $n - 2$, obeying the quadratic relations

$$\sum_{n=0}^{N-1} (-1)^{n(N+1)} l_{n+1} l_{N-n} = 0, \quad N = 1, 2, ..., \infty, \tag{5.17}$$

where N is the total number of arguments involved. The left hand side is meant to act on the graded antisymmetrized tensor algebra so that, for instance, the $N = 3$ relation $l_1 l_3 + l_2 l_2 + l_3 l_1 = 0$ explicitly reads as (5.5). The set of differential forms organizes naturally in terms of a double grading (p, d) given by the form degree and the L_∞ degree, respectively. A generic differential p-form of L_∞ degree d will be denoted by

$$\omega_{\mu_1...\mu_p}^d \equiv \omega_{\mu[p]}^d. \tag{5.18}$$

This two-dimensional array has the structure of a bi-complex with respect to two separate differentials, the de Rham differential d, increasing the form degree by one, and the l_1 operator, decreasing the L_∞ degree by one. The two gradings, in spite of being independent, can be linked in the tensor hierarchy thanks to the physical interpretation of the fields: A p-form curvature has L_∞ degree $p - 2$, $F_{\mu[p]}^{p-2}$, a p-form gauge field $A_{\mu[p]}^{p-1}$ has degree[10] $p - 1$, a p-form gauge parameter has degree p, $\xi_{\mu[p]}^p$ and so on. One can notice that a new, diagonal, degree can be defined as the difference between L_∞ and form degree: $N := d - p$, and that this degree is only sensitive to the physical role of a given differential form. Indeed, it is immediate to see that every curvature $F_{\mu[p]}^{p-2}$ has N-degree -2, every gauge field $A_{\mu[p]}^{p-1}$ has N-degree -1 and any gauge parameter $\xi_{\mu[p]}^p$ has N-degree 0. The main advantage of classifying the fields in terms of the N-degree (that is identified with their physical interpretation) is that gauge fields (as well as curvatures, gauge parameters etc.) of arbitrary form degree can be treated on equal footing, and eventually dealt with at once in a single, string field-like, object. In the following, we will show that it is possible to define new maps, that we will denote by ℓ_n, that obey the usual L_∞ symmetry properties and quadratic relations in terms of the single degree N. The new maps differ from the l_n's only by phases that are quite lengthy to determine. In order to deal efficiently with such phases we introduce anticommuting generating elements θ^μ, obeying

$$\theta^\mu \theta^\nu + \theta^\nu \theta^\mu = 0. \tag{5.19}$$

They differ from ordinary one-form basis elements dx^μ in that they are formally assigned L_∞ degree -1. By this we mean that they pick up a sign when commuting with a differential form: $\theta^\mu \omega_{\nu[p]}^d = (-1)^d \omega_{\nu[p]}^d \theta^\mu$, where d is the L_∞ degree, as well as when (formally) going through an l_n map: $\theta^\mu l_n = (-1)^n l_n \theta^\mu$. This allows us to define objects

$$\omega_p^N := \frac{1}{p!} \theta^{\mu_1}...\theta^{\mu_p} \omega_{\mu_1...\mu_p}^d \equiv \theta^{\mu[p]} \omega_{\mu[p]}^d \tag{5.20}$$

---

[10] This is chosen to give L_∞ degree zero to the vector field, its scalar gauge parameter and curvature two-form.

that have (so far formally) the N-degree as their L_∞ degree. The precise way to determine the signs defining the new ℓ_n maps is to move all the θ generating elements to the left and outside the maps according to the above commutation relations, *e.g.*

$$\ell_1(\omega_p^N) = \ell_1(\theta^{\mu[p]} \omega_{\mu[p]}^d) := (-1)^p \theta^{\mu[p]} l_1(\omega_{\mu[p]}^d)$$

$$\ell_2(\omega_{p_1}^{N_1}, \omega_{p_2}^{N_2}) = \ell_2\left(\theta^{\mu[p_1]} \omega_{\mu[p_1]}^{d_1}, \theta^{\nu[p_2]} \omega_{\nu[p_2]}^{d_2}\right) \tag{5.21}$$

$$:= (-1)^{p_2 d_1} \theta^{\mu[p_1]} \theta^{\nu[p_2]} l_2\left(\omega_{\mu[p_1]}^{d_1}, \omega_{\nu[p_2]}^{d_2}\right).$$

This allows us to give a precise definition of the ℓ_n maps as

$$\ell_n(\omega_{p_1}^{N_1}, ..., \omega_{p_n}^{N_n}) := (-1)^\Sigma \theta^{\mu[p_1]}...\theta^{\nu[p_n]} l_n\left(\omega_{\mu[p_1]}^{d_1}, ..., \omega_{\mu[p_n]}^{d_n}\right),$$

$$\Sigma = n \sum_{i=1}^n p_i + \sum_{i=2}^n \sum_{j=1}^{i-1} p_i d_j. \tag{5.22}$$

According to this definition, the ℓ_n maps are graded antisymmetric w.r.t. the N-degree, as can be proven by direct computation:

$$\ell_n(\omega_{p_1}^{N_1}, ..., \omega_{p_i}^{N_i}, \omega_{p_{i+1}}^{N_{i+1}}, ..., \omega_{p_n}^{N_n})$$

$$= (-1)^{n \sum_{k=1}^n p_k + \sum_{k=2}^n \sum_{l=1}^{k-1} p_k d_l} \theta^{\mu[p_1]}...\theta^{\nu[p_i]} \theta^{\lambda[p_{i+1}]}...\theta^{\rho[p_n]}$$

$$\times l_n\left(\omega_{\mu[p_1]}^{d_1}, ..., \omega_{\nu[p_i]}^{d_i}, \omega_{\lambda[p_{i+1}]}^{d_{i+1}}, ..., \omega_{\rho[p_n]}^{d_n}\right)$$

$$= (-1)^{n \sum_{k=1}^n p_k + \sum_{k=2}^n \sum_{l=1}^{k-1} p_k d_l} (-1)^{1 + p_i p_{i+1} + d_i d_{i+1}} \theta^{\mu[p_1]}...\theta^{\lambda[p_{i+1}]} \theta^{\nu[p_i]}...\theta^{\rho[p_n]}$$

$$\times l_n\left(\omega_{\mu[p_1]}^{d_1}, ..., \omega_{\lambda[p_{i+1}]}^{d_{i+1}}, \omega_{\nu[p_i]}^{d_i}, ..., \omega_{\rho[p_n]}^{d_n}\right)$$

$$= (-1)^{1 + N_i N_{i+1}} \ell_n(\omega_{p_1}^{N_1}, ..., \omega_{p_{i+1}}^{N_{i+1}}, \omega_{p_i}^{N_i}, ..., \omega_{p_n}^{N_n}). \tag{5.23}$$

The n-dependent part of the sign factor in the definition (5.22) (that corresponds to the formal property of θ's picking a phase to go through the maps themselves) is needed in order to ensure the correct symmetry property of nested ℓ_n maps, for instance

$$\ell_2(\ell_n(\omega_{p_1}^{N_1}, ..., \omega_{p_n}^{N_n}), \eta_q^{N_q}) = (-1)^{1 + N_q(\sum_i N_i + n - 2)} \ell_2(\eta_q^{N_q}, \ell_n(\omega_{p_1}^{N_1}, ..., \omega_{p_n}^{N_n})), \tag{5.24}$$

and in general

$$\ell_n(\omega_{p_1}^{N_1}, ..., \ell_m(\eta_{q_1}^{M_1}, ..., \eta_{q_m}^{M_m}), \omega_{p_k}^{N_k}, ..., \omega_{p_{n-1}}^{N_{n-1}})$$

$$= (-1)^{1 + N_k(\sum_i M_i + m - 2)} \ell_n(\omega_{p_1}^{N_1}, ..., \omega_{p_k}^{N_k}, \ell_m(\eta_{q_1}^{M_1}, ..., \eta_{q_m}^{M_m}), ..., \omega_{p_{n-1}}^{N_{n-1}}), \tag{5.25}$$

