

THE COMPLEXITY OF APPROXIMATING THE COMPLEX-VALUED POTTS MODEL

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Abstract. We study the complexity of approximating the partition function of the q -state Potts model and the closely related Tutte polynomial for complex values of the underlying parameters. Apart from the classical connections with quantum computing and phase transitions in statistical physics, recent work in approximate counting has shown that the behaviour in the complex plane, and more precisely the location of zeros, is strongly connected with the complexity of the approximation problem, even for positive real-valued parameters. Previous work in the complex plane by Goldberg and Guo focused on $q = 2$, which corresponds to the case of the Ising model; for $q > 2$, the behaviour in the complex plane is not as well understood and most work applies only to the real-valued Tutte plane. Our main result is a complete classification of the complexity of the approximation problems for all non-real values of the parameters, by establishing #P-hardness results that apply even when restricted to planar graphs. Our techniques apply to all $q \geq 2$ and further complement/refine previous results both for the Ising model and the Tutte plane, answering in particular a question raised by Bordewich, Freedman, Lovász and Welsh in the context of quantum computations.

Keywords. Counting Complexity, Potts model, Ising model, Tutte polynomial, Approximate Counting

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1. Introduction

The q -state Potts model is a classical model of ferromagnetism in statistical physics (Potts 1952; Welsh 1993) which generalises the well-known Ising model. On a (multi)graph $G = (V, E)$, configurations of the model are all possible assignments $\sigma : V \rightarrow [q]$ where $[q] = \{1, \dots, q\}$ is a set of q spins with $q \geq 2$. The model is parameterised by y , which is a function of the temperature of the model and is also known as the *edge interaction*. Each configuration σ is assigned weight $y^{m(\sigma)}$ where $m(\sigma)$ denotes the number of monochromatic edges of G under σ . The partition function of the model is the aggregate weight over all configurations, i.e.,

$$Z_{\text{Potts}}(G; q, y) = \sum_{\sigma: V \rightarrow [q]} y^{m(\sigma)},$$

When $q = 2$, this model is known as the Ising model, and we sometimes use the notation $Z_{\text{Ising}}(G; y)$ to denote its partition function.

The Ising/Potts models have an extremely useful generalisation to non-integer values of q via the so-called random-cluster formulation and the closely related Tutte polynomial. In particular, for numbers q and γ , the Tutte polynomial of a graph G is given by

$$(1.1) \quad Z_{\text{Tutte}}(G; q, \gamma) = \sum_{A \subseteq E} q^{k(A)} \gamma^{|A|},$$

where $k(A)$ denotes the number of connected components in the graph (V, A) (isolated vertices do count). When q is an integer with $q \geq 2$, we have $Z_{\text{Potts}}(G; q, y) = Z_{\text{Tutte}}(G; q, y - 1)$, see, for instance, the paper of Sokal (2005). The Tutte polynomial on planar graphs is particularly relevant in quantum computing since it corresponds to the Jones polynomial of an ‘‘alternating link’’ (Welsh 1993, Chapter 5), and polynomial-time quantum computation can be simulated by additively approximating the Jones polynomial at a suitable value, as we will explain later in more detail, see also (Bordewich *et al.* 2005) for details.

In this paper, we study the complexity of approximating the partition function of the Potts model and the Tutte polynomial on planar graphs as the parameter y ranges in the complex plane.

Traditionally, this problem has been mainly considered in the case where y is a positive real; however recent developments have shown that for various models, including the Ising and Potts models, there is a close interplay between the location of zeros of the partition function in the complex plane and the approximability of the problem, even for positive real values of y .

The framework of viewing partition functions as polynomials in the complex plane of the underlying parameters has been well-explored in statistical physics and has recently gained traction in computer science as well in the context of approximate counting. On the positive side, zero-free regions in the complex plane translate into efficient algorithms for approximating the partition function (Barvinok 2017; Patel & Regts 2017) and this scheme has led to a broad range of new algorithms even for positive real values of the underlying parameters (Guo *et al.* 2021, 2020; Harrow *et al.* 2020; Liu *et al.* 2019; Liu *et al.* 2019a,b; Peters & Regts 2018, 2019). On the negative side, the presence of zeros poses a barrier to this approach and, in fact, it has been demonstrated that zeros mark the onset of computational hardness for the approximability of the partition function (Bezáková *et al.* 2020; Bezáková *et al.* 2021; Goldberg & Guo 2017; Goldberg & Jerrum 2014). These new algorithmic and computational complexity developments stemming from the complex plane mesh with the statistical physics perspective where zeros have long been studied in the context of pinpointing phase transitions, see e.g., (Bena *et al.* 2005; Heilmann & Lieb 1972; Lieb & Sokal 1981; Sokal 2005; Welsh 1993; Yang & Lee 1952).

For the problem of exactly computing the partition function of the Potts model, Jaeger *et al.* (1990), as a corollary of a more general classification theorem for the Tutte polynomial, established $\#P$ -hardness unless (q, y) is one of seven exceptional points, see Section 6.3 for more details; Vertigan (2005) further showed that the same classification applies on planar graphs with the exception of the Ising model ($q = 2$), where the problem is in FP.

For the approximation problem, the only known result that applies for general values y in the complex plane is by Goldberg & Guo (2017), which addresses the case $q = 2$; the case $q \geq 3$ is

largely open apart from the case when y is real which has been studied extensively even for planar graphs (Goldberg & Guo 2017; Goldberg & Jerrum 2008, 2012a, 2014; Jerrum & Sinclair 1993; Kuperberg 2015). We will review all these results more precisely in the next section, where we also state our main theorems.

1.1. Our main results. In this work, we completely classify the complexity of approximating $Z_{\text{Potts}}(G; q, y)$ for $q \geq 2$ and non-real y , even on planar graphs G ; in fact, our results also classify the complexity of the Tutte polynomial on planar graphs for reals $q \geq 2$ and non-real γ . Along the way, we also answer a question for the Jones polynomial raised by Bordewich, Freedman, Lovász & Welsh (2005).

To formally state our results, we define the computational problems we consider. Let K and ρ be real algebraic numbers with $K > 1$ and $\rho > 0$. We investigate the complexity of the following problems for any integer q with $q \geq 2$ and any algebraic number y .¹

Name: FACTOR- K -NORMPOTTS(q, y)

Instance: A (multi)graph G .

Output: If $Z_{\text{Potts}}(G; q, y) = 0$, the algorithm may output any rational number. Otherwise, it must output a rational number \hat{N} such that $\hat{N}/K \leq |Z_{\text{Potts}}(G; q, y)| \leq K\hat{N}$.

A well-known fact is that the difficulty of the computational problem FACTOR- K -NORMPOTTS(q, γ) does not depend on the constant $K > 1$. This can be proved using standard powering techniques, see (Goldberg & Guo 2017, Lemma 11) for a proof when $q = 2$. In fact, the complexity of the problem is the same even for $K = 2^{n^{1-\epsilon}}$ for any constant $\epsilon > 0$ where n is the size of the input.

¹For $z \in \mathbb{C} \setminus \{0\}$, we denote by $|z|$ the norm of z , by $\text{Arg}(z) \in [0, 2\pi)$ the principal argument of z and by $\arg(z)$ the set $\{\text{Arg}(z) + 2\pi j : j \in \mathbb{Z}\}$ of all the arguments of z , so that for any $a \in \arg(z)$ we have $z = |z| \exp(ia)$.

Name: DISTANCE- ρ -ARGPOTTS(q, y)

Instance: A (multi)graph G .

Output: If $Z_{\text{Potts}}(G; q, y) = 0$, the algorithm may output any rational number. Otherwise, it must output a rational \hat{A} such that, for some $a \in \arg(Z_{\text{Potts}}(G; q, \gamma))$, $|\hat{A} - a| \leq \rho$.

In the special case that q equals 2, we omit the argument q and write ISING instead of POTTS in the name of the problem. Similarly, when the input of the problems is restricted to planar graphs, we write PLANARPOTTS instead of POTTS. We also consider the problem FACTOR- K -NORMTUTTE(q, γ) and the problem DISTANCE- $\pi/3$ -ARGTUTTE(q, γ) for the Tutte polynomial when q, γ are algebraic numbers. Note also that, when q, γ are real, the latter problem is equivalent to finding the sign of the Tutte polynomial, and we sometimes write SIGNTUTTE(q, γ) (and SIGNPLANARTUTTE(q, γ) for the planar version of the problem).

Our first and main result is a full resolution of the complexity of approximating $Z_{\text{Potts}}(G; q, y)$ for $q \geq 3$ and non-real y . More precisely, we show the following.

THEOREM 1.2. *Let $q \geq 3$ be an integer, $y \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic number, and $K > 1$. Then, FACTOR- K -NORMPLANARPOTTS(q, y) and DISTANCE- $\pi/3$ -ARGPLANARPOTTS(q, y) are #P-hard, unless $q = 3$ and $y \in \{e^{2\pi i/3}, e^{4\pi i/3}\}$ when both problems can be solved exactly in polynomial time.*

We remark that, for real $y > 0$, the complexity of approximating $Z_{\text{Potts}}(G; q, y)$ on planar graphs is not fully known, though on general graphs the problem is #BIS-hard (Goldberg & Jerrum 2012a) and NP-hard for $y \in (0, 1)$ (Goldberg & Jerrum 2008), for all $q \geq 3$. For real $y < 0$, the problem is NP-hard on general graphs when $y \in (-\infty, 1 - q]$ for all $q \geq 3$ (Goldberg & Jerrum 2014)² and

²Note, for $y \in (-\infty, 1 - q) \cup [0, \infty)$, #P-hardness is impossible (assuming $\text{NP} \neq \text{\#P}$): finding the sign of $Z_{\text{Potts}}(G; q, y)$ is easy, even on non-planar graphs (Goldberg & Jerrum 2014), and $Z_{\text{Potts}}(G; q, y)$ can be approximated using an NP-oracle. For $y = 1 - q$, the same applies when $q \geq 6$; the cases $q \in \{3, 4, 5\}$ are not fully resolved though Goldberg & Jerrum (2014) showed that $q = 3, 4$ are NP-hard, whereas $q = 5$ should be easy unless Tutte's 5-flow conjecture is false (Welsh 1993, Section 3.5).

#P-hard on planar graphs when $y \in (1 - q, 0)$ and $q \geq 5$ (Kuperberg (2015), see also Goldberg & Jerrum (2012b)). Our techniques for proving Theorem 1.2 allow us to resolve the remaining cases $q = 3, 4$ for $y \in (1 - q, 0)$ on planar graphs, as a special case of the following theorem that applies for general $q \geq 3$. This is our second main result.

THEOREM 1.3. *Let $q \geq 3$ be an integer, $y \in (-q + 1, 0)$ be a real algebraic number, and $K > 1$. Then the computational problems FACTOR- K -NORMPLANARPOTTS(q, y) and DISTANCE- $\pi/3$ -ARGPLANARPOTTS(q, y) are #P-hard, unless we are in the case $(q, y) = (4, -1)$ when both problems can be solved exactly in polynomial time.*

Our third main contribution is a full classification of the range of the parameters where approximating the partition function of the Ising model is #P-hard. Note, on planar graphs G , $Z_{\text{Ising}}(G; y)$ can be computed in polynomial time for all y . For general (non-planar) graphs and non-real y , Goldberg and Guo show #P-hardness on the non-real unit circle ($|y| = 1$) with $y \neq \pm i$, and establish NP-hardness elsewhere. Our next result shows that the NP-hardness results of Goldberg & Guo (2017) for non-real y can be elevated to #P-hardness.

THEOREM 1.4. *Let $y \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic number, and $K > 1$. Then, FACTOR- K -NORMISING(y) and DISTANCE- $\pi/3$ -ARGISING(y) are #P-hard, unless $y = \pm i$ when both problems can be solved exactly in polynomial time.*

For real y , we remark that the problems of approximating $Z_{\text{Ising}}(G; y)$ and determining its sign (when non-trivial) are well-understood:³ the problem is FPRASable for $y > 1$ and NP-hard for $y \in (0, 1)$ (Jerrum & Sinclair 1993), #P-hard for $y \in (-1, 0)$ Goldberg & Guo (2017); Goldberg & Jerrum (2014), and equivalent to approximating #PERFECTMATCHINGS for $y < -1$ Goldberg & Jerrum (2008). For $y = 0, \pm 1$, $Z_{\text{Ising}}(G; y)$ can be computed exactly in polynomial time.

³Analogously to Footnote 2, for $y \in (-\infty, -1) \cup (0, 1)$ #P-hardness is unlikely since the problem can be approximated with an NP-oracle.

1.2. Consequences of our techniques for the Tutte/Jones polynomials. While our main results are on the Ising/Potts models, in order to prove them it is convenient to work in the “Tutte world”; this simplifies the proofs and has also the benefit of allowing us to generalise our results to non-integer q . The following result generalises Theorem 1.2 to non-integer $q > 2$.

THEOREM 1.5. *Let $q > 2$ be a real, $\gamma \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic number, and $K > 1$. Then, $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma)$ and $\text{DISTANCE-}\pi/3\text{-ARGPLANARTUTTE}(q, \gamma)$ are $\#P$ -hard, unless $q = 3$ and $\gamma + 1 \in \{e^{2\pi i/3}, e^{4\pi i/3}\}$ when both problems can be solved exactly in polynomial time.*

Our techniques can further be used to elevate previous NP-hardness results of Goldberg & Jerrum (2008, 2014) in the Tutte plane to $\#P$ -hardness for planar graphs, and answer a question for the Jones polynomial raised by Bordewich *et al.* (2005). A more detailed discussion can be found in Section 7.

2. Proof outline

In this section we provide some insight on the proofs of our main results. As mentioned earlier, the proofs are performed in the context of the Tutte polynomial.

In previous $\#P$ -hardness results (Goldberg & Guo 2017; Goldberg & Jerrum 2014) for the Tutte polynomial, the authors reduce the exact counting $\#$ MINIMUMCARDINALITY (s, t) -CUT problem to the problem of approximating $Z_{\text{Tutte}}(G; q, \gamma)$ using an elaborate binary search based on suitable oracle calls. Key to these oracle calls are gadget constructions which are mainly based on series-parallel graphs which “implement” points (q', γ') ; this means that, by pasting the gadgets appropriately onto a graph G , the computation of $Z_{\text{Tutte}}(G; q', \gamma')$ reduces to the computation of $Z_{\text{Tutte}}(G; q, \gamma)$.

Much of the work in (Goldberg & Guo 2017; Goldberg & Jerrum 2014), and for us as well, is understanding what values (q', γ') can be implemented starting from (q, γ) .

For planar graphs, while the binary-search technique from (Goldberg & Guo 2017) is still useful, we have to use a different overall reduction scheme since $\#\text{MINIMUMCARDINALITY}(s, t)\text{-CUT}$ is not $\#\text{P}$ -hard when the input is restricted to planar graphs (Provan & Ball 1983). To obtain our $\#\text{P}$ -hardness results our plan instead is to reduce the problem of exactly evaluating the Tutte polynomial for some appropriately selected parameters q', γ' to the problem of computing its sign and the problem of approximately evaluating it at parameters q, γ ; note, this gives us the freedom to use any parameters q', γ' we wish as long as the corresponding exact problem is $\#\text{P}$ -hard. Then, much of the work consists of understanding what values (q', γ') can be implemented starting from (q, γ) , so we focus on that component first.

We first review previous constructions in the literature, known as shifts, and then introduce our refinement of these constructions, which we call polynomial-time approximate shifts, and state our main result about them.

2.1. Shifts in the Tutte plane. We say that there is a *shift* from (q, γ_1) to (q, γ_2) if there is a graph $H = (V, E)$ and vertices s, t such that

$$\gamma_2 = q \frac{Z_{st}(H; q, \gamma_1)}{Z_{s|t}(H; q, \gamma_1)},$$

where $Z_{st}(H; q, \gamma_1)$ is the contribution to $Z_{\text{Tutte}}(H; q, \gamma_1)$ from configurations $A \subseteq E$ in which s, t belong to the same connected component in (V, A) , while $Z_{s|t}(H; q, \gamma_1)$ is the contribution from all other configurations A . In the following, we will usually encounter shifts in the (x, y) -parametrisation of the Tutte plane, rather than the (q, γ) -parameterisation which was used for convenience here. To translate between these, set $y = \gamma + 1$ and $(x - 1)(y - 1) = q$, see (Welsh 1993, Chapter 3). We denote by \mathcal{H}_q the hyperbola $\{(x, y) \in \mathbb{C}^2 : (x - 1)(y - 1) = q\}$, and we will use both parametrisations as convenient. Section 3.2 has a more detailed description of shifts that apply to the multivariate Tutte polynomial.

As described earlier, shifts can be used to “move around” the complex plane. If one knows hardness for some $(x_2, y_2) \in \mathcal{H}_q$, and there is a shift from $(x_1, y_1) \in \mathcal{H}_q$ to (x_2, y_2) , then one also obtains hardness for (x_1, y_1) . This approach has been very effective when attention is restricted to real parameters (Goldberg & Jerrum 2008, 2012b, 2014); however, when it comes to non-real parameters, the success of this approach has been limited. To illustrate this, in (Goldberg & Guo 2017), the authors established #P-hardness of the Ising model when $y_2 \in (-1, 0)$, and used this to obtain #P-hardness for y_1 on the unit circle by constructing appropriate shifts. However, their shift construction does not extend to general complex numbers, and this kind of result seems unreachable with those techniques.

2.2. Polynomial-time approximate shifts. To obtain our main theorems, we instead need to consider what we call polynomial-time approximate shifts; such a shift from $(x_1, y_1) \in \mathcal{H}_q$ to $(x_2, y_2) \in \mathcal{H}_q$ is an algorithm that, for any positive integer n , computes in time polynomial in n a graph G_n that (x_1, y_1) -implements (\hat{x}_2, \hat{y}_2) with $|y_2 - \hat{y}_2| \leq 2^{-n}$. In fact, our constructions need to maintain planarity, and we will typically ensure this by either making every G_n a series-parallel graph, in which case we call the algorithm a *polynomial-time approximate series-parallel shift*, or by making every G_n a theta graph, in which case we call the algorithm a *polynomial-time approximate theta shift*.⁴

These generalised shifts allow us to overcome the challenges mentioned above and are key ingredients in our reduction. Our main technical theorem about them is the following.

THEOREM 2.1. *Let $q \geq 2$ be a real algebraic number. Let x and y be algebraic numbers such that $(x, y) \in \mathcal{H}_q$, $y \in (-1, 0) \cup (\mathbb{C} \setminus \mathbb{R})$ and $(x, y) \notin \{(i, -i), (-i, i), (\omega_3, \omega_3^2), (\omega_3^2, \omega_3)\}$, where $\omega_3 = \exp(2\pi i/3)$. Then, for any pair of real algebraic numbers $(x', y') \in$*

⁴A theta graph consists of two terminals s and t joined by internally disjoint paths Brown *et al.* (2001). A series-parallel graph with terminals s and t can be obtained from the single-edge graph with edge (s, t) by repeatedly subdividing edges or adding parallel edges (Brandstädt *et al.* 1999, Chapter 11).

\mathcal{H}_q there is a polynomial-time approximate series-parallel shift from (x, y) to (x', y') .

The exceptions $\{(i, -i), (-i, i), (\omega_3, \omega_3^2), (\omega_3^2, \omega_3)\}$ are precisely the non-real points of the (x, y) plane where the Tutte polynomial of a graph can be evaluated in polynomial time (see Section 6.3). As we will see, being able to (x, y) -implement approximations of any number in $(-1, 0)$ is essentially the property that makes the approximation problem $\#P$ -hard at (x, y) .

We remark that the idea of implementing approximations of a given weight or edge interaction has been explored in the literature, though only when all the edge interactions involved are real. We review these results in Section 4.

We study the properties of polynomial-time approximate shifts in Section 4 and prove Theorem 2.1 in Section 5. In the next section, we describe some of the techniques used.

2.2.1. Proof Outline of Theorem 2.1. Shifts, as defined in Section 2.1, have a transitivity property: if there is a shift from (x_1, y_1) to (x_2, y_2) and from (x_2, y_2) to (x_3, y_3) , then there is a shift from (x_1, y_1) to (x_3, y_3) , see Section 3.2 for more details.

The polynomial-time approximate shift given in Theorem 2.1 is constructed in a similar way. First, we construct a polynomial-time approximate shift from (x, y) to some (x_2, y_2) such that $y_2 \in (-1, 0)$, where x_2 and y_2 depend on x, y . Then, we construct a polynomial-time approximate shift from (x_2, y_2) to (x', y') . Finally, we combine both polynomial-time approximate shifts using an analogue of the transitivity property.

However, when this approach is put into practice, there is a difficulty that causes various technical complications: we only have mild control in our constructions over the intermediate shift (x_2, y_2) . In particular, even if the numbers x and y are algebraic, we cannot guarantee that x_2 and y_2 are algebraic, and this causes problems with obtaining the required transitivity property. Instead, we have to work with a wider class of numbers, the set $\mathbf{P}_{\mathbb{C}}$ of *polynomial-time computable numbers*. These are numbers that can be approximated efficiently, i.e., for $y \in \mathbf{P}_{\mathbb{C}}$ there is an algorithm that computes $\hat{y}_n \in \mathbb{Q}[i]$ with $|y - \hat{y}_n| \leq 2^{-n}$ in time polynomial

in n (Ko 1991, Chapter 2). We denote by $\mathbb{P}_{\mathbb{R}} = \mathbb{R} \cap \mathbb{P}_{\mathbb{C}}$ the set of polynomial-time computable real numbers.

Our polynomial-time approximate shifts are constructed in Section 5. The first of these polynomial-time approximate shifts is provided by Lemma 2.2.

LEMMA 2.2. *Let q be a real algebraic number with $q \geq 2$. Let x and y be algebraic numbers such that $(x, y) \in \mathcal{H}_q$, $y \in (-1, 0) \cup (\mathbb{C} \setminus \mathbb{R})$ and $(x, y) \notin \{(i, -i), (-i, i), (\omega_3, \omega_3^2), (\omega_3^2, \omega_3)\}$, where $\omega_3 = \exp(2\pi i/3)$. Then there is a polynomial-time approximate series-parallel shift from (x, y) to (x', y') for some $(x', y') \in \mathcal{H}_q$ with $x', y' \in \mathbb{P}_{\mathbb{R}}$ and $y' \in (0, 1)$.*

The construction in Lemma 2.2 is obtained using a theta graph and trying to get a shift that is very close to the real line. However, we cannot control the point (x', y') that we are approximating, and as mentioned, x', y' might not be algebraic. The proof of Lemma 2.2 requires the most technical work in the paper and is given in Section 5.4.

Using Lemma 2.2, we have a series-parallel polynomial-time approximate shift from (x, y) to some $(x', y') \in \mathcal{H}_q$ with $x', y' \in \mathbb{P}_{\mathbb{R}}$ and $y' \in (0, 1)$. Next, we have to construct a polynomial-time approximate shift from (x', y') to (\hat{x}, \hat{y}) , where (\hat{x}, \hat{y}) is the point that we want to shift to in Theorem 2.1. In fact, we actually use a theta shift, which also facilitates establishing the required transitivity property later on. Note that since y' is not necessarily algebraic, we cannot directly apply the results that have already appeared in the literature on implementing approximations of edge interactions. In the next lemma, we generalise these results to the setting of polynomial-time computable numbers, where we need to address some further complications that arise from computing with polynomial-time computable numbers instead of algebraic numbers. The proof of the lemma is given in Section 5.5.

LEMMA 2.3. *Let $q, x, y \in \mathbb{P}_{\mathbb{R}}$ such that $q > 0$, $(x, y) \in \mathcal{H}_q$, y is positive and $1 - q/2 < y < 1$. There is a polynomial-time algorithm that takes as an input:*

- two positive integers k and n , in unary;

- a real algebraic number $w \in [y^k, 1]$.

The algorithm produces a theta graph J that (x, y) -implements (\hat{x}, \hat{y}) such that $|\hat{y} - w| \leq 2^{-n}$. The size of J is at most a polynomial in k and n , independently of w .

Then, we are able to combine the shifts in Lemmas 2.2 and 2.3 via a transitivity property for polynomial-time approximate shifts (see Lemma 4.8 in Section 4), and therefore prove Theorem 2.1, see Section 5 for the details.

2.3. The reductions. In Section 6.6 we show how to use a polynomial-time approximate shift from (x_1, y_1) to (x_2, y_2) to reduce the problem of approximating the Tutte polynomial at (x_2, y_2) to the same problem at (x_1, y_1) . The following lemma gives such a reduction for the problem of approximating the norm, we also give an analogous result for approximating the argument.

LEMMA 2.4. *Let $q \neq 0$, γ_1 and $\gamma_2 \neq 0$ be algebraic numbers, and $K > 1$. For $j \in \{1, 2\}$, let $y_j = \gamma_j + 1$ and $x_j = 1 + q/\gamma_j$. If there is a polynomial-time series-parallel approximate shift from (x_1, y_1) to (x_2, y_2) , then we have a reduction from FACTOR- K -NORMTUTTE(q, γ_2) to FACTOR- K -NORMTUTTE(q, γ_1). This reduction also holds for the planar version of the problem.*

In order to prove Lemma 2.4, we need some lower bounds on the norm of the partition function $Z_{\text{Tutte}}(G; q, \gamma)$. This kind of lower bound plays an important role in several hardness results on the complexity of approximating partition functions (Bezáková *et al.* 2020; Goldberg & Guo 2017). Here, we have to work a bit harder than usual since we have two (algebraic) underlying parameters (in the case of Tutte), and we need to use results in algebraic number theory, see Section 6.1 for details.

By combining Theorem 2.1 and Lemma 2.4 with existing hardness results, we obtain our hardness results for non-real edge interactions in Section 6.8. On the way, we collect some hardness on real parameters as well that strengthen previous results in the literature, and part of Section 6 is devoted to this. The main reason behind these improvements is that previous work on real parameters

used reductions from approximately counting minimum cardinality (s, t) -cuts (Goldberg & Guo 2017; Goldberg & Jerrum 2014), the minimum 3-way cut problem (Goldberg & Jerrum 2008), or maximum independent set for planar cubic graphs (Goldberg & Jerrum 2012b), which are either easy on planar graphs or the parameter regions they cover are considerably smaller or cannot be used to conclude #P-hardness. We instead reduce the exact computation of $Z_{\text{Tutte}}(G; q, \gamma)$ to its approximation, which has the advantage that the problem that we are reducing from is #P-hard for planar graphs (Vertigan 2005). Interestingly, our reduction requires us to apply an algorithm of Kannan, Lenstra and Lovász (Kannan *et al.* 1988) to reconstruct the minimal polynomial of an algebraic number from an additive approximation of the number. The lower bounds on the partition function $Z_{\text{Tutte}}(G; q, \gamma)$ that are gathered in Section 6.1 also play a role in this reduction, the details will be given in Section 6.5.

3. Preliminaries

3.1. The multivariate Tutte polynomial. The random cluster formulation of the multivariate Tutte polynomial is particularly convenient when working with implementations (as we will see in Section 3.2), and is defined as follows. Let $G = (V, E)$ be a graph. For any $\gamma: E \rightarrow \mathbb{C}$ and $q \in \mathbb{C}$, the *multivariate Tutte polynomial of G* is

$$(3.1) \quad Z_{\text{Tutte}}(G; q, \gamma) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} \gamma_e.$$

We will make use of the following notation. Let s and t be two distinct vertices of G . We define

$$Z_{st}(G; q, \gamma) = \sum_{\substack{A \subseteq E: \\ s \text{ and } t \text{ in the same component}}} q^{k(A)} \prod_{e \in A} \gamma_e.$$

Analogously, let $Z_{s|t}$ be the contribution to $Z_{\text{Tutte}}(G; q, \gamma)$ from the configurations $A \subseteq E$ such that s and t are in different connected components in (V, A) . That is, $Z_{s|t}(G; q, \gamma) = Z_{\text{Tutte}}(G; q, \gamma) - Z_{st}(G; q, \gamma)$.

3.2. Implementing weights, series compositions and parallel compositions. In this section, we define implementations, shifts, series compositions and parallel compositions. The definitions and results that we give are standard and can also be found, for instance, in (Jaeger *et al.* 1990, Section 4), (Goldberg & Jerrum 2012b, Section 2.1) or (Sokal 2005, Section 4).

Let $q \in \mathbb{C}$ with $q \neq 0$. The value of q is fixed across all this section. Let H be a weighted graph with weight function $\hat{\gamma}$. Let s and t be two distinct vertices of H , which are usually referred to as terminals. We say that the graph H $\hat{\gamma}$ -implements the weight w with respect to the terminals s and t if

$$w = q \frac{Z_{st}(H; q, \hat{\gamma})}{Z_{s|t}(H; q, \hat{\gamma})}.$$

We say that H $\hat{\gamma}$ -implements the weight w if there are terminals s and t such that H $\hat{\gamma}$ -implements the weight w with respect to s and t . These definitions are motivated by Lemma 3.2, whose proof is a straightforward computation involving the definitions of implementations and the multivariate Tutte polynomial.

LEMMA 3.2 (Goldberg & Jerrum 2012b, Equation 2.2). *Let G and H be two graphs with weight functions γ and $\hat{\gamma}$, respectively. Let f be an edge of G with weight γ_f such that H $\hat{\gamma}$ -implements γ_f with respect to terminals s and t . Let G_f be the graph constructed by considering the union of G and H , identifying the terminals s and t with the endpoints of f in G and removing f . Let γ' be the weight function on G_f that inherits the weights from γ and $\hat{\gamma}$. Then*

$$\begin{aligned} Z_{st}(G_f; q, \gamma') &= \frac{Z_{s|t}(H; q, \hat{\gamma})}{q^2} Z_{st}(G; q, \gamma), \\ Z_{s|t}(G_f; q, \gamma') &= \frac{Z_{s|t}(H; q, \hat{\gamma})}{q^2} Z_{s|t}(G; q, \gamma). \end{aligned}$$

In particular, we have $Z_{\text{Tutte}}(G_f; q, \gamma') = \frac{Z_{s|t}(H; q, \hat{\gamma})}{q^2} Z_{\text{Tutte}}(G; q, \gamma)$. Moreover, if G γ -implements a weight w , then G_f also γ' -implements w .

