Towards a theory of arithmetic degrees

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1 Introduction

The aim of this paper is to start a systematic investigation of the arithmetic degree of projective schemes as introduced in [1]. One main theme concerns itself with the behaviour of this arithmetic degree under hypersurface sections, see Theorem 2.1.

The classical intersection theory only considers the top-dimensional (or isolated) primary components. However, the notion of arithmetic degree involves the new concept of length-multiplicity of embedded primary ideals as considered in [1], [4], [7], [9], [13]. Therefore it is much harder to control the arithmetic degree under a hypersurface section than in the case for the classical degree theory. We describe in §3 an upper bound for the arithmetic degree in terms of the Castelnuovo-Mumford regularity, see Theorem 3.1. In addition, we generalize Bezout's theorem via iterated hypersurface sections, see Theorem 4.1. We conclude in §5 by studying two examples.

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2 Arithmetic degree and hypersurface sections

Before stating our main result of this section we need to introduce the concept of the length-multiplicity of (embedded) primary components.

Let K be an arbitrary field and S the polynomial ring $K[x_0, \cdots, x_n]$. Let $\mathsf{m} = (x_0, \cdots, x_n)$ be the homogeneous maximal ideal of S. Let I be a homogeneous ideal of S.

Definition ([1]): Let p be a homogeneous prime ideal belonging to I. For a primary decomposition $I = \cap \mathsf{q}$ we take the primary ideal q with $\sqrt{\mathsf{q}} = \mathsf{p}$. Let J be the intersection of all primary components of I with associated prime ideals p_1 such that $\mathsf{p}_1 \underset{\neq}{\subset} \mathsf{p}$. If the prime ideal p is an isolated component of I, we set $J = S$. We define the length-multiplicity of q denoted by $\mathrm{mult}_I(\mathsf{p})$, to be the length ℓ of a maximal strictly increasing chain of ideals

$$\mathsf{q} \cap J =: J_\ell \subset J_{\ell-1} \subset \cdots \subset J_1 \subset J_0 := J$$

where J_k, $1 \leq k \leq \ell - 1$, equals $\mathsf{q}_k \cap J$ for some p-primary ideal q_k.

Despite the non-uniqueness of embedded components, the number $\ell = \mathrm{mult}_I(\mathsf{p})$ is well-defined.

For a finitely generated graded S-module M, let $H(M, \ell)$ be the Hilbert function of M for all integers ℓ, that is, $H(M, \ell)$ is the dimension of the vector space $[M]_\ell$ over K. It is well-known that the Hilbert function of M is a polynomial in ℓ for ℓ large enough. We denote this polynomial by $P(M, \ell)$.

We set $\Delta(H(M, \ell)) = H(M, \ell) - H(M, \ell - 1)$, $\Delta^0 H(M, \ell) = H(M, \ell)$, and $\Delta^r(H(M, \ell)) = \Delta^{r-1}(\Delta H(M, \ell))$ for all integers $r \geq 2$. Moreover, we set

$$\Delta_\tau(H(M, \ell)) = H(M, \ell) - H(M, \ell - \tau)$$

for all integers $\tau \geq 1$. Further, the Hilbert polynomial $P(M, \ell)$ is written as

$$P(M, \ell) = \frac{e}{d!}\ell^d + (\text{lower order terms}), \quad e \neq 0.$$

Then we define h-dim $M = d$ (homogeneous dimension) and degree of M by $\deg M := e$. Also, we write, for any ideal I of S, $\dim I$ and $\deg I$ for h-dim S/I and $\deg S/I$ respectively. In case $P(M, \ell) = 0$, we define h-dim $M = -1$ and $\deg M = \sum_{\ell \in \mathbf{Z}} H(M, \ell)$. In particular, $\dim \mathsf{m} = -1$ and $\deg \mathsf{m} = 1$.

Definition ([1]): For an integer $r \geq -1$, we define

$$\begin{aligned} \text{arith-deg}_r(I) &= \sum_{\substack{\mathsf{p} \text{ is a prime ideal} \\ \text{such that } \dim \mathsf{p} = r}} \mathrm{mult}_I(\mathsf{p}) \cdot \deg \mathsf{p} \\ &= \sum_{\substack{\mathsf{p} \in \mathrm{Ass}\ S/I \\ \text{such that } \dim \mathsf{p} = r}} \mathrm{mult}_I(\mathsf{p}) \cdot \deg \mathsf{p} \end{aligned}$$

Definition ([7]): Let I be a homogeneous ideal of S. Let r be an integer with $r \geq -1$. We define the ideal $I_{\geq r}$ as the intersection of all primary components q of I with $\dim \mathsf{q} \geq r$.