thereby confirming that the ℓ_n maps have intrinsic degree $n - 2$. In order to prove that the new maps ℓ_n obey the same quadratic relations as the l_n, but w.r.t. the N-degree, we have to show that they pick the correct sign under the permutation of arguments between the two maps involved: The original maps obey the quadratic relations

$$l_n(l_m(\omega_{\mu[p_1]}^{d_1}, ..., \omega_{\mu[p_m]}^{d_m}), \omega_{\mu[p_{m+1}]}^{d_{m+1}}, ..., \omega_{\mu[p_{n+m-1}]}^{d_{n+m-1}})$$

$$+ (-1)^{1+d_m d_{m+1}} l_n (l_m (\omega^{d_1}_{\mu[p_1]}, ..., \omega^{d_{m+1}}_{\mu[p_{m+1}]}), \omega^{d_m}_{\mu[p_m]}, ..., \omega^{d_{n+m-1}}_{\mu[p_{n+m-1}]}) + \cdots = 0.$$

$$(5.26)$$

Multiplying this relation by

$$(-1)^{(n+m) \sum_{i=1}^{n+m-1} p_i + \sum_{i=2}^{n+m-1} \sum_{j=1}^{i-1} p_i d_j} \theta^{\mu[p_1]} ... \theta^{\mu[p_{n+m-1}]}$$

one obtains

$$\ell_n (\ell_m (\omega^{N_1}_{p_1}, ..., \omega^{N_m}_{p_m}), \omega^{N_{m+1}}_{p_{m+1}}, ..., \omega^{N_{n+m-1}}_{p_{n+m-1}})$$

$$+ (-1)^{1+N_m N_{m+1}} \ell_n (\ell_m (\omega^{N_1}_{p_1}, ..., \omega^{N_{m+1}}_{p_{m+1}}), \omega^{N_m}_{p_m}, ..., \omega^{N_{n+m-1}}_{p_{n+m-1}}) + \cdots = 0,$$

$$(5.27)$$

thus proving that the ℓ_n maps obey the L_∞ relations with respect to the N-degree. The de Rham differential is now written as

$$d := \theta^\mu \partial_\mu \qquad (5.28)$$

and thus possesses N-degree -1. Accordingly, it obeys a graded Leibniz rule that only sees the N-degree:

$$d \, \ell_n (\omega^{N_1}_{p_1}, \omega^{N_2}_{p_2}, ..., \omega^{N_n}_{p_n})$$

$$= (-1)^n \{ \ell_n (d\omega^{N_1}_{p_1}, \omega^{N_2}_{p_2}, ..., \omega^{N_n}_{p_n}) + (-1)^{N_1} \ell_n (\omega^{N_1}_{p_1}, d\omega^{N_2}_{p_2}, ..., \omega^{N_n}_{p_n}) +$$

$$... + (-1)^{\sum_{i=1}^{n-1} N_i} \ell_n (\omega^{N_1}_{p_1}, \omega^{N_2}_{p_2}, ..., d\omega^{N_n}_{p_n}) \}. \qquad (5.29)$$

In particular, it obeys $d\ell_1 + \ell_1 d = 0$, that allows one to define a new nilpotent operator:

$$\tilde{\ell}_1 := \ell_1 + d \qquad (5.30)$$

that still obeys the L_∞ relations.

5.3. Topological gauge theory.

We are now ready to construct a topological higher gauge theory with the above ingredients: The gauge fields $A_{\mu[p]}$ can be packaged in a single string field-like object of N-degree -1

$$\mathcal{A}(x, \theta) := \sum_{p=1}^{\infty} \frac{1}{p!} \theta^{\mu_1} ... \theta^{\mu_p} A_{\mu_1 ... \mu_p}(x), \quad N_{\mathcal{A}} = -1, \qquad (5.31)$$

and similarly one can define a degree zero gauge parameter

$$\Xi(x, \theta) := \sum_{p=0}^{\infty} \frac{1}{p!} \theta^{\mu_1} ... \theta^{\mu_p} \xi_{\mu_1 ... \mu_p}(x), \quad N_{\Xi} = 0, \qquad (5.32)$$

and degree -2 curvature

$$\mathcal{F}(x, \theta) := \sum_{p=2}^{\infty} \frac{1}{p!} \theta^{\mu_1} ... \theta^{\mu_p} F_{\mu_1 ... \mu_p}(x), \quad N_{\mathcal{F}} = -2. \qquad (5.33)$$

For the field strength and gauge transformations, we make the ansatz

$$\delta \mathcal{A} = d\Xi + \sum_{n=1}^{\infty} \alpha_n \, \ell_n(\mathcal{A}, ..., \mathcal{A}, \Xi)$$

$$\mathcal{F} = d\mathcal{A} + \sum_{n=1}^{\infty} \gamma_n \, \ell_n(\mathcal{A}, ..., \mathcal{A}),$$

(5.34)

where the coefficients α_n and γ_n have to be fixed by demanding that the curvatures transform into themselves. For the gauge variation of the curvature we compute

$$\delta \mathcal{F} = \sum_{n=1}^{\infty} (-1)^n (n-1)\alpha_n \, \ell_n(d\mathcal{A}, \mathcal{A}, ..., \mathcal{A}, \Xi) - \alpha_n \, \ell_n(\mathcal{A}, ..., \mathcal{A}, d\Xi)$$

$$+ \sum_{n=1}^{\infty} n\gamma_n \left\{ \ell_n(d\Xi, \mathcal{A}, ..., \mathcal{A}) + \sum_{m=1}^{\infty} \alpha_m \, \ell_n(\ell_m(\mathcal{A}, ..., \mathcal{A}, \Xi), \mathcal{A}, ..., \mathcal{A}) \right\}, \quad (5.35)$$

and demanding that the $d\Xi$ terms cancel fixes $\gamma_n = \frac{1}{n}\alpha_n$, yielding

$$\delta \mathcal{F} = \sum_{n=1}^{\infty} (-1)^n (n-1)\alpha_n \, \ell_n(d\mathcal{A}, \mathcal{A}, ..., \mathcal{A}, \Xi)$$

$$+ \sum_{n,m=1}^{\infty} \alpha_n \alpha_m \, \ell_n(\ell_m(\mathcal{A}, ..., \mathcal{A}, \Xi), \mathcal{A}, ..., \mathcal{A})$$

$$= \sum_{n=1}^{\infty} (-1)^n (n-1)\alpha_n \, \ell_n(\mathcal{F}, \mathcal{A}, ..., \mathcal{A}, \Xi)$$

$$+ \sum_{n,m=1}^{\infty} \alpha_n \alpha_m \left\{ \ell_n(\ell_m(\mathcal{A}, ..., \mathcal{A}, \Xi), \mathcal{A}, ..., \mathcal{A}) \right.$$

$$\left. - (-1)^n \frac{n-1}{m} \, \ell_n(\ell_m(\mathcal{A}, ..., \mathcal{A}), \mathcal{A}, ..., \mathcal{A}, \Xi) \right\}.$$

(5.36)

For the curvature to transform proportionally to itself the last line above has to vanish. The L_∞ relations for $N-1$ $\mathcal{A}$'s of degree -1 and one Ξ of degree zero read

$$\sum_{k=0}^{N-1} (-1)^{k(N+1)} \left[\frac{1}{(k-1)!(N-k)!} \, \ell_{k+1}(\ell_{N-k}(\mathcal{A}, ..., \mathcal{A}), \mathcal{A}, ..., \mathcal{A}, \Xi) \right.$$

$$\left. + \frac{(-1)^k}{k!(N-1-k)!} \, \ell_{k+1}(\ell_{N-k}(\mathcal{A}, ..., \mathcal{A}, \Xi), \mathcal{A}, ..., \mathcal{A}) \right] = 0,$$