Therefore, if we can compute $Z_{s|t}(H; q, \hat{\gamma})$ efficiently, then computing $Z_{\text{Tutte}}(G; q, \gamma)$ is as hard as computing $Z_{\text{Tutte}}(G_f; q, \gamma')$. This observation leads to some of the reductions that appear in this paper.

In the remaining sections we usually assume that the weights are constant, that is, each edge of the graph has the same weight, and we will make it clear when this is not the case. In the constant weight function case Lemma 3.2 can be applied to each edge of the graph constructed by copying G and substituting each edge f in G by a copy of H (identifying the endpoints of f with s and t). Let $\alpha_1, \alpha_2 \in \mathbb{C}$. We say that there is a *shift* from (q, α_1) to (q, α_2) if there is a graph H that α_1 -implements α_2 . An important property of shifts is transitivity; if there are shifts from (q, α_1) to (q, α_2) and from (q, α_2) to (q, α_3) , then there is a shift from (q, α_1) to (q, α_3) . This is a consequence of Lemma 3.2. Let $y_1 = \alpha_1 + 1$ and $y_2 = \alpha_2 + 1$. We define x_1 and x_2 by $q = (x_1 - 1)(y_1 - 1) = (x_2 - 1)(y_2 - 1)$, which is the change of variables that relates the Tutte polynomial and Z_{Tutte} . We equivalently refer to the shift from (q, α_1) to (q, α_2) as a shift from (x_1, y_1) to (x_2, y_2) , and we also say that H (x_1, y_1) -implements (x_2, y_2) . This notation is convenient to express many of the shifts considered in this paper.

To conclude this section we introduce two tools that will provide us with many examples of implementations and shifts: parallel compositions and series compositions. For each $j \in \{1, 2\}$, let G_j be a graph, let s_j and t_j be two terminals of G_j , and let γ_j be a weight function such that G_j γ_j -implements a weight w_j with respect to s_j and t_j .

Parallel compositions. The parallel composition of (G_1, s_1, t_1) and (G_2, s_2, t_2) is the graph G constructed by considering the union of G_1 and G_2 and identifying s_1 with s_2 and t_1 with t_2 . Let $\hat{\gamma}$ be the weight function on G inherited from γ_1 and γ_2 . It is well known and easy to check that G $\hat{\gamma}$ -implements the weight

$$w = (1 + w_1)(1 + w_2) - 1$$

with respect to the terminals s_1 and t_1 . Let (x_1, y_1) and (x_2, y_2) be the Tutte coordinates of (q, w_1) and (q, w_2) , respectively, (so $y_j = w_j + 1$ and $(x_j - 1)(y_j - 1) = q$). Then the Tutte coordinates

of (q, w) are (x, y) with $y = y_1 y_2$ and $(x - 1)(y - 1) = q$. Let Υ be a graph with two vertices s, t and one edge joining them, and let Υ^n be the parallel composition of n copies of (Υ, s, t) (so Υ^n has two vertices and n edges joining them). Then Υ^n (x, y) -implements (x', y') with $y' = y^n$ and $(x' - 1)(y' - 1) = q$. This is known as an n -thickening of (x, y) and it yields a shift from (x, y) to (x', y^n) .

Series compositions. The series composition of (G_1, s_1, t_1) and (G_2, s_2, t_2) is the graph G constructed by considering the union of G_1 and G_2 and identifying t_1 with s_2 . Let $\hat{\gamma}$ be the weight function on G inherited from γ_1 and γ_2 . It is well known and easy to check that G $\hat{\gamma}$ -implements the weight

$$w = \frac{w_1 w_2}{w_1 + w_2 + q}$$

with respect to the terminals s_1 and t_2 . Note that w satisfies

$$(3.3) \quad \left(1 + \frac{q}{w}\right) = \left(1 + \frac{q}{w_1}\right) \left(1 + \frac{q}{w_2}\right).$$

Let (x_1, y_1) and (x_2, y_2) be the Tutte coordinates of (q, w_1) and (q, w_2) , respectively (so $y_j = w_j + 1$ and $(x_j - 1)(y_j - 1) = q$). Then, in view of (3.3), the Tutte coordinates of (q, w) are (x, y) with $x = x_1 x_2$ and $(x - 1)(y - 1) = q$. Let Υ be a graph with two vertices s, t and one edge joining them, and let Υ_n be the series composition of n copies of (Υ, s, t) (so Υ_n is a path graph with n edges). Then Υ_n (x, y) -implements (x', y') with $x' = x^n$ and $(x' - 1)(y' - 1) = q$. This is known as an n -stretching of (x, y) and it yields a shift from (x, y) to (x^n, y') .

For series-parallel and theta graphs (see Footnote 4), these constructions give that either $Z_{s|t}(G; q, \gamma) = 0$, or the series-parallel graph G (with terminals s and t) γ -implements a weight $w(G, s, t; q, \gamma)$ that can be computed from the recursive definition of series-parallel graphs in polynomial time. In particular, let $\Theta_{(l_1, \dots, l_m)}$ be the theta graph with m internal paths of lengths l_1, \dots, l_m . In this case,⁵ we

⁵We should mention that we will make use of the Θ asymptotic notation in this paper and this notation should not be confused with that of theta graphs.

have that

$$(3.4) \quad w(\Theta_{(l_1, \dots, l_m)}, s, t; q, \gamma) = \prod_{j=1}^m \left(1 + \frac{q}{x^{l_j} - 1}\right) - 1,$$

where $x = 1 + q/\gamma$. Series-parallel graphs can be built using series and parallel compositions. The following definition is equivalent to the one in Footnote 4. A graph G is *series-parallel* (with terminals s and t) if either G is the graph with two vertices s and t and one edge joining them, or G is the parallel or series composition of (G_1, s_1, t_1) and (G_2, s_2, t_2) , where $s = s_1$, $t = t_2$ and G_j is a series-parallel graph with terminals s_j and t_j (Brandstädt *et al.* 1999, Chapter 11).

Finally, the *size of a graph* $G = (V, E)$ is the integer $\text{size}(G) = |V| + |E|$. Note that the size of $\Theta_{(l_1, \dots, l_m)}$ is $2 \sum_{j=1}^m l_j - m + 2$.

4. Polynomial-time approximate shifts

Implementing a specific weight cannot always be achieved. Nonetheless, sometimes we can implement an approximation of the desired weight with as much precision as we need. These implementations have been exploited several times in the literature on Tutte polynomials and the Ising model; see (Goldberg & Jerrum 2008, 2012b, 2014, 2019). Here we collect some of these results appearing in (Goldberg & Jerrum 2014), which in turn are based on arguments in (Goldberg & Jerrum 2012b); here, we follow the presentation in (Goldberg & Jerrum 2019) (that was stated for $q = 2$).

LEMMA 4.1. (Goldberg & Jerrum 2019, Lemma 22 Goldberg & Jerrum 2014, Lemma 5) *Let x and y be real algebraic numbers such that $y \notin [-1, 1]$ and $(x - 1)(y - 1) = q > 0$. There is a polynomial-time algorithm that takes as an input:*

- two positive integers n and k , in unary;
- a real algebraic number $y' \in [1, |y|^k]$.

This algorithm produces a theta graph G that (x, y) -implements (\hat{x}, \hat{y}) such that $|y' - \hat{y}| \leq 2^{-n}$. The size of G is at most a polynomial in n and k , independently of y' .

In Lemma 2.3 (Section 5), we give a similar result to Lemma 4.1 where the numbers x and y may not be algebraic. The fact that the graph G computed in Lemma 4.1 is a theta graph is not directly stated in the statement of (Goldberg & Jerrum 2014, Lemma 5) but it can easily be inferred from the proof. This also applies to Lemma 4.2.

LEMMA 4.2. (Goldberg & Jerrum (2019), Lemma 22, Goldberg & Jerrum (2014), Lemma 7)] *Let x_1, y_1, x_2, y_2 be real algebraic numbers such that $y_1 \in (-1, 1)$, $y_2 \notin [-1, 1]$, and $(x_1 - 1)(y_1 - 1) = (x_2 - 1)(y_2 - 1) = q < 0$. There is a polynomial-time algorithm that takes as an input:*

- *two positive integers n and k , in unary;*
- *a real algebraic number $y' \in [1, |y_1|^{-k}]$.*

This algorithm produces a theta graph $G = (V, E)$ and a weight function $\hat{\gamma}: E \rightarrow \{y_1 - 1, y_2 - 1\}$ such that G $\hat{\gamma}$ -implements (\hat{x}, \hat{y}) with $|y' - \hat{y}| \leq 2^{-n}$. The size of G is at most a polynomial in n and k , independently of y' .

COROLLARY 4.3. *Let x_1, y_1, x_2, y_2 be real algebraic numbers such that $y_1 \in (-1, 0) \cup (0, 1)$, $y_2 \notin [-1, 1]$, $(x_1 - 1)(y_1 - 1) = (x_2 - 1)(y_2 - 1) = q$, $q \neq 0$. There is a polynomial-time algorithm that takes as an input:*

- *two positive integers n and k , in unary;*
- *a positive real algebraic number y' such that $|y'| \in [|y_1|^k, |y_1|^{-k}]$.*

This algorithm produces a theta graph $G = (V, E)$ and a weight function $\hat{\gamma}: E \rightarrow \{y_1 - 1, y_2 - 1\}$ such that G $\hat{\gamma}$ -implements (\hat{x}, \hat{y}) with $|y' - \hat{y}| \leq 2^{-n}$. The size of G is at most a polynomial in n and k , independently of y' . Moreover, if either $y_1 < 0$ or $y_2 < 0$, then the restriction that y' is positive can be replaced with a restriction that y' is non-zero.

PROOF. This result easily follows from Lemmas 4.1 and 4.2 by an argument of Goldberg and Jerrum (see the proof of (Goldberg & Jerrum 2014, Lemma 2)). We include here their argument for completeness. The case when $y' \geq 1$ has been covered in Lemmas 4.1 and 4.2. First, let us assume that $y' \in (0, 1)$. We have $1 \leq y' \cdot y_1^{-2k} \leq |y_1|^{-2k}$ and using Lemmas 4.1 and 4.2 we can implement \tilde{y} with $|\tilde{y} - y' \cdot y_1^{-2k}| \leq 2^{-n}$. We have $|y_1^{2k}\tilde{y} - y'| \leq 2^{-n}$, so we set $\hat{y} = y_1^{2k}\tilde{y}$. The graph G is the parallel composition of the graph used to implement \tilde{y} and $2k$ edges with weight y_1 . Finally, let us assume that there is $i \in \{1, 2\}$ such that $y_i < 0$, and let us consider the case where y' is negative. We implement an approximation \hat{y}' of $y'/y_i > 0$, and return $\hat{y} = \hat{y}'y_i$. \square

The graphs G produced by the algorithms given in Lemma 4.1, Lemma 4.2 and Corollary 4.3 are theta graphs. One may wonder which weights can be approximated as in these results. This leads to the following definition. Let $(x_1, y_1), (x_2, y_2) \in \mathcal{H}_q$. Let $\gamma_1 = y_1 - 1$ and $\gamma_2 = y_2 - 1$. We say that there is a *polynomial-time approximate shift* from (q, γ_1) to (q, γ_2) or, equivalently, from (x_1, y_1) to (x_2, y_2) , if there is an algorithm that, for any positive integer n , computes in polynomial time in n a graph G_n that (x_1, y_1) -implements (\hat{x}_2, \hat{y}_2) with $|y_2 - \hat{y}_2| \leq 2^{-n}$. If the graph G_n computed by this algorithm is always a theta graph (resp. a series-parallel graph), then we say that this is a *polynomial-time approximate theta shift* (resp. *polynomial-time approximate series-parallel shift*). Lemma 4.1 gives polynomial-time approximate theta shifts from (x_1, y_1) to (x_2, y_2) when the considered numbers are real algebraic, $y_1 \notin [-1, 1]$, $y_2 \in [1, \infty)$ and $q > 0$. Note that shifts are a particular case of polynomial-time approximate shifts. Moreover, due to the transitivity property of shifts, if there is a shift from (x_1, y_1) to (x_2, y_2) and there is a polynomial-time approximate shift from (x_2, y_2) to (x_3, y_3) , then there is a polynomial-time approximate shift from (x_1, y_1) to (x_3, y_3) . In fact, polynomial-time approximate shifts exhibit some of the properties of shifts; in Lemma 4.4 we show that they behave well with respect to parallel and series compositions and in Lemma 4.8 we show that they are transitive under certain conditions. In Section 5 we give more examples of polynomial-time approximate shifts, some of which will

be constructed by transitivity. These approximate shifts play an important role in our hardness proofs.

LEMMA 4.4. *Let $q \in \mathbb{C} \setminus \{0\}$ and let $(x_j, y_j) \in \mathcal{H}_q$ for each $j \in \{1, 2, 3\}$. Let us assume that there are polynomial-time approximate shifts from (x_1, y_1) to (x_2, y_2) , and from (x_1, y_1) to (x_3, y_3) . Let $(x_4, y_4), (x_5, y_5) \in \mathcal{H}_q$ with $y_4 = y_2y_3$ and $x_5 = x_2x_3$. Then:*

- (i) *there is a polynomial-time approximate shift from (x_1, y_1) to (x_4, y_4) ;*
- (ii) *there is a polynomial-time approximate shift from (x_1, y_1) to (x_5, y_5) .*

Moreover, if the polynomial-time approximate shifts from (x_1, y_1) to (x_2, y_2) and (x_3, y_3) are series-parallel, then the obtained polynomial-time approximate shifts are also series-parallel.

PROOF. For $j \in \{2, 3\}$, let $G_{n,j}$ be the graph computed by the polynomial-time approximate shift from (x_1, y_1) to (x_j, y_j) , so $G_{n,j}$ (x_1, y_1) -implements (\hat{x}_j, \hat{y}_j) with $|y_j - \hat{y}_j| \leq 2^{-n}$, for certain terminals t_j and s_j .

For Item i, let P_n be the parallel composition of $(G_{n,2}, s_2, t_2)$ and $(G_{n,3}, s_3, t_3)$. The graph P_n gives a shift from (x_1, y_1) to $(\hat{x}_4, \hat{y}_2\hat{y}_3) \in \mathcal{H}_q$. Since $|y_3 - \hat{y}_3| \leq 2^{-n}$, we have $|\hat{y}_3| \leq |y_3| + 1$ and

$$|y_2y_3 - \hat{y}_2\hat{y}_3| \leq |y_2 - \hat{y}_2| |\hat{y}_3| + |y_3 - \hat{y}_3| |y_2| \leq 2^{-n} (|y_3| + 1 + |y_2|).$$

Therefore, for k large enough, the graphs P_{n+k} give a polynomial-time approximate shift from (x_1, y_1) to (x_4, y_4) with $y_4 = y_2y_3$.

For Item ii, the proof is analogous but now we define the graph S_n as the series composition of $(G_{n,2}, s_2, t_2)$ and $(G_{n,3}, s_3, t_3)$, which gives a shift from (x_1, y_1) to $(\hat{x}_2\hat{x}_3, \hat{y}_4) \in \mathcal{H}_q$.

Note that if the original polynomial-time approximate shifts are series-parallel, then the obtained ones are also series-parallel by the definition of series-parallel graphs. \square

When it comes to hardness results, we are only interested in algebraic numbers. However, we will have to consider polynomial-time approximate shifts from (x_1, y_1) to (x_2, y_2) such that the numbers involved are not algebraic. This is due to the fact that, even

if x_1 and y_1 are algebraic, x_2 and y_2 might not be. Nonetheless, in that case we can ensure that x_2 and y_2 are polynomial-time computable. A real number x is *polynomial-time computable* if there is a function $\phi: \mathbb{N} \rightarrow \mathbb{Q}$ that is computable in polynomial time (with the input written in unary notation, i.e., 0^n) such that $|x - \phi(n)| \leq 2^{-n}$ for all $n \in \mathbb{N}$ (Ko 1991, Chapter 2). The definition given in (Ko 1991, Chapter 2) uses dyadic rational numbers instead of rational numbers, but these two definitions are easily seen to be equivalent. We denote the set of polynomial-time computable real numbers by $P_{\mathbb{R}}$. One can easily show that the set $P_{\mathbb{R}}$ is a field. Real algebraic numbers are in $P_{\mathbb{R}}$ because we can approximate them as closely as we want by applying Sturm sequences and binary search (Emiris *et al.* 2008). We say that a complex number z is *polynomial-time computable* if $z = x + iy$ for some $x, y \in P_{\mathbb{R}}$. We denote the set of polynomial-time computable complex numbers by $P_{\mathbb{C}}$. Algebraic numbers are in $P_{\mathbb{C}}$ (their real and imaginary parts are real algebraic numbers). It turns out that $P_{\mathbb{C}}$ is an algebraically closed field (Ko 1991, Chapter 2). In particular, for $z \in P_{\mathbb{C}}$, we have $|z| \in P_{\mathbb{R}}$.

If there is a polynomial-time approximate theta shift from (x_1, y_1) to (x_2, y_2) and x_1 and y_1 are algebraic, then we can compute in polynomial time in n an algebraic number that additively approximates y_2 up to an additive error 2^{-n} . Since we can approximate algebraic numbers by rational numbers efficiently, it follows that x_2 and y_2 are polynomial-time computable. However, if we only know that x_1 and y_1 are polynomial-time computable, then it is not clear if x_2 and y_2 are polynomial-time computable or not. Lemma 4.7 gives a partial answer to this question and plays a key role in our transitivity result for polynomial-time approximate shifts (Lemma 4.8). First, we need to prove some lemmas on polynomial-time computable numbers.

LEMMA 4.5. *Let $z \in P_{\mathbb{C}}$. There is an algorithm that computes $b_1 \in \mathbb{Q}$ with $|z| \leq b_1$. Moreover, if $z \neq 0$, then there is an algorithm that computes $b_2 \in \mathbb{Q}$ with $0 < b_2 \leq |z|$.*

PROOF. Let $x = |z|$. From $x \in P_{\mathbb{R}}$, it follows that we can compute a sequence $\hat{x}_n \in \mathbb{Q}$ such that $|x - \hat{x}_n| \leq 2^{-n}$, that is, we have

$x \in [\hat{x}_n - 2^{-n}, \hat{x}_n + 2^{-n}]$. This computation for $n = 1$ gives the upper bound $\hat{x}_1 + 1/2$. Note that the sequences $\hat{x}_n - 2^{-n}$ and $\hat{x}_n + 2^{-n}$ converge to x . Hence, if $x \neq 0$, then there must be n such that $0 < \hat{x}_n - 2^{-n} \leq x$. We compute \hat{x}_n until this inequality happens, obtaining the desired lower bound. \square

LEMMA 4.6. *Let $z \in \mathbb{P}_{\mathbb{C}}$ with $|z| \neq 1$. There is a polynomial-time algorithm that takes as inputs two positive integers n and k and computes a positive integer $r(n, k)$ such that*

- (i) $r(n, k)$ is increasing in k ;
- (ii) $r(n, k) = n + \Theta(k)$;
- (iii) if $|z - \hat{z}| \leq 2^{-r(n, k)}$, then $\left| \frac{1}{z^k - 1} - \frac{1}{\hat{z}^k - 1} \right| \leq 2^{-n}$.

PROOF. By Lemma 4.5, we can compute an integer $t \geq 0$ such that $2^{-t} \leq ||z| - 1|$ and $|z| \leq 2^t$. Note that for every integer $k \geq 1$ we have the bound $|z^k - 1| \geq 2^{-t}$. Indeed, if $|z| < 1$, then

$$2^{-t} \leq 1 - |z| \leq 1 - |z|^k \leq |z^k - 1|$$

and when $|z| > 1$, we analogously find that $2^{-t} \leq |z| - 1 \leq |z|^k - 1 \leq |z^k - 1|$.

Let n and k be the inputs of our algorithm. Let $r(n, k) = n + (t + 1)(k + 1)$, and note that r is increasing in k and $r(n, k) = n + \Theta(k)$, establishing Items i and ii.

For Item iii, consider \hat{z} such that $|z - \hat{z}| \leq 2^{-r(n, k)}$. Since $|\hat{z}| \leq |z| + 2^{-r(n, k)} \leq 2^{t+1}$, for every $j \in \{0, \dots, k - 1\}$ we have $|\hat{z}|^j |z|^{k-1-j} \leq 2^{t(k-1)+j}$ and hence

$$\begin{aligned} |z^k - \hat{z}^k| &= \left| (z - \hat{z}) \sum_{j=0}^{k-1} \hat{z}^j z^{k-1-j} \right| \leq |z - \hat{z}| \sum_{j=0}^{k-1} |\hat{z}|^j |z|^{k-1-j} \\ &\leq |z - \hat{z}| \sum_{j=0}^{k-1} 2^{t(k-1)+j} < |z - \hat{z}| 2^{t(k-1)+k} \leq 2^{-(n+2t+1)}. \end{aligned}$$

Moreover, we have that $||z^k - 1| - |\hat{z}^k - 1|| \leq |z^k - \hat{z}^k| < 2^{-(t+1)}$ and, thus,

$$|\hat{z}^k - 1| \geq |z^k - 1| - 2^{-(t+1)} \geq 2^{-(t+1)},$$

where we used that $|z^k - 1| \geq 2^{-t}$. Therefore, we find that

$$\left| \frac{1}{z^k - 1} - \frac{1}{\hat{z}^k - 1} \right| = \left| \frac{z^k - \hat{z}^k}{(z^k - 1)(\hat{z}^k - 1)} \right| \leq 2^{2t+1} |z^k - \hat{z}^k| \leq 2^{-n}.$$

□

LEMMA 4.7. *Let $q \in \mathbb{P}_{\mathbb{C}}$ with $q \neq 0$ and let $\gamma \in \mathbb{P}_{\mathbb{C}}$ with $\gamma \notin \{0\} \cup -q/2 + iq\mathbb{R}$. There is a polynomial-time algorithm that takes as an input:*

- a positive integer n ;
- a theta graph $G = \Theta_{(l_1, \dots, l_m)}$ with terminals s and t .

This algorithm computes $f(n, G)$ such that

- (i) $f(n, G) = n + \Theta(\text{size}(G))$;
- (ii) for any $\hat{\gamma}$ with $|\gamma - \hat{\gamma}| \leq 2^{-f(n, G)}$, we have $|w(G, s, t; q, \gamma) - w(G, s, t; q, \hat{\gamma})| \leq 2^{-n}$.

PROOF. Let $y = \gamma + 1$ and $x = 1 + q/\gamma$. Note that $|x| = 1$ if and only if $|\gamma + q| = |\gamma|$. By basic geometry, the latter statement is equivalent to $\gamma \in -q/2 + iq\mathbb{R}$. Hence, by hypothesis, $|x| \neq 1$. There are two cases:

- $|x| < 1$. Then for any positive integer k we have

$$\left| 1 + \frac{q}{x^k - 1} \right| \leq 1 + \frac{|q|}{1 - |x|^k} \leq 1 + \frac{|q|}{1 - |x|} = 1 + \frac{|q|}{|1 - |x||}.$$

- $|x| > 1$. Then for any positive integer k we have

$$\left| 1 + \frac{q}{x^k - 1} \right| \leq 1 + \frac{|q|}{|x|^k - 1} \leq 1 + \frac{|q|}{|x| - 1} = 1 + \frac{|q|}{|1 - |x||}.$$

Since $q, x \in \mathbb{P}_{\mathbb{C}}$, we can apply Lemma 4.5 along with the above bounds to compute a non-negative integer t_x such that

$|1 + q/(x^k - 1)| \leq 2^{t_x}$ for every positive integer k . Lemma 4.5

also allows us to compute non-negative integers t_q and t_γ such that $|q| \leq 2^{t_q}$ and $2^{-t_\gamma} \leq |\gamma|$.

Let n and $G = \Theta_{(l_1, \dots, l_m)}$ be the inputs of our algorithm. Let $k = \max\{l_1, \dots, l_m\}$. Since $|x| \neq 1$, we can compute $g(n, G) = r(n + (t_x + 1)(m + 1) + t_q, k)$, where r is as in Lemma 4.6 for the polynomial-time computable number x . We compute $f(n, G) = g(n, G) + t_q + 2t_\gamma + 1$. We claim that f satisfies the statement. In view of the properties of r , we have

$$f(n, G) = g(n, G) + \Theta(1) = n + \Theta(\text{size}(G)).$$

We define $y_j = 1 + q/(x^{l_j} - 1)$ for every $j \in \{1, \dots, m\}$. Recall that in (3.4) we argued that

$$w(G, s, t; q, \gamma) = \prod_{j=1}^m y_j - 1.$$

Let $\hat{\gamma}$ with $|\gamma - \hat{\gamma}| \leq 2^{-f(n, G)}$. Let $\hat{y} = \hat{\gamma} + 1$ and $\hat{x} = 1 + q/(\hat{y} - 1)$. Then

$$w(G, s, t; q, \hat{\gamma}) = \prod_{j=1}^m \hat{y}_j - 1,$$

where $\hat{y}_j = 1 + q/(\hat{x}^{l_j} - 1)$. Since $|\gamma - \hat{\gamma}| \leq 2^{-f(n, G)} \leq 2^{-t_\gamma - 1}$, we have $|\hat{\gamma}| \geq |\gamma| - 2^{-t_\gamma - 1} \geq 2^{-t_\gamma - 1}$ and

$$\begin{aligned} |x - \hat{x}| &= \left| \frac{q}{\gamma} - \frac{q}{\hat{\gamma}} \right| = \left| q \frac{\hat{\gamma} - \gamma}{\gamma \hat{\gamma}} \right| \\ &\leq |q| |\hat{\gamma} - \gamma| 2^{2t_\gamma + 1} \leq 2^{t_q + 2t_\gamma + 1 - f(n, G)} = 2^{-g(n, G)}. \end{aligned}$$

In light of the properties of r (Lemma 4.6) and the fact that $l_j \leq k$, it follows that

$$\begin{aligned} |y_j - \hat{y}_j| &= \left| \frac{q}{x^{l_j} - 1} - \frac{q}{\hat{x}^{l_j} - 1} \right| \leq |q| 2^{-n - (t_x + 1)(m + 1) - t_q} \\ &\leq 2^{-n - (t_x + 1)(m + 1)} \end{aligned}$$

for every $j \in \{1, \dots, m\}$. Thus, we have $|\hat{y}_j| \leq |y_j| + 1 \leq 2^{t_x + 1}$.

We obtain

$$\begin{aligned}
 \left| \prod_{j=1}^m y_j - \prod_{j=1}^m \hat{y}_j \right| &= \left| \sum_{j=1}^m (y_j - \hat{y}_j) \prod_{s=1}^{j-1} \hat{y}_s \prod_{s=j+1}^m y_s \right| \\
 &< \sum_{j=1}^m |y_j - \hat{y}_j| 2^{t_x(m-1)+j-1} \\
 &\leq 2^{t_x(m-1)} \sum_{j=1}^m 2^{-n-(t_x+1)(m+1)+j-1} \\
 &\leq 2^{-n-m-2} \sum_{j=1}^m 2^j < 2^{-n}.
 \end{aligned}$$

Equivalently, $|w(G, s, t; q, \gamma) - w(G, s, t; q, \hat{\gamma})| < 2^{-n}$ as we wanted to prove. \square

We now prove the main transitivity property of polynomial-time approximate shifts that we will use in our constructions.

LEMMA 4.8. *Let $q \in \mathbb{P}_{\mathbb{C}}$ with $q \neq 0$ and let $(x_j, y_j) \in \mathcal{H}_q$ for each $j \in \{1, 2, 3\}$. Let us assume that the following hypotheses hold:*

- (i) x_2 and y_2 are polynomial-time computable;
- (ii) $y_2 \notin \{1\} \cup (1 - q/2 + iq\mathbb{R})$;
- (iii) there is a polynomial-time approximate shift from (x_1, y_1) to (x_2, y_2) ;
- (iv) there is a polynomial-time approximate theta shift from (x_2, y_2) to (x_3, y_3) .

Then there is a polynomial-time approximate shift from (x_1, y_1) to (x_3, y_3) . Moreover, if the polynomial-time approximate shift from (x_1, y_1) to (x_2, y_2) is series-parallel, then the polynomial-time approximate shift from (x_1, y_1) to (x_3, y_3) is also series-parallel.

PROOF. Let $\gamma_j = y_j - 1$ for every $j \in \{1, 2, 3\}$. Let n be a positive integer. We give an algorithm that constructs a graph J_n , in polynomial time in n , such that J_n γ_1 -implements $\hat{\gamma}_3$ with $|\gamma_3 - \hat{\gamma}_3| \leq 2^{-n}$. This algorithm is as follows. First, we use the

approximate theta shift from (x_2, y_2) to (x_3, y_3) to compute a theta graph G_2 with terminals s_2 and t_2 such that

$$(4.9) \quad |\gamma_3 - w(G_2, s_2, t_2; q, \gamma_2)| \leq 2^{-n-1}.$$

The size of G_2 is at most polynomial in n . In light of Lemma 4.7, we can compute, in polynomial time in n , a positive integer $f(n + 1, G_2)$ such that for any $\hat{\gamma}_2$ with $|\gamma_2 - \hat{\gamma}_2| \leq 2^{-f(n+1, G_2)}$, we have

$$(4.10) \quad |w(G_2, s_2, t_2; q, \gamma_2) - w(G_2, s_2, t_2; q, \hat{\gamma}_2)| \leq 2^{-n-1}.$$

We also have $f(n + 1, G_2) = n + \Theta(\text{size}(G_2))$, so $f(n + 1, G_2)$ is bounded by a polynomial in n . Now we use the approximate shift from (x_1, y_1) to (x_2, y_2) to compute, in polynomial time in n , a graph G_1 such that G_1 γ_1 -implements $\hat{\gamma}_2$ with $|\gamma_2 - \hat{\gamma}_2| \leq 2^{-f(n+1, G_2)}$. Combining (4.9) and (4.10) with the triangle inequality, we obtain $|\gamma_3 - w(G_2, s_2, t_2; q, \hat{\gamma}_2)| \leq 2^{-n}$.