The aim of this section is to prove the following theorem.

Theorem 2.1: *Let r be an integer with $r \geq 0$. Let I be a homogeneous ideal of S. Let F be a homogeneous polynomial of S with degree $(F) = \tau \geq 1$. Assume that F does not belong to any associated prime ideal p of I with $\dim \mathsf{p} \geq r$. Then we have*

$$\textit{arith-deg}_{r-1}(I, F) - \textit{arith-deg}_{r-1}(I_{\geq r+1}, F) \geq \tau \cdot \textit{arith-deg}_r(I)$$

and the equality holds if and only if F does not belong to any associated prime ideal p of I with $\dim \mathsf{p} = r - 1$.

Corollary 2.2: *Under the above condition,*

$$\textit{arith-deg}_{r-1}(I, F) \geq \tau \cdot \textit{arith-deg}_r(I)$$

and the equality holds if and only if F does not belong to any associated prime ideal p of I with $\dim \mathsf{p} = r - 1$ and the ideal $(I_{\geq r+1}, F)$ has no associated prime ideals of dimension $(r - 1)$.

Proof. Corollary 2.2 follows immediately from Theorem 2.1.

We note that Corollary 2.2 and lemma 3 of [8] yield Theorem 2.3 of [13].

We want to consider generic hyperplane sections. The following useful lemma is obtained from [2], (4.2) and [5], (5.2), which was pointed out to us by H. Flenner.

Lemma 2.3: *Let I be a homogeneous ideal of S. We set $A = S/I$. Let $h = 0$ be the defining equation of a generic hyperplane of $\mathbf{P}^n_K$ (K: infinite field). Then we have*

$$\textit{Ass}(A/h) \backslash \{\mathsf{m}\} \subseteqq \bigcup_{\mathsf{p} \in \text{Ass } A} \textit{Min}(A/(\mathsf{p}, h)),$$

where $\textit{Min}(A/(\mathsf{p}, h))$ is the set of minimal primes belonging to (p, h).

Corollary 2.4: *Let r be an integer ≥ 1. Let H be a generic hyperplane in $\mathbf{P}^n_K$, given by $h = 0$. Then we have:*

$$\textit{arith-deg}_r(I) = \textit{arith-deg}_{r-1}(I, h).$$

Proof. Corollary 2.4 follows immediately from Theorem 2.2 and Lemma 2.3..

We note that Corollary 2.4 is stated in [1], page 33 without proof.

Before we turn to the proof of Theorem 2.1, two technical results are needed. First we state a more or less known result describing a different characterization of the arithmetic degree, which is purely algebraic and, in fact, serves as the definition in [7].

Lemma 2.5: *Let r be a non-negative integer. Let I be a homogeneous ideal of S. Then we have*

$$\textit{arith-deg}_r(I) = \Delta^r(P(S/I,\ell) - P(S/I_{\geq r+1},\ell))$$

for all integers ℓ.

Lemma 2.6: *Let r be an integer with $r \geq 1$. Let I be a homogeneous ideal of S and F a homogeneous polynomial of S with $\deg(F) = \tau \geq 1$. Assume that F does not belong to any associated prime ideal $\mathfrak{p}$ of I with $\dim \mathfrak{p} \geq r+1$. Then we have*

$$\begin{aligned} &\textit{arith-deg}_{r-1}(I,F) - \Delta^{r-1}P(S/(I_{\geq r+1},F),\ell) + \Delta^{r-1}P(S/(I,F)_{\geq r},\ell) \\ &\qquad = \tau \cdot \textit{arith-deg}_r(I) + \Delta^{r-1}P([0:F]_{S/I},\ell-\tau) \end{aligned}$$

for all integers ℓ.

Proof. ¿From the exact sequences

$$0 \to [0:F]_{S/I}(-\tau) \to S/I(-\tau) \xrightarrow{F} S/I \to S/(I,F) \to 0$$

and

$$0 \to S/I_{\geq r+1}(-\tau) \xrightarrow{F} S/I_{\geq r+1} \to S/(I_{\geq r+1},F) \to 0,$$