(5.37)

while the last line of (5.36) can be rewritten as

$$
\begin{aligned}
\sum_{n,m=1}^{\infty} \alpha_n \alpha_m & \Big\{ \ell_n(\ell_m(\mathcal{A}, ..., \mathcal{A}, \Xi), \mathcal{A}, ..., \mathcal{A}) \\
& - (-1)^n \tfrac{n-1}{m} \, \ell_n(\ell_m(\mathcal{A}, ..., \mathcal{A}), \mathcal{A}, ..., \mathcal{A}, \Xi) \Big\} \\
= \sum_{N=1}^{\infty} \sum_{k=0}^{N-1} \alpha_{k+1} \alpha_{N-k} & \Big\{ \ell_{k+1}(\ell_{N-k}(\mathcal{A}, ..., \mathcal{A}, \Xi), \mathcal{A}, ..., \mathcal{A}) \\
& + (-1)^k \tfrac{k}{N-k} \, \ell_{k+1}(\ell_{N-k}(\mathcal{A}, ..., \mathcal{A}), \mathcal{A}, ..., \mathcal{A}, \Xi) \Big\}.
\end{aligned}
\tag{5.38}
$$

For this to be proportional to the L_∞ relations one has to demand that

$$
\alpha_{k+1} \alpha_{N-k} = f(N) \frac{(-1)^{Nk}}{k!(N-1-k)!}
\tag{5.39}
$$

for an arbitrary $f(N)$, which can be solved by $\alpha_n = \frac{(-1)^{\frac{n(n\pm 1)}{2}}}{(n-1)!}$. By choosing the solution with the minus sign we get

$$
\begin{aligned}
\delta \mathcal{A} &= d\Xi + \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!} \, \ell_n(\mathcal{A}, ..., \mathcal{A}, \Xi), \\
\mathcal{F} &= d\mathcal{A} + \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \, \ell_n(\mathcal{A}, ..., \mathcal{A}),
\end{aligned}
\tag{5.40}
$$

with the curvature transforming as

$$
\delta \mathcal{F} = \sum_{n=2}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{(n-2)!} \, \ell_n(\mathcal{F}, \mathcal{A}, ..., \mathcal{A}, \Xi).
\tag{5.41}
$$

As anticipated in the Introduction, the curvatures are not gauge covariant, due to the presence of bare gauge fields in (5.41) that cannot be avoided. However, the zero curvature condition

$$
\mathcal{F} = 0
\tag{5.42}
$$

is a consistent, gauge invariant, topological field equation. The curvature obeys the generalized Bianchi identity

$$
d\mathcal{F} + \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!} \, \ell_n(\mathcal{A}, ..., \mathcal{A}, \mathcal{F}) \equiv 0,
\tag{5.43}
$$

that suggests to define a generalized covariant derivative as

$$
Dx := dx + \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!} \, \ell_n(\mathcal{A}, ..., \mathcal{A}, x),
\tag{5.44}
$$

such that the Bianchi identity and gauge transformation of $\mathcal{A}$ take the form

$$\delta \mathcal{A} = D\Xi, \quad D\mathcal{F} \equiv 0. \tag{5.45}$$

Establishing the form of trivial gauge parameters is facilitated by the identity

$$D^2 x = \sum_{n=2}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{(n-2)!} \ell_n(\mathcal{F}, \mathcal{A}, ..., \mathcal{A}, x), \tag{5.46}$$

that generalizes the usual Yang-Mills relation $D^2 = F$. It can be proven by direct computation:

$$D^2 x = \sum_{n=2}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{(n-2)!} \ell_n(d\mathcal{A}, \mathcal{A}, ..., \mathcal{A}, x) - \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!} \ell_n(\mathcal{A}, ..., \mathcal{A}, dx)$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!} \ell_n(\mathcal{A}, ..., \mathcal{A}, dx)$$

$$+ \sum_{n,m=1}^{\infty} \frac{(-1)^{\frac{n(n-1)+m(m-1)}{2}}}{(n-1)!(m-1)!} \ell_n(\mathcal{A}, ..., \mathcal{A}, \ell_m(\mathcal{A}, ..., \mathcal{A}, x))$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{(n-2)!} \ell_n(\mathcal{F}, \mathcal{A}, ..., \mathcal{A}, x)$$

$$- \sum_{n,m=1}^{\infty} \frac{(-1)^{\frac{n(n+1)+m(m-1)}{2}}(n-1)}{(n-1)!m!} \ell_n(\ell_m(\mathcal{A}, ..., \mathcal{A}), \mathcal{A}, ..., \mathcal{A}, x) \tag{5.47}$$

$$+ \sum_{n,m=1}^{\infty} \frac{(-1)^{\frac{n(n-1)+m(m-1)}{2}}(-1)^{|x|(n-1)}}{(n-1)!(m-1)!} \ell_n(\ell_m(\mathcal{A}, ..., \mathcal{A}, x)\mathcal{A}, ..., \mathcal{A})$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{(n-2)!} \ell_n(\mathcal{F}, \mathcal{A}, ..., \mathcal{A}, x) + \sum_{N=1}^{\infty} (-1)^{\frac{N(N-1)}{2}}$$

$$\times \sum_{k=0}^{N-1} \frac{(-1)^{k(N+1)}}{k!(N-k-1)!} \left[\frac{k}{N-k} \ell_{k+1}(\ell_{N-k}(\mathcal{A}, ..., \mathcal{A}), \mathcal{A}, ..., \mathcal{A}, x) \right.$$

$$\left. + (-1)^{k(|x|+1)} \ell_{k+1}(\ell_{N-k}(\mathcal{A}, ..., \mathcal{A}, x), \mathcal{A}, ..., \mathcal{A}) \right]$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{(n-2)!} \ell_n(\mathcal{F}, \mathcal{A}, ..., \mathcal{A}, x),$$

where we denoted the degree of x by $|x|$ and used the L_∞ relations in the last line. As promised, the property (5.46) immediately shows that a gauge parameter of the form $\Xi = D\Lambda$ generates transformations that are trivial on-shell:

$$\delta_{D\Lambda} \mathcal{A} = D^2 \Lambda = \sum_{n=2}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{(n-2)!} \ell_n(\mathcal{F}, \mathcal{A}, ..., \mathcal{A}, \Lambda) \overset{\mathcal{F}=0}{=} 0. \tag{5.48}$$

The above construction is closely related to the Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) [46] formalism to construct topological sigma models from graded source manifolds (for a review see *e.g.* [47]). In particular, the quantity

$$Q(\mathcal{A}) := \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \, \ell_n(\mathcal{A}, ..., \mathcal{A}) \tag{5.49}$$

appearing in the field equation

$$\mathcal{F} = d\mathcal{A} + Q(\mathcal{A}) = 0, \tag{5.50}$$

is related to the power series expansion of the Q-structure of the target space, in AKSZ terms. The present construction, however, only produces the ghost-number zero classical fields of the theory. In fact, contrary to the usual AKSZ construction, the oscillators θ^μ do not carry ghost number and all component fields of $\mathcal{A}$ are classical gauge fields. One can indeed think about constructing the entire BV spectrum of the model, by further endowing the θ^μ oscillators with ghost number, and enlarge the field content by promoting every component A_p in $\mathcal{A} = \sum_p A_p$ to an arbitrary function of θ^μ.

6. Conclusions and Outlook

In this paper we have developed the general gauge theory of Leibniz-Loday algebras. We introduced the structure of an 'infinity enhanced Leibniz algebra' and proved that there is an associated tensor hierarchy of p-form gauge potentials that is consistent to arbitrary levels. Our proposal is that the 'infinity enhanced Leibniz algebra' yields the proper mathematical axiomatization of the notion of 'tensor hierarchy' developed in theoretical physics.

There are numerous potential applications and further extensions of the general framework developed here, which we briefly list in the following:

- Various examples of Leibniz algebras and their associated gauge theories have already been discussed in the recent literature [5,6], notably in [43], where the notation of this paper is employed. However, the tensor hierarchies have typically only been developed up to the form degree needed in order to write a gauge invariant action. It would be important to use the general mathematical machinery defined here to construct exact tensor hierarchies. In particular, this would allow one to formulate dynamical equations in terms of a hierarchy of duality relations between curvatures and their duals [49].