Finally, we construct a graph J_n as a copy of G_2 where every edge is substituted by a copy of G_1 as in Lemma 3.2. Since the sizes of G_1 and G_2 are polynomial in n , the size of J_n also is polynomial in n . Recall that G_2 $\hat{\gamma}_2$ -implements $\hat{\gamma}_3 = w(G_2, s_2, t_2; q, \hat{\gamma}_2)$ and G_1 γ_1 -implements $\hat{\gamma}_2$. Therefore, the graph J_n γ_1 -implements $\hat{\gamma}_3$, and $|\gamma_3 - \hat{\gamma}_3| \leq 2^{-n}$, as we wanted to obtain. Finally, if the polynomial-time approximate shift from (x_1, y_1) to (x_2, y_2) is series-parallel, then the graphs J_n are easily seen to be series-parallel, and the result follows. \square

5. Polynomial-time approximate shifts with complex weights

In this section we show how to implement approximations of real weights when the original weight is a non-real algebraic number. As a consequence of our results, for any real algebraic number q with $q \geq 2$ and any pair of algebraic numbers $(x, y) \in \mathcal{H}_q$ with $y \notin \mathbb{R}$ and $(x, y) \notin \{(-i, i), (i - i), (\omega_3^2, \omega_3), (\omega_3, \omega_3^2)\}$, where $\omega_3 = \exp(2\pi i/3)$, there is a polynomial-time approximate shift from (x, y) to any pair of real algebraic numbers $(x', y') \in \mathcal{H}_q$ (see Theorem 2.1). Our approach to prove Theorem 2.1 is as follows. First, we show that there is $(x', y') \in \mathcal{H}_q$ with $y' \in (0, 1)$ such that there is a

polynomial-time approximate theta shift from (x, y) to (x', y') (see Lemma 5.23). Since x and y are algebraic, it follows that x' and y' are polynomial-time computable. Secondly, we extend part of Lemma 4.1 to the case where the numbers involved are only known to be polynomial-time computable (see Lemma 2.3). Finally, we use the transitivity property given in Lemma 4.8 to combine both results in the proof of Theorem 2.1.

5.1. Computing with algebraic numbers. In our proofs we use and develop some algorithms on algebraic numbers. We gather these algorithms in this section. We represent an algebraic number z as its minimal polynomial p and a rectangle R of the complex plane such that z is the only root of p in R . We can compute the addition, subtraction, multiplication, division and conjugation of algebraic numbers in polynomial time in the length of their representations, see (Strzeboński 1997) for details. As a consequence, we can also compute the real and imaginary parts of z and the norm of z , which are algebraic numbers themselves, in polynomial time. Note that an algebraic number is 0 if and only if its minimal polynomial is x , which can be easily checked in this representation. Hence, we can also determine in polynomial time whether two algebraic numbers z_1 and z_2 are equal by checking if $z_1 - z_2$ is 0.

When z is a real algebraic number, we can simply represent it as its minimal polynomial p and an interval I with rational endpoints such that z is the only root of p in I . If we are given a real algebraic number z with this representation, then we can approximate it as closely as we want by applying Sturm sequences and binary search (Emiris *et al.* 2008). In fact, for z_1 and z_2 real algebraic numbers, Sturm sequences also allow us to check whether $z_1 \geq z_2$ in time polynomial in the length of the representations of z_1 and z_2 . See (Emiris *et al.* 2008) for more details and complexity analysis.

A *root of unity* is a complex number z such that $z^k = 1$ for some positive integer k . The smallest positive integer n such that $z^n = 1$ is the *order of z* . Note that roots of unity are algebraic numbers. The roots of unity of order n share the same minimal polynomial, known as the n -th cyclotomic polynomial, whose degree is $\varphi(n)$, the

Euler phi function. We can determine whether an algebraic number z is a root of unity by checking whether its minimal polynomial is cyclotomic, see (Bradford & Davenport 1989) for a polynomial-time algorithm. If z is a root of unity, then we can easily compute its order from its representation; we compute the smallest n such that the minimal polynomial of z divides $z^n - 1$. This computation runs in polynomial time in the length of the representation of z as a consequence of the elementary bound $\varphi(n) \geq \sqrt{n}/2$.

Another operation that we can perform in polynomial time is checking if the argument of an algebraic number is in a fixed interval.

LEMMA 5.1. *Let $a, b \in \mathbb{Q} \cap [0, 1]$ with $a \leq b$. Then for any algebraic number z we can check whether $\text{Arg}(z) \in [2\pi a, 2\pi b]$ in time polynomial in the length of the representation of z .*

PROOF. We can split the interval $[2\pi a, 2\pi b]$ into intervals of length at most $\pi/2$ and check if $\text{Arg}(z)$ belongs to any of those intervals. Hence, let us assume for the sake of simplicity that $[2\pi a, 2\pi b] \subseteq [0, \pi/2]$. The other cases are analogous. Note that $e^{2\pi i a}$ and $e^{2\pi i b}$ are roots of unity and, in particular, algebraic. Thus, we can compute $z_a = ze^{-2\pi i a}$ and $z_b = ze^{2\pi i(1/4-b)}$. We have $\text{Arg}(z_a) \in [0, \pi/2]$ if and only if $\text{Arg}(z) \in [2\pi a, \pi/2 + 2\pi a]$, and $\text{Arg}(z_b) \in [0, \pi/2]$ if and only if $\text{Arg}(z) \in [-\pi/2 + 2\pi b, 2\pi b]$. We conclude that $\text{Arg}(z) \in [2\pi a, 2\pi b]$ if and only if $\text{Arg}(z_a) \in [0, \pi/2]$ and $\text{Arg}(z_b) \in [0, \pi/2]$. Finally, note that, for any algebraic number y , since $\text{Re}(y)$ and $\text{Im}(y)$ are algebraic, we can determine if $\text{Arg}(y) \in [0, \pi/2]$ or not by checking the inequalities $\text{Re}(y) \geq 0$ and $\text{Im}(y) \geq 0$. \square

In the rest of this section we show how to efficiently compute a sequence $\sigma(n)$ such that $\text{Arg}(z^{\sigma(n)}) \in [2\pi a, 2\pi b]$ for every n . We will use the following well-known result, see, e.g., (Brin & Stuck 2002, Section 1.2): if $z \in \mathbb{C}$ is not a root of unity and $|z| = 1$, then $\{z^j : j \in \mathbb{N}\}$ is dense in the unit circle.

LEMMA 5.2. *Let $a, b \in \mathbb{Q} \cap [0, 1]$ with $a < b$. Let z be an algebraic number such that $|z| = 1$ and z is not a root of unity. Then there*

exists a sequence of positive integers $\{\sigma(n)\}$ and a positive integer k such that such that:

- (i) k can be computed from z ;
- (ii) $\sigma(n)$ can be computed in polynomial time in n ;
- (iii) $n \leq \sigma(n) \leq n + k - 1$ for every positive integer n ;
- (iv) $\text{Arg}(z^{\sigma(n)}) \in [2a\pi, 2b\pi] + 2\pi\mathbb{Z}$ for every positive integer n .

PROOF. Our algorithm to compute $\sigma(n)$ is as follows. Set $\sigma(0) = 0$. We compute $\sigma(n)$ as the smallest integer such that $n \leq \sigma(n)$ and $\text{Arg}(z^{\sigma(n)}) \in [2a\pi, 2b\pi]$. We can check whether $\text{Arg}(z^{\sigma(n)}) \in [2a\pi, 2b\pi]$ or not by applying the procedure given in Lemma 5.1.

We show that $\sigma(n)$ is well-defined. Let $\theta = \text{Arg}(z)$. Since z is not a root of unity, $\{z^j : j \in \mathbb{N}\}$ is dense in the unit circle, as we have discussed in the previous paragraph. Therefore, there is $q \in \mathbb{N}$ such that $\text{Arg}(z^q) \in [0, 2(b-a)\pi]$. Note that we can compute q in constant time with the help of Lemma 5.1. Let $\tau = \text{Arg}(z^q)$. Since z is not a root of unity, we find that $\tau \neq 0$. Let $t = \lceil 2\pi/\tau \rceil$. Since t is the smallest positive integer such that $t\tau \geq 2\pi$, t can be computed by sequentially determining which of the following intervals contains the argument of z^{qj} : $(0, \pi/2)$, $(\pi/2, \pi)$, $(\pi, 3\pi/2)$ or $(3\pi/2, 2\pi)$. Hence, we can compute $k = tq$. For each positive integer n , since $t\tau \geq 2\pi$ and $\tau < 2(b-a)\pi$, there is $p_n \in \{0, \dots, t-1\}$ such that $n\theta + p_n\tau \in [2a\pi, 2b\pi] + 2\pi\mathbb{Z}$. The integer $m_n = n + p_nq$ satisfies $n \leq m_n \leq n + k - 1$ and

$$m_n\theta \in n\theta + p_n\tau + 2\pi\mathbb{Z} \subseteq [2a\pi, 2b\pi] + 2\pi\mathbb{Z}.$$

We conclude that $\sigma(n)$ is well-defined and $n \leq \sigma(n) \leq m_n \leq n + k - 1$, so our algorithm computes $\sigma(n)$ in polynomial time in n . \square

LEMMA 5.3. Let z be a root of unity of order k with $k \notin \{1, 2, 4\}$. Then there exists a sequence of positive integers $\{\sigma(n)\}$ and an integer l such that:

- (i) $\sigma(n)$ can be computed in polynomial time in n ;

- (ii) $n \leq \sigma(n) \leq n + k - 1$ for every positive integer n ;
- (iii) $z^{\sigma(n)} = e^{2\pi il/k}$ for every positive integer n ;
- (iv) $\pi < 2\pi l/k < 3\pi/2$.

PROOF. Let $\theta = \text{Arg}(z)$. Since $\theta \neq 0$, we can write $\theta = 2\pi j/k$ for some integer j coprime with k . We consider two cases.

Case I: $k = 3$. Then either we have $\theta = 2\pi/3$ and we compute $\sigma(n) \in \{n, n + 1, n + 2\}$ with $\sigma(n) \equiv 2 \pmod{3}$, or we have $\theta = 4\pi/3$ and we compute $\sigma(n) \in \{n, n + 1, n + 2\}$ with $\sigma(n) \equiv 1 \pmod{3}$. In any case, we have $\sigma(n)\theta \in 4\pi/3 + 2\pi\mathbb{Z}$, that is, $z^{\sigma(n)} = e^{4\pi i/3}$ for any positive integer n .

Case II: $k \geq 5$. Then there is an integer l such that $k/2 < l < 3k/4$, that is, $2\pi l/k \in (\pi, 3\pi/2)$. The Euclidean algorithm gives two integers t_1, t_2 such that $t_1 j + t_2 k = 1$. We compute $\sigma(n) \in \{n, \dots, n + k - 1\}$ such that $\sigma(n) \equiv t_1 l \pmod{k}$. We can write $\sigma(n) = t_1 l + q_n k$ for some integer q_n . We have

$$\sigma(n)\theta = t_1 l \frac{2\pi j}{k} + q_n 2\pi j = l(1 - t_2 k) \frac{2\pi}{k} + q_n 2\pi j = \frac{2\pi l}{k} + (q_n j - l t_2) 2\pi$$

and, equivalently, $z^{\sigma(n)} = e^{2\pi il/k}$ for every positive integer n . \square

COROLLARY 5.4. *Let z be an algebraic number such that $z \notin \mathbb{R} \cup i\mathbb{R}$. Let $\theta = \text{Arg}(z)$. Then there exists a sequence of positive integers $\{\sigma(n)\}$, a positive integer k and a positive rational number C such that such that:*

- (i) k and C can be computed from z ;
- (ii) $\sigma(n)$ can be computed in polynomial time in n ;
- (iii) $n \leq \sigma(n) \leq n + k - 1$ for every positive integer n ;
- (iv) $\cos(\sigma(n)\theta) \leq -C$ and $\sin(\sigma(n)\theta) \leq -C$ for every positive integer n .

PROOF. We may assume that $|z| = 1$ since, otherwise, we can compute the algebraic number $z/|z|$ and apply the following algorithm to this quantity. We invoke either Lemma 5.2 for $a = 7/12$ and $b = 8/12$ or Lemma 5.3, depending on whether z is a root of unity or not, which can be checked as explained at the beginning of this section. In any case, we find a sequence σ and a positive integer k that satisfy the first three assertions announced in the statement. In the non-root of unity case, we have $\cos(\sigma(n)\theta) \leq \cos(2\pi b) < 0$ and $\sin(\sigma(n)\theta) \leq \sin(2\pi a) < 0$ for every positive integer n . In the root of unity case, the sequences $\cos(\sigma(n)\theta)$ and $\sin(\sigma(n)\theta)$ are negative constants. In any case, we can compute a positive rational number C such that $\cos(\sigma(n)\theta) \leq -C$ and $\sin(\sigma(n)\theta) \leq -C$ for every positive integer n . \square

COROLLARY 5.5. *Let z be an algebraic number with $|z| > 1$. Then for any $x \in \mathbb{Q}$ with $x > 0$, we can compute n such that $\operatorname{Re}(z^n) \geq x$. Moreover, if $z \notin [0, \infty)$, then we can compute m such that $\operatorname{Re}(z^m) \leq -x$.*

PROOF. Let $z = Re^{i\theta}$ for some $\theta \in [0, 2\pi)$ and $R > 1$. We determine if $z/|z| = e^{i\theta}$ is a root of unity or not, and compute its order as explained before. If $e^{i\theta}$ is a root of unity of order k , then $z^k \in (1, \infty)$, so computing n is straightforward. If $e^{i\theta}$ is not a root of unity, then, in view of Lemma 5.3 for $a = 1/12$ and $b = 1/6$, we can compute a sequence σ such that $\sigma(j) \geq j$ and $\sigma(j)\theta \in [\pi/6, \pi/3] + 2\pi\mathbb{Z}$ for every positive integer j . We find that $\operatorname{Re}(z^{\sigma(j)}) \geq R^{\sigma(j)} \cos(\pi/3) \geq R^j/2$. Hence, we can compute j large enough such that $\operatorname{Re}(z^{\sigma(j)}) \geq x$ and we choose $n = \sigma(j)$.

Now let us assume that $z \notin [0, \infty)$. Note that $e^{i\theta} \neq 1$. If $e^{i\theta}$ is a root of unity of order 2 or 4, then the result is trivial. If $\theta \notin \{0, \pi/2, \pi, 3\pi/2\}$, then, by invoking Corollary 5.4, we compute σ and a positive rational number C such that $\sigma(j) \geq j$ and $\cos(\sigma(j)\theta) \leq -C$ for every positive integer j . We find that $\operatorname{Re}(z^{\sigma(j)}) \leq -CR^{\sigma(j)} \leq -CR^j$. Hence, we can compute j large enough such that $\operatorname{Re}(z^{\sigma(j)}) \leq -x$ and we choose $m = \sigma(j)$. \square

5.2. Some shifts for non-real algebraic numbers. In this section we gather some of the shifts that we use in our proofs. Let

q be a real algebraic number with $q \geq 2$ and let $(x, y) \in \mathcal{H}_q$ be a pair of algebraic numbers. We are interested in computing a shift from (x, y) to $(x_1, y_1) \in \mathcal{H}_q$ with $x_1 \notin \mathbb{R}$ and $|x_1| > 1$ whenever possible. The existence of this shift turns out to be closely related to the hardness of approximating $|Z_{\text{Tutte}}(G; q, \gamma)|$ with $\gamma = y - 1$; when we can compute such a shift the approximation problem is $\#P$ -hard, as we will see in Section 6. Recall that one can evaluate the Tutte polynomial of a graph in polynomial time at any of the points in $\{(-i, i), (i, -i), (\omega_3^2, \omega_3), (\omega_3, \omega_3^2)\}$, where $\omega_3 = \exp(2\pi i/3)$ (see Section 6.3). These are the points for which our results fail to construct the desired shift.

The results of this section involve computations that might not run in polynomial time in the length of the representation of the algebraic numbers q, x and y involved. However, when applying these results, the numbers q, x and y are constants and, hence, this will not affect the complexity of our algorithms.

REMARK 5.6. *Let q be a positive real number and let $(x, y) \in \mathcal{H}_q$. From $(x - 1)(y - 1) = q$ it follows that x is real if and only if y is real. Note that $x = 1 + q/(y - 1) = (y + q - 1)/(y - 1)$. As noted in the proof of Lemma 4.7, we find that $|x| = 1$ if and only if $|y + q - 1| = |y - 1|$, that is, y is on the line $1 - q/2 + i\mathbb{R}$. Moreover, $|x| > 1$ if and only if $\text{Re}(y) > 1 - q/2$. Note that when $q \geq 2$ and $\text{Re}(y) > 0$, we have $\text{Re}(y) > 1 - q/2$ and, thus, $|x| > 1$. These observations will be applied several times in this section.*

LEMMA 5.7. *Let q be a real algebraic number with $q \geq 2$. Let x and y be algebraic numbers such that $(x, y) \in \mathcal{H}_q$ and $\text{Arg}(y) \notin \{0, \pi/2, 2\pi/3, \pi, 4\pi/3, 3\pi/2\}$. Then we can compute a theta graph J that (x, y) -implements (x_1, y_1) with $|x_1| > 1$ and $x_1 \notin \mathbb{R}$.*

PROOF. We show how to compute n such that $\text{Re}(y^n) > 0$ and $\text{Im}(y^n) > 0$. For such a n , we let $y_1 = y^n$ and $x_1 = 1 + q/(y_1 - 1)$, so Remark 5.6 ensures that $|x_1| > 1$ and $x_1 \notin \mathbb{R}$. Hence, we can return J as the graph with two vertices and n edges joining them. Since y and $|y|$ are algebraic numbers, we can compute the algebraic number $y/|y|$. We can detect if $y/|y|$ is a root of unity or not as explained in Section 5.1. There are two cases:

- (i) $y/|y|$ is not a root of unity. Then we can apply Lemma 5.2 with $a = 1/12, b = 1/6$ and $z = y^n$ to compute the smallest positive integer n such that $\text{Arg}(y^n) \in [\pi/6, \pi/3]$. Recall that such an integer exists because $\{(y/|y|)^j : j \in \mathbb{N}\}$ is dense in the unit circle. Finally, since $\text{Arg}(y^n) \in [\pi/6, \pi/3]$, we have $\text{Re}(y^n) > 0$ and $\text{Im}(y^n) > 0$.
- (ii) $y/|y|$ is a root of unity of order r with $r \geq 5$. Recall that we can compute r by sequentially computing the powers of $y/|y|$ until we obtain 1. Then we have $(y/|y|)^{r+1} = e^{i2\pi/r}$. Note that the real and imaginary parts of $e^{i2\pi/r} = \cos(2\pi/r) + i \sin(2\pi/r)$ are positive. □

Note that the argument given in Lemma 5.7 strongly uses the fact that $q \geq 2$, that is, $1 - q/2 \leq 0$. A proof of a version of Lemma 5.7 with $q \in (0, 2)$ is unknown to us. Now we deal with the cases $\text{Arg}(y) \in \{\pi/2, 2\pi/3, 4\pi/3, 3\pi/2\}$, where the exemptions $(-i, i), (i, -i), (\omega_3^2, \omega_3), (\omega_3, \omega_3^2)$ arise. Note that $(-i, i), (i, -i) \in \mathcal{H}_2$ and $(\omega_3^2, \omega_3), (\omega_3, \omega_3^2) \in \mathcal{H}_3$. In fact, one can easily check that these are the only pairs (x, y) such that $|y| = 1$ and $q \in \{2, 3\}$.

LEMMA 5.8. *Let q be a real algebraic number with $q \geq 2$. Let x and y be algebraic numbers such that $(x, y) \in \mathcal{H}_q, y \neq 0$ and $\text{Arg}(y) \in \{2\pi/3, 4\pi/3\}$. If $q \neq 3$ or $|y| \neq 1$, then we can compute a series-parallel graph J that (x, y) -implements (x_1, y_1) with $|x_1| > 1$ and $x_1 \notin \mathbb{R}$.*

PROOF. Note that $y/|y|$ is a root of unity of order 3. We have $\text{Re}(y) = |y| \cos(2\pi/3) = -|y|/2 < 0$. Let $x = 1 + q/(y - 1)$. We consider three cases.

Case I: $\text{Re}(y) > 1 - q/2$. Then, by Remark 5.6, $|x| > 1$. We return J as the graph with 2 vertices and one edge joining them.

Case II: $\text{Re}(y) < 1 - q/2$. Then $|x| < 1$. Let $y_n = 1 + q/(x^n - 1)$. An n -stretch gives a shift from (x, y) to (x^n, y_n) . Since $x \notin \mathbb{R}$, there are infinitely many values of n such that $y_n \notin \mathbb{R}$. Note that y_n converges to $1 - q \in (-\infty, -1]$, and the distance between $1 - q$ and the set of complex points $\{z \in \mathbb{C} : \text{Arg}(z) \in \{\pi/2, 2\pi/3, 4\pi/3, 3\pi/2\}\}$ is larger than 0. Hence, we can compute n such that $\text{Arg}(y_n) \notin$

$\{0, \pi/2, 2\pi/3, \pi, 4\pi/3, 3\pi/2\}$. Since $(x^n, y_n) \in \mathcal{H}_q$, the result follows from applying Lemma 5.7 to (x^n, y_n) , the transitivity property of shifts and noticing that the obtained graph is series-parallel.

Case III: $\operatorname{Re}(y) = 1 - q/2$. Note that $q > 2$ because for $q = 2$ we would obtain $\operatorname{Re}(y) = 0$. We distinguish three subcases:

- $|y| > 1$. We compute the smallest positive integer n such that $\operatorname{Arg}(y^n) = 2\pi/3$ and $\operatorname{Re}(y^n) = -|y|^n/2 < 1 - q/2$. The proof is concluded by applying Case II to (x_n, y^n) , where $x_n = 1 + q/(y^n - 1)$, the transitivity property of shifts and noticing that the obtained graph is series-parallel.
- $|y| < 1$. We compute the smallest positive integer n such that $|y|^n < q - 2$ and $\operatorname{Arg}(y^n) = 2\pi/3$. We have $\operatorname{Re}(y^n) > 1 - q/2$ (otherwise by applying $\operatorname{Re}(y^n) = -|y|^n/2$ we would find that $|y|^n \geq q - 2$), so $|x_n| > 1$ for $x_n = 1 + q/(x^n - 1)$. We return J as the graph with two vertices and n edges joining them.
- $|y| = 1$. Then $1 - q/2 = \operatorname{Re}(y) = -|y|/2 = -1/2$. It follows that $q = 3$, but this case ($|y| = 1$ and $q \neq 3$) was excluded in the hypothesis.

This finishes the proof. □

LEMMA 5.9. *Let q be a real algebraic number with $q \geq 2$. Let y be an algebraic number such that $y \neq 0$ and $\operatorname{Arg}(y) \in \{\pi/2, 3\pi/2\}$.*

- (i) *If $q > 2$, then we can compute a theta graph J that (x, y) -implements (x_1, y_1) with $|x_1| > 1$ and $x_1 \notin \mathbb{R}$.*
- (ii) *If $q = 2$ and $|y| \neq 1$, then we can compute a series-parallel graph J that (x, y) -implements (x_2, y_2) with $y_2 \in (-1, 0)$.*

PROOF. The hypotheses $y \neq 0$ and $\operatorname{Arg}(y) \in \{\pi/2, 3\pi/2\}$ are equivalent to $y \neq 0$ and $\operatorname{Re}(y) = 0$. Let $x = 1 + q/(y - 1)$. If $q > 2$, then $1 - q/2 < 0 = \operatorname{Re}(y)$ and $|x| > 1$ as a consequence of Remark 5.6, so we return the graph with two vertices and one edge joining them as J . The second claim (case $q = 2$) has been studied in (Goldberg & Guo 2017, Lemma 3.15), where the graph constructed is a 2-thickening of a k -stretching. □

COROLLARY 5.10. *Let q be a real algebraic number with $q \geq 2$. Let x and y be algebraic numbers such that $(x, y) \in \mathcal{H}_q$, $y \notin (-\infty, -1] \cup [0, \infty)$ and $(x, y) \notin \{(i, -i), (-i, i), (\omega_3, \omega_3^2), (\omega_3^2, \omega_3)\}$, where $\omega_3 = \exp(2\pi i/3)$. Then we can compute $(x_2, y_2) \in \mathcal{H}_q$ and a series-parallel graph J such that $|x_2| < 1$ and J (x, y) -implements (x_2, y_2) .*

PROOF. First, we assume that $y \notin \mathbb{R}$. The case $q = 2$ and $y \in \mathbb{R}i$ is covered in Lemma 5.9, so we assume that $q \neq 2$ or $y \notin \mathbb{R}i$. By applying Lemmas 5.7, 5.8 or 5.9 (depending on the argument of y), we can compute a graph J that (x, y) -implements (x_1, y_1) with $|x_1| > 1$ and $x_1 \notin \mathbb{R}$. We apply Corollary 5.5 with $z = x_1$ in order to compute n such that $\text{Re}(x_1^n) > 1$. A n -stretching of (x_1, y_1) gives a shift from (x_1, y_1) to (\hat{x}, \hat{y}) , where $\hat{x} = x_1^n$ and $\hat{y} = 1 + q/(\hat{x} - 1)$. We have $\text{Re}(\hat{y}) = 1 + q(\text{Re}(\hat{x}) - 1)/|\hat{x} - 1|^2 > 1$, so $|\hat{y}| > 1$. There are two cases:

- $\hat{y} \notin \mathbb{R}$. We apply Corollary 5.5 with $z = \hat{y}$ to compute t such that $\text{Re}(\hat{y}^t) < 1 - q/2 < 0$. We set $y_2 = \hat{y}^t$ and $x_2 = 1 + q/(y_2 - 1)$. By the transitivity property of shifts, we have a shift from (x, y) to (x_2, y_2) . Since $\text{Re}(y_2) < 1 - q/2$, we conclude that $|x_2| < 1$ (Remark 5.6).
- $\hat{y} \in \mathbb{R}$. Hence, we have $\hat{y} \in (1, \infty)$. We can compute a positive integer l such that the norm of $y' = \hat{y}^l y$ is larger than 1. Note that $y' = \hat{y}^l y \notin \mathbb{R}$. A parallel composition yields a shift from (x, y) to (x', y') , where $x' = 1 + q/(y' - 1)$. We compute the graph J by applying the previous case to (x', y') .

Now we deal with the case $y \in (-1, 0)$. A 2-thickening gives us a shift from (x, y) to (a_1, b_1) , where $b_1 = y^2 \in (0, 1)$ and $a_1 = 1 + q/(b_1 - 1) < 1 - q \leq -1$. A 2-stretching gives us a shift from (a_1, b_1) to (a_2, b_2) , where $a_2 = a_1^2 > 1$ and $b_2 = 1 + q/(a_2 - 1) > 1$. We compute a positive integer j such that $b_2^j y < -q$ and, with the help of a j -thickening, construct a shift from (a_2, b_2) to (a_3, b_3) with $b_3 = b_2^j$. The transitivity property of shifts allows us to construct a shift from (x, y) to (a_3, b_3) . To conclude the proof, we apply a parallel composition between the latter shift and the identity shift

from (x, y) to (x, y) , obtaining a shift from (x, y) to (x_2, y_2) with $y_2 = b_3y$. Recall that $b_3y = b_2^jy < -q$, so $q/(y_2 - 1) \in (-1, 0)$ and $x_2 = 1 + q/(y_2 - 1) \in (0, 1)$.

Finally, note that the graphs considered in this proof are series-parallel. \square

5.3. An approximate shift to $(0, 1 - q)$. In Lemma 5.11 and Corollary 5.13 we give a polynomial-time approximate series-parallel shift from (x, y) to $(0, 1 - q)$ under certain conditions.