we have

$$\Delta_\tau P(S/I,\ell) = P(S/(I,F),\ell) - P([0:F]_{S/I},\ell-\tau)$$

and

$$\Delta_\tau P(S/I_{\geq r+1},\ell) = P(S/(I_{\geq r+1},F),\ell)$$

for all integers ℓ. From Lemma 2.5, we have

$$\begin{aligned} &\text{arith-deg}_{r-1}(I,F) - \Delta^{r-1}P(S/(I_{\geq r+1},F),\ell) + \Delta^{r-1}P(S/(I,F)_{\geq r},\ell) \\ &\quad = \Delta^{r-1}(P(S/(I,F),\ell) - P(S/(I,F)_{\geq r},\ell)) \\ &\qquad -\Delta^{r-1}P(S/(I_{\geq r+1},F),\ell) + \Delta^{r-1}P(S/(I,F)_{\geq r},\ell) \\ &\quad = \Delta^{r-1}(\Delta_\tau P(S/I,\ell) + P([0:F]_{S/I},\ell-\tau)) \\ &\qquad -\Delta^{r-1}\Delta_\tau P(S/I_{\geq r+1},\ell) \\ &\quad = \Delta_\tau(\Delta^{r-1}(P(S/I,\ell) - P(S/I_{\geq r+1},\ell))) \\ &\qquad +\Delta^{r-1}P([0:F]_{S/I},\ell-\tau) \\ &\quad = \tau \cdot \Delta^r(P(S/I,\ell) - P(S/I_{\geq r+1},\ell)) + \Delta^{r-1}P([0:F]_{S/I},\ell-\tau) \\ &\quad = \tau \cdot \text{arith-deg}_r(I) + \Delta^{r-1}P([0:F]_{S/I},\ell-\tau). \end{aligned}$$

Thus the assertion is proved.

The following Lemma is used in the proof of Theorem 2.1 and Lemma 4.5.

Lemma 2.7: *Let I be a homogeneous ideal of S. Let r be an integer. Let F be a homogeneous polynomial of S with degree $(F) \geq 1$ such that F does not belong to any associated prime ideal $\mathfrak{p}$ of I with $\dim \mathfrak{p} \geq r$. Then we have*

$$(I_{\geq u},F)_{\geq r} = (I,F)_{\geq r}$$

for all integers $u = -1, 0, \cdots, r+1$.

Proof. We want to prove that

$$(I_{\geq u}, F)_{\mathsf{p}} = (I, F)_{\mathsf{p}}$$

for all prime ideals p with $\dim \mathsf{p} \geq r$. We may assume that $F \in \mathsf{p}$. If p does not contain any primary component q of I with $\dim(\mathsf{q}) < u$, the proof is done. Now assume that there is a primary component q of I with $\dim(\mathsf{q}) < u$ such that $\mathsf{p} \supseteqq \mathsf{q}$. Since $r \leq \dim(\mathsf{p}) \leq \dim(\mathsf{q}) < u$, we see that $u = r+1$ and $\mathsf{p} = \sqrt{\mathsf{q}}$. Thus we have that p is an associated prime ideal of I with $\dim \mathsf{p} = r$ and $F \in \mathsf{p}$, which contradicts the hypothesis.

Proof of Theorem 2.1. First we prove the case $r \geq 1$. Applying (2.5) and (2.6), we have

$$\begin{aligned}
&\text{arith-deg}_{r-1}(I, F) - \text{arith-deg}_{r-1}(I_{\geq r+1}, F) \\
&\quad = \tau \cdot \text{arith-deg}_r(I) + \Delta^{r-1} P(S/(I_{\geq r+1}, F), \ell) \\
&\qquad - \Delta^{r-1} P(S/(I, F)_{\geq r}, \ell) + \Delta^{r-1} P([0 : F]_{S/I}, \ell - \tau) \\
&\qquad - \Delta^{r-1} (P(S/(I_{\geq r+1}, F), \ell) - P(S/(I_{\geq r+1}, F)_{\geq r}, \ell)) \\
&\quad = \tau \cdot \text{arith-deg}_r(I) + \Delta^{r-1}[P(S/(I_{\geq r+1}, F)_{\geq r}, \ell) - P(S/(I, F)_{\geq r}, \ell)] \\
&\qquad + \Delta^{r-1} P([0 : F]_{S/I}, \ell - \tau).
\end{aligned}$$

By the assumption, $[0 : F]_{S/I}$ has at most $(r-1)$-dimensional support, which means $\Delta^{r-1} P([0 : F]_{S/I}, \ell - \tau) \geq 0$. Further, $\Delta^{r-1} P([0 : F]_{S/I}, \ell - \tau) = 0$ if and only if F does not belong to any associated prime ideal p with $\dim \mathsf{p} = r-1$. On the other hand, $S/(I_{\geq r+1}, F)_{\geq r} = S/(I, F)_{\geq r}$ by Lemma 2.7. Therefore we have

$$\text{arith-deg}_{r-1}(I, F) - \text{arith-deg}_{r-1}(I_{\geq r+1}, F) \geq \tau \cdot \text{arith-deg}_r(I)$$

and the equality holds if and only if F does not belong to any associated prime ideal p with $\dim \mathsf{p} = r-1$.