- Apart from the applications in string and M-theory that motivated the formulation of Leibniz-Loday gauge theories, it is to be expected that they will play a role in other areas, too. For instance, it has recently been shown that these structures are needed for a local formulation of gauge theories based on the algebra of volume-preserving diffeomorphisms [44], in turn suggesting potential applications in hydrodynamics [48].

- Another area where applications are quite likely is that of higher-spin gauge theories as introduced by Vasiliev, whose formulation relies on the unfolded approach, which is closely related to L_∞ algebras [50–52], although interactions are more directly understood as governed by deformations of associative algebras [52,53]. One may suspect that the even further generalized algebraic structures discussed here will be useful for higher-spin gravity, particularly in reference to the issue of modding out ideals that is crucial for higher-spin theories in arbitrary dimensions [54–57].

- We believe to have identified a new and rich mathematical structure, but the formulation found here leaves something to be desired. For instance, it would be useful to understand the infinity enhanced Leibniz algebras as the 'homotopy version' of some simpler algebraic structure — in the same sense that an L_∞ algebra is the homotopy version of a Lie algebra. Moreover, the arguably most efficient and useful formulation of A_∞ or L_∞ algebras is in terms of co-derivations on suitable tensor algebras that square to zero [28]. It would be helpful to find a similar formulation for the structures identified here.

- An open problem in exceptional field theory, the duality covariant formulation of the spacetime actions of string/M-theory, is the question of whether there is a 'universal' formulation unifying all U-duality groups, combining $E_{n(n)}$, $n = 2, \ldots, 9$, into a single algebraic structure. So far, these theories are based on a split into 'external' and 'internal' spaces, with the latter governed by the Leibniz algebra of generalized diffeomorphisms and the former by p-forms building a tensor hierarchy for this Leibniz algebra. Is there, perhaps, a formulation without split, based on a larger algebraic structure from which one would recover the presently understood exceptional field theories by choosing a Leibniz subalgebra and decomposing according to a $\mathbb{Z} \oplus \mathbb{Z}$ grading as in (5.2)?

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A. Proof of the CS Bianchi Identity

In the construction of gauge covariant curvatures for the tensor hierarchy one is led to introduce Chern-Simons-like forms that are built from the one-form $A_1 \equiv A$ alone, and are instrumental to prove the Bianchi identities. We define the pseudo Chern-Simons (CS) n-form by

$$\Omega_n(A) = \tfrac{(-1)^n}{(n-1)!} \, (\iota_A)^{n-2}\big[dA - \tfrac{1}{n} \, A \circ A\big], \quad |\Omega_n| = n - 2, \qquad (A.1)$$

where $\iota_A x := A \bullet x$ and we recall that the pure Yang-Mills two-form F_2 is included as Ω_2. We will now prove the identity (4.7)

$$D\Omega_n + \tfrac{1}{2} \sum_{k=2}^{n-1} \Omega_k \bullet \Omega_{n+1-k} = \mathfrak{D}\Omega_{n+1}, \quad n \geqslant 2, \qquad (A.2)$$

that was used in Sect. 4 to prove the Bianchi identity for the curvatures. We will prove (A.2) by induction. To this end, let us define the quantities

$$\omega_n := \frac{(-1)^n}{(n-1)!} A \bullet (A \bullet (...(A \bullet dA))) \sim A^{n-2} dA, \quad \omega_2 \equiv dA,$$

$$a_n := \frac{(-1)^{n+1}}{n!} A \bullet (A \bullet (...(A \circ A))) \sim A^n, \quad a_2 \equiv -\frac{1}{2} A \circ A.$$

$$(A.3)$$

The pseudo CS form can then be written as

$$\Omega_n = \omega_n + a_n, \tag{A.4}$$

and one has

$$D\Omega_n = d\omega_n + \left[da_n - \mathcal{L}_A \omega_n \right] - \mathcal{L}_A a_n , \tag{A.5}$$

where we grouped terms with two, one and zero spacetime derivatives. Notice that both ω_n and a_n have form degree n and intrinsic degree $n-2$, making them $\bullet$-commutative, i.e.

$$\omega_k \bullet \omega_l = \omega_l \bullet \omega_k, \quad \omega_k \bullet a_l = a_l \bullet \omega_k, \quad a_k \bullet a_l = a_l \bullet a_k, \quad \forall \, k, l \geqslant 2. \tag{A.6}$$

From assumption 6) of (3.31), upon combining internal and form degrees, one can derive

$$- \iota_A(\omega_k \bullet \omega_l) = (\iota_A \omega_k) \bullet \omega_l + \omega_k \bullet (\iota_A \omega_l), \tag{A.7}$$

that holds also for (a_k, a_l) and the mixed case (ω_k, a_l). Finally, by definition, they obey the recursive relation

$$\omega_{n+1} = -\frac{1}{n} A \bullet \omega_n, \quad a_{n+1} = -\frac{1}{n+1} A \bullet a_n, \quad \forall \, n \geqslant 2, \tag{A.8}$$

on which the proof is based. We first show that

$$d\omega_n + \frac{1}{2} \sum_{k=2}^{n-1} \omega_k \bullet \omega_{n+1-k} = 0, \quad \forall \, n \geqslant 2. \tag{A.9}$$

One has

$$\begin{aligned} n &= 2 \quad d\omega_2 = d^2 A = 0 \quad \text{(degenerate case)} \\ n &= 3 \quad d\omega_3 = -\frac{1}{2} d(A \bullet dA) = -\frac{1}{2} dA \bullet dA = -\frac{1}{2} \omega_2 \bullet \omega_2. \end{aligned} \tag{A.10}$$

Supposing that (A.9) holds for n we can derive

$$\begin{aligned} d\omega_{n+1} &= -\frac{1}{n} d(A \bullet \omega_n) = -\frac{1}{n} \omega_2 \bullet \omega_n \\ &\quad -\frac{1}{2n} A \bullet \left[\sum_{k=2}^{n-1} \omega_k \bullet \omega_{n+1-k} \right] \\ &= -\frac{1}{n} \omega_2 \bullet \omega_n - \frac{1}{2n} \sum_{k=2}^{n-1} \left[k \, \omega_{k+1} \bullet \omega_{n+1-k} + (n+1-k) \, \omega_k \bullet \omega_{n+2-k} \right] \\ &= -\frac{1}{2} \sum_{i=2}^{n} \omega_i \bullet \omega_{n+2-i}, \end{aligned} \tag{A.11}$$

thus proving (A.9). By using it in (A.5) we can write

$$
D\Omega_n + \tfrac{1}{2} \sum_{k=2}^{n-1} \Omega_k \bullet \Omega_{n+1-k} = da_n - \mathcal{L}_A \omega_n
$$

$$
+ \sum_{k=2}^{n-1} \omega_k \bullet a_{n+1-k} - \mathcal{L}_A a_n + \tfrac{1}{2} \sum_{k=2}^{n-1} a_k \bullet a_{n+1-k}, \tag{A.12}
$$

where we repeatedly used (A.4). We now claim that

$$
da_n - \mathcal{L}_A \omega_n + \sum_{k=2}^{n-1} \omega_k \bullet a_{n+1-k} = \mathfrak{D}\omega_{n+1}. \tag{A.13}
$$