LEMMA 5.11. *Let $q \in \mathbb{P}_{\mathbb{R}}$ with $q > 0$. Let $(x, y) \in \mathcal{H}_q$ such that $x, y \in \mathbb{P}_{\mathbb{C}}$ and $\text{Re}(y) < 1 - q/2$. Then there is a polynomial-time approximate theta shift from (x, y) to $(0, 1 - q)$.*

PROOF. Let $x = 1 + q/(y - 1)$. In light of Remark 5.6, we have $|x| < 1$. Therefore, the weight $y_j = 1 + q/(x^j - 1)$ implemented by an j -stretch converges to $1 - q$ as $j \rightarrow \infty$. We have

$$(5.12) \quad |q - 1 + y_j| = \left| \frac{qx^j}{x^j - 1} \right| \leq \frac{q|x|^j}{1 - |x|^j} \leq \frac{q|x|^j}{1 - |x|}.$$

We use (5.12) to give a a polynomial-time approximate theta shift from (x, y) to $(0, 1 - q)$. Let n be a positive integer, so the desired accuracy of the quantity in (5.12) is 2^{-n} . We are going to return a path graph with j edges for j large enough. It remains to show how to compute j from n . Since $q, |x| \in \mathbb{P}_{\mathbb{R}}$, we can compute $b, c \in \mathbb{Q}$ such that $q \leq c$ and $0 < b \leq 1 - |x|$ (Lemma 4.5). Hence, $|x| \leq 1 - b < 1$, and it suffices to compute j with $j \geq \log_{1-b}(2^{-n}b/c)$. \square

COROLLARY 5.13. *Let q be a real algebraic number with $q \geq 2$. Let x and y be algebraic numbers such that $(x, y) \in \mathcal{H}_q$, $y \notin (-\infty, -1] \cup [0, \infty)$ and $(x, y) \notin \{(-i, i), (i, -i), (\omega_3^2, \omega_3), (\omega_3, \omega_3^2)\}$, where $\omega_3 = \exp(2\pi i/3)$. Then there is a polynomial-time approximate series-parallel shift from (x, y) to $(0, 1 - q)$.*

PROOF. From Corollary 5.10 we obtain a shift from (x, y) to (x_2, y_2) with $|x_2| < 1$ or, equivalently, $\text{Re}(y_2) < 1 - q/2$. The result follows from applying Lemma 5.11 to (x_2, y_2) and the transitivity property of shifts. \square

5.4. An approximate shift to (x', y') with $y' \in (0, 1)$. In Lemma 5.16 we show that if a sequence z_n of complex numbers has certain properties, then there is $w \in (0, 1) \cap \mathbb{P}_{\mathbb{R}}$ that is the limit of $\prod_{j=1}^n z_j^{e_j}$ for some non-negative integers e_1, e_2, \dots that we can compute. Then we apply this result to a subsequence of $\{y_n\}$, where (x^n, y_n) is the pair implemented by an n -stretch of (x, y) , obtaining a polynomial-time approximate theta shift from (x, y) to some (x', y') with $y' \in (0, 1)$ (Lemma 5.23). First, we need the following elementary results.

LEMMA 5.14. *We have $\sin(x) \leq x \leq \pi \sin(x)/2$ for every $x \in [0, \pi/2]$.*

PROOF. First, we prove that $\sin(x) \leq x$ for every $x \in [0, \pi/2]$. Let $f(x) = x - \sin(x)$. We have $f'(x) = 1 - \cos(x) > 0$ for every $x \in [0, \pi/2]$. Hence, f is strictly increasing in $[0, \pi/2]$. Since $f(0) = 0$, we obtain $x - \sin(x) \geq 0$ for every $x \in [0, \pi/2]$.

Now we prove that $x \leq \pi \sin(x)/2$ for every $x \in [0, \pi/2]$. Let $g(x) = \pi \sin(x)/2 - x$ for every $x \in [0, \pi/2]$. We have $g'(x) = \pi \cos(x)/2 - 1$. Let $y \in [0, \pi/2]$ such that $\cos(y) = 2/\pi$. Note that $g'(x) > 0$ in $[0, y]$, $g(y) = 0$ and $g'(x) < 0$ in $(y, \pi/2]$. Hence, g only reaches a minimum at $x \in \{0, \pi/2\}$. Since $g(0) = g(\pi/2) = 0$, we conclude that $0 \leq \pi \sin(x)/2 - x$ for every $x \in [0, \pi/2]$. \square

LEMMA 5.15. *Let $\{z_n\}$ be a sequence of algebraic complex numbers such that:*

- (i) *we can compute two rational numbers C and R such that $C > 0$, $R \in (0, 1)$ and $|z - z_n| \leq CR^n$ for every positive integer n ;*
- (ii) *we can compute the representation of the algebraic number z_n in polynomial time in n .*

Then $z \in \mathbb{P}_{\mathbb{C}}$, i.e., z is polynomial-time computable.

PROOF.

Let n be an arbitrary positive integer. For $j = \lceil \log_R(2^{-n-1}/C) \rceil$ we have $|z - z_j| \leq 2^{-n-1}$. Note that $j = \Theta(n)$ and hence z_j is

an algebraic number whose representation we can compute in time polynomial in n . So, we can also compute $\hat{z}_j \in \mathbb{Q}[i]$ such that $|z_j - \hat{z}_j| \leq 2^{-n-1}$ in time polynomial in n . Then, we have that

$$|z - \hat{z}_n| \leq |z_n - z| + |z_n - \hat{z}_n| \leq 2^{-n}.$$

Since n was arbitrary, we have that z is polynomial-time computable. \square

LEMMA 5.16. *Let $r, c \in (0, 1) \cap \mathbb{Q}$. Let $\{z_n\}$ be a sequence of algebraic complex numbers with:*

- (i) $|z_n| < 1$ for every positive integer n ;
- (ii) $z_n = 1 - f(n) + ig(n)$ with $f, g: \mathbb{Z}^+ \rightarrow (0, 1)$;
- (iii) $cr^n \leq f(n) \leq r^n/2$ and $cr^n \leq g(n) \leq r^n/2$ for every positive integer n .

Then there is $w \in (0, 1)$ and a bounded sequence of positive integers $\{e_n\}$ such that

$$\left| \prod_{j=1}^n z_j^{e_j} - w \right| \leq \left(\frac{\pi}{2} + \frac{\pi}{c(1-r)} \right) r^n$$

for every positive integer n . Moreover, if the representation of the algebraic number z_n can be computed in polynomial time in n , then $w \in \mathbb{P}_{\mathbb{R}}$ and e_n can be computed in polynomial time in n .

PROOF. We can write $z_n = \rho_n e^{i\theta_n}$ for some $\rho_n \in (0, 1)$ and $\theta_n \in (0, \pi/2)$. Note that $1 - f(n) < \rho_n$. Let $h(n) = 1 - \rho_n$. We obtain

$$(5.17) \quad 0 < h(n) < f(n) \leq r^n/2$$

for every positive integer n . We have

$$\sin(\theta_n) = \frac{\text{Im}(z_n)}{\rho_n} = \frac{g(n)}{1 - h(n)}.$$

In view of Lemma 5.14, we obtain

$$\frac{g(n)}{1 - h(n)} \leq \theta_n \leq \frac{\pi g(n)}{2(1 - h(n))}.$$

Since $0 < h(n) \leq 1/2$ (see (5.17)), it follows that

$$(5.18) \quad g(n) \leq \theta_n \leq \pi g(n).$$

As a consequence, we find that, for any integer n with $n \geq 2$,

$$(5.19) \quad \frac{\theta_{n-1}}{\theta_n} \leq \pi \frac{g(n-1)}{g(n)} \leq \frac{\pi}{2cr},$$

where we used the fact that $cr^n \leq g(n) \leq r^n/2$. The bounds (5.17), (5.18) and (5.19) will be used several times in this proof.

Let $\tau_0 = 0$. We define τ_n and e_n by induction on n . Let e_n be the largest integer such that $\tau_{n-1} + e_n\theta_n \leq 2\pi$ and let $\tau_n = \tau_{n-1} + e_n\theta_n$. By definition, $\{\tau_n\}$ is an increasing sequence that is bounded above by 2π . Moreover, we have $2\pi - \theta_n < \tau_n$, since $\tau_n + \theta_n \leq 2\pi$ contradicts the definition of e_n . That is, we have $0 \leq 2\pi - \tau_n < \theta_n$. We show that e_n is bounded. Note that $e_1 \leq 2\pi/\theta_1 \leq 2\pi/(cr)$, where we used that $cr \leq g(1) \leq \theta_1$ (recall (5.18)). For $n \geq 2$ we have

$$0 \leq e_n = \frac{\tau_n - \tau_{n-1}}{\theta_n} \leq \frac{2\pi - \tau_{n-1}}{\theta_n} < \frac{\theta_{n-1}}{\theta_n} \leq \frac{\pi}{2cr},$$

where we applied (5.19). By combining the latter inequality with the case $n = 1$ we conclude that

$$(5.20) \quad 0 \leq e_n \leq \frac{2\pi}{cr}$$

for every positive integer n .

The sequence $\{e^{i\tau_n}\}$ converges to 1. In fact, we show that it does so exponentially fast. Note that the derivative of e^{it} has constant norm 1. Therefore, e^{it} is a Lipschitz function with constant 1, that is, $|e^{it} - e^{is}| \leq |s - t|$ for every $s, t \in \mathbb{R}$. It follows that

$$(5.21) \quad |1 - e^{i\tau_n}| = |e^{i2\pi} - e^{i\tau_n}| \leq |2\pi - \tau_n| < \theta_n \leq \pi g(n) \leq \frac{\pi}{2} r^n$$

for every positive integer n , where we applied (5.18).

Now we study the sequence $\{x_n\}$ for $x_n = \prod_{j=1}^n \rho_j^{e_j}$. Since $\rho_j \in (0, 1)$, $\{x_n\}$ is decreasing and has a limit $w \in [0, 1)$. We claim that this is the real number in $(0, 1)$ announced in the statement.

First, we prove that $w > 0$. Let $b = \lceil 2\pi/(cr) \rceil$. In view of (5.20), we have

$$x_n \geq \prod_{j=1}^n \rho_j^b = \left(\prod_{j=1}^n (1 - h(j)) \right)^b.$$

Recall that a product of the form $\prod_{j=1}^n (1 - a_n)$ with $a_n \in [0, 1)$ converges to a positive number if and only if $\sum_{j=1}^n a_n$ converges (Stein & Shakarchi 2010, Proposition 3.1). From (5.17) we obtain

$$\sum_{n=1}^{\infty} h(n) \leq \frac{1}{2} \sum_{n=1}^{\infty} r^n = \frac{r}{2(1-r)}$$

and, thus, $\prod_{j=1}^n (1 - h(j))$ converges to a real number L with $L > 0$. We conclude that $w \geq L^b > 0$, as we wanted to prove. Now we show that $\{x_n\}$ converges exponentially fast to w . Note that $x_n = (1 - h(n))^{e_n} x_{n-1}$ and, thus, for $n \geq 2$, we have

$$\begin{aligned} 0 \leq x_{n-1} - x_n &= x_{n-1} (1 - (1 - h(n))^{e_n}) \\ &\leq 1 - (1 - h(n))^{e_n} \leq h(n)e_n \leq \frac{\pi}{cr} r^n, \end{aligned}$$

where we used the fact that $(1 - x)^k \geq 1 - kx$ for every $x \in (0, 1)$ and $k \in \mathbb{Z}^+$, and the bounds on $h(n)$ and e_n (see (5.17) and (5.20)). We obtain

$$|x_{n+q} - x_n| \leq \sum_{j=1}^q |x_{n+j} - x_{n+j-1}| \leq \frac{\pi}{cr} \sum_{j=1}^q r^{n+j} = \frac{\pi(1 - r^q)}{c(1 - r)} r^n$$

for any positive integers n and q . Hence, by making q tend to ∞ we conclude that

$$(5.22) \quad |x_n - w| \leq \frac{\pi}{c(1 - r)} r^n$$

for every positive integer n .

In light of (5.21) and (5.22), we obtain for every positive integer n that

$$\begin{aligned} \left| \prod_{j=1}^n z_j^{e_j} - w \right| &\leq \left| \prod_{j=1}^n z_j^{e_j} - x_n \right| + |x_n - w| \\ &= |x_n| \left| \prod_{j=1}^n e^{ie_j \theta_j} - 1 \right| + |x_n - w| \\ &\leq \left| \prod_{j=1}^n e^{ie_j \theta_j} - 1 \right| + |x_n - w| \\ &= |e^{i\tau_n} - 1| + |x_n - w| \leq \frac{\pi}{2} r^n + \frac{\pi}{c(1-r)} r^n. \end{aligned}$$

Finally, we argue that if the representation of z_n can be computed in polynomial time in n , then e_n can be computed in polynomial time in n and we have $w \in \mathbb{P}_{\mathbb{R}}$. Note that e_1 is the smallest positive integer such that $\text{Arg}(z_1^{e_1}) \in [3\pi/2, 2\pi) \cup \{0\}$ and $\text{Arg}(z_1^{e_1+1}) \in (0, \pi/2]$ and, thus, e_1 can be computed by sequentially applying Lemma 5.1 with intervals $[3\pi/2, 2\pi]$ and $[0, \pi/2]$, with the z of Lemma 5.1 equal to z^k for every positive integer $k \leq e_1 + 1$. This takes constant time since the quantities and objects involved are constant. For $n \geq 2$, let us assume that we have computed e_1, \dots, e_{n-1} , and let $y_{n-1} = \prod_{j=1}^{n-1} z_j^{e_j}$ (so $\tau_{n-1} = \text{Arg}(y_{n-1})$). Since the sequence $\{e_n\}$ is bounded and the length of the representation of z_n is bounded by a polynomial in n , the computation of y_{n-1} takes polynomial time in n . Then e_n is the smallest non-negative integer such that $\text{Arg}(y_{n-1} z_n^{e_n}) \in [3\pi/2, 2\pi) \cup \{0\}$ and $\text{Arg}(y_{n-1} z_n^{e_n+1}) \in (0, \pi/2]$, and we can compute e_n again by sequentially applying Lemma 5.1 with intervals $[3\pi/2, 2\pi]$ and $[0, \pi/2]$, with the z of Lemma 5.1 equal to z^k for every positive integer $k \leq e_n + 1$. There is a bounded number of applications of Lemma 5.1 because e_n is bounded, and each application takes polynomial time in n because the length of the representation of $y_{n-1} z_n^k$ is polynomial in n for any $k \in \{1, 2, \dots, e_n\}$. We conclude that w is the limit of a sequence of algebraic numbers that converges exponentially fast and the representation of its n -th element can be computed in polynomial time in n . As a consequence, we have $w \in \mathbb{P}_{\mathbb{R}}$ by Lemma 5.15. □

LEMMA 5.23. *Let q be a real algebraic number with $q > 0$. Let x and y be algebraic numbers such that $(x, y) \in \mathcal{H}_q$, $y \notin \mathbb{R}$ and $|x| > 1$. Then there is a polynomial-time approximate theta shift from (x, y) to (x', y') for some $(x', y') \in \mathcal{H}_q$ with $y' \in (0, 1) \cap \mathbb{P}_{\mathbb{R}}$.*

PROOF. Since $y \notin \mathbb{R}$, we have $x \notin \mathbb{R}$ (Remark 5.6). Let us write $x = Re^{i\theta}$ for some $R > 1$ and $\theta \in (0, 2\pi)$. An m -stretch gives a shift from (x, y) to (x^m, y_m) with $y_m = (x^m + q - 1)/(x^m - 1)$. By plugging $x = Re^{i\theta}$ in the definition of y_m and multiplying by $R^m e^{-im\theta} - 1$ in the numerator and denominator, we obtain

$$(5.24) \quad y_m = \frac{R^{2m} - q + 1 + (q - 2)R^m \cos(m\theta) - iqR^m \sin(m\theta)}{1 + R^{2m} - 2R^m \cos(m\theta)}.$$

If $\theta \in \{\pi/2, 3\pi/2\}$, that is, $x \in i\mathbb{R}$, then for $m \equiv 2 \pmod{4}$ we have $\cos(m\theta) = -1$, $\sin(m\theta) = 0$ and

$$y_m = \frac{(1 + R^m)^2 - q(1 + R^m)}{(1 + R^m)^2} = \frac{1 + R^m - q}{1 + R^m}.$$

Hence, for $m \equiv 2 \pmod{4}$ such that $1 + R^m > q$, we have $y_m \in (0, 1)$, so we can choose $y' = y_m$ and we are done.

In the rest of the proof we assume that $\theta \notin \{\pi/2, 3\pi/2\}$. We are going to apply Lemma 5.16 to a subsequence of y_m . First, we invoke Corollary 5.4 with $z = x$ in order to find a sequence $\sigma(m)$, a positive integer k and a positive rational C that satisfies:

- $\sigma(m)$ can be computed in polynomial time in m ;
- k and C can be computed in constant time from x ;
- $m \leq \sigma(m) \leq m + k - 1$ for every positive integer m ;
- $\sin(\sigma(m)\theta) \leq -C$ and $\cos(\sigma(m)\theta) \leq -C$ for every positive integer m .

It follows that

$$\operatorname{Re}(x^{\sigma(m)}) = \operatorname{Re}(R^{\sigma(m)} e^{i\sigma(m)\theta}) \leq -CR^{\sigma(m)} \leq -CR^m.$$

Since $R > 1$, we can compute a positive integer m_1 such that for $m \geq m_1$ we have $\operatorname{Re}(x^{\sigma(m)}) < 1 - q/2$ and, thus, $|y_{\sigma(m)}| < 1$ (recall that $y_m = (x^m + q - 1) / (x^m - 1)$ and Remark 5.6). Let

$$a_m = 1 - \operatorname{Re}(y_m) = \frac{q - qR^m \cos(m\theta)}{1 + R^{2m} - 2R^m \cos(m\theta)};$$

$$b_m = \operatorname{Im}(y_m) = \frac{-qR^m \sin(m\theta)}{1 + R^{2m} - 2R^m \cos(m\theta)};$$

that is, $y_m = 1 - a_m + ib_m$. We have

$$R^{2\sigma(m)} \leq 1 + R^{2\sigma(m)} - 2R^{\sigma(m)} \cos(\sigma(m)\theta) \leq 4R^{2\sigma(m)}.$$

Therefore, we obtain

$$(5.25) \quad \begin{aligned} \frac{qC}{4} R^{-\sigma(m)} &\leq a_{\sigma(m)} \leq 2qR^{-\sigma(m)}, \\ \frac{qC}{4} R^{-\sigma(m)} &\leq b_{\sigma(m)} \leq qR^{-\sigma(m)}. \end{aligned}$$

We compute a positive integer m_2 such that $m_2 \geq \log_R(4q)$ and $m_2 \geq m_1$. We also compute a rational number c with $c \in (0, qCR^{-m_2-k-1}/4)$. Note that computing these quantities takes constant time. Let $f(m) = a_{\sigma(m+m_2)}$ and $g(m) = b_{\sigma(m+m_2)}$. In view of (5.25) and the inequalities $R^{-m-k+1} \leq R^{-\sigma(m)} \leq R^{-m}$, we find that

$$(5.26) \quad cR^{-m} \leq f(m) \leq \frac{1}{2}R^{-m}, \quad cR^{-m} \leq g(m) \leq \frac{1}{2}R^{-m},$$

for any positive integer m . The sequence $\{z_m\} = \{y_{\sigma(m+m_2)}\}$ satisfies

- $|z_m| < 1$ for every positive integer m ;
- $z_m = 1 - f(m) + ig(m)$ with $f, g: \mathbb{Z}^+ \rightarrow (0, 1)$;
- f and g are bounded as in (5.26).
- z_m is an algebraic number whose representation can be computed in polynomial time in m . This is due to the facts that $z_m = (x^{\sigma(m+m_2)} + q - 1) / (x^{\sigma(m+m_2)} - 1)$, $\sigma(m)$ can be computed in polynomial time in m , and $\sigma(m) = O(m)$.

Therefore, we can apply Lemma 5.16 to the sequence $\{z_m\}$ for $r = R^{-1}$. There are $y' \in (0, 1) \cap \mathbb{P}_{\mathbb{R}}$ and a bounded sequence of positive integers $\{e_m\}$ such that

$$\left| \prod_{j=1}^m z_j^{e_j} - y' \right| \leq \left(\frac{\pi}{2} + \frac{\pi}{c(1 - 1/R)} \right) R^{-m}$$

for every positive integer m . Moreover, we can compute e_m in polynomial time in m . Let $M = \pi/2 + \pi/(c(1 - 1/R))$. For any positive integer n , we can compute an integer m with $m \geq \log_{1/R}(2^{-n}/M)$ and $m = \Theta(n)$ in polynomial time in n . We obtain

$$\left| \prod_{j=1}^m z_j^{e_j} - y' \right| \leq 2^{-n}.$$

This gives the following polynomial-time approximate theta shift from (x, y) to (x', y') , where $x' = 1 + q/(y' - 1)$. For each positive integer n we return a graph J_n that is the parallel composition of the path graphs that are used to implement the weights $y_{\sigma(j+m_2)}$, each one repeated e_j times, for $j \in \{1, \dots, m\}$. The graph J_n (x, y) -implements $(\hat{x}, \hat{y}) \in \mathcal{H}_q$ for $\hat{y} = \prod_{j=1}^m z_j^{e_j} = \prod_{j=1}^m y_{\sigma(j+m_2)}^{e_j}$. \square

LEMMA 2.2. *Let q be a real algebraic number with $q \geq 2$. Let x and y be algebraic numbers such that $(x, y) \in \mathcal{H}_q$, $y \in (-1, 0) \cup (\mathbb{C} \setminus \mathbb{R})$ and $(x, y) \notin \{(i, -i), (-i, i), (\omega_3, \omega_3^2), (\omega_3^2, \omega_3)\}$, where $\omega_3 = \exp(2\pi i/3)$. Then there is a polynomial-time approximate series-parallel shift from (x, y) to (x', y') for some $(x', y') \in \mathcal{H}_q$ with $x', y' \in \mathbb{P}_{\mathbb{R}}$ and $y' \in (0, 1)$.*

PROOF. If $y \in (-1, 0)$, then a 2-thickening of (x, y) gives the result. Hence, let us assume that $y \notin (-1, 0)$ in the rest of the proof. There are two cases:

- $q \neq 2$ or $y \notin i\mathbb{R}$. We apply either Lemma 5.7, Lemma 5.8 or Lemma 5.9, depending on $\text{Arg}(y)$, to find a shift from (x, y) to $(x_1, y_1) \in \mathcal{H}_q$ with $y_1 \notin \mathbb{R}$ and $|x_1| > 1$. The graph of this shift is series-parallel. Then we apply Lemma 5.23 to obtain a polynomial-time approximate theta shift from (x_1, y_1) to some $(x', y') \in \mathcal{H}_q$ with $y' \in (0, 1) \cap \mathbb{P}_{\mathbb{R}}$. The result follows from the transitivity property of shifts.

- $q = 2$ and $y \in i\mathbb{R}$. Since $y \neq \pm i$, Lemma 5.9 gives a shift from (x, y) to (x', y') for some $(x', y') \in \mathcal{H}_q$ with $y' \in (-1, 0)$. A 2-thickening of (x', y') gives the result.

The fact that $x' \in \mathbb{P}_{\mathbb{R}}$ follows from $x' = 1 + q/(y' - 1)$ and $y' \in \mathbb{P}_{\mathbb{R}}$. □

5.5. Approximate shifts for polynomial-time computable real numbers. In this subsection we show how we can obtain a polynomial-time approximate shift from (x, y) to (x', y') for any $(x, y), (x', y') \in \mathcal{H}_q$ when $q \geq 2$, $y \in (0, 1) \cap \mathbb{P}_{\mathbb{R}}$ and y' is a positive real algebraic number (Lemma 2.3). This extends a particular case of Lemma 4.1 to polynomial-time computable numbers. Our proof follows the same approach as that of (Goldberg & Jerrum 2019, Lemma 22) but we have to overcome some difficulties that arise when working with the class of numbers $\mathbb{P}_{\mathbb{R}}$. These difficulties will become apparent in the proof, but the reader that is familiar with the literature might want to skip the proof. Then we combine this result and Lemma 2.2 to prove Theorem 2.1, the main result of Section 5.

LEMMA 2.3. *Let $q, x, y \in \mathbb{P}_{\mathbb{R}}$ such that $q > 0$, $(x, y) \in \mathcal{H}_q$, y is positive and $1 - q/2 < y < 1$. There is a polynomial-time algorithm that takes as an input:*

- two positive integers k and n , in unary;
- a real algebraic number $w \in [y^k, 1]$.

The algorithm produces a theta graph J that (x, y) -implements (\hat{x}, \hat{y}) such that $|\hat{y} - w| \leq 2^{-n}$. The size of J is at most a polynomial in k and n , independently of w .

PROOF. If $w = 1$, then J is the graph with vertices s and t and no edges. In the rest of the proof we focus on the case $w \in (0, 1)$.

Recall that $x = 1 + q/(y - 1)$. Since $q > 0$ and $y \in (1 - q/2, 1)$, we find that $x \in (-\infty, -1)$. Let $y_j = 1 + q/(x^j - 1)$. A j -stretch gives a shift from (x, y) to (x^j, y_j) . If j is even, then $x^j > 1$ and $y_j > 1$. Otherwise, $x^j < -1$ and $y_j \in (1 - q/2, 1)$. Moreover, the sequences $\{y_{2j+1}\}$ and $\{y_{2j}\}$ are increasing and decreasing, respectively, and

$|y_j - 1|$ can be made exponentially small as a function of j . We use these properties of y_j to show that we can compute $y_{(e_1, \dots, e_m)} = \prod_{j=1}^m y_j^{e_j}$ such that $|y_{(e_1, \dots, e_m)} - w| \leq 2^{-n}$. Let J be the parallel composition of the path graphs that (x, y) -implement (x^j, y_j) , each one repeated e_j times, for $j \in \{1, \dots, m\}$. Then J is a theta graph and, in view of (3.4), we have $w(G; q, y - 1) = y_{(e_1, \dots, e_m)} - 1$, that is, J (x, y) -implements $(\hat{x}, \hat{y}) \in \mathcal{H}_q$ with $\hat{y} = y_{(e_1, \dots, e_m)}$. The graph J is the theta graph output by our algorithm.

First, we define a sequence $\{d_j\}$ that will be related to the exponents e_1, \dots, e_m . Since $q, x \in \mathbb{P}_{\mathbb{R}}$, we can compute rational upper bounds of q and x (Lemma 4.5) and, with the help of these bounds, a positive integer j_0 such that $j_0 > \log_{|x|} q$. Let $d_j = 0$ for every positive integer j with $j < j_0$ and let $d_j = 0$ for every even positive integer j . For j odd with $j \geq j_0$ we define d_j recursively as the largest non-negative integer such that $y_{(d_1, \dots, d_j)} \geq w$. The integer d_j is well-defined because $0 < y_j < 1$ when j is odd and $j \geq j_0$. An equivalent definition is that $\{d_j\}$ satisfies

$$(5.27) \quad y_j < w/y_{(d_1, \dots, d_j)} \leq 1$$

for every odd integer j with $j \geq j_0$. A similar sequence $\{d_j\}$ is used in the proofs of (Goldberg & Jerrum 2019, Lemma 22) and (Goldberg & Jerrum 2012b, Lemma 3.28). For any odd integer m with $m \geq \log_{|x|} (q2^n - 1)$ we have $0 \leq 1 - y_m \leq 2^{-n}$ and, in light of (5.27),

$$0 \leq 1 - w/y_{(d_1, \dots, d_m)} \leq 1 - y_m \leq 2^{-n}.$$

Since $1 \geq y_{(d_1, \dots, d_m)} \geq w$, it follows that

$$(5.28) \quad |w - y_{(d_1, \dots, d_m)}| \leq y_{(d_1, \dots, d_m)} 2^{-n} \leq 2^{-n}.$$

Now we study the size of the integers d_1, \dots, d_m . We bound d_j using an argument given in (Goldberg & Jerrum 2012b, Lemma 3.28). First, we show that d_{j_0} is $O(k)$. We have $y_{j_0}^{d_{j_0}} \geq w$. We obtain

$$d_{j_0} \leq \log_{y_{j_0}}(w) = \log_y(w) \log_{y_{j_0}}(y).$$

Since $w \in [y^k, 1)$ and $\log_{y_{j_0}}(y) > 0$, it follows that $0 < \log_y(w) \leq k$ and $d_{j_0} \in O(k)$. Now we show that d_j is bounded for any $j > j_0$.

By applying (5.27) twice, we find that

$$y_{j-2} < w/y_{(d_1, \dots, d_{j-2})} = wy_{j-2}^{d_j}/y_{(d_1, \dots, d_j)} \leq y_j^{d_j}$$

for every odd integer j with $j > j_0$. It follows that $d_j \leq \log(y_{j-2})/\log(y_j)$ (here and in the rest of this paper \log is taken in base e). For every $x \in (1, 5/4)$, we have $3(x - 1)/4 \leq \log(x) \leq x - 1$. Hence, we obtain

$$\begin{aligned} d_j &\leq \frac{\log(y_{j-2})}{\log(y_j)} = \frac{\log(1/y_{j-2})}{\log(1/y_j)} \leq \frac{4}{3} \frac{1/y_{j-2} - 1}{1/y_j - 1} \\ &= \frac{4y_j}{3y_{j-2}} \frac{1 - y_{j-2}}{1 - y_j} = \frac{4y_j}{3y_{j-2}} \frac{|x|^j + 1}{|x|^{j-2} + 1} \leq \frac{4y_j}{3y_{j-2}} |x|^2, \end{aligned}$$

where the last inequality is a consequence of $|x|^2 (|x|^{j-2} + 1) \geq |x|^j + 1$. Since y_j/y_{j-2} converges to 1 and, thus, is bounded, it follows that d_j is bounded. We conclude that $\sum_{j=1}^m d_j = O(k + m)$.

Let us assume that we can compute d_1, \dots, d_m for $m = \lceil 1 + \log_{|x|}(q2^n - 1) \rceil$. In light of (5.28), we can return J as the theta graph that implements the weight $w(J; q, y - 1) = y_{(d_1, \dots, d_m)} - 1$. Since $\sum_{j=1}^m d_j = O(k + m)$ and $m = \Theta(n)$, the size of J is at most a polynomial in k and n .

If y were algebraic, computing d_1, \dots, d_m in polynomial time would be straightforward from their definition because we can efficiently check inequalities between real algebraic numbers as explained in Section 5.1. This is the approach followed in (Goldberg & Jerrum 2019, Lemma 22). However, we only know that $y \in \mathbb{P}_{\mathbb{R}}$ and, thus, it is not clear how to efficiently determine whether $y_{(d_1, \dots, d_{m-1}, d)} \geq w$ or not for any given d . In the rest of this proof, we show how to overcome this difficulty.