For the lowest values of n one computes

$$
\begin{aligned}
n = 2 \quad da_2 - \mathcal{L}_A \omega_2 &= -\tfrac{1}{2} d(A \circ A) - A \circ dA = -\tfrac{1}{2} (A \circ dA + dA \circ A) \\
&= -\tfrac{1}{2} \mathfrak{D}(A \bullet dA) = \mathfrak{D}\omega_3 \quad \text{degenerate case,} \\
n = 3 \quad da_3 - \mathcal{L}_A \omega_3 + \omega_2 \bullet a_2 &= \tfrac{1}{6} d(A \bullet (A \circ A)) + \tfrac{1}{2} \mathcal{L}_A (A \bullet dA) - \tfrac{1}{2} (A \circ A) \bullet dA \\
&= -\tfrac{1}{3} (A \circ A) \bullet dA - \tfrac{1}{6} A \bullet (dA \circ A - A \circ dA) + \tfrac{1}{2} \mathcal{L}_A (A \bullet dA) \\
&= -\tfrac{1}{3} (\mathcal{L}_A A) \bullet dA + \tfrac{1}{3} A \bullet (\mathcal{L}_A dA) - \tfrac{1}{6} A \bullet \mathfrak{D}(A \bullet dA) + \tfrac{1}{2} \mathcal{L}_A (A \bullet dA) \\
&= \tfrac{1}{6} \mathfrak{D}(A \bullet (A \bullet dA)) = \mathfrak{D}\omega_4.
\end{aligned} \tag{A.14}
$$

Supposing that (A.13) is valid for n, we deduce

$$
da_{n+1} - \mathcal{L}_A \omega_{n+1} + \sum_{k=2}^{n} \omega_k \bullet a_{n+2-k} = -\tfrac{1}{n+1} d(A \bullet a_n) - \mathcal{L}_A \omega_{n+1} + \sum_{k=2}^{n} \omega_k \bullet a_{n+2-k}
$$

$$
= -\tfrac{1}{n+1} \omega_2 \bullet a_n + \tfrac{1}{n+1} A \bullet \left[\mathcal{L}_A \omega_n + \mathfrak{D}\omega_{n+1} - \sum_{k=2}^{n-1} \omega_k \bullet a_{n+1-k} \right] - \mathcal{L}_A \omega_{n+1}
$$

$$
+ \sum_{k=2}^{n} \omega_k \bullet a_{n+2-k} \tag{A.15}
$$

$$
= -\tfrac{1}{n+1} \omega_2 \bullet a_n - \tfrac{1}{n+1} [\mathcal{L}_A (A \bullet \omega_n) + 2 a_2 \bullet \omega_n] + \tfrac{1}{n+1} [\mathcal{L}_A \omega_{n+1} - \mathfrak{D}(A \bullet \omega_{n+1})]
$$

$$
- \mathcal{L}_A \omega_{n+1}
$$

$$
- \tfrac{1}{n+1} \sum_{k=2}^{n-1} [k\, \omega_{k+1} \bullet a_{n+1-k} + (n+2-k)\, \omega_k \bullet a_{n+2-k}] + \sum_{k=2}^{n} \omega_k \bullet a_{n+2-k}
$$

$$
= -\tfrac{1}{n+1} \mathfrak{D}(A \bullet \omega_{n+1}) = \mathfrak{D}\omega_{n+2},
$$

thus proving the relation by induction. The last step is to prove

$$
\mathcal{L}_A a_n - \tfrac{1}{2} \sum_{k=2}^{n-1} a_k \bullet a_{n+1-k} = -\mathfrak{D}a_{n+1}. \tag{A.16}
$$

The lowest values of n give[11] ($n = 2$ is degenerate as usual)

$$
\begin{aligned}
n = 2 \quad & \mathcal{L}_A a_2 = -\tfrac{1}{2} \mathcal{L}_A (A \circ A) = -\tfrac{1}{2} A \circ (A \circ A) = -\tfrac{1}{6} [A \circ (A \circ A) + (A \circ A) \circ A] \\
& = -\tfrac{1}{6} \mathfrak{D}(A \bullet (A \circ A)) = -\mathfrak{D} a_3 \\
n = 3 \quad & \mathcal{L}_A a_3 - \tfrac{1}{2} a_2 \bullet a_2 = -\tfrac{1}{3} \mathcal{L}_A (A \bullet a_2) - \tfrac{1}{2} a_2 \bullet a_2 = \tfrac{1}{6} a_2 \bullet a_2 - \tfrac{1}{3} A \bullet \mathfrak{D} a_3 \\
& = -\tfrac{1}{3} [\mathcal{L}_A a_3 - \tfrac{1}{2} a_2 \bullet a_2] + \tfrac{1}{3} \mathfrak{D}(A \bullet a_3) = \tfrac{1}{4} \mathfrak{D}(A \bullet a_3) = -\mathfrak{D} a_4 .
\end{aligned}
\tag{A.17}
$$

Supposing that (A.16) holds for n one has

$$
\begin{aligned}
\mathcal{L}_A a_{n+1} - \tfrac{1}{2} \sum_{k=2}^{n} a_k \bullet a_{n+2-k} &= -\tfrac{1}{n+1} \mathcal{L}_A (A \bullet a_n) - \tfrac{1}{2} \sum_{k=2}^{n} a_k \bullet a_{n+2-k} \\
&= \tfrac{2}{n+1} a_2 \bullet a_n + \tfrac{1}{n+1} A \bullet \left[\tfrac{1}{2} \sum_{k=2}^{n-1} a_k \bullet a_{n+1-k} - \mathfrak{D} a_{n+1} \right] - \tfrac{1}{2} \sum_{k=2}^{n} a_k \bullet a_{n+2-k} \\
&= \tfrac{2}{n+1} a_2 \bullet a_n + \tfrac{1}{2(n+1)} \sum_{k=2}^{n-1} [(k+1)\, a_{k+1} \bullet a_{n+1-k} \\
&\quad + (n+2-k)\, a_k \bullet a_{n+2-k}] \\
&\quad - \tfrac{1}{n+1} [\mathcal{L}_A a_{n+1} - \mathfrak{D}(A \bullet a_{n+1})] - \tfrac{1}{2} \sum_{k=2}^{n} a_k \bullet a_{n+2-k} \\
&= -\tfrac{1}{n+1} \left[\mathcal{L}_A a_{n+1} - \tfrac{1}{2} \sum_{k=2}^{n} a_k \bullet a_{n+2-k} \right] + \tfrac{1}{n+1} \mathfrak{D}(A \bullet a_{n+1}) = \tfrac{1}{n+2} \mathfrak{D}(A \bullet a_{n+1}) \\
&= -\mathfrak{D} a_{n+2} ,
\end{aligned}
\tag{A.18}
$$

proving (A.16). Using now the two results (A.13), (A.16) in (A.12) establishes the relation (4.7).

B. L_∞ Algebra from Infinity Enhanced Leibniz Algebra

Having discussed topological higher gauge theories based on a general L_∞ algebra, we will show here how to construct a family of L_∞ algebras from the data $(\circ, \mathfrak{D}, \bullet)$ of an infinity enhanced Leibniz algebra. Rather than discussing the most general L_∞ algebra that can be constructed this way, we will make the choices that yield the simplest form for the l_n brackets, and present them explicitly, acting on elements of arbitrary degree in the graded vector space X, up to the four-bracket l_4.

From now on we are going to distinguish elements x, y, z, ... in the X_0 subspace (that is the only one endowed with the Leibniz product $\circ$) from elements of higher degrees in $\bar{X} = \oplus_{n=1}^{\infty} X_n$, that will be denoted as u_n, with degree $|u_n| = n > 0$. The degree -1 nilpotent operator l_1 of the L_∞ algebra will be identified throughout this section with the $\mathfrak{D}$ operator of the Leibniz algebra, $l_1 := \mathfrak{D}$, that does not act on the subspace X_0.

l_2 brackets and $N = 2$ relations

---

[11] Recall that, by using the Leibniz property, one has $(A \circ A) \circ A = 2 A \circ (A \circ A)$.