Let n be a positive integer, so 2^{-n} is the desired accuracy for our algorithm. Let us assume that we have computed the integers d_1, \dots, d_{j-1} and we want to compute d_j for an odd positive integer j with $j \geq j_0$. We are going to sequentially try all the values $d = 0, 1, \dots$ until we have

$$y_j < \frac{w}{y_{(d_1, \dots, d_{j-1}, d)}} \leq 1,$$

in which case we have found the value d_j (see (5.27)). Recall that $y_{(d_1, \dots, d_{j-1}, d)} - 1$ is the weight implemented by a theta graph J_d whose size is bounded by a polynomial in k and j . Therefore, by applying Lemma 4.7 with $G = J_d$ and $\gamma = y - 1$, we can compute in polynomial time in n and the size of J_d , a positive integer $f(n + 2, J_d)$ with $f(n + 2, J_d) = n + \Theta(\text{size}(J_d))$ such that if $|\gamma - \hat{\gamma}| \leq 2^{-f(n+2, J_d)}$, then $|w(G; q, \gamma) - w(G; q, \hat{\gamma})| \leq 2^{-n-2}$. Since $y \in \mathbb{P}_{\mathbb{R}}$, we can compute a rational number $\hat{\gamma}$ such that $|\gamma - \hat{\gamma}| \leq 2^{-f(n+2, J_d)}$ in polynomial time in n and the size of J_d . Let $\hat{y}_{(d_1, \dots, d_{j-1}, d)} = w(G; q, \hat{\gamma}) + 1$. Then we have computed in polynomial time in k, j and n a rational number $\hat{y}_{(d_1, \dots, d_{j-1}, d)}$ such that

$$|\hat{y}_{(d_1, \dots, d_{j-1}, d)} - y_{(d_1, \dots, d_{j-1}, d)}| \leq 2^{-n-2}.$$

Because $|\hat{y}_{(d_1, \dots, d_{j-1}, d)} - w|$ is a real algebraic number, we can check if the following inequality holds in polynomial time,

$$(5.29) \quad |\hat{y}_{(d_1, \dots, d_{j-1}, d)} - w| \leq 2^{-n-1}.$$

If that is the case, then

$$\begin{aligned} |y_{(d_1, \dots, d_{j-1}, d)} - w| &\leq |y_{(d_1, \dots, d_{j-1}, d)} - \hat{y}_{(d_1, \dots, d_{j-1}, d)}| + |\hat{y}_{(d_1, \dots, d_{j-1}, d)} - w| \\ &\leq 3 \cdot 2^{-n} / 4 < 2^{-n}, \end{aligned}$$

so $y_{(d_1, \dots, d_{j-1}, d)}$ is a good enough approximation of w and we can stop the algorithm (even though we have not computed d_j). Otherwise, we claim that $\hat{y}_{(d_1, \dots, d_{j-1}, d)} \geq w$ if and only if $y_{(d_1, \dots, d_{j-1}, d)} \geq w$. If $\hat{y}_{(d_1, \dots, d_{j-1}, d)} \geq w$ and $w > y_{(d_1, \dots, d_{j-1}, d)}$, then

$$|\hat{y}_{(d_1, \dots, d_{j-1}, d)} - w| \leq |\hat{y}_{(d_1, \dots, d_{j-1}, d)} - y_{(d_1, \dots, d_{j-1}, d)}| \leq 2^{-n-2}$$

and (5.29) holds, a contradiction. The same reasoning applies when $\hat{y}_{(d_1, \dots, d_{j-1}, d)} < w$ and $w \leq y_{(d_1, \dots, d_{j-1}, d)}$. Hence, we can check whether $y_{(d_1, \dots, d_{j-1}, d)} \geq w$ or not by checking $\hat{y}_{(d_1, \dots, d_{j-1}, d)} \geq w$, provided that (5.29) does not hold. This gives a procedure to compute d_j for odd j with $j \geq j_0$:

1. Set $d = 0$.
2. If (5.29) holds, then return d . We have failed to compute d_j , but we have succeeded in finding an approximation of w .

3. If $\hat{y}_{(d_1, \dots, d_{j-1}, d+1)} \geq w$, then increase d by 1 and go to step 2. Else, we have $d_j = d$.

We repeat this procedure to compute d_j sequentially until (5.29) holds, in which case we stop and return the graph J associated to $y_{(d_1, \dots, d_{j-1}, d)}$.

It remains to show that this procedure always halts and runs in polynomial time. In light of (5.28), we find that, for odd $m \geq \log_{|x|}(q2^{n+2} - 1)$,

$$|\hat{y}_{(d_1, \dots, d_m)} - w| \leq |\hat{y}_{(d_1, \dots, d_m)} - y_{(d_1, \dots, d_m)}| + |y_{(d_1, \dots, d_m)} - w| \leq 2^{-n-1},$$

that is, (5.29) holds. Therefore, our procedure that computes non-negative integers d_1, \dots, d_{m-1}, d with $|y_{(d_1, \dots, d_{m-1}, d)} - w| \leq 2^{-n}$ halts for $m = O(n)$. As a consequence, the whole procedure runs in polynomial time in k and n . \square

The proof of Lemma 2.3 can be adapted to to the case $w \in (1, y^{-k}]$. The main difference is that this time we work with the decreasing sequence $\{y_{2j}\}$. We set $d_j = 0$ for odd j and, for even j , we define d_j recursively as the largest non-negative integer such that $y_{(d_1, \dots, d_j)} \leq w$. The details of the proof are left to the reader. When studying the hardness of approximating $Z_{\text{Tutte}}(G; q, \gamma)$ we only need the version stated in Lemma 2.3.

THEOREM 2.1. *Let $q \geq 2$ be a real algebraic number. Let x and y be algebraic numbers such that $(x, y) \in \mathcal{H}_q$, $y \in (-1, 0) \cup (\mathbb{C} \setminus \mathbb{R})$ and $(x, y) \notin \{(i, -i), (-i, i), (\omega_3, \omega_3^2), (\omega_3^2, \omega_3)\}$, where $\omega_3 = \exp(2\pi i/3)$. Then, for any pair of real algebraic numbers $(x', y') \in \mathcal{H}_q$ there is a polynomial-time approximate series-parallel shift from (x, y) to (x', y') .*

PROOF. First, let us assume that $y' \in (0, 1]$. By Lemma 2.2, there is a polynomial-time approximate series-parallel shift from (x, y) to (\tilde{x}, \tilde{y}) for some $(\tilde{x}, \tilde{y}) \in \mathcal{H}_q$ with $\tilde{x}, \tilde{y} \in \mathbb{P}_{\mathbb{R}}$ and $\tilde{y} \in (0, 1)$. Since $q \geq 2$, we have $1 - q/2 \leq 0$ and $\tilde{y} \in (1 - q/2, 1)$. Hence, Lemma 2.3 gives us a polynomial-time approximate theta shift from (\tilde{x}, \tilde{y}) to (x', y') . Since $\tilde{y} \notin 1 - q/2 + i\mathbb{R} = 1 - q/2 + iq\mathbb{R}$ and $\tilde{x}, \tilde{y} \in \mathbb{P}_{\mathbb{R}}$, the transitivity property of polynomial-time approximate shifts, Lemma 4.8, for $(x_1, y_1) = (x, y)$, $(x_2, y_2) = (\tilde{x}, \tilde{y})$ and

$(x_3, y_3) = (x', y')$ gives us a polynomial-time approximate series-parallel shift from (x, y) to (x', y') .

Now we treat the case $y' = 0$. As a consequence of what we have just shown in the paragraph above, there is a polynomial-time approximate series-parallel shift from (x, y) to $(1 - 2q, 1/2) \in \mathcal{H}_q$. An n -thickening gives a shift from $(1 - 2q, 1/2)$ to $(x_n, 2^{-n})$, where $x_n = 1 + q/(2^{-n} - 1)$, so there is also a polynomial-time approximate theta shift from $(1 - 2q, 1/2)$ to $(1 - q, 0)$. We conclude that there is a polynomial-time approximate series-parallel shift from (x, y) to $(1 - q, 0)$ by applying Lemma 4.8 with $(x_1, y_1) = (x, y)$, $(x_2, y_2) = (1 - 2q, 1/2)$ and $(x_3, y_3) = (1 - q, 0)$. Note that we can indeed apply Lemma 4.8 because $1 - 2q, 1/2 \in \mathbb{P}_{\mathbb{R}}$ and $1/2 \notin 1 - q/2 + iq\mathbb{R}$.

Now we deal with the case $y' > 1$. We use again the polynomial-time approximate series-parallel shift from (x, y) to $(x_1, y_1) = (1 - 2q, 1/2) \in \mathcal{H}_q$. We use a 2-stretch to $(1 - 2q, 1/2)$ -implement (x_2, y_2) with $x_2 = (1 - 2q)^2 \geq 9$ and $y_2 = 1 + q/(x_2 - 1) > 1$. Hence, there is a polynomial-time approximate series-parallel shift from (x, y) to (x_2, y_2) . Since x_2 and y_2 are real algebraic numbers with $y_2 > 1$ and $(x_2 - 1)(y_2 - 1) = q > 0$, in view of Lemma 4.1, we have a polynomial-time approximate theta shift from (x_2, y_2) to (x', y') . Note that $y_2 \notin \{1\} \cup (1 - q/2 + iq\mathbb{R})$. Hence, we can apply the transitivity property shown in Lemma 4.8 with $(x_1, y_1) = (x, y)$, $(x_2, y_2) = (x_2, y_2)$ and $(x_3, y_3) = (x', y')$ and find a polynomial-time approximate series-parallel shift from (x, y) to (x', y') .

Finally, we study the case $y' < 0$. In light of Corollary 5.13, there is a polynomial-time approximate series-parallel shift from (x, y) to $(0, 1 - q)$. Note that $1 - q \leq -1$. In this proof we have already shown that there is a polynomial-time approximate series-parallel shift from (x, y) to $(x_3, y_3) \in \mathcal{H}_q$ for $y_3 = y'/(1 - q) > 0$. Since $y' = y_3(1 - q)$, by Lemma 4.4 with parameters $(x_1, y_1) = (x, y)$, $(x_2, y_2) = (0, 1 - q)$ and $(x_3, y_3) = (x_3, y_3)$, we conclude that there is a polynomial-time approximate series-parallel shift from (x, y) to (x', y') . \square

6. Hardness results

We begin with obtaining lower bounds on $Z_{\text{Tutte}}(G; q, \gamma)$ for algebraic numbers q and γ . In Section 6.2, we review the algorithm

of Kannan *et al.* (1988) for computing algebraic representations, and in Section 6.3 the exact #P-hardness results that we will use. The rest of the section gives various ingredients that are needed in the reduction, which are put together in Section 6.8 where we prove all of our main theorems.

6.1. Properties of $Z_{\text{Tutte}}(G; q, \gamma)$ for algebraic numbers q and γ . In this section we give a lower bound on $Z_{\text{Tutte}}(G; q, \gamma)$ and study the degree and height of $Z_{\text{Tutte}}(G; q, \gamma)$ when q and γ are algebraic numbers. First, we have to introduce some concepts and results from algebraic number theory. The *degree* of an algebraic number γ is the degree of its minimal polynomial p , and we denote it by $d(\gamma)$. Recall that the *degree of a field extension* F/K is the dimension of F as a K -vector space, and it is denoted by $[F : K]$. It is well known that if γ is algebraic, then $[K(\gamma) : K]$ is the degree of the minimal polynomial of γ over K (Stewart 2004, Chapter 5). In particular, we have $[\mathbb{Q}(\gamma) : \mathbb{Q}] = d(\gamma)$. The *usual height* of a polynomial $f \in \mathbb{Z}[x_1, \dots, x_m]$ is the largest value among the absolute values of its coefficients and it is denoted by $H(f)$. The *usual height* of γ is $H(\gamma) = H(p)$. One can find several (non-equivalent) definitions of the height of an algebraic number in the literature. Another one of these definitions is the absolute logarithmic height. First, we have to introduce the *Mahler's measure* of a polynomial $f \in \mathbb{Z}[x]$, which is given by

$$M(f) = |a_d| \prod_{i=1}^d \max\{1, |\alpha_i|\},$$

where $f(x) = \sum_{j=0}^d a_j x^j$, $a_d \neq 0$, and $\alpha_1, \dots, \alpha_d$ are the roots of f . It is well known that

$$(6.1) \quad 2^{-d(f)} H(f) \leq M(f) \leq H(f) \sqrt{d(f) + 1},$$

see (Waldschmidt 2000, Lemma 3.11). The *Mahler's measure* of an algebraic number γ with minimal polynomial p is $M(\gamma) = M(p)$. The *absolute logarithmic height* of γ is $h(\gamma) = d(\gamma)^{-1} \log M(\gamma)$. Note that $h(\gamma) \geq 0$ because $M(\gamma) \geq 1$. Now we can state a lower bound for the evaluation of a polynomial at algebraic numbers.

LEMMA 6.2 (Waldschmidt 2000, Section 3.5.4).

Let $f \in \mathbb{Z}[x_1, \dots, x_m]$ be a polynomial in m variables and let $\gamma_1, \dots, \gamma_m$ be algebraic numbers. If $f(\gamma_1, \dots, \gamma_m) \neq 0$, then we have

$$|f(\gamma_1, \dots, \gamma_m)| \geq e^{-cT},$$

where $T = \deg f + \log H(f)$, $c = D(2 + h(\gamma_1) + \dots + h(\gamma_m))$ and $D = [\mathbb{Q}(\gamma_1, \dots, \gamma_m) : \mathbb{Q}]$.

COROLLARY 6.3. Let q and γ be algebraic numbers. We can compute a rational number $C_{q,\gamma}$ with $C_{q,\gamma} > 1$ such that, for any graph G , either $Z_{\text{Tutte}}(G; q, \gamma) = 0$ or $|Z_{\text{Tutte}}(G; q, \gamma)| \geq C_{q,\gamma}^{-\text{size}(G)}$.

PROOF. Recall that we represent an algebraic number γ as its minimal polynomial p and a rectangle of the complex plane where γ is the only root of p . Let $G = (V, E)$ be a graph. Let $n = |V|$ and $m = |E|$. Let us assume that $Z_{\text{Tutte}}(G; q, \gamma) \neq 0$. We can apply Lemma 6.2 for $f(q, \gamma) = Z_{\text{Tutte}}(G; q, \gamma)$ to find that $|Z_{\text{Tutte}}(G; q, \gamma)| \geq e^{-cT}$, where c and T are as in Lemma 6.2. We have $c = D(2 + h(q) + h(\gamma))$ and $D = [\mathbb{Q}(q, \gamma) : \mathbb{Q}]$, so $c \geq 2$. Note that, by definition of Z_{Tutte} , we have $H(f) \leq 2^m$ and $\deg f \leq n + m$. Hence, we find that $|Z_{\text{Tutte}}(G; q, \gamma)| \geq e^{-2c \text{size}(G)}$. It remains to compute a rational number $C_{q,\gamma}$ in (e^{2c}, ∞) to conclude the result. From $D = [\mathbb{Q}(q, \gamma) : \mathbb{Q}]$, we can compute D exactly. Moreover, we can apply (6.1) to upper bound $h(q)$ and $h(\gamma)$ in terms of the usual heights and degrees of q and γ , and compute an appropriate rational number $C_{q,\gamma}$ with the help of these upper bounds. \square

The case $q = 2$ (Ising model) of Corollary 6.3 has previously been shown in (Goldberg & Guo 2017, Lemma 6.4). Note that the approach followed in this section can be applied to obtain lower bounds for other partition functions.

In the rest of this section we upper bound the degree and the usual height of the algebraic number $Z_{\text{Tutte}}(G; q, \gamma)$ in terms of the usual heights and degrees of q and γ . We will make use of these bounds in the proof of Lemma 6.13.

Let q and γ be two algebraic numbers. By the tower law, we have $[\mathbb{Q}(q, \gamma) : \mathbb{Q}] = [\mathbb{Q}(q, \gamma) : \mathbb{Q}(q)][\mathbb{Q}(q) : \mathbb{Q}] \leq d(q)d(\gamma)$, where

we used that the degree of the minimal polynomial of γ over $\mathbb{Q}(q)$ is bounded by $d(\gamma)$. Since $Z_{\text{Tutte}}(G; q, \gamma)$ is in $\mathbb{Q}(q, \gamma)$, it follows that its degree is bounded by $d(q)d(\gamma)$.

Now we argue how we can bound the usual height of $Z_{\text{Tutte}}(G; q, \gamma)$. A well-known property of the absolute logarithmic height is that $h(\alpha\beta) \leq h(\alpha) + h(\beta)$, $h(\alpha + \beta) \leq \log 2 + h(\alpha) + h(\beta)$ and $h(1/\alpha) = h(\alpha)$ (Waldschmidt 2000, Property 3.3). Moreover, if n is an integer, then $h(n) = \log |n|$. A more general property is the following one.

LEMMA 6.4 (Waldschmidt 2000, Lemma 3.7).

Let $f \in \mathbb{Z}[x_1, \dots, x_t]$ be a non-zero polynomial in t variables with integer coefficients. Let $\gamma_1, \dots, \gamma_t$ be algebraic numbers. Then

$$h(f(\gamma_1, \dots, \gamma_t)) \leq \log L(f) + \sum_{j=1}^t \deg_{x_j}(f)h(\gamma_j),$$

where $L(f)$ is the sum of the absolute values of the coefficients of f and $\deg_{x_j}(f)$ is the degree of f with respect to the j -th variable.

COROLLARY 6.5. Let q and γ be algebraic numbers. Then, for any graph $G = (V, E)$ with $n = |V|$ and $m = |E|$, we have

$$\begin{aligned} d\left(\frac{Z_{s|t}(G; q, \gamma)}{Z_{st}(G; q, \gamma)}\right) &\leq d(q) d(\gamma), \\ H\left(\frac{Z_{s|t}(G; q, \gamma)}{Z_{st}(G; q, \gamma)}\right) &\leq (2^{m+1/2} e^{nh(q)+mh(\gamma)})^{2d(q)d(\gamma)}. \end{aligned}$$

PROOF. The degree bound on $Z_{s|t}(G; q, \gamma)/Z_{st}(G; q, \gamma)$ follows from the fact that that it is in $\mathbb{Q}(q, \gamma)$. For its absolute logarithmic height, we have

$$h\left(\frac{Z_{s|t}(G; q, \gamma)}{Z_{st}(G; q, \gamma)}\right) \leq h(Z_{st}(G; q, \gamma)) + h(Z_{s|t}(G; q, \gamma)).$$

Note that $L(Z_{st}(G; q, \gamma)) + L(Z_{s|t}(G; q, \gamma)) = 2^m$. As a consequence of Lemma 6.4, we find that

$$h(Z_{st}(G; q, \gamma)) + h(Z_{s|t}(G; q, \gamma)) \leq 2(m \log 2 + nh(q) + mh(\gamma)).$$

Recall that $M(\alpha) = \exp(d(\alpha)h(\alpha))$. Thus, the bounds on the Mahler's measure (6.1) yield the inequality $H(\alpha) \leq (2 \exp(h(\alpha)))^{d(\alpha)}$. We conclude that

$$\begin{aligned} H\left(\frac{Z_{st}(G; q, \gamma)}{Z_{st}(G; q, \gamma)}\right) &\leq (2e^{2(m \log 2 + nh(q) + mh(\gamma))})^{d(q)d(\gamma)} \\ &= (2^{m+1/2} e^{nh(q) + mh(\gamma)})^{2d(q)d(\gamma)}. \quad \square \end{aligned}$$

One could derive analogous bounds to those of Corollary 6.5 for the algebraic number $Z_{\text{Tutte}}(G; q, \gamma)$ by applying the same argument.

6.2. Computing representations of algebraic numbers via approximations. Kannan, Lenstra & Lovász (1988) showed how to reconstruct the minimal polynomial of an algebraic number from a certain number of digits of its binary expansion, and we will use their algorithm as a black-box in our reduction of Section 6.5, in the following form.

LEMMA 6.6 (Kannan *et al.* 1988, Theorem 1.19). *Let α be an algebraic number and let d and U be upper bounds on the degree and usual height, respectively, of α . Suppose that we are given a rational approximation $\bar{\alpha}$ to α such that $|\alpha - \bar{\alpha}| \leq 2^{-b}/(12d)$, where b is the smallest positive integer such that*

$$2^b \geq 2^{d^2/2} (d+1)^{(3d+4)/2} U^{2d}.$$

Then the minimal polynomial of α can be determined in $O(d^5(d + \log U))$ arithmetic operations on integers having $O(d^2(d + \log U))$ binary bits.

The algorithm in Lemma 6.6 is based on the Lenstra-Lenstra-Lovász lattice basis reduction algorithm, we refer the reader to (Yap 2000) for more details.

6.3. Exact Hardness results. We will use the following hardness results from (Jaeger *et al.* 1990) regarding the problem of exactly evaluating $Z_{\text{Tutte}}(G; q, \gamma)$, given a graph G . We refer to this problem as TUTTE(q, γ). Jaeger *et al.* (1990) identify the following 9 “special” points of the Tutte plane: $(1, -1)$, $(0, 0)$, $(4, -2)$,

$(2, -2)$, $(2, -1)$, $(2, -i - 1)$, $(2, i - 1)$, $(3, \omega_3^2 - 1)$, and $(3, \omega_3 - 1)$, where $i = \sqrt{-1}$ and $\omega_3 = \exp(2\pi i/3)$.⁶ With these special points in mind, their main result on the complexity of $\text{TUTTE}(q, \gamma)$ can be stated as follows.

THEOREM 6.7 (Jaeger *et al.* 1990, Proposition 1). *Let q and γ be algebraic numbers. Then $\text{TUTTE}(q, \gamma)$ is $\#\text{P}$ -hard unless $q = 1$ or (q, γ) is a special point, in which case $\text{TUTTE}(q, \gamma)$ is in FP .*

Vertigan (2005) studied the complexity of the computational problem $\text{PLANAR}\text{TUTTE}(q, \gamma)$, which also turns out to be hard for most parameters q and γ .

THEOREM 6.8 (Vertigan 2005, Theorem 5.1). *Let q and γ be algebraic numbers. Then $\text{PLANAR}\text{TUTTE}(q, \gamma)$ is $\#\text{P}$ -hard unless $q \in \{1, 2\}$ or (q, γ) is a special point, in which case $\text{PLANAR}\text{TUTTE}(q, \gamma)$ is in FP .*

6.4. Computational problems. In this section, we define a few computational problems that will be useful in our reductions; these were also considered in (Goldberg & Guo 2017). Let q be a real algebraic number, $\gamma_1, \dots, \gamma_k$ be algebraic numbers, and K, ρ be real numbers with $K > 1, \rho > 0$.

Name: $\text{SIGN-TUTTE}(q, \gamma_1, \dots, \gamma_k)$.

Instance: A (multi)graph G and a weight function $\hat{\gamma}: E \rightarrow \{\gamma_1, \dots, \gamma_k\}$.

Output: A correct statement of the form $Z_{\text{Tutte}}(G; q, \hat{\gamma}) \geq 0$ or $Z_{\text{Tutte}}(G; q, \hat{\gamma}) \leq 0$.

Name: $\text{FACTOR-}K\text{-NORMTUTTE}(q, \gamma_1, \dots, \gamma_k)$.

Instance: A (multi)graph G and a weight function $\hat{\gamma}: E \rightarrow \{\gamma_1, \dots, \gamma_k\}$.

Output: If $Z_{\text{Tutte}}(G; q, \hat{\gamma}) = 0$, the algorithm may output any rational number. Otherwise, it must output $\hat{N} \in \mathbb{Q}$ such that $\hat{N}/K \leq |Z_{\text{Tutte}}(G; q, \hat{\gamma})| \leq K\hat{N}$.

⁶In the (x, y) -parametrisation, the special points are $(0, 0)$, $(1, 1)$, $(-1, -1)$, $(0, -1)$, $(-1, 0)$, $(i, -i)$, $(-i, i)$, (ω_3, ω_3^2) , and (ω_3^2, ω_3) .

Name: DISTANCE- ρ -ARGTUTTE(q, γ).

Instance: A (multi)graph G .

Output: If $Z_{\text{Tutte}}(G; q, \gamma) = 0$, the algorithm may output any rational number. Otherwise, it must output $\hat{A} \in \mathbb{Q}$ such that, for some $a \in \arg(Z_{\text{Tutte}}(G; q, \gamma))$, we have $|\hat{A} - a| \leq \rho$.

We also consider these problems for the Potts model (with parameters q and $y = \gamma + 1$), and we write POTTS instead of TUTTE in the name of these problems when we refer to the Potts ones. We also consider all these problems restricted to planar graphs, in which case we write PLANARTUTTE instead of TUTTE in the name of the problem. It is a trivial observation that the planar case reduces to the general case.

6.5. Reducing exact computation to sign and approximate computation. In this section, we first review the binary search technique of Goldberg & Guo (2017), which we will refer to as “interval-shrinking”. Then, we use this to obtain several of our inapproximability theorems.

Let $f(\varepsilon) = -\varepsilon A + B$ be a linear function, where A and B are real algebraic numbers with $A \neq 0$. Let $\varepsilon^* = B/A$ be the zero of f . Let $(\varepsilon', \varepsilon'')$ be an open interval with length $l > 0$ such that ε^* is in $(\varepsilon', \varepsilon'')$ or, equivalently, $f(\varepsilon')f(\varepsilon'') < 0$. We want to find a small open subinterval of $(\varepsilon', \varepsilon'')$ that contains ε^* .

First, assume that we have an oracle that, on input ε , outputs the sign of $f(\varepsilon)$, unless when $f(\varepsilon) = 0$, in which case the output of the oracle is unreliable. Let $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_4$ be a partition of the interval $(\varepsilon', \varepsilon'')$ such that $\varepsilon_0 = \varepsilon'$, $\varepsilon_4 = \varepsilon''$ and $\varepsilon_{i+1} - \varepsilon_i \geq l/10$ for every $i \in \{0, \dots, 3\}$. We invoke the oracle with input ε_i to determine the sign of $f(\varepsilon_i)$ for every $i \in \{0, \dots, 4\}$; let s_i be the answer of the oracle. Then, we have a monotone sequence s_0, \dots, s_4 of positive and negative signs with $s_0 \neq s_4$. Hence, there are two possibilities: either $s_0 = s_1 = s_2$, in which case $\varepsilon_1 < \varepsilon^*$ and we can recurse on $(\varepsilon_1, \varepsilon_4)$, or $s_2 = s_3 = s_4$, in which case $\varepsilon^* < \varepsilon_3$ and we can recurse on $(\varepsilon_0, \varepsilon_3)$. In any of these two cases, we can shrink the interval $(\varepsilon', \varepsilon'')$ to at most 9/10 of its original length. Then, recursively, we can find an open subinterval of arbitrarily small length containing the zero of f .

Next, assume that we have an oracle that returns a multiplicative approximation to the norm of f . More accurately, let $\eta = 1/41$ and suppose that we have an oracle that, on input ε , returns a value $\hat{f}(\varepsilon)$ satisfying

$$(1 - \eta) |f(\varepsilon)| < \frac{1}{1 + \eta} |f(\varepsilon)| \leq \hat{f}(\varepsilon) \leq (1 + \eta) |f(\varepsilon)|$$

when $f(\varepsilon) \neq 0$ (otherwise the value $\hat{f}(\varepsilon)$ is unreliable). The approach given in [Goldberg & Guo \(2017\)](#) by Goldberg and Guo to shrink $(\varepsilon', \varepsilon'')$ is as follows. First, let us assume that $A > 0$, so f is strictly decreasing. Let $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{10}$ be a partition of the interval $(\varepsilon', \varepsilon'')$ such that $\varepsilon_0 = \varepsilon'$, $\varepsilon_{10} = \varepsilon''$ and $\varepsilon_{i+1} - \varepsilon_i \geq l/20$ for every $i \in \{0, \dots, 9\}$. These numbers are not chosen to be optimal but they suffice. We invoke our oracle to compute $\hat{f}(\varepsilon_i)$ for $i \in \{0, \dots, 10\}$. Let s_i be the sign (positive, negative, or zero) of $\hat{f}(\varepsilon_i) - \hat{f}(\varepsilon_{i+1})$ for each $i \in \{0, \dots, 9\}$. We analyse the signs s_i for $i \in \{0, \dots, 9\}$. First, we consider the case $\varepsilon_i < \varepsilon_{i+1} < \varepsilon^*$. Note that we have $f(\varepsilon_i) > f(\varepsilon_{i+1}) > 0$. Moreover,

$$\begin{aligned} \hat{f}(\varepsilon_i) - \hat{f}(\varepsilon_{i+1}) &\geq (1 - \eta) f(\varepsilon_i) - (1 + \eta) f(\varepsilon_{i+1}) \\ &= A(\varepsilon_{i+1} - \varepsilon_i - \eta(2\varepsilon^* - \varepsilon_i - \varepsilon_{i+1})). \end{aligned}$$

Note that $\varepsilon^* - \varepsilon_i$ and $\varepsilon^* - \varepsilon_{i+1}$ are both at most l and, thus, we obtain $2\varepsilon^* - \varepsilon_i - \varepsilon_{i+1} \leq 2l$. So since $\eta = 1/41$ and $\varepsilon_{i+1} - \varepsilon_i \geq l/20$, we conclude that s_i is positive. Now we consider the case $\varepsilon^* < \varepsilon_i < \varepsilon_{i+1}$. This time we have $f(\varepsilon_{i+1}) < f(\varepsilon_i) < 0$,

$$\begin{aligned} \hat{f}(\varepsilon_i) - \hat{f}(\varepsilon_{i+1}) &\leq (1 + \eta)(-f(\varepsilon_i)) - (1 - \eta)(-f(\varepsilon_{i+1})) \\ &= -A(\varepsilon_{i+1} - \varepsilon_i - \eta(\varepsilon_i + \varepsilon_{i+1} - 2\varepsilon^*)), \end{aligned}$$

and $0 < \varepsilon_i + \varepsilon_{i+1} - 2\varepsilon^* < 2l$. We conclude that s_i is negative. If $\varepsilon_i \leq \varepsilon^*$ and $\varepsilon^* \leq \varepsilon_{i+1}$, then we do not know what the value of s_i will be. However, this is true for at most two consecutive values of i . With these properties of the signs s_i in mind, let us study the sequence s_0, \dots, s_9 . There are two possibilities. The first one is that s_0, s_1, s_2, s_3 are all positive, in which case $\varepsilon_2 < \varepsilon^*$ and we can recurse on $(\varepsilon_2, \varepsilon_{10})$. The second possibility is that s_6, s_7, s_8, s_9 are all negative, in which case $\varepsilon^* < \varepsilon_8$ and we can recurse on $(\varepsilon_0, \varepsilon_8)$.

In any of these two cases, we can shrink the interval $(\varepsilon', \varepsilon'')$ to at most $9/10$ of its original length. Again using binary search it is possible to find a small open subinterval containing the zero of f . Let us now assume that $A < 0$. In this case, one can analogously prove that the sign s_i is positive when $\varepsilon_i < \varepsilon_{i+1} < \varepsilon^*$ and negative when $\varepsilon^* < \varepsilon_i < \varepsilon_{i+1}$, so the same procedure allows us to shrink $(\varepsilon', \varepsilon'')$.

Let q and γ be real algebraic numbers with $q \notin \{0, 1\}$ and $\gamma > 0$. Let H be a graph and let s and t be two distinct connected vertices of H . We are going to apply these interval stretching techniques to the linear function

$$(6.9) \quad f(\varepsilon; H, \gamma) = Z_{st}(H; q, \gamma) \left(1 - \frac{1}{q}\right) + \varepsilon \left(Z_{st}(H; q, \gamma) + \frac{1}{q} Z_{s|t}(H; q, \gamma) \right).$$

Let us write this function as $f(\varepsilon; H, \gamma) = B(H, \gamma) - \varepsilon A(H, \gamma)$, where $B(H, \gamma) = Z_{st}(H; q, \gamma)(1 - 1/q)$ and $A(H, \gamma) = -Z_{st}(H; q, \gamma) - q^{-1}Z_{s|t}(H; q, \gamma)$. We have

$$(6.10) \quad \begin{aligned} f(0; H, \gamma) &= Z_{st}(H; q, \gamma) \left(1 - \frac{1}{q}\right); \\ f(1 - q; H, \gamma) &= (1 - q)Z_{st}(H; q, \gamma). \end{aligned}$$

Under certain hypotheses, we are going to prove that $f(0; H, \gamma)f(1 - q; H, \gamma) < 0$, so $A(H, \gamma) \neq 0$ and $f(-; H, \gamma)$ has a zero between 0 and $1 - q$. This allows us to find a suitable interval where we can perform interval-shrinking. For this purpose we will also need Lemma 6.11, that tells us that the zero of $f(-; H, \gamma)$ is not close to either 0 or $1 - q$.

LEMMA 6.11. *Let q and γ be real algebraic numbers with $q \notin \{0, 1\}$ and $\gamma > 0$. Let $H = (V, E)$ be a graph and let s and t be two distinct connected vertices of H . Let $n = |V|$, $m = |E|$, $r = \max\{n, m\}$ and $c = 2 \max\{|q|, 1/|q|\} \max\{\gamma, 1/\gamma\}$. Let ε^* be the zero of the function $f(\varepsilon; H, \gamma) = B(H, \gamma) - A(H, \gamma)$, defined as in (6.9). Let us assume that $|Z_{st}(H; q, \gamma)| \geq c^{-r}$, $|Z_{s|t}(H; q, \gamma)| \geq c^{-r}$ and $A(H, \gamma) \neq 0$. Then we have $|1 - q - \varepsilon^*| \geq |1 - q| c^{-2r}$ and $|\varepsilon^*| \geq |1 - 1/q| c^{-2r}$.*

PROOF. In view of the definition of $f(\varepsilon; H, \gamma)$ and equation (6.10), we have

$$|1 - q - \varepsilon^*| = \frac{|f(\varepsilon^*; H, \gamma) - f(1 - q)|}{|A(H, \gamma)|} = \frac{|1 - q| |Z_{st}(H; q, \gamma)|}{|A(H, \gamma)|}.$$

Note that

$$(6.12) \quad |A(H, \gamma)| \leq \sum_{A \subseteq E} \max\{|q|, 1/|q|\} |q|^{k(A)-1} |\gamma|^{|A|} \leq c^r.$$

Moreover, we have $|Z_{st}(H; q, \gamma)| \geq c^{-r}$ by hypothesis, so we conclude that $|1 - q - \varepsilon^*| \geq |1 - q|c^{-2r}$. Analogously, we find that

$$\begin{aligned} |\varepsilon^*| &= \frac{|f(\varepsilon^*; H, \gamma) - f(0)|}{|A(H, \gamma)|} \\ &= \frac{|1 - 1/q| |Z_{s|t}(H; q, \gamma)|}{|A(H, \gamma)|} \geq \left|1 - \frac{1}{q}\right| c^{-2r}. \quad \square \end{aligned}$$

LEMMA 6.13. *Let K be a real number with $K > 1$. Let q, γ_1 and γ_2 be real algebraic numbers such that $q > 1, \gamma_1 \in (-2, -1)$ and $\gamma_2 > 0$. Let us assume that we have access to an oracle for FACTOR- K -NORMPLANARTUTTE(q, γ_1, γ_2). Then there exists an algorithm that takes as input a positive integer ρ and a planar graph H along with two distinct connected vertices s and t of H , and, for $\gamma = (\gamma_2 + 1)^\rho - 1$, this algorithm computes a representation of the algebraic number $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma)$ in polynomial time in ρ and the size of H . Moreover, if we have access to the more powerful oracle FACTOR- K -NORMTUTTE(q, γ_1, γ_2), then we can remove the constraint that H is planar.*

PROOF. Since FACTOR- K -NORMPLANARTUTTE(q, z) is equally hard for any $K > 1$ (see Section 1.1), we may assume that $K = 1 + \eta$ for $\eta = 1/41$.

Let $\rho, H = (V, E)$ and s, t be the inputs of our algorithm. Let $n = |V|$ and $m = |E|$. Let $c = 2 \max\{|q|, 1/|q|\} \max\{\gamma, 1/\gamma\}$, so $c \geq 2$, and let $r = \max\{n, m\}$. Let H' be a copy of H with an extra edge from s to t . Let γ' be a weight that we can implement and let $\varepsilon = \gamma' + 1$, so the pair $(1 + q/(\varepsilon - 1), \varepsilon)$ is (q, γ') written

in (x, y) coordinates. We will choose γ' and argue how we can implement γ' later in the proof. When we say we implement ε , we mean that we implement the pair $(1 + q/(\varepsilon - 1), \varepsilon)$ in (x, y) notation or, equivalently, γ' .

Let τ be the weight function on H' that assigns the weight γ to the edges of H and the weight γ' to the new edge. Then, as was observed in (Goldberg & Jerrum 2014, Lemma 2), we have

$$\begin{aligned}
 Z_{\text{Tutte}}(H'; q, \tau) &= Z_{st}(H; q, \gamma)(1 + \gamma') \\
 &\quad + Z_{s|t}(H; q, \gamma) \left(1 + \frac{\gamma'}{q}\right) \\
 (6.14) \qquad \qquad &= Z_{s|t}(H; q, \gamma) \left(1 - \frac{1}{q}\right) \\
 &\quad + \varepsilon \left(Z_{st}(H; q, \gamma) + \frac{1}{q} Z_{s|t}(H; q, \gamma) \right) \\
 &= f(\varepsilon; H, \gamma),
 \end{aligned}$$

where $f(\varepsilon; H, \gamma)$ was introduced in (6.9). Hence, $Z_{\text{Tutte}}(H'; q, \tau)$ can be seen as a function, with variable ε , of the form $f(\varepsilon; H, \gamma) = B(H, \gamma) - \varepsilon A(H, \gamma)$, where $B(H, \gamma) = Z_{s|t}(H; q, \gamma)(1 - 1/q)$ and $A(H, \gamma) = -Z_{st}(H; q, \gamma) - q^{-1}Z_{s|t}(H; q, \gamma)$. This construction will be used several times in this section. Now we analyse $f(-; H, \gamma)$ for our particular setting ($q > 1$). Since q and γ are positive, the quantities $Z_{st}(H; q, \gamma)$ and $Z_{s|t}(H; q, \gamma)$ are positive, so $A(H, \gamma)$ is negative. From $q > 1$ and (6.10), it follows that $f(0; H, \gamma) = B(H, \gamma) > 0$ and $f(1 - q; H, \gamma) < 0$, so $f(0; H, \gamma)f(1 - q; H, \gamma) < 0$ as we wanted. We conclude that the zero ε^* of $f(\varepsilon; H, \gamma)$ is in $(1 - q, 0)$. Note that $\varepsilon \in (1 - q, 0)$ if and only if $\gamma' \in (-q, -1)$. Moreover, we have

$$\begin{aligned}
 (6.15) \qquad \qquad Z_{st}(H; q, \gamma) &\geq q\gamma^m \geq c^{-r}, \\
 Z_{s|t}(H; q, \gamma) &\geq q^n \geq c^{-r}.
 \end{aligned}$$

This allow us to apply Lemma 6.11. Once we have all these properties of $f(\varepsilon; H, \gamma)$ at our disposal, we can proceed to describe our algorithm. Our algorithm also works for $q \in (-\infty, 0) \cap (0, 1)$ as long as $f(0; H, \gamma)f(1 - q; H, \gamma) < 0$ and the hypotheses of Lemma 6.11 hold. In the rest of the proof we will only use the fact that $q > 1$

one more time, but this will be made explicit and can easily be adapted to the case $q < 1$ as we will explain in Lemma 6.19.

Our algorithm computes a positive integer j_0 such that $c^{-j_0} \leq |q - 1|/2$. Let j be an integer with $j \geq j_0$. We will first show how to additively approximate $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma)$ with error at most $2|q|c^{-j}/|q - 1|$.

If we could efficiently implement the point $(1 - q/(\varepsilon - 1), \varepsilon)$ (in (x, y) coordinates) for any $\varepsilon \in [1 - q, 0]$ using only planar graphs, then our algorithm could perform the interval-shrinking technique explained at the beginning of this section. This would allow us to compute an interval of length at most c^{-j-4r} where the linear function $f(\varepsilon)$ has a zero, which would, in turn, provide us with the desired additive approximation, as we will see later. However, some difficulties arise since we do not know how to implement any specific real algebraic weight. This difficulty was overcome by Goldberg and Jerrum by developing Lemmas 4.1 and 4.2. Here we use the version of these lemmas given in Corollary 4.3. Let $y_1 = \gamma_1 + 1$, $x_1 = 1 + q/(y_1 - 1)$, $y_2 = \gamma_2 + 1$ and $x_2 = 1 + q/(y_2 - 1)$. Note that $y_1 \in (-1, 0)$, $y_2 > 1$ and $q \neq 0$. Hence, Corollary 4.3 allows us to efficiently implement approximations of real algebraic numbers when applied with the parameters x_1, y_1, x_2, y_2 . Every time we invoke Corollary 4.3 we will be using these parameters. We are going to use this corollary to implement approximations of $\varepsilon \in (1 - q, 0)$. This is the only point where our algorithm uses the fact that $\gamma_1 \in (-2, -1)$ or, equivalently, $y_1 \in (-1, 0)$. In further lemmas where we study the case $q < 1$, we will have to implement approximations of $\varepsilon \in (0, 1 - q)$ and, hence, we will get away with the weaker hypothesis $\gamma_1 \in (-1, 0)$, or, equivalently, $y_1 \in (0, 1)$. (This hypothesis is “weaker” in the sense that a 2-thickening of a $y_1 \in (-1, 0)$ implements a $y_1 \in (0, 1)$.)

We want to implement numbers ε' and ε'' so that $\varepsilon^* \in (\varepsilon', \varepsilon'') \subseteq (1 - q, 0)$. Note that here we are using that $q > 1$. When $q < 1$ our algorithm would work on the interval $(0, 1 - q)$ instead of $(1 - q, 0)$. This paragraph is the last time that we use the hypothesis $q > 1$ in this proof. The argument given in this paragraph will be revisited when we deal with the case $q < 1$ in further lemmas. Our algorithm first applies the algorithm given in Corollary 4.3

with $y' = -(1 - 1/q)c^{-2r}/2$, k such that $|y_1|^k < |y'| < |y_1|^{-k}$ and $n = \lceil 2r \log_2(c) - \log_2(1 - 1/q) + 2 \rceil$. Note that $k = O(r)$ and $n = O(r)$. This procedure computes a theta graph and a weight function taking weights in $\{\gamma_1, \gamma_2\}$ that implement a point $(1 + q/(\varepsilon'' - 1), \varepsilon'')$ such that $|y' - \varepsilon''| \leq 2^{-n} \leq (1 - 1/q)c^{-2r}/4$ in polynomial time in $r = O(\text{size}(H))$. We have $-3(1 - 1/q)c^{-2r}/4 \leq \varepsilon'' \leq -(1 - 1/q)c^{-2r}/4$, so, by Lemma 6.11, we find that $\varepsilon^* < \varepsilon'' < 0$. Now our algorithm invokes again Corollary 4.3, this time with inputs $y' = 1 - q + (q - 1)c^{-2r}/2$, k such that $|y_1|^k < |y'| < |y_1|^{-k}$ and $n = \lceil 2r \log_2(c) - \min\{0, \log_2(q - 1)\} + 2 \rceil$. This implements $(1 + q/(\varepsilon' - 1), \varepsilon')$ with $|y' - \varepsilon'| \leq (q - 1)c^{-2r}/4$, which gives $1 - q + (q - 1)c^{-2r}/4 \leq \varepsilon' \leq 1 - q + 3(q - 1)c^{-2r}/4$. Again by Lemma 6.11, we find that $1 - q < \varepsilon' < \varepsilon^*$. The interval $(\varepsilon', \varepsilon'')$ is the starting interval for the interval-shrinking procedure.

Let us assume that we are carrying out the interval-shrinking technique explained at the beginning of this section, so we have an interval $(\varepsilon', \varepsilon'')$ of length l where f changes sign. Let us also assume that we can implement the endpoints ε' and ε'' . We want to find a subinterval of length at most $9l/10$ where f changes sign. We can assume that $l > c^{-j-4r}$, since otherwise we do not need to shrink the interval further. Let $p = 10$ be the number of subintervals into which $(\varepsilon', \varepsilon'')$ is partitioned by the interval-shrinking technique. We want to find numbers $\varepsilon_1, \dots, \varepsilon_{p-1}$ such that we can implement the point $(1 + q/(\varepsilon_i - 1), \varepsilon_i)$ for every $i \in \{1, \dots, p - 1\}$ and, for $\varepsilon_0 = \varepsilon'$ and $\varepsilon_p = \varepsilon''$, we have $\varepsilon_i - \varepsilon_{i-1} \geq l/2p$ for every $i \in \{1, \dots, p\}$, which is what is required to perform interval-shrinking. For each $i \in \{1, \dots, p - 1\}$, our algorithm computes $\varepsilon'_i = \varepsilon' + il/p$ and then it applies the algorithm given in Corollary 4.3 with $y' = \varepsilon'_i$, k such that $|y_1|^k < |y'| < |y_1|^{-k}$ and $n = \lceil (j + 4r) \log_2(c) + \log_2(4p) \rceil$. This procedure computes a graph and a weight function taking weights in $\{\gamma_1, \gamma_2\}$ that implement a point $(1 + q/(\varepsilon_i - 1), \varepsilon_i)$ such that $|\varepsilon'_i - \varepsilon_i| \leq 2^{-n} \leq c^{-j-4r}/(4p)$. This application of the procedure given in Corollary 4.3 takes polynomial time in j , r and k . Note that k is polynomial in r and j because $|1 - q| \geq |\varepsilon'_i| \geq l/p \geq c^{-j-4r}/p$ for any $i \in \{1, \dots, p - 1\}$. The algebraic numbers $\varepsilon', \varepsilon_1, \dots, \varepsilon_{p-1}, \varepsilon''$ form a partition the interval $(\varepsilon', \varepsilon'')$. Our algorithm has computed theta (and, thus, planar)

graphs that implement $(1 + q/(\varepsilon_i - 1), \varepsilon_i)$, so it can use the oracle $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma_1, \gamma_2)$ to multiplicatively approximate $f(\varepsilon_i)$ for every $i \in \{0, \dots, p\}$. Note that

$$\varepsilon_i - \varepsilon_{i-1} \geq \varepsilon'_i - \varepsilon'_{i-1} - c^{-j-4r} \frac{1}{2p} \geq \frac{l}{2p}$$

for every $i \in \{1, \dots, p\}$. Therefore, our algorithm can apply the interval-shrinking technique discussed at the beginning of this section to shrink $(\varepsilon', \varepsilon'')$.

To guarantee that this interval-shrinking technique computes an interval of length at most c^{-j-4r} , it suffices to subdivide the original interval $\lceil (j + 4r) \log_{10/9}(c) + \log_{10/9} |1 - q| \rceil$ times due to the fact that each iteration shrinks the interval to 9/10 of its size. In (Goldberg & Guo 2017; Goldberg & Jerrum 2014) the authors used the information provided by this interval-shrinking procedure to solve the problem $\# \text{MINIMUM CARDINALITY } (s, t)\text{-CUT}$ for arbitrary graphs (not-necessarily planar). Here we follow a different approach that allows us to compute the representation of $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma)$.

Once our algorithm has computed an interval of length at most c^{-j-4r} where f has a zero, it implements a point $(1 + q/(\hat{\varepsilon} - 1), \hat{\varepsilon})$ such that $\hat{\varepsilon}$ is in this interval. This can be done by applying Corollary 4.3 with the same parameters as before other than y' , which is set as the middle point of the computed interval. Let ε^* be the zero of f . Note that $|\hat{\varepsilon} - \varepsilon^*| \leq c^{-j-4r}$. Recall that $f(\varepsilon; H, \gamma) = B(H, \gamma) - A(H, \gamma)\varepsilon$. For a graph H' and a weight function τ as in (6.14), with $\gamma' = \hat{\varepsilon} - 1$ (which we can now implement as promised before (6.14)), we obtain

$$(6.16) \quad \begin{aligned} |Z_{\text{Tutte}}(H'; q, \tau)| &= |f(\hat{\varepsilon})| = |f(\hat{\varepsilon}) - f(\varepsilon^*)| \\ &\leq |A(H, \gamma)|c^{-j-4r} \leq c^{-j-3r}, \end{aligned}$$

where we used the elementary bound $|A(H, \gamma)| \leq c^r$, which has been established in (6.12). By dividing by $Z_{st}(H; q, \gamma)$ in (6.14), which is non-zero, and rearranging the terms we find that

$$\frac{Z_{\text{Tutte}}(H'; q, \tau)}{Z_{st}(H; q, \gamma)} = \hat{\varepsilon} + \left(1 + \frac{\hat{\varepsilon} - 1}{q}\right) \frac{Z_{s|t}(H; q, \gamma)}{Z_{st}(H; q, \gamma)}.$$

Dividing by $1 + (\hat{\varepsilon} - 1)/q = (q - 1 + \hat{\varepsilon})/q$ yields

$$(6.17) \quad \frac{qZ_{\text{Tutte}}(H'; q, \tau)}{(q - 1 + \hat{\varepsilon})Z_{st}(H; q, \gamma)} = -\frac{\hat{\varepsilon}q}{1 - q - \hat{\varepsilon}} + \frac{Z_{st}(H; q, \gamma)}{Z_{st}(H; q, \gamma)}.$$

We claim that $|1 - q - \hat{\varepsilon}| \geq |1 - q|c^{-2r}/2$. Recall that in view of Lemma 6.11, we have $|1 - q - \varepsilon^*| \geq |1 - q|c^{-2r}$. Hence, we obtain

$$\begin{aligned} |1 - q - \hat{\varepsilon}| &\geq |1 - q - \varepsilon^*| - |\varepsilon^* - \hat{\varepsilon}| \\ &\geq |1 - q|c^{-2r} - c^{-j-4r} \geq \frac{|1 - q|}{2}c^{-2r}, \end{aligned}$$

where we used that $c^{-j-4r} \leq c^{-j_0}c^{-4r} \leq |q - 1|c^{-4r}/2$ by definition of j_0 . Therefore, we can apply this lower bound in conjunction with (6.15), (6.16) and (6.17) to conclude that

$$\left| \frac{Z_{st}(H; q, \gamma)}{Z_{st}(H; q, \gamma)} - \frac{\hat{\varepsilon}q}{1 - q - \hat{\varepsilon}} \right| \leq \frac{2|q| |Z_{\text{Tutte}}(H'; q, \tau)|}{|1 - q|} c^{3r} \leq \frac{2|q|}{|1 - q|} c^{-j}.$$

Our algorithm then computes $\hat{\varepsilon}q/(1 - q - \hat{\varepsilon})$ as an approximation of $\alpha = Z_{st}(H; q, \gamma)/Z_{st}(H; q, \gamma)$. We have shown that α is a real algebraic number that we can additively approximate up to an error at most $2|q|c^{-j}/|1 - q|$ in polynomial time in j and the size of H . Technically, our approximation $\hat{\varepsilon}q/(1 - q - \hat{\varepsilon})$ is another algebraic number. For this reason, our algorithm approximates $\hat{\varepsilon}q/(1 - q - \hat{\varepsilon})$ by a rational number $\bar{\alpha}$ (with additive error at most $2|q|c^{-j}/|q - 1|$) and uses this rational number as our approximation of α . The overall error that we make is then $|\alpha - \bar{\alpha}| \leq 4|q|c^{-j}/|1 - q|$.

In view of Corollary 6.5, we have $d(\alpha) \leq d(q)d(\gamma) \leq d(q)d(\gamma_2)$, where we have used that $\gamma \in \mathbb{Q}(\gamma_2)$ and, thus, $d(\gamma) \leq d(\gamma_2)$. Moreover, Corollary 6.5 yields

$$H(\alpha) \leq (2^{m+1/2}e^{nh(q)+mh(\gamma)})^{2d(q)d(\gamma)}.$$

Since $h(\gamma) = h((\gamma_2 - 1)^\rho - 1) \leq \rho(1 + h(\gamma_2))$ by Lemma 6.4, our algorithm can compute a rational number D_{q,γ_2} with $D_{q,\gamma_2} > 1$ such that $H(\alpha) \leq D_{q,\gamma_2}^{\rho \text{size}(H)}$. The only non-trivial step of this computation is upper bounding $h(q)$ and $h(\gamma_2)$ in terms of the degrees and

usual heights of q and γ_2 as in (6.1). Let $d = d(q)d(\gamma_2) = O(1)$ and $U = D_{q,\gamma_2}^{\rho \text{size}(H)}$. Let b be as in Lemma 6.6. Then we have $2^b = O(D_{q,\gamma_2}^{2d\rho \text{size}(H)})$, so $b = O(\rho \text{size}(H))$. By choosing j appropriately, we can use the algorithm that we have developed in this proof to find a rational approximation $\bar{\alpha}$ with $|\alpha - \bar{\alpha}| \leq 2^{-b}/(12d)$. As we have argued, this takes polynomial time in b and $\text{size}(H)$. Since $b = O(\rho \text{size}(H))$, we conclude that the computation of $\bar{\alpha}$ runs in polynomial time in ρ and $\text{size}(H)$. Once we have computed this approximation, our algorithm invokes the algorithm given in Lemma 6.6 to determine the minimal polynomial of α in time $O(d^5(d + \log U)) = O(\rho \text{size}(H))$. Finally, it remains to compute an interval of the real line where α is the only root of its minimal polynomial. Since α is a real algebraic number and we know its minimal polynomial, our algorithm can use Sturm sequences to isolate the real roots of this minimal polynomial. Then, by approximating α it decides which one of the computed intervals corresponds to α .

Finally, note that our algorithm also works for arbitrary graphs (not-necessarily planar) as long as our oracle provides us with reliable answers for any graph. \square

LEMMA 6.18. *Let q, γ_1 and γ_2 be real algebraic numbers such that $q > 1, \gamma_1 \in (-2, -1)$ and $\gamma_2 > 0$. Let us assume that we have access to an oracle for the problem SIGN-PLANAR-TUTTE(q, γ_1, γ_2). Then there exists an algorithm that takes as input a positive integer ρ and a planar graph H along with two distinct connected vertices s and t of H , and, for $\gamma = (\gamma_2 + 1)^\rho - 1$, this algorithm computes a representation of the algebraic number $Z_{st}(H; q, \gamma)/Z_{st}(H; q, \gamma)$ in polynomial time in ρ and the size of H . Moreover, if we have access to the more powerful oracle SIGN-TUTTE(q, γ_1, γ_2), then we can remove the constraint that H is planar.*

PROOF. The algorithm is exactly the same one of Lemma 6.13. The proof is analogous too. The only difference is in the interval-shrinking technique, where we split $(\varepsilon', \varepsilon'')$ into 4 intervals instead of 10 (so $p = 4$ in the proof), but this has been discussed at the beginning of this section. \square

LEMMA 6.19. *Let K be a real number with $K > 1$. Let q , γ_1 and γ_2 be real algebraic numbers such that $0 < q < 1$, $\gamma_1 \in (-1, 0)$ and $\gamma_2 > 0$. Let us assume that we have access to an oracle for $\text{FACTOR-}K\text{-NORMPLANAR-TUTTE}(q, \gamma_1, \gamma_2)$. Then there exists an algorithm that takes as input a positive integer ρ and a planar graph H along with two distinct connected vertices s and t of H , and, for $\gamma = (\gamma_2 + 1)^\rho - 1$, this algorithm computes a representation of the algebraic number $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma)$ in polynomial time in ρ and the size of H . Moreover, if we have access to the more powerful oracle $\text{FACTOR-}K\text{-NORMTUTTE}(q, \gamma_1, \gamma_2)$, then we can remove the constraint that H is planar.*

PROOF. We claim that the algorithm presented in Lemma 6.13 also works in this setting. Let $f(\varepsilon; H, \gamma) = B(H, \gamma) - \varepsilon A(H, \gamma)$ as in (6.9). As we pointed out in the proof of Lemma 6.13, the algorithm works as long as $f(0; H, \gamma)f(1 - q; H, \gamma) < 0$ and the hypothesis of Lemma 6.11 hold. First, since q and γ are positive, equations (6.15) hold. It follows that $A(H, \gamma) = -Z_{st}(H; q, \gamma) - q^{-1}Z_{st}(H; q, \gamma) \neq 0$. Hence, the hypothesis of Lemma 6.11 hold. In view of (6.10) and the fact that $q \in (0, 1)$ and γ is positive, we have $f(0; H, \gamma) < 0$ and $f(1 - q; H, \gamma) > 0$. We conclude that $f(0; H, \gamma)f(1 - q; H, \gamma) < 0$, as we wanted.

This time the interval-stretching technique applied in Lemma 6.13 runs on a subinterval $(\varepsilon', \varepsilon'')$ of $(0, 1 - q)$, so we only need to implement positive values of ε . For this reason, we can get away with the hypothesis $\gamma_1 \in (-1, 0)$ instead of the hypothesis $\gamma_1 \in (-2, -1)$, as was announced in the proof of Lemma 6.13. Finally, we must indicate how our algorithm implements the numbers ε' and ε'' so that $\varepsilon^* \in (\varepsilon', \varepsilon'') \subseteq (0, 1 - q)$, as this was only done in Lemma 6.13 for $q > 1$. The argument that we give here also applies when $q < 0$. Let $y_1 = \gamma_1 + 1$, $x_1 = 1 + q/(y_1 - 1)$, $y_2 = \gamma_2 + 1$ and $x_2 = 1 + q/(y_2 - 1)$. We have $y_1 \in (0, 1)$, $y_2 > 1$, $q < 1$ and $q \neq 0$. Our algorithm first applies the algorithm given in Corollary 4.3 with $y' = |1 - 1/q|c^{-2r}/2$, k such that $|y_1|^k < |y'| < |y_1|^{-k}$ and $n = \lceil 2r \log_2(c) - \min\{0, \log_2 |1 - 1/q|\} + 2 \rceil$. Note that $k = O(r)$ and $n = O(r)$. This procedure computes a theta graph and a weight function taking weights in $\{\gamma_1, \gamma_2\}$ that implement a point $(1 + q/(\varepsilon' - 1), \varepsilon')$ such that $|y' - \varepsilon'| \leq 2^{-n} \leq |1 - 1/q|c^{-2r}/4$ in

polynomial time in $r = O(\text{size}(H))$. We obtain $|1 - 1/q|c^{-2r}/4 \leq \varepsilon' \leq 3|1 - 1/q|c^{-2r}/4$, so, by Lemma 6.11, we find that $0 < \varepsilon' < \varepsilon^*$. Next our algorithm invokes again Corollary 4.3, this time with inputs $y' = 1 - q - (1 - q)c^{-2r}/2$, k such that $|y_1|^k < |y'| < |y_1|^{-k}$ and $n = \lceil 2r \log_2(c) - \min\{0, \log_2(1 - q)\} + 2 \rceil$. This implements $(1 + q/(\varepsilon'' - 1), \varepsilon'')$ with $|y' - \varepsilon''| \leq (1 - q)c^{-2r}/4$, which gives $1 - q - 3(1 - q)c^{-2r}/4 \leq \varepsilon'' \leq 1 - q + (1 - q)c^{-2r}/4$. Again by Lemma 6.11, we find that $\varepsilon^* < \varepsilon'' < 1 - q$. The interval $(\varepsilon', \varepsilon'')$ is the starting interval for the interval-shrinking procedure that we needed. \square

LEMMA 6.20. *Let q, γ_1 and γ_2 be real algebraic numbers such that $0 < q < 1, \gamma_1 \in (-1, -0)$ and $\gamma_2 > 0$. Let us assume that we have access to an oracle for the computational problem SIGN-PLANAR TUTTE(q, γ_1, γ_2). Then there exists an algorithm that takes as input a positive integer ρ and a planar graph H along with two distinct connected vertices s and t of H , and, for $\gamma = (\gamma_2 + 1)^\rho - 1$, this algorithm computes a representation of the algebraic number $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma)$ in polynomial time in ρ and the size of H . Moreover, if we have access to the more powerful oracle SIGN-TUTTE(q, γ_1, γ_2), then we can remove the constraint that H is planar.*

PROOF. The algorithm is exactly the same the one of Lemma 6.19, the only difference being in the interval-stretching technique as we have already explained. \square

LEMMA 6.21. *Let K be a real number with $K > 1$. Let q, γ_1 and γ_2 be real algebraic numbers such that $q < 0, \gamma_1 \in (-1, 0)$ and $\gamma_2 > 0$. Let us assume that we have access to an oracle for FACTOR- K -NORMPLANAR TUTTE(q, γ_1, γ_2). Then there exists an algorithm that takes as input:*

- a positive integer ρ ;
- a planar graph $H = (V, E)$ such that, for $\gamma = (\gamma_2 + 1)^\rho - 1$, we have $\gamma \geq (8 \max\{|q|, 1/|q|\})^r$, where $r = \max\{|V|, |E|\}$;

- two distinct connected vertices s and t of H .