We start from the l_2 bracket acting on two degree zero elements x and y, that is completely fixed, up to an overall normalization, by degree and antisymmetry:

$$l_2(x, y) := [x, y] \equiv \tfrac{1}{2}(x \circ y - y \circ x). \tag{B.1}$$

Since $\mathfrak{D}$ does not act on X_0, there is no nontrivial $N = 2$ relation at this level. The most general ansatz for the remaining l_2 brackets is given by

$$l_2(x, u_n) := k_n \mathcal{L}_x u_n + j_n \mathfrak{D}(x \bullet u_n), \quad l_2(u_n, x) := -l_2(x, u_n), \quad n > 0,$$
$$l_2(u_n, u_m) := A_{nm}(u_n \bullet \mathfrak{D}u_m - (-1)^{nm} u_m \bullet \mathfrak{D}u_n), \quad A_{nm} = A_{mn}, \quad n, m > 0, \tag{B.2}$$

and is determined by degree and graded antisymmetry. The above brackets have to obey the $N = 2$ relation $l_1 l_2 = l_2 l_1$, that takes the explicit form

$$\mathfrak{D}l_2(x, u_n) = l_2(x, \mathfrak{D}u_n), \quad \mathfrak{D}l_2(u_n, u_m) = l_2(\mathfrak{D}u_n, u_m) + (-1)^n l_2(u_n, \mathfrak{D}u_m). \tag{B.3}$$

By using covariance of the Lie derivative and nilpotency of $\mathfrak{D}$ one finds

$$\mathfrak{D}l_2(x, u_n) = k_n \mathcal{L}_x \mathfrak{D}u_n,$$
$$l_2(x, \mathfrak{D}u_n) = \begin{cases} \tfrac{1}{2} \mathcal{L}_x \mathfrak{D}u_n, & n = 1 \\ k_{n-1} \mathcal{L}_x \mathfrak{D}u_n + j_{n-1} \mathfrak{D}(x \bullet \mathfrak{D}u_n) = (k_{n-1} + j_{n-1})\mathcal{L}_x \mathfrak{D}u_n, & n > 1, \end{cases} \tag{B.4}$$

from which one concludes that

$$k_1 = \tfrac{1}{2}, \quad k_n = k_{n-1} + j_{n-1} = \ldots = \tfrac{1}{2} + \sum_{k=1}^{n-1} j_k, \quad n > 1, \tag{B.5}$$

with all the j_k parameters left free. We see that at each stage one has to single out the special cases of one (or multiple) u_n elements having degree $+1$, since the corresponding $\mathfrak{D}u_n$ terms have degree zero, and thus have brackets of a different form. The same happens in order to verify the second relation in (B.3):

$$\mathfrak{D}l_2(u_n, u_m) = -[1 + (-1)^{n+m}] A_{nm} \mathfrak{D}u_n \bullet \mathfrak{D}u_m,$$
$$l_2(\mathfrak{D}u_n, u_m) - (-1)^{nm} l_2(\mathfrak{D}u_m, u_n)$$
$$= \begin{cases} -2j_1 \mathfrak{D}u_n \bullet \mathfrak{D}u_m, & n = m = 1 \\ -[j_m + (-1)^m A_{1\,m-1}] \mathfrak{D}u_n \bullet \mathfrak{D}u_m, & n = 1, m > 1 \\ [A_{n-1\,m} + (-1)^{n+m} A_{n\,m-1}] \mathfrak{D}u_n \bullet \mathfrak{D}u_m, & n, m > 1 \end{cases} \tag{B.6}$$

where we used $\mathcal{L}_{\mathfrak{D}u} = 0$ and the twisted Leibniz property 5) of (3.31). The above result enforces $A_{11} = j_1$ and

$$[1 - (-1)^m] A_{1m} = j_m + (-1)^m A_{1\,m-1}, \quad m > 1,$$
$$[1 + (-1)^{n+m}] A_{nm} = -[A_{n-1\,m} + (-1)^{n+m} A_{n\,m-1}], \quad n, m > 1. \tag{B.7}$$

The first equation is solved by

$$A_{1n} = (-1)^n j_{n+1} + [1 + (-1)^n] j_{n+2}, \quad n \geqslant 1, \quad j_2 = -j_1, \tag{B.8}$$

where $j_2 = -j_1$ is required from matching $A_{11} = j_1$ with the first equation for $m = 2$, and the general solution for A_{1n} is found upon splitting the first equation above for the cases of even and odd values of m. Similarly, one can split the second equation into

$$
\begin{aligned}
A_{n\,m-1} &= A_{n-1\,m}, \quad n+m \text{ odd}, \\
A_{n\,m-1} + A_{n-1\,m} &= -A_{nm}, \quad n+m \text{ even}.
\end{aligned}
\tag{B.9}
$$

The first case yields

$$
n+m \text{ even} \;\to\; A_{nm} = A_{n-1\,m+1} = \ldots = A_{1\,n+m-1} = -j_{n+m}
\tag{B.10}
$$

that, used in the second equation, gives

$$
n+m \text{ odd} \;\to\; A_{nm} = -A_{n-1\,m+1} + 2\,j_{n+m+1}.
\tag{B.11}
$$

This imposes the simultaneous conditions

$$
n+m \text{ odd} \;\to\; A_{nm} = j_{n+m} + 2\,j_{n+m+1} = -j_{n+m},
\tag{B.12}
$$

determining the most general solution for the l_2 brackets:

$$
\begin{aligned}
j_{2k} &= -j_{2k-1}, \quad k \geqslant 1, \quad j_{2k-1} \text{ are free parameters} \\
k_n &= \tfrac{1}{2} + \tfrac{1+(-1)^n}{2}\,j_{n-1}, \quad n \geqslant 1, \\
A_{nm} &= -j_{n+m}, \quad n,m \geqslant 1.
\end{aligned}
\tag{B.13}
$$

Rather than using the general solution (B.13), in the following we will fix the free parameters $j_n = 0$ for all $n \geqslant 1$ yielding the simplest set of l_2 brackets:

$$
\begin{aligned}
l_2(x, y) &= [x, y] = \tfrac{1}{2}(\mathcal{L}_x y - \mathcal{L}_y x), \quad x, y \in X_0, \\
l_2(x, u_n) &= \tfrac{1}{2}\mathcal{L}_x u_n, \quad n > 0, \\
l_2(u_n, u_m) &= 0, \quad n, m > 0.
\end{aligned}
\tag{B.14}
$$

Before moving to the l_3 brackets, we use (B.14) to compute the Jacobiators, *i.e.* the failure of the graded Jacobi identity:

$$
\begin{aligned}
\mathrm{Jac}(x_1, x_2, x_3) &:= 3\,l_2(l_2(x_{[1}, x_2), x_{3]}) = \tfrac{1}{2}\,\mathfrak{D}[x_{[1} \bullet (x_2 \circ x_{3]})], \\
\mathrm{Jac}(x_1, x_2, u_n) &:= l_2(l_2(x_1, x_2), u_n) + 2\,l_2(l_2(u_n, x_{[1}), x_{2]}) = \tfrac{1}{4}\,\mathcal{L}_{[x_1, x_2]}u_n, \quad n > 0, \\
\mathrm{Jac}(x, u_n, u_m) &:= l_2(l_2(u_n, u_m), x) + 2\,l_2(l_2(x, u_{[n}), u_{m]}) = 0, \quad n, m > 0, \\
\mathrm{Jac}(u_n, u_m, u_l) &:= 3\,l_2(l_2(u_{[n}, u_m), u_{l]}) = 0.
\end{aligned}
\tag{B.15}
$$

To derive the first relation we have used the identity

$$
x_{[1} \circ (x_2 \circ x_{3]}) = \tfrac{1}{2}\,(x_{[1} \circ x_2) \circ x_{3]} = 2\,[[x_{[1}, x_2], x_{3]}] = \tfrac{1}{3}\,\mathfrak{D}(x_{[1} \bullet (x_2 \circ x_{3]})),
\tag{B.16}
$$

that originates from the Leibniz property, and we have used the shorthand notation $[nm)$ and $[nml)$ for total graded antisymmetrization with strength one, *i.e.* $T_{[nm)} := \tfrac{1}{2}\,(T_{nm} - (-1)^{nm}\,T_{mn})$ and so on.

l_3 brackets and $N = 3$ relations

The first Jacobiator in (B.15) uniquely fixes the three-bracket on three degree zero elements x_1, x_2 and x_3 from the $N = 3$ relation

$$\mathfrak{D}l_3(x_1, x_2, x_3) + \text{Jac}(x_1, x_2, x_3) = 0, \tag{B.17}$$

to be

$$l_3(x_1, x_2, x_3) = -\tfrac{1}{2} x_{[1} \bullet (x_2 \circ x_{3]}). \tag{B.18}$$