This algorithm computes a representation of the algebraic number $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma)$ in polynomial time in ρ and the size of H . Moreover, for such inputs ρ, H and s, t , we have $Z_{st}(H; q, \gamma) \neq 0$ and $Z_{\text{Tutte}}(H; q, \gamma) \neq 0$. If we have access to the more powerful oracle $\text{FACTOR-}K\text{-NORMTUTTE}(q, \gamma_1, \gamma_2)$, then we can remove the constraint that H is planar.

PROOF. We claim that the algorithm presented in Lemmas 6.13 and 6.19 also works in this setting. Let $n = |V|$ and $m = |E|$. Let $c = 2 \max\{|q|, 1/|q|\}\gamma$. We may assume that $r \geq 2$. First, let us assume that H is connected. Let $f(\varepsilon; H, \rho) = B(H, \gamma) - \varepsilon A(H, \gamma)$ as in (6.9), so $B(H, \gamma) = Z_{s|t}(H; q, \gamma)(1 - 1/q)$ and $A(H, \gamma) = -Z_{st}(H; q, \gamma) - q^{-1}Z_{s|t}(H; q, \gamma)$. Recall that we have to prove that the conditions of Lemma 6.11 hold, as well as the inequality $f(0; H, \gamma)f(1 - q; H, \gamma) < 0$. Let $\delta = (2 \max\{|q|, 1/|q|\})^r/\gamma$. Note that $0 < \delta \leq 1/4$ because $\gamma \geq (8 \max\{|q|, 1/|q|\})^r$. Each one of the (at most 2^m) terms in $Z_{st}(H; q, \gamma)$, other than the term with all edges in A , has absolute value at most $\gamma^{m-1}|q| \max\{|q|, 1\}^{n-1} \leq \delta 2^{-m}\gamma^m|q|$. Since H is connected, the term with all edges in A is $q\gamma^m$. Thus, we have the inequalities

$$(6.22) \quad \gamma^m q - \delta \gamma^m |q| \leq Z_{st}(H; q, \gamma) \leq \gamma^m q + \delta \gamma^m |q| < 0.$$

In particular, $Z_{st}(H; q, \gamma) \neq 0$. It also follows that

$$|Z_{st}(H; q, \gamma)| \geq \gamma^m |q| (1 - \delta) \geq \gamma^m |q| 3/4 \geq c^{-r},$$

which is one of the conditions of Lemma 6.11. Recall that an (s, t) -cut of H is a subset A of edges of H such that any path from s to t in H has an edge in A . The size of this (s, t) -cut is the cardinality of A . Let k be the size of a minimum cardinality (s, t) -cut in H , and let C be the number of (s, t) -cuts of size k . We study the terms $q^{k(A)}\gamma^{|A|}$ appearing in $Z_{s|t}(H; q, \gamma)$, so A is a subset of E such that s and t are not connected in (V, A) . Note that such an A is the complement of an (s, t) -cut and, hence, $|A| \leq m - k$. Moreover, if A is not the complement of an (s, t) -cut of size k , then the absolute value of

$q^{k(A)}\gamma^{|A|}$ is at most $\gamma^{m-k-1}q^2 \max\{1, |q|\}^{n-2} \leq \delta 2^{-m}\gamma^{m-k}q^2$. Thus, we have the inequalities

$$(6.23) \quad \begin{aligned} 0 < C\gamma^{m-k}q^2 - \delta\gamma^{m-k}q^2 &\leq Z_{s|t}(H; q, \gamma) \\ &\leq C\gamma^{m-k}q^2 + \delta\gamma^{m-k}q^2. \end{aligned}$$

The inequalities (6.22) and (6.23) have been previously given in the proof of (Goldberg & Jerrum 2014, Lemma 2). As a consequence, we find that

$$|Z_{s|t}(H; q, \gamma)| \geq C\gamma^{m-k}q^2(1-\delta) \geq C\gamma^{m-k}q^2 3/4 \geq \gamma^{m-k}q^2 3/4 \geq c^{-r},$$

which is another one of the conditions of Lemma 6.11. In view of (6.10) and the facts that $q < 0$ and we know the signs of $Z_{s|t}(H; q, \gamma)$ and $Z_{st}(H; q, \gamma)$, it follows that $f(0; H, \gamma) > 0$ and $f(1-q; H, \gamma) < 0$. Hence, we find that $f(0; H, \gamma)f(1-q; H, \gamma) < 0$, as we wanted. Note that $A(H, \gamma)$ has to be non-zero because $f(-; H, \gamma)$ is non-constant as $f(0; H, \gamma)f(1-q; H, \gamma) < 0$. This is the last condition of Lemma 6.11 that we had to check. We conclude that we can apply the algorithm given in the proof of Lemma 6.19 to compute $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma)$ in polynomial time in ρ and the size of H . Finally, we show that $Z_{\text{Tutte}}(H; q, \gamma) \neq 0$. This is not needed for the algorithm of Lemma 6.19, but is part of the statement of the current lemma. In light of (6.22) and (6.23), we have $|Z_{st}(H; q, \gamma)| \geq \gamma^m|q|(1-\delta)$ and $|Z_{s|t}(H; q, \gamma)| \leq C\gamma^{m-k}q^2(1+\delta)$. Note that

$$\gamma^m|q|(1-\delta) \geq \frac{3}{4}\gamma^m|q| > \frac{5}{4}C\gamma^{m-k}q^2 \geq \gamma^{m-k}q^2(1+\delta),$$

where we used that $\gamma \geq (8 \max\{|q|, 1/|q|\})^r \geq 8 \cdot 2^m|q| > 5C|q|$ since $r \geq 2$. Therefore, we find that $|Z_{st}(H; q, \gamma)| > |Z_{s|t}(H; q, \gamma)|$. We conclude that

$$Z_{\text{Tutte}}(H; q, \gamma) = Z_{st}(H; q, \gamma) + Z_{s|t}(H; q, \gamma) < 0.$$

It remains to consider the case where H is not connected. Let H_1, \dots, H_l be the connected components of H , and let us assume

that the vertices s and t are in H_1 without loss of generality. We have

$$\begin{aligned} Z_{st}(H; q, \gamma) &= Z_{st}(H_1; q, \gamma) Z_{\text{Tutte}}(H_2; q, \gamma) \cdots Z_{\text{Tutte}}(H_l; q, \gamma); \\ Z_{s|t}(H; q, \gamma) &= Z_{s|t}(H_1; q, \gamma) Z_{\text{Tutte}}(H_2; q, \gamma) \cdots Z_{\text{Tutte}}(H_l; q, \gamma); \\ Z_{\text{Tutte}}(H; q, \gamma) &= Z_{\text{Tutte}}(H_1; q, \gamma) Z_{\text{Tutte}}(H_2; q, \gamma) \cdots Z_{\text{Tutte}}(H_l; q, \gamma). \end{aligned}$$

We have already shown that the quantities $Z_{st}(H_1; q, \gamma)$, $Z_{st}(H_j; q, \gamma)$ and $Z_{\text{Tutte}}(H_j; q, \gamma)$ are non-zero for all j . Hence, we obtain $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma) = Z_{s|t}(H_1; q, \gamma)/Z_{st}(H_1; q, \gamma)$, and we can apply our algorithm to H_1 instead of H . Moreover, we have $Z_{st}(H; q, \gamma) \neq 0$ and $Z_{\text{Tutte}}(H; q, \gamma) \neq 0$ as we wanted. This finishes the proof. \square

LEMMA 6.24. *Let q, γ_1 and γ_2 be real algebraic numbers such that $q < 0, \gamma_1 \in (-1, 0)$ and $\gamma_2 > 0$. Let us assume that we have access to an oracle for $\text{SIGN-PLANAR-TUTTE}(q, \gamma_1, \gamma_2)$. Then there exists an algorithm that takes as input:*

- a positive integer ρ ;
- a planar graph $H = (V, E)$ such that, for $\gamma = (\gamma_2 + 1)^\rho - 1$, we have $\gamma \geq (8 \max\{|q|, 1/|q|\})^r$, where $r = \max\{|V|, |E|\}$;
- two distinct connected vertices s and t of H .

This algorithm computes a representation of the algebraic number $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma)$ in polynomial time in ρ and the size of H . Moreover, for such inputs ρ, H and s, t , we have $Z_{st}(H; q, \gamma) \neq 0$ and $Z_{\text{Tutte}}(H; q, \gamma) \neq 0$. If we have access to the more powerful oracle $\text{SIGN-TUTTE}(q, \gamma_1, \gamma_2)$, then we can remove the constraint that H is planar.

PROOF. The algorithm is the same one as that of Lemma 6.21, the only difference being in the interval-stretching technique, as we have already explained. \square

Now we deal with the last part of our reduction, where we reduce the computation of $Z_{\text{Tutte}}(G; q, \gamma)$ to the computation of $Z_{s|t}(H; q, \gamma)/Z_{st}(H; q, \gamma)$ on the subgraphs H of G . First, let us introduce some notation.

DEFINITION 6.25. We say that a pair (q, γ) of algebraic numbers is zero-free for a graph G if $q \neq 0$ and, for every subgraph H of G and every pair of distinct vertices s and t in the same connected component of H , the quantities $Z_{st}(H, q, \gamma)$ and $Z_{\text{Tutte}}(H, q, \gamma)$ are non-zero.

Note that if (q, γ) is zero-free for G , then (q, γ) is also zero-free for any subgraph of H . We consider the following computational problems.

Name: $\text{RATIO}\text{TUTTE}(q, \gamma)$.

Instance: A (multi)graph $G = (V, E)$ such that (q, γ) is zero-free for G and two distinct vertices s and t in the same connected component of G .

Output: A representation of the algebraic number $Z_{s|t}(G; q, \gamma)/Z_{st}(G; q, \gamma)$.

Name: $\text{ZERO}\text{FREE}\text{TUTTE}(q, \gamma)$.

Instance: A (multi)graph $G = (V, E)$ such that (q, γ) is zero-free for G .

Output: A representation of the algebraic number $Z_{\text{Tutte}}(G; q, \gamma)$.

We also consider the planar versions of these problems, $\text{RATIO}\text{PLANAR}\text{TUTTE}(q, \gamma)$ and $\text{ZERO}\text{FREE}\text{PLANAR}\text{TUTTE}(q, \gamma)$. Then we can express the last part of our reduction as a reduction between these two computational problems.

LEMMA 6.26. Let q and γ be algebraic numbers with $q \neq 0$. Then we have the reductions

$$\begin{aligned} \text{ZERO}\text{FREE}\text{PLANAR}\text{TUTTE}(q, \gamma) &\leq_T \text{RATIO}\text{PLANAR}\text{TUTTE}(q, \gamma), \\ \text{ZERO}\text{FREE}\text{TUTTE}(q, \gamma) &\leq_T \text{RATIO}\text{TUTTE}(q, \gamma). \end{aligned}$$

PROOF. First, we show that $\text{ZERO}\text{FREE}\text{PLANAR}\text{TUTTE}(q, \gamma)$ reduces to $\text{RATIO}\text{PLANAR}\text{TUTTE}(q, \gamma)$. Let G be the input of the problem $\text{ZERO}\text{FREE}\text{PLANAR}\text{TUTTE}(q, \gamma)$. The reduction computes a representation of $Z_{\text{Tutte}}(G; q, \gamma)$ as follows. We assume that G is not a tree since it is known how to compute the Tutte polynomial of a tree in polynomial time (Sokal 2005, Example 2.1). Then we

can find an edge $e = (s, t)$ of G that is not a bridge. We are going to use the oracle for $\text{RATIOPLANARTUTTE}(q, \gamma)$ to reduce the computation of $Z_{\text{Tutte}}(G; q, \gamma)$ to that of $Z_{\text{Tutte}}(G \setminus e; q, \gamma)$, where $G \setminus e$ is formed from G by deleting e . Note that if G is planar, then $G \setminus e$ is also planar. Since (q, γ) is zero-free for G , we have $Z_{st}(G; q, \gamma) \neq 0$. Let $\alpha = Z_{s|t}(G; q, \gamma)/Z_{st}(G; q, \gamma)$. First, note that $Z_{\text{Tutte}}(G; q, \gamma) = Z_{st}(G; q, \gamma) + Z_{s|t}(G; q, \gamma) = Z_{st}(G; q, \gamma)(1 + \alpha)$.

By calling the oracle the algorithm obtains a representation of the factor $1 + \alpha$. Since e is not a bridge, s and t are connected in $G \setminus e$, so, by calling the oracle again, the algorithm has access to a representation of the algebraic number $\beta = Z_{s|t}(G \setminus e; q, \gamma)/Z_{st}(G \setminus e; q, \gamma)$. We have

$$\begin{aligned} Z_{st}(G; q, \gamma) &= Z_{st}(G \setminus e; q, \gamma)(1 + \gamma) + \gamma q^{-1} Z_{s|t}(G \setminus e; q, \gamma) \\ &= Z_{\text{Tutte}}(G \setminus e; q, \gamma) \left(\frac{1 + \gamma}{1 + \beta} + \gamma q^{-1} \frac{\beta}{1 + \beta} \right), \end{aligned}$$

where we multiplied and divided by $Z_{\text{Tutte}}(G \setminus e; q, \gamma) = Z_{st}(G \setminus e; q, \gamma)(1 + \beta)$, which is non-zero since (q, γ) is zero-free for G . Note that the fact that $Z_{\text{Tutte}}(G \setminus e; q, \gamma) \neq 0$ is equivalent to $\beta \neq -1$. We obtain

$$Z_{\text{Tutte}}(G; q, \gamma) = Z_{\text{Tutte}}(G \setminus e; q, \gamma) (1 + \gamma + \gamma q^{-1} \beta) \frac{1 + \alpha}{1 + \beta}.$$

The algorithm then computes a representation of $Z_{\text{Tutte}}(G \setminus e; q, \gamma)$ recursively. Note that this reduction also works between the non-planar versions of the problems. \square

In the rest of this section we put our reduction together. There is one result for each one of the cases $q > 1$, $0 < q < 1$ and $q < 0$ (see Lemmas 6.27, 6.28 and 6.31).

LEMMA 6.27. *Let K be a real number with $K > 1$. Let q, γ_1 and γ_2 be real algebraic numbers such that $q > 1$, $\gamma_1 \in (-2, -1)$ and $\gamma_2 > 0$. Then we have the following reductions:*

$$\begin{aligned} \text{PLANARTUTTE}(q, \gamma_2) &\leq_T \\ &\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma_1, \gamma_2), \\ \text{PLANARTUTTE}(q, \gamma_2) &\leq_T \text{SIGN-PLANARTUTTE}(q, \gamma_1, \gamma_2), \end{aligned}$$

where \leq_T denotes a Turing reduction. Moreover, these reductions also hold for the analogous non-planar problems.

PROOF. We claim that the problems $\text{PLANARTUTTE}(q, \gamma_2)$ and $\text{ZEROFREEPLANARTUTTE}(q, \gamma_2)$ are equivalent. This follows from the fact that (q, γ_2) is zero-free for every graph G . Lemma 6.26 gives us a reduction from $\text{ZEROFREEPLANARTUTTE}(q, \gamma_2)$ to $\text{RATIOPLANARTUTTE}(q, \gamma_2)$. Recall that we have $q > 0$, $\gamma_1 \in (-2, -1)$ and $\gamma_2 > 0$. Thus, we can apply Lemma 6.13 with $\rho = 1$ to obtain a reduction from the problem $\text{RATIOPLANARTUTTE}(q, \gamma_2)$ to the problem $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma_1, \gamma_2)$, which gives the first reduction of the statement. The second reduction is derived analogously, but this time we apply Lemma 6.18 instead of Lemma 6.13. Finally, note that our reductions also hold for the non-planar version of the problems since the algorithms given in Lemma 6.13 and Lemma 6.18 work for arbitrary graphs (non-necessarily planar) as long as the oracle does. \square

LEMMA 6.28. *Let K be a real number with $K > 1$. Let q, γ_1 and γ_2 be real algebraic numbers such that $0 < q < 1$, $\gamma_1 \in (-1, 0)$ and $\gamma_2 > 0$. Then we have the following reductions:*

$$\begin{aligned} \text{PLANARTUTTE}(q, \gamma_2) &\leq_T \text{SIGN-PLANARTUTTE}(q, \gamma_1, \gamma_2), \\ \text{PLANARTUTTE}(q, \gamma_2) &\leq_T \\ &\quad \text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma_1, \gamma_2). \end{aligned}$$

Moreover, these reductions also hold for the analogous non-planar problems.

PROOF. The proof is analogous to that of Lemma 6.27; now, we instead combine Lemmas 6.26, 6.19 and 6.20. \square

So far we have obtained reductions when $q > 1$ or $0 < q < 1$. To obtain a similar result when $q < 0$ we have to introduce the following variant of $\text{TUTTE}(q, \gamma)$, where q is an algebraic number and γ is a positive real algebraic number.

Name: THICKENEDTUTTE(q, γ).

Instance: A (multi)graph $G = (V, E)$.

Output: A representation of the algebraic number $Z_{\text{Tutte}}(G; q, (\gamma + 1)^{\rho(G)} - 1)$, where $\rho(G)$ is the smallest positive integer such that $(\gamma + 1)^{\rho(G)} - 1 > M(G)$ for $M(G) = (8 \max\{|q|, 1/|q|\})^r$ and $r = \max\{|V|, |E|\}$.

We also consider the planar version of this problem, THICKENEDPLANARTUTTE(q, γ), where the input graph is promised to be planar.

LEMMA 6.29. *Let q be an algebraic number and let γ be a real algebraic number with $\gamma > 0$. Then THICKENEDPLANARTUTTE(q, γ) is #P-hard unless $q \in \{1, 2\}$, and THICKENEDTUTTE(q, γ) is #P-hard unless $q = 1$.*

PROOF. We are going to reduce the problem PLANARTUTTE($q, 2$) to THICKENEDPLANARTUTTE(q, γ). The result then follows from the #P-hardness of PLANARTUTTE($q, 2$), cf. Theorem 6.8.

Let G be an m -edge instance of PLANARTUTTE($q, 2$). For $j = 1, \dots, m$, let G_j be the graph obtained from G by j -thickening each of its edges. We have $M(G_j) = (8 \max\{|q|, 1/|q|\})^{\max\{n, jm\}}$ so $M(G_j)$, and therefore $\rho(G_j)$, are non-decreasing in j . Let $\gamma_j = (\gamma + 1)^{j\rho(G_j)} - 1$ and note that $Z_{\text{Tutte}}(G_j; q, (\gamma + 1)^{\rho(G_j)} - 1) = Z_{\text{Tutte}}(G; q, \gamma_j)$. Note that the points $\gamma_1, \dots, \gamma_m$ are distinct because $j\rho(G_j) \leq j\rho(G_{j+1}) < (j+1)\rho(G_{j+1})$ for every j . Moreover, their representation is polynomial in the size of G , and hence so is the representation of $Z_{\text{Tutte}}(G; q, \gamma_j)$.

The reduction constructs G_1, \dots, G_m and computes $Z_{\text{Tutte}}(G; q, \gamma_j)$ using the oracle for THICKENEDPLANARTUTTE(q, γ) with input G_j . By interpolation, we then recover the polynomial $Z_{\text{Tutte}}(G; q, x)$, whose degree is m when q is viewed as a constant, in time polynomial in the size of G . The reduction is then completed by evaluating $Z_{\text{Tutte}}(G; q, x)$ at $x = 2$.

Finally note that this reduction also works from TUTTE($q, 2$) to THICKENEDTUTTE(q, γ). The only difference is that TUTTE($q, 2$) is also #P-hard for $q = 2$ (see Theorem 6.7), so we also get #P-hardness in this case. \square

We are going to reduce $\text{THICKENEDPLANARTUTTE}(q, \gamma_2)$ to $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma_1, \gamma_2)$ for appropriate γ_1 and γ_2 . In order to do so, we need to adapt Lemma 6.26 to this context. For this purpose, we consider the following computational problems.

Name: $\text{RATIOThICKENEDTUTTE}(q, \gamma)$.

Instance: A (multi)graph $G = (V, E)$, two distinct connected vertices s and t of G , and a positive integer ρ such that, for $\gamma_\rho = (\gamma + 1)^\rho - 1$, (q, γ_ρ) is zero-free for G and $\gamma_\rho > M(G)$, where $M(G) = (8 \max\{|q|, 1/|q|\})^r$ and $r = \max\{|V|, |E|\}$.

Output: A representation of the algebraic number $Z_{st}(G; q, \gamma_\rho) / Z_{st}(G; q, \gamma)$.

Name: $\text{ZEROFreeThICKENEDTUTTE}(q, \gamma)$.

Instance: A (multi)graph $G = (V, E)$ and a positive integer ρ such that, for $\gamma_\rho = (\gamma + 1)^\rho - 1$, (q, γ_ρ) is zero-free for G and $\gamma_\rho > M(G)$, where $M(G) = (8 \max\{|q|, 1/|q|\})^r$ and $r = \max\{|V|, |E|\}$.

Output: A representation of the algebraic number $Z_{\text{Tutte}}(G; q, \gamma_\rho)$.

We also consider the planar versions of these problems, $\text{RATIOThICKENEDPLANARTUTTE}(q, \gamma)$ and $\text{ZEROFreeThICKENEDPLANARTUTTE}(q, \gamma)$.

LEMMA 6.30. *Let q and γ be algebraic numbers with $q \neq 0$. Then we have the reductions*

$$\begin{aligned} \text{ZEROFreeThICKENEDPLANARTUTTE}(q, \gamma) &\leq_T \\ &\quad \text{RATIOThICKENEDPLANARTUTTE}(q, \gamma), \\ \text{ZEROFreeThICKENEDTUTTE}(q, \gamma) &\leq_T \\ &\quad \text{RATIOThICKENEDTUTTE}(q, \gamma). \end{aligned}$$

PROOF. The reduction is almost exactly the one explained in Lemma 6.26. The only difference is that, for an input (G, ρ) , each call to the oracle has as parameters a subgraph H of G , two vertices s and t determined in the reduction, and the same positive integer ρ . \square

LEMMA 6.31. *Let K be a real number with $K > 1$. Let q , γ_1 and γ_2 be real algebraic numbers such that $q < 0$, $\gamma_1 \in (-1, 0)$ and $\gamma_2 > 0$. Then we have the following reductions:*

$$\begin{aligned} \text{THICKENEDPLANARTUTTE}(q, \gamma_2) &\leq_T \\ &\quad \text{SIGN-PLANARTUTTE}(q, \gamma_1, \gamma_2), \\ \text{THICKENEDPLANARTUTTE}(q, \gamma_2) &\leq_T \\ &\quad \text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma_1, \gamma_2). \end{aligned}$$

Moreover, these reductions also hold for the analogous non-planar problems.

PROOF. Let G and ρ be the inputs of the problem $\text{THICKENEDPLANARTUTTE}(q, \gamma_2)$. Let H be a subgraph of G and let s and t be two distinct connected vertices of H . By applying Lemma 6.21 we find that $Z_{st}(H; q, (\gamma_2 + 1)^\rho - 1)$ and $Z_{\text{Tutte}}(H; q, (\gamma_2 + 1)^\rho - 1)$ are non-zero. Hence, $(q, (\gamma_2 + 1)^\rho - 1)$ is zero-free for G . This shows that $\text{THICKENEDPLANARTUTTE}(q, \gamma_2)$ reduces to $\text{ZEROFREETHICKENEDPLANARTUTTE}(q, \gamma_2)$.

Lemma 6.30 gives us a reduction from the problem $\text{ZEROFREETHICKENEDPLANARTUTTE}(q, \gamma_2)$ to $\text{RATIOHICKENEDPLANARTUTTE}(q, \gamma_2)$. Recall that we have $q < 0$, $\gamma_1 \in (-1, 0)$ and $\gamma_2 > 0$. Thus, Lemma 6.21 gives a reduction from $\text{RATIOHICKENEDPLANARTUTTE}(q, \gamma_2)$ to $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma_1, \gamma_2)$, which completes the proof for the first reduction of the statement. The second reduction is analogous, but this time we apply Lemma 6.24 instead of Lemma 6.21. Finally, note that our reductions also hold for the analogous non-planar problems since the algorithms given in Lemma 6.21 and Lemma 6.24 work for arbitrary graphs (non-necessarily planar) as long as the oracle does. \square

6.6. The connection between approximate shifts and reductions. In this section we show how a polynomial-time approximate shift from (q, γ_1) to (q, γ_2) may allow us to reduce the problems of approximating the norm of the Tutte polynomial at (q, γ_2) to the same problem at (q, γ_1) (see Lemma 2.4). We also derive a similar result for the problem $\text{DISTANCE-}\rho\text{-ARGTUTTE}(q, \gamma)$ in Lemma 6.34.

LEMMA 6.32. *Let q , γ_1 and γ_2 be algebraic numbers with $q \neq 0$ such that there is a polynomial-time series-parallel approximate shift from (q, γ_1) to (q, γ_2) . Then there is an algorithm that has as input a graph G and a positive integer k and computes, in polynomial time in k and the size of G , a graph H and a representation of an algebraic number D with $D \neq 0$ such that*

$$\left| Z_{\text{Tutte}}(G; q, \gamma_2) - \frac{Z_{\text{Tutte}}(H; q, \gamma_1)}{D} \right| \leq 2^{-k}.$$

Moreover, if the graph G is planar, then the graph H is also planar, and if q and γ_1 are real, then D is also real.

PROOF. Let $G = (V, E)$ and k be the inputs of the algorithm. Let $n = |V|$ and $m = |E|$. By the definition of series-parallel polynomial-time approximate shifts, for any positive integer j , one can compute, in polynomial time in j , a series-parallel graph J_j that γ_1 -implements $\hat{\gamma}$ with $|\gamma_2 - \hat{\gamma}| \leq 2^{-j}$ for terminals s and t . By definition of implementations, we have $\hat{\gamma} = qZ_{st}(J_j; q, \gamma_1)/Z_{s|t}(J_j; q, \gamma_1)$ and $Z_{s|t}(J_j; q, \gamma_1) \neq 0$. We construct a graph G_j that is a copy of G where every edge f in G has been replaced by a copy of J_j as in Lemma 3.2, identifying the endpoints of f with s and t . In light of Lemma 3.2, we have

$$Z_{\text{Tutte}}(G_j; q, \gamma_1) = \left(\frac{Z_{s|t}(J_j; q, \gamma_1)}{q^2} \right)^m Z_{\text{Tutte}}(G; q, \hat{\gamma}).$$

We can compute a representation of $D_j = Z_{s|t}(J_j; q, \gamma_1)/q^2$ in polynomial time in the size of J_j because J_j is a series-parallel graph. However, note that this hypothesis is not essential as long as there is some way to compute a representation of D_j while constructing J_j . Note that $|\hat{\gamma}| \leq |\gamma_2| + 2^{-j}$, so $|Z_{\text{Tutte}}(G; q, \gamma_2) - Z_{\text{Tutte}}(G; q, \hat{\gamma})|$ is upper bounded by

$$\begin{aligned} \sum_{A \subseteq E} |q|^{k(A)} \left| \gamma_2^{|A|} - \hat{\gamma}^{|A|} \right| &\leq \sum_{A \subseteq E} |q|^{k(A)} |\gamma_2 - \hat{\gamma}| \sum_{t=0}^{|A|-1} \left| \gamma_2^{|A|-1-t} \hat{\gamma}^t \right| \\ &\leq \sum_{A \subseteq E} |q|^{k(A)} |\gamma_2 - \hat{\gamma}| (|A|-1) \left(|\gamma_2| + s \frac{1}{2} \right)^{|A|-1} \\ &\leq |\gamma_2 - \hat{\gamma}| |q|^n 2^m (m-1) \left(|\gamma_2| + \frac{1}{2} \right)^{m-1}. \end{aligned}$$

Hence, for j such that $2^{-j}|q|^n 2^m(m-1)(|\gamma_2| + 1/2)^{m-1} \leq 2^{-k}$, which can be achieved for $j = O(\text{size}(G) + k)$, we obtain

$$\left| Z_{\text{Tutte}}(G; q, \gamma_2) - \frac{Z_{\text{Tutte}}(G_j; q, \gamma_1)}{D_j^m} \right| = |Z_{\text{Tutte}}(G; q, \gamma_2) - Z_{\text{Tutte}}(G; q, \hat{\gamma})| \leq 2^{-k}.$$

The algorithm returns $H = G_j$ and $D = D_j^m \neq 0$. Note that if G is planar, then $H = G_j$ is also planar by construction. If q and γ_1 are real, then the number $D = (Z_{\text{st}}(J_j; q, \gamma_1)/q^2)^m$ is clearly real too. \square

In the rest of this section we use Lemma 6.32 to translate information about the function $Z_{\text{Tutte}}(-; q, \gamma_1)$ for certain graphs to information about $Z_{\text{Tutte}}(G; q, \gamma_2)$. This leads to the reductions given in Lemmas 2.4 and 6.34. These results are stated for polynomial series-parallel approximate shifts, but they would also hold even if the shifts are not series-parallel as long as, in the proof of Lemma 6.32, the graphs J_j are planar and we can compute $D_j = Z_{\text{st}}(J_j; q, \gamma_1)/q^2$ in polynomial time in the size of J_j .

We are now ready to prove Lemma 2.4, which was stated in the Proof Outline, and which we restate here for convenience.

LEMMA 2.4. *Let $q \neq 0$, γ_1 and $\gamma_2 \neq 0$ be algebraic numbers, and $K > 1$. For $j \in \{1, 2\}$, let $y_j = \gamma_j + 1$ and $x_j = 1 + q/\gamma_j$. If there is a polynomial-time series-parallel approximate shift from (x_1, y_1) to (x_2, y_2) , then we have a reduction from FACTOR- K -NORMTUTTE(q, γ_2) to FACTOR- K -NORMTUTTE(q, γ_1). This reduction also holds for the planar version of the problem.*

PROOF. We are going to solve the problem FACTOR- $4K$ -NONZERO-NORMTUTTE(q, γ_2) in polynomial time with the help of an oracle for FACTOR- K -NORMTUTTE(q, γ_1). Recall that hardness of these problems does not depend on K (see Section 1.1). Let $C_{q, \gamma_2} > 1$ be the constant computed in Corollary 6.3 for the algebraic numbers q and $\gamma = \gamma_2$; so, for any graph G , either $Z_{\text{Tutte}}(G; q, \gamma_2) = 0$ or $|Z_{\text{Tutte}}(G; q, \gamma_2)| \geq C_{q, \gamma_2}^{-\text{size}(G)}$. Let $G = (V, E)$ be the input of FACTOR- $4K$ -NONZERO-NORMTUTTE(q, γ_2). We

assume that $Z_{\text{Tutte}}(G; q, \gamma_2) \neq 0$ since otherwise we can output anything. Let k be the smallest integer such that $2^{-k} \leq C_{q, \gamma_2}^{-\text{size}(G)}/2$. The reduction uses the algorithm given in Lemma 6.32 to compute a graph H and a representation of an algebraic number D with $D \neq 0$ such that

$$\begin{aligned}
 (6.33) \quad \left| Z_{\text{Tutte}}(G; q, \gamma_2) - \frac{Z_{\text{Tutte}}(H; q, \gamma_1)}{D} \right| &\leq 2^{-k} \\
 &\leq \frac{C_{q, \gamma_2}^{-\text{size}(G)}}{2} \\
 &\leq \frac{|Z_{\text{Tutte}}(G; q, \gamma_2)|}{2}.
 \end{aligned}$$

Therefore, we have

$$\frac{1}{2} \leq \frac{|Z_{\text{Tutte}}(H; q, \gamma_1)|}{D |Z_{\text{Tutte}}(G; q, \gamma_2)|} \leq \frac{3}{2}.$$

By invoking the oracle for FACTOR- K -NORMTUTTE(q, γ_1), the reduction computes a rational number N such that we have $N/K \leq |Z_{\text{Tutte}}(H; q, \gamma_1)| \leq KN$. The reduction also computes a non-zero rational number \hat{D} such that $1/2 \leq D/\hat{D} \leq 2$. Then $\hat{N} = N/\hat{D}$ satisfies

$$\frac{\hat{N}}{4K} \leq |Z_{\text{Tutte}}(G; q, \gamma_2)| \leq 4K\hat{N},$$

so the reduction outputs \hat{N} as solution of the computational problem FACTOR- $4K$ -NONZERO-NORMTUTTE(q, γ_2). Note that this reduction analogously applies to the planar case since the graph H is planar when G is planar (see Lemma 6.32). \square

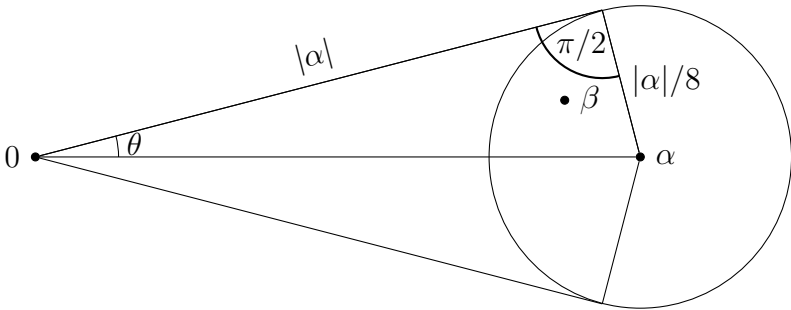
We next give the analogue of Lemma 2.4 for the argument.

LEMMA 6.34. *Let q, γ_1 and γ_2 be algebraic numbers with $q \neq 0$. If there is a polynomial-time series-parallel approximate shift from (q, γ_1) to (q, γ_2) , then DISTANCE- $5\pi/12$ -ARGTUTTE(q, γ_2) reduces to DISTANCE- $\pi/3$ -ARGTUTTE(q, γ_1). This reduction also holds for the planar version of the problem.*

PROOF. Let C_{q,γ_2} be the constant computed in Corollary 6.3 for $\gamma = \gamma_2$. Let $G = (V, E)$ be the input of the problem DISTANCE- $\pi/2$ -ARGTUTTE(q, γ_2). We assume that $Z_{\text{Tutte}}(G; q, \gamma_2) \neq 0$ since otherwise we can output anything. The reduction proceeds again similarly to that of Lemma 2.4. First, it applies Lemma 6.32 for appropriate k as in (6.33) to compute a graph H and a representation of a real algebraic number D with $D \neq 0$ such that

$$\begin{aligned}
 (6.35) \quad \left| Z_{\text{Tutte}}(G; q, \gamma_2) - \frac{Z_{\text{Tutte}}(H; q, \gamma_1)}{D} \right| &\leq 2^{-k-2} \\
 &\leq \frac{C_{q,\gamma_2}^{-\text{size}(G)}}{8} \\
 &\leq \frac{|Z_{\text{Tutte}}(G; q, \gamma_2)|}{8}.
 \end{aligned}$$

Let $\alpha = Z_{\text{Tutte}}(G; q, \gamma_2)$ and $\beta = Z_{\text{Tutte}}(H; q, \gamma_1)/D$, so (6.35) can be rewritten as $|\alpha - \beta| \leq |\alpha|/8$. We claim that $|\text{Arg}(\alpha) - \text{Arg}(\beta)| \leq \pi/24$. Since β is in the disc of centre α and radius $|\alpha|/8$, by basic geometry, we have



so $\sin(\theta) = 1/8$, where θ is the angle between $0, \alpha$ and the intersection of the circle of radius $|\alpha|/8$ and center α with the tangent line that goes through 0 . Since $\sin(\pi/24) > 1/8$, we conclude that $|\text{Arg}(\alpha) - \text{Arg}(\beta)| \leq \theta \leq \pi/24$ as we claimed. By invoking the oracle for DISTANCE- $\pi/3$ -ARGTUTTE(q, γ_1), the reduction computes a rational number \hat{A}_1 such that, for some $a_1 \in \arg(Z_{\text{Tutte}}(H; q, \gamma_1))$, we have $|a_1 - \hat{A}_1| \leq \pi/3$. Since the reduction has at its disposal a representation of the algebraic number D , it can compute (in polynomial time in the length of this representation) a

rational number \hat{A}_2 such that, for some $a_2 \in \arg(D)$, we have $|a_2 - \hat{A}_2| \leq \pi/24$. The reduction outputs $\hat{A} = \hat{A}_1 - \hat{A}_2$. We claim that there is an argument a of α such that $|a - \hat{A}| \leq 5\pi/12$. Note that $b = a_1 - a_2$ is an argument of β . By the triangle inequality, we have $|b - \hat{A}| \leq |a_1 - \hat{A}_1| + |\hat{A}_2 - a_2| \leq 9\pi/24$. Let $a = \text{Arg}(\alpha) + (b - \text{Arg}(\beta))$, which is an argument of α . We conclude that

$$|a - \hat{A}| \leq |b - \hat{A}| + |a - b| = |b - \hat{A}| + |\text{Arg}(\alpha) - \text{Arg}(\beta)| \leq 5\pi/12.$$

This reduction analogously applies to the planar case since the graph H is planar when G is planar (see Lemma 6.32). \square

One could actually change the angles $\rho_2 = 5\pi/12$ and $\rho_1 = \pi/3$ in the statement of Lemma 6.34 as long as $\rho_1 < \rho_2$, but $\rho_2 = 5\pi/12$ and $\rho_1 = \pi/3$ will suffice for our purposes.

6.7. Hardness for the Tutte polynomial. In this section we use the reductions of Section 6.5 to obtain intermediate hardness results that will be used to obtain our main theorems in the upcoming sections. We start with the following corollary which strengthens previous results of Goldberg & Jerrum (2014) (that applied to general graphs rather than planar).

COROLLARY 6.36. *Let $K > 1$ be a real number. Let $q \neq 0, 2$, and γ_1, γ_2 be real algebraic numbers with $\gamma_2 \in (-\infty, -2) \cup (0, \infty)$ and either*

- $\circ q > 1, \gamma_1 \in (-2, -1),$ or
- $\circ q < 1, \gamma_1 \in (-1, 0).$

Then, the problems $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma_1, \gamma_2)$ and $\text{SIGN-PLANARTUTTE}(q, \gamma_1, \gamma_2)$ are $\#P$ -hard.

PROOF. We consider first the case when $\gamma_2 > 0$. For q, γ_1, γ_2 as in the first item, the conclusion follows from Theorem 6.8 and the reductions given in Lemma 6.27. For the second item: when $q \in (0, 1)$, the result follows from the reductions given in Lemma 6.28 and Theorem 6.8, while for $q < 0$, the result follows from Lemmas 6.29 and 6.31.

The other case is when $\gamma_2 < -2$. Then, we can γ_2 -implement $(\gamma_2 + 1)^2 - 1 > 0$ with a 2-thickening and proceed as in the previous case. \square

LEMMA 6.37. *Let K be a real number with $K > 1$. Let x, y be a real algebraic numbers such that $(x, y) \neq (-1, -1)$, $\min\{x, y\} \leq -1$ and $\max\{x, y\} < 0$. Let $q = (x - 1)(y - 1)$ and $\gamma = y - 1$. Then the problems $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma)$ and $\text{SIGN-PLANARTUTTE}(q, \gamma)$ are $\#\text{P}$ -hard.*

PROOF. Note that $q > 2$. We claim that we can (x, y) -implement (x_1, y_2) with $y_1 \in (-1, 0)$, and (x_2, y_2) with $|y_2| > 1$ using planar (in fact, series-parallel) graphs. The result then follows by invoking Corollary 6.36 with $\gamma_1 = y_1 - 1$ and $\gamma_2 = y_2 - 1$.

The case $\min\{x, y\} < -1$ is treated in (Goldberg & Jerrum 2014, Lemmas 8–11). Hence, we may assume that $-1 \leq x < 0$ and $-1 \leq y < 0$. Since $(x, y) \neq (-1, -1)$ by hypothesis, there are two cases:

- $x = -1$ and $-1 < y < 0$. As pointed out in (Goldberg & Jerrum 2014, Corollary 26), a 3-thickening from (x, y) implements the point $(x', y') = \left(1 - \frac{2}{1+y+y^2}, y^3\right)$ with $x' < -1$ and $y' \in (-1, 0)$, so the point (x', y') has already been studied in this proof.
- $-1 < x < 0$ and $y = -1$. This time we perform a 3-stretching from (x, y) to implement a point (x', y') with $x' \in (-1, 0)$ and $y' < -1$. \square

LEMMA 6.38. *Let $K > 1$ be a real number and q, x, y be real algebraic numbers with $\max\{|x|, |y|\} < 1$ and $q = (x - 1)(y - 1) > 32/27$. Then, for $\gamma = y - 1$, $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma)$ and $\text{SIGN-PLANARTUTTE}(q, \gamma)$ are $\#\text{P}$ -hard, unless $q = 2$.*

PROOF. In view of (Goldberg & Jerrum 2014, Lemmas 12 and 15), we can (x, y) -implement points (x_1, y_1) and (x_2, y_2) with $y_1 \in (-1, 0)$ and $y_2 > 1$. These implementations only use series-parallel graphs. Hence, we can apply (the first item of) Corollary 6.36 with $\gamma_1 = y_1 - 1$ and $\gamma_2 = y_2 - 1$ to finish the proof. \square

6.8. Proofs of our main theorems. In this section we show how our main Theorems 1.2, 1.3, 1.4 and 1.5 follow from the #P-hardness results of Section 6.7. We start with Theorem 1.5.

THEOREM 1.5. *Let $q > 2$ be a real, $\gamma \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic number, and $K > 1$. Then, FACTOR- K -NORMPLANARTUTTE(q, γ) and DISTANCE- $\pi/3$ -ARGPLANARTUTTE(q, γ) are #P-hard, unless $q = 3$ and $\gamma + 1 \in \{e^{2\pi i/3}, e^{4\pi i/3}\}$ when both problems can be solved exactly in polynomial time.*

PROOF. Let $(x, y) \in \mathcal{H}_q$ be such that $y = \gamma + 1$. Consider the point $(x_2, y_2) \in H_q$ with $y_2 = -1/2$ and $x_2 = 1 + q/(y_2 - 1)$. Note that $x_2 = 1 - 2q/3 \leq 1 - 4/3 < 0$. There are two cases. Either $x_2 \leq -1$ and the point (x_2, y_2) satisfies the hypothesis of Lemma 6.37, or $-1 < x_2 < 0$ and the point (x_2, y_2) satisfies the hypothesis of Lemma 6.38. In any case, we conclude that the computational problems SIGN-PLANARTUTTE(q, γ_2) and FACTOR- K -NORMPLANARTUTTE(q, γ_2) are #P-hard for $\gamma_2 = y_2 - 1$ when $q > 2$.

By Lemma 2.4 (for $\gamma_1 = \gamma$ and $\gamma_2 = \gamma_2$), we see that the problem FACTOR- K -NORMPLANARTUTTE(q, γ_2) reduces to the problem FACTOR- K -NORMPLANARTUTTE(q, γ), proving that the latter is #P-hard too. The proof for DISTANCE- $\pi/3$ -ARGPLANARTUTTE(q, γ) is analogous: first observe that since q, γ_2 are real and $5\pi/12 < \pi/2$, the problem SIGN-PLANARTUTTE(q, γ_2) reduces (trivially) to DISTANCE- $5\pi/12$ -ARGPLANARTUTTE(q, γ_2). Moreover, applying Theorem 2.1 with x and y as above, $y' = y_2 \in (-1, 0)$ and $x' = x_2$, we have a polynomial-time series-parallel approximate shift from (x, y) to (x', y') or, equivalently, from (q, γ) to (q, γ_2) . Using Lemma 6.34 with $\gamma_1 = \gamma$ and $\gamma_2 = \gamma_2$, we conclude that the problem DISTANCE- $5\pi/12$ -ARGPLANARTUTTE(q, γ_2) reduces to DISTANCE- $\pi/3$ -ARGPLANARTUTTE(q, γ), proving that the latter is #P-hard, as wanted. \square

THEOREM 1.4. *Let $y \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic number, and $K > 1$. Then, FACTOR- K -NORMISING(y) and DISTANCE- $\pi/3$ -ARGISING(y) are #P-hard, unless $y = \pm i$ when both problems can be solved exactly in polynomial time.*

PROOF. Let $q = 2$, $\gamma = y - 1$, $y_2 = -1/2$, $\gamma_2 = y_2 - 1$. From the result of Goldberg and Guo [Goldberg & Guo \(2017\)](#), the problems $\text{FACTOR-}K\text{-NORMISING}(y_2)$ and $\text{DISTANCE-}\pi/3\text{-ARGISING}(y_2)$ are $\#P$ -hard, hence the problems $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma_2)$ and $\text{DISTANCE-}\pi/3\text{-ARGPLANARTUTTE}(q, \gamma_2)$ are $\#P$ -hard as well, using that $Z_{\text{Ising}}(G; y_2) = Z_{\text{Tutte}}(G; 2, \gamma_2)$.

By applying Lemma [2.4](#) and Theorem [2.1](#) analogously to the proof of Theorem [1.5](#), we conclude that the problems $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma)$ and $\text{DISTANCE-}\pi/3\text{-ARGPLANARTUTTE}(q, \gamma)$ are $\#P$ -hard, and hence $\text{FACTOR-}K\text{-NORMISING}(y)$ and $\text{DISTANCE-}\pi/3\text{-ARGISING}(y)$, using that $Z_{\text{Ising}}(G; y) = Z_{\text{Tutte}}(G; 2, \gamma)$. \square

THEOREM [1.2](#). *Let $q \geq 3$ be an integer, $y \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic number, and $K > 1$. Then, $\text{FACTOR-}K\text{-NORMPLANARPOTTS}(q, y)$ and $\text{DISTANCE-}\pi/3\text{-ARGPLANARPOTTS}(q, y)$ are $\#P$ -hard, unless $q = 3$ and $y \in \{e^{2\pi i/3}, e^{4\pi i/3}\}$ when both problems can be solved exactly in polynomial time.*

PROOF. Just apply Theorem [1.5](#) to the integer q , and use the equality $Z_{\text{Potts}}(G; q, y) = Z_{\text{Tutte}}(G; q, y - 1)$. \square

THEOREM [1.3](#). *Let $q \geq 3$ be an integer, $y \in (-q + 1, 0)$ be a real algebraic number, and $K > 1$. Then the computational problems $\text{FACTOR-}K\text{-NORMPLANARPOTTS}(q, y)$ and $\text{DISTANCE-}\pi/3\text{-ARGPLANARPOTTS}(q, y)$ are $\#P$ -hard, unless we are in the case $(q, y) = (4, -1)$ when both problems can be solved exactly in polynomial time.*

PROOF. Let $y \in (-q + 1, 0)$. The point (x, y) with $x = 1 + q/(y - 1)$ satisfies $x \in (1 - q, 0)$, $(x, y) \neq (-1, -1)$ and $y < 0$. If $x \leq -1$ or $y \leq -1$, $\#P$ -hardness follows from Lemma [6.37](#). Otherwise, we have $q \geq 3$ and $x, y \in (-1, 0)$, so hardness follows from Lemma [6.38](#). \square

7. Further consequences of our results

In this final section, we discuss some further consequences of our techniques, as mentioned in Section 1.2. First, in Section 7.1, we explain how our results can be used to obtain hardness for the computational problems $\text{SIGN-PLANARTUTTE}(q, \gamma)$ and $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma)$ (and the non-planar version of these problems) at other parameters than the ones studied in Section 6.7, building on work of [Goldberg & Jerrum \(2014\)](#). Secondly, in Section 7.2, we apply our results to the problem of approximating the Jones polynomial of an alternating link, which is connected to the quantum complexity class BQP as explained in ([Bordewich et al. 2005](#)).

7.1. Hardness results for real algebraic parameters in the Tutte plane. The regions studied in Lemmas 6.37 and 6.38 have been studied by [Goldberg & Jerrum \(2014\)](#), where they showed $\#P$ -hardness of $\text{SIGNPLANARTUTTE}(q, \gamma)$ at several regions of the real algebraic plane. As we explained in Section 6.7, we obtain hardness at a point (q, γ) as long as we can γ -implement algebraic numbers γ_1 and γ_2 as in Corollary 6.36. Goldberg and Jerrum came up with multiple implementations that achieve the conditions of Corollary 6.36. By applying their implementations, we obtain $\#P$ -hardness for $\text{FACTOR-}K\text{-NORMTUTTE}(q, y-1)$ in the same regions where they obtained $\#P$ -hardness of $\text{SIGNPLANARTUTTE}(q, \gamma)$ in ([Goldberg & Jerrum 2014](#), Theorem 1).

Some of the implementations developed in ([Goldberg & Jerrum 2014](#)) consist of planar graphs (as those used in Lemmas 6.37 and 6.38), so we can extend their results to the planar version of the problems for some of the previous regions.

THEOREM 7.1. *Let q and γ be real algebraic numbers with $q \neq 0, 1, 2$. Let $y = \gamma + 1$ and $x = 1 + q/(y - 1)$. The computational problems $\text{SIGN-PLANARTUTTE}(q, \gamma)$ and $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, \gamma)$ are $\#P$ -hard when x, y are real algebraic numbers satisfying one of the following:*

- (i) $\min(x, y) \leq -1$, $\max(x, y) < 0$ and $(x, y) \neq (-1, -1)$,

- (ii) $|x| > 1$, $|y| > 1$ and $xy < 0$,
- (iii) $\max(|x|, |y|) < 1$ and $q > 32/27$,
- (iv) $\max(|x|, |y|) < 1$, $q \leq 32/27$ and $x < -2y - 1$,
- (v) $\max(|x|, |y|) < 1$, $q \leq 32/27$ and $y < -2x - 1$.

PROOF. The proof follows from the following results of [Goldberg & Jerrum \(2014\)](#), which show how to implement γ_1 and γ_2 with a planar (actually series-parallel) graph as in Corollary 6.36 for each of the regions in the statement.

Item i follows from Lemma 6.37. For Item ii, note that $q < 0$, so we have to implement $\gamma_1 \in (-1, 0)$ and $\gamma_2 \notin [-2, 0]$. We choose $\gamma_2 = y - 1$ and γ_1 as implemented in ([Goldberg & Jerrum 2014](#), Lemma 16). Item iii follows from Lemma 6.37. For Item iv, we implement $\gamma_1 \in (-1, 0)$ and $\gamma_2 \notin [-2, 0]$; the implementations are as in ([Goldberg & Jerrum 2014](#), Lemmas 14 and 15). For Item v, we implement $\gamma_1 \in (-1, 0)$ and $\gamma_2 \notin [-2, 0]$; the implementations are as in ([Goldberg & Jerrum 2014](#), Lemmas 13 and 15). \square

The complexity of approximating the Tutte polynomial of a planar graph has previously been studied in ([Goldberg & Jerrum 2012b](#)) and ([Kuperberg 2015](#)). Our result on this matter (Theorem 7.1) strengthens the results of ([Goldberg & Jerrum 2012b](#)) in three directions. First, we also study the complexity of determining the sign of the Tutte polynomial. Secondly, we find new regions where the approximation problem is hard. These regions are 3, 4 and 5, as well as the points in region 1 such that $q \leq 5$ and $q \neq 3$. Finally, we prove $\#P$ -hardness, whereas in ([Goldberg & Jerrum 2012b](#)) hardness was obtained under the hypothesis that $RP \neq NP$.

For $q \in \mathbb{Z}^+$, let $P(G; q)$ count the number of proper q -colourings of a graph G . The *chromatic polynomial* of G is the only polynomial that agrees with $P(G; q)$ on positive integers. It is well known that $P(G; q) = Z_{\text{Tutte}}(G; q, -1)$, see for instance ([Sokal 2005](#)). The value $q = 32/27$ appearing in Theorem 7.1 is, in some sense, a phase transition for the complexity of computing the sign of $P(G, q)$: this sign depends upon G in an essentially trivial way for $q < 32/27$

([Jackson 1993](#), Theorem 5) and its computation is $\#P$ -hard for $q > 32/27$, see ([Goldberg & Jerrum 2014](#)) for an in detail discussion of the relevance of the phase transition $q = 32/27$.

7.2. Hardness results for the Jones polynomial. We briefly review some relevant facts about links and the Jones polynomial that relate it to the Tutte polynomial on graphs, see ([Welsh 1993](#)) for their definitions. Let $V_L(T)$ denote the Jones polynomial of a link L . By a result of Thistlethwaite, when L is an alternating link with associated planar graph $G(L)$, we have $V_L(t) = f_L(t)T(G(L); -t, -t^{-1})$, where $f_L(t)$ is an easily computable factor that is plus or minus a half integer power of t , and $T(G; x, y)$ is the Tutte polynomial of G in the (x, y) -parametrisation ([Thistlethwaite 1987](#); [Welsh 1993](#)). Moreover, every planar graph is the graph of an alternating link ([Welsh 1993](#), Chapter 2). Hence, we can translate our results on the complexity of approximating the Tutte polynomial of a planar graph to the complexity of approximating the Jones polynomial of an alternating link, and obtain $\#P$ -hardness results for approximating $V_L(t)$. More formally, we consider the following problems.

Name: FACTOR- K -NORMJONES(t).

Instance: A link L .

Output: If $V_L(t) = 0$, the algorithm may output any rational number. Otherwise, it must output $\hat{N} \in \mathbb{Q}$ such that $\hat{N}/K \leq |V_L(t)| \leq K\hat{N}$.

Name: DISTANCE- ρ -ARGJONES(q, γ).

Instance: A link L .

Output: If $V_L(t) = 0$, the algorithm may output any rational number. Otherwise, it must output $\hat{A} \in \mathbb{Q}$ such that, for some $a \in \arg(V_L(t))$, we have $|\hat{A} - a| \leq \rho$.

COROLLARY 7.2. *Let K be a real number with $K > 1$. Let t be an algebraic number with $\operatorname{Re}(t) > 0$. Then FACTOR- K -NORMJONES(t) and DISTANCE- $\pi/3$ -ARGJONES(t) are $\#P$ -hard unless $t \in \{1, -e^{2\pi i/3}, -e^{4\pi i/3}\}$ when both problems can be solved exactly.*

PROOF. Let us consider the point $(x, y) = (-t, -t^{-1})$ in the Tutte plane. Note that $t \in \{1, -e^{2\pi i/3}, -e^{4\pi i/3}\}$ if and only if (x, y) is one of the special points $(-1, -1)$, $(e^{4\pi i/3}, e^{2\pi i/3})$ and $(e^{2\pi i/3}, e^{4\pi i/3})$, where the Jones polynomial of a link can be exactly evaluated in polynomial time in the size of the link (Jaeger *et al.* 1990). Let us assume that t is not one of these three values. We have $q = (-t-1)(-t^{-1}-1) = 2+2\operatorname{Re}(t) > 2$. When t is non-real, in view of Theorem 1.5, $\text{FACTOR-}K\text{-NORMPLANARTUTTE}(q, y-1)$ and $\text{DISTANCE-}\pi/3\text{-ARGPLANARTUTTE}(q, y-1)$ are $\#\text{P}$ -hard and the result follows. When t is real, note that $y < 0$, $x < 0$ and $q > 2$. Thus, either (x, y) is such that $\max\{|x|, |y|\} \geq 1$ and $(x, y) \neq (-1, -1)$, so hardness is covered in region 1 of Theorem 7.1, or $\max\{|x|, |y|\} < 1$, so hardness is covered in region 3 of Theorem 7.1. \square

The case $t = e^{2\pi i/5}$ of Corollary 7.2 is particularly relevant due to its connection with quantum computation. This connection between approximate counting and the quantum complexity class BQP was explored by Bordewich, Freedman, Lovász and Welsh in (Bordewich *et al.* 2005), where they posed the question of determining the complexity of the following problem:

Name: SIGN-REAL-PLANARTUTTE(q, γ)

Instance: A planar (multi)graph G .

Output: Determine whether $\operatorname{Re}(Z_{\text{Tutte}}(G; q, \gamma)) \geq 0$
or $\operatorname{Re}(Z_{\text{Tutte}}(G; q, \gamma)) \leq 0$.

The non-planar version of SIGN-REAL-PLANARTUTTE(q, γ) has been studied in (Goldberg & Guo 2017, Section 5), where it was shown that determining the sign of the real part of the Tutte polynomial is $\#\text{P}$ -hard in certain cases that include $t = e^{2\pi i/5}$. Our results on the complexity of SIGN-PLANARTUTTE(q, γ) allow us to adapt the argument in (Goldberg & Guo 2017) to answer the question asked in (Bordewich *et al.* 2005).

COROLLARY 7.3. Consider $(x, y) = (\exp(-a\pi i/b), \exp(a\pi i/b))$, where a and b are positive integers such that $1/2 < a/b < 3/2$ and $a \neq b$. Let $q = (x-1)(y-1)$ and $\gamma = y-1$. Then $q \in (2, 4)$ and SIGN-REAL-PLANARTUTTE(q, γ) is $\#\text{P}$ -hard.

PROOF. The proof is essentially the same one as that of (Goldberg & Guo 2017, Theorem 1.7). First, note that

$$\begin{aligned} q &= (x - 1)(y - 1) = 2 - x - y \\ &= 2 - \exp(-a\pi i/b) - \exp(a\pi i/b) = 2 - 2 \cos(a\pi/b), \end{aligned}$$

which is real. Since $1/2 < a/b < 3/2$ and $a \neq b$, we have $q \in (2, 4)$. A b -thickening allows us to (x, y) -implement $(1 - q/2, -1)$. Since $\text{SIGN-PLANAR-TUTTE}(q, -2)$ is $\#\text{P}$ -hard (see Theorem 7.1), we conclude that $\text{SIGN-REAL-PLANAR-TUTTE}(q, \gamma)$ is $\#\text{P}$ -hard. \square

Corollary 7.3 includes the case where $a = 3$ and $b = 5$. In this case, we have $x = \exp(-a\pi i/b) = -\exp(\pi i) \exp(-3\pi i/5) = -\exp(2\pi i/5)$ and $y = x^{-1}$. That is, $(x, y) = (-t, -t^{-1})$ for $t = \exp(2\pi i/5)$, which is the point of interest in (Bordewich *et al.* 2005).

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