For elements of higher degree, graded antisymmetry and the properties 5) and 6) of (3.31), together with $|l_3| = +1$ fix the most general ansatz to be

$$l_3(x_1, x_2, u_n) := \alpha_n [x_1, x_2] \bullet u_n + \beta_n x_{[1} \bullet \mathcal{L}_{x_2]} u_n + \gamma_n \mathfrak{D}(x_{[1} \bullet (x_2] \bullet u_n)), \quad n > 0,$$

$$l_3(x, u_n, u_m) := a_{nm} u_{[n} \bullet \mathcal{L}_x u_{m)} + b_{nm} u_{[n} \bullet (x \bullet \mathfrak{D} u_{m)})$$

$$+ c_{nm} \mathfrak{D}(u_{[n} \bullet (u_{m)} \bullet x)), \quad n, m > 0, \tag{B.19}$$

$$l_3(u_k, u_n, u_m) := A_{knm} u_{[k} \bullet (u_n \bullet \mathfrak{D} u_{m)}), \quad k, n, m > 0.$$

The relevant $N = 3$ relations to be satisfied read

$$\mathfrak{D}l_3(x_1, x_2, u_n) + l_3(x_1, x_2, \mathfrak{D} u_n) + \tfrac{1}{4} \mathcal{L}_{[x_1, x_2]} u_n = 0, \quad n > 0,$$

$$\mathfrak{D}l_3(x, u_n, u_m) + 2 \, l_3(x, \mathfrak{D} u_{[n}, u_{m)}) = 0, \quad n, m > 0, \tag{B.20}$$

$$\mathfrak{D}l_3(u_k, u_n, u_m) + 3 \, l_3(\mathfrak{D} u_{[k}, u_n, u_{m)}) = 0, \quad k, n, m > 0.$$

In order to compute the above expressions from the ansatz (B.19), one has to treat separately the cases of one or more u's having degree $+1$, just as it has been shown explicitly for the l_2 bracket. A straightforward but tedious computation shows that the only free parameter left, after imposing (B.20), is γ_1. Instead of showing the entire proof, we rather choose $\gamma_1 = 0$, that gives the simplest realization of the brackets, and show that this choice is indeed consistent with (B.20). With γ_1 fixed to zero, the l_3 brackets read

$$l_3(x_1, x_2, x_3) = -\tfrac{1}{2} x_{[1} \bullet (x_2 \circ x_{3]}),$$

$$l_3(x_1, x_2, u_n) = -\tfrac{1}{6} [x_1, x_2] \bullet u_n - \tfrac{1}{6} x_{[1} \bullet \mathcal{L}_{x_2]} u_n, \quad n > 0,$$

$$l_3(x, u_n, u_m) = \tfrac{1}{6} u_{[n} \bullet \mathcal{L}_x u_{m)}, \quad n, m > 0, \tag{B.21}$$

$$l_3(u_k, u_n, u_m) = 0, \quad k, n, m > 0.$$

For any $n > 0$ one has

$$\mathfrak{D}l_3(x_1, x_2, u_n) = -\tfrac{1}{6} \mathfrak{D}\{[x_1, x_2] \bullet u_n + x_{[1} \bullet \mathcal{L}_{x_2]} u_n\}, \tag{B.22}$$

and

$$l_3(x_1, x_2, \mathfrak{D} u_n) = -\tfrac{1}{6} [x_1, x_2] \bullet \mathfrak{D} u_n - \tfrac{1}{6} x_{[1} \bullet \mathcal{L}_{x_2]} \mathfrak{D} u_n, \tag{B.23}$$

where, for $n > 1$, this is computed from $l_3(x_1, x_2, u_{n-1})$ with $u_{n-1} = \mathfrak{D} u_n$, while for $n = 1$ we used $l_3(x_1, x_2, x_3)$ with $x_3 = \mathfrak{D} u_1$. Summing the two contributions one obtains

$$\mathfrak{D}l_3(x_1, x_2, u_n) + l_3(x_1, x_2, \mathfrak{D} u_n)$$

$$= -\tfrac{1}{6} \mathcal{L}_{[x_1, x_2]} u_n - \tfrac{1}{6} \left[x_{[1} \bullet \mathfrak{D}(\mathcal{L}_{x_2]} u_n) + \mathfrak{D}(x_{[1} \bullet \mathcal{L}_{x_2]} u_n) \right] \tag{B.24}$$

$$= -\tfrac{1}{6} \mathcal{L}_{[x_1, x_2]} u_n - \tfrac{1}{6} \mathcal{L}_{x_{[1}} \mathcal{L}_{x_2]} u_n = -\tfrac{1}{4} \mathcal{L}_{[x_1, x_2]} u_n,$$

thus proving the first of the relations (B.20). Similarly, for any $n, m > 0$ one has

$$\mathfrak{D}l_3(x, u_n, u_m) = \tfrac{1}{6}\mathfrak{D}\big[u_{[n} \bullet \mathcal{L}_x u_{m)}\big], \tag{B.25}$$

while, for $n, m > 1$ one has

$$2\, l_3(x, \mathfrak{D}u_{[n}, u_{m)}) \overset{[nm]}{=} \tfrac{1}{6}\big[\mathfrak{D}u_n \bullet \mathcal{L}_x u_m - (-1)^{(n-1)m} u_m \bullet \mathcal{L}_x \mathfrak{D}u_n\big]$$
$$\overset{[nm]}{=} \tfrac{1}{6}\big[\mathfrak{D}u_n \bullet \mathcal{L}_x u_m + (-1)^n u_n \bullet \mathcal{L}_x \mathfrak{D}u_m\big] \tag{B.26}$$
$$\overset{[nm]}{=} -\tfrac{1}{6}\mathfrak{D}\big[u_n \bullet \mathcal{L}_x u_m\big]$$

thus proving the relation for $n, m > 1$. When $n = 1$ and $m > 1$ one obtains the same result:

$$l_3(x, \mathfrak{D}u_1, u_m) - l_3(x, u_1, \mathfrak{D}u_m) = -\tfrac{1}{12}\mathcal{L}_x\mathfrak{D}u_1 \bullet u_m + \tfrac{1}{12}\mathfrak{D}u_1 \bullet \mathcal{L}_x u_m$$
$$- \tfrac{1}{12}\big[u_1 \bullet \mathcal{L}_x\mathfrak{D}u_m + (-1)^m \mathfrak{D}u_m \bullet \mathcal{L}_x u_1\big]$$
$$= tfrac112\big[\mathfrak{D}u_1 \bullet \mathcal{L}_x u_m - u_1 \bullet \mathcal{L}_x\mathfrak{D}u_m\big] \tag{B.27}$$
$$- (-1)^m \tfrac{1}{12}\big[\mathfrak{D}u_m \bullet \mathcal{L}_x u_1 + (-1)^m u_m \bullet \mathcal{L}_x\mathfrak{D}u_1\big]$$
$$= -\tfrac{1}{12}\mathfrak{D}\big[u_1 \bullet \mathcal{L}_x u_m - (-1)^m u_m \bullet \mathcal{L}_x u_1\big],$$

with the first term computed from $l_3(x_1, x_2, u_m)$ for $x_2 = \mathfrak{D}u_1$. Similarly, the same is also obtained for $n = m = 1$:

$$2\, l_3(x, \mathfrak{D}u_{(1}, u_{2)}) = -\tfrac{1}{6}\mathcal{L}_x\mathfrak{D}u_{(1} \bullet u_{2)} + \tfrac{1}{6}\mathfrak{D}u_{(1} \bullet \mathcal{L}_x u_{2)}$$
$$= \tfrac{1}{6}\big[\mathfrak{D}u_{(1} \bullet \mathcal{L}_x u_{2)} - u_{(1} \bullet \mathcal{L}_x\mathfrak{D}u_{2)}\big] \tag{B.28}$$
$$= -\tfrac{1}{6}\mathfrak{D}\big[u_{(1} \bullet \mathcal{L}_x u_{2)}\big],$$

where both $|u_1| = |u_2| = 1$ and we used the subscripts to distinguish the elements, rather than denoting the degree. Finally, the last relation in (B.20) does not need any computation to be proved, since either every term is identically zero or, if any element of degree one is present, the corresponding degree zero object $\mathfrak{D}u$ only acts through a Lie derivative, and thus vanishes. With this we have thus proved that (B.21) provides a consistent set of three-brackets on the entire space.

l_4 brackets and $N = 4$ relations

As the last explicit realization of the l_n brackets, we will now show that, given the two- and three-brackets as in (B.14) and (B.21), *all* the four brackets l_4 vanish. The abstract $N = 4$ relation reads $l_1 l_4 - l_4 l_1 = l_2 l_3 - l_3 l_2$ so, rather than giving a general ansatz for the l_4 maps, we will prove that $l_2 l_3 = l_3 l_2$ on the entire space. From the initial relation

$$\mathfrak{D}l_4(x_1, x_2, x_3, x_4) = 4\, l_2(l_3(x_{[1}, x_2, x_3), x_{4]}) - 6\, l_3(l_2(x_{[1}, x_2), x_3, x_{4]}) = 0 \tag{B.29}$$

one can then prove recursively that all l_4 maps vanish. In order to prove $l_2 l_3 = l_3 l_2$, we start indeed with all four elements in X_0, giving

$$4\, l_2(l_3(x_{[1}, x_2, x_3), x_{4]}) - 6\, l_3(l_2(x_{[1}, x_2), x_3, x_{4]})$$
$$\overset{[1234]}{=} \mathcal{L}_{x_4}\big[x_1 \bullet \mathcal{L}_{x_2} x_3\big] + [x_1, x_2] \bullet [x_3, x_4] + 2\, x_3 \bullet [x_4, [x_1, x_2]] \tag{B.30}$$
$$\overset{[1234]}{=} -\mathcal{L}_{x_1}\big[x_2 \bullet \mathcal{L}_{x_3} x_4\big] + (\mathcal{L}_{x_1} x_2) \bullet (\mathcal{L}_{x_3} x_4) + x_2 \bullet [x_1 \circ (x_3 \circ x_4)] = 0,$$

that proves $l_4(x_1, x_2, x_3, x_4) = 0$. The next quadratic relation reads (antisymmetrization [123] is understood)

$$
\begin{aligned}
l_2&(l_3(x_1, x_2, x_3), u_n) - 3\, l_2(l_3(x_1, x_2, u_n), x_3) - 3\, l_3(l_2(x_1, x_2), x_3, u_n) \\
&- 3\, l_3(l_2(x_1, u_n), x_2, x_3) \\
&= tfrac32\, \mathcal{L}_{x_3} l_3(x_1, x_2, u_n) + \tfrac{1}{4}\big\{[(x_1 \circ x_2) \circ x_3 - x_3 \circ (x_1 \circ x_2)] \bullet u_n \\
&\quad + (x_1 \circ x_2) \bullet \mathcal{L}_{x_3} u_n \\
&\quad - x_3 \bullet \mathcal{L}_{[x_1, x_2]} u_n + (x_2 \circ x_3) \bullet \mathcal{L}_{x_1} u_n + x_2 \bullet \mathcal{L}_{x_3}\mathcal{L}_{x_1} u_n\big\} \\
&= tfrac32\, \mathcal{L}_{x_3} l_3(x_1, x_2, u_n) + \tfrac{1}{4}\big\{[x_3 \circ (x_1 \circ x_2)] \bullet u_n + (x_1 \circ x_2) \bullet \mathcal{L}_{x_3} u_n \\
&\quad + (x_3 \circ x_1) \bullet \mathcal{L}_{x_2} u_n + x_1 \bullet \mathcal{L}_{x_3}\mathcal{L}_{x_2} u_n\big\} \\
&= \tfrac{3}{2}\, \mathcal{L}_{x_3} l_3(x_1, x_2, u_n) + \tfrac{1}{4}\, \mathcal{L}_{x_3}\big\{(x_1 \circ x_2) \bullet u_n + x_1 \bullet \mathcal{L}_{x_2} u_n\big\} = 0,
\end{aligned}
\tag{B.31}
$$

proving $l_4(x_1, x_2, x_3, u_n) = 0$. Next, when two elements have degree higher than zero one has

$$
\begin{aligned}
2\, l_2&(l_3(x_1, x_2, u_n), u_m) + 2\, l_2(l_3(x_1, u_n, u_m), x_2) \\
&- l_3(l_2(x_1, x_2), u_n, u_m) + 4\, l_3(l_2(x_1, u_n), x_2, u_m) - l_3(l_2(u_n, u_m), x_1, x_2) \\
&= \mathcal{L}_{x_1} l_3(x_2, u_n, u_m) - \tfrac{1}{6}\, u_n \bullet \mathcal{L}_{[x_1, x_2]} u_m + 2\, l_3(x_1, \mathcal{L}_{x_2} u_n, u_m) \\
&= \mathcal{L}_{x_1} l_3(x_2, u_n, u_m) - \tfrac{1}{6}\, u_n \bullet \mathcal{L}_{[x_1, x_2]} u_m + \tfrac{1}{6}\, \mathcal{L}_{x_2} u_n \bullet \mathcal{L}_{x_1} u_m \\
&\quad - (-1)^{mn} \tfrac{1}{6}\, u_m \bullet \mathcal{L}_{x_1}\mathcal{L}_{x_2} u_n \\
&= \mathcal{L}_{x_1} l_3(x_2, u_n, u_m) - \tfrac{1}{6}\, \mathcal{L}_{x_1}\big[u_n \bullet \mathcal{L}_{x_2} u_m\big] = 0
\end{aligned}
\tag{B.32}
$$

with [12] and [nm] left implicit, yielding $l_4(x_1, x_2, u_n, u_m) = 0$. The last two cases, namely $l_4(x, u_k, u_n, u_m)$ and $l_4(u_k, u_l, u_m, u_n)$ do not need any computation, since any term in $l_2 l_3$ and $l_3 l_2$ vanishes identically. This finally proves that, with the choice (B.14) and (B.21) for the lower brackets, all the l_4's vanish. Moreover, it can be shown that the l_4 brackets vanish for any choice of the l_3's, provided that the l_2 maps are given by (B.14).

We summarize here the list of non-vanishing L_∞ brackets explicitly constructed from the Leibniz algebra thus far:[12]

$$
\begin{aligned}
l_2(x, y) &= [x, y], \quad x, y \in X_0, \\
l_2(x, u_n) &= \tfrac{1}{2}\, \mathcal{L}_x u_n, \quad n > 0, \\
l_3(x_1, x_2, x_3) &= -\tfrac{1}{2}\, x_{[1} \bullet (x_2 \circ x_{3]}), \\
l_3(x_1, x_2, u_n) &= -\tfrac{1}{6}\, [x_1, x_2] \bullet u_n - \tfrac{1}{6}\, x_{[1} \bullet \mathcal{L}_{x_2]} u_n, \quad n > 0, \\
l_3(x, u_n, u_m) &= \tfrac{1}{6}\, u_{[n} \bullet \mathcal{L}_x u_{m)}, \quad n, m > 0.
\end{aligned}
\tag{B.33}
$$

Having shown that all l_4 brackets vanish does not mean that higher brackets vanish, and indeed one can prove that there is no allowed choice of coefficients for which all l_5 maps are zero. In particular, the richer structure on the Leibniz side hints at the existence of special points in the "moduli space" of L_∞ maps, for which infinitely many brackets vanish, even though the L_∞ algebra is not truncated to a finite degree.

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[12] *Note added in proof*: In the meantime, it was shown in [41] how to construct the associated L_∞ brackets in general by using the derived bracket construction.

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