Next we prove the case $r = 0$. Now we see that

$$\begin{aligned}
\text{arith-deg}_{-1}(I, F) &= \text{length}_S (I, F)_{\geq 0}/(I, F) \\
&= \sum_{\ell=0}^{N} (\dim_K [S/(I, F)]_\ell - \dim_K [S/(I, F)_{\geq 0}]_\ell) \\
&= \sum_{\ell=0}^{N} (\dim_K [S/I]_\ell - \dim_K [S/I]_{\ell-\tau} + \dim_K \left[[0 : F]_{S/I}\right]_{\ell-\tau} \\
&\quad - \dim_K [S/(I, F)_{\geq 0}]_\ell)
\end{aligned}$$

for large N. Similarly, we see that

$$\begin{aligned}
\text{arith-deg}_{-1}(I_{\geq 1}, F) &= \sum_{\ell=0}^{N} (\dim_K [S/I_{\geq 0}]_\ell - \dim_K [S/I_{\geq 1}]_{\ell-\tau} \\
&\quad - \dim_K [S/(I_{\geq 1}, F)_{\geq 0}]_\ell \big)
\end{aligned}$$

for large N. Hence we have

$$\begin{aligned}
&\text{arith-deg}_{-1}(I,F) - \text{arith-deg}_{-1}(I_{\geq 1},F) \\
&\quad = \sum_{\ell=0}^{N}(\dim_K [I_{\geq 1}/I]_\ell - \dim_K [I_{\geq 1}/I]_{\ell-\tau}) \\
&\qquad - \sum_{\ell=0}^{N} \dim_K [(I_{\geq 1},F)_{\geq 0}/(I,F)_{\geq 0}]_\ell + \sum_{\ell=0}^{N} \dim_K [[0:F]_{S/I}]_{\ell-\tau} \\
&\quad = \sum_{\ell=N-\tau+1}^{N} \dim_K [I_{\geq 1}/I]_\ell - \sum_{\ell=0}^{N} \dim_K [(I_{\geq 1},F)_{\geq 0}/(I,F)_{\geq 0}]_\ell \\
&\qquad + \sum_{\ell=0}^{N} \dim_K [[0:F]_{S/I}]_{\ell-\tau}
\end{aligned}$$

for large N. By the assumption, $\sum_{\ell=0}^{N} \dim_K [[0:F]_{S/I}]_{\ell-\tau} = \text{length}_S([0:F]_{S/I}) \geq 0$ for large N, and $[0:F]_{S/I} = 0$ if and only if F is a non-zero-divisor of S/I. On the other hand, we see $\dim_K [I_{\geq 1}/I]_\ell = P(I_{\geq 1}/I,\ell) = \text{arith-deg}_0(I)$ for large ℓ. Further, we have $(I_{\geq 0},F)_{\mathsf{p}} = (I,F)_{\mathsf{p}}$ for all prime ideals p with $\dim \mathsf{p} = 0$ by Lemma 2.7. Hence we have

$$\text{arith-deg}_{-1}(I,F) - \text{arith-deg}_{-1}(I_{\geq 1},F) \geq \tau \cdot \text{arith-deg}_0(I)$$

and the equality holds if and only if F is a non-zero-divisor of S/I. This completes the proof of Theorem 2.1.

3 Castelnuovo-Mumford regularity

Bayer and Mumford [1] give a bound for the arithmetic degree in terms of the Castelnuovo-Mumford regularity. The aim of this section is to describe improved bound on this degree.

Let $m = m(I)$ be the Castelnuovo-Mumford regularity (see, e.g., [1], [3], [10]) for a homogeneous ideal I of the polynomial ring $S = K[x_0,\cdots,x_n]$. Then our main result is the following theorem.

Theorem 3.1: *Let I be a homogeneous ideal of S. Let $m = m(I)$ be the Castelnuovo-Mumford regularity of I. Then we have, for any integer $r \geq 0$*

$$\textit{arith-deg}_r(I) \leq \Delta^r P(S/I,\ell)$$

for all integers $\ell \geq m-1$.

We want to give two corollaries. The first one shows that (3.1) improves the bound given in [1], Proposition 3.6.

Corollary 3.2: *For all $\ell \geq m$, we have*

$$\begin{aligned}
\textit{arith-deg}_r(I) &\leq \Delta^r P(S/I,\ell-1) \\
&\leq \binom{m+n-r-1}{n-r} \\
&\leq m^{n-r}
\end{aligned}$$

Corollary 3.3: *Let $t = depth\ S/I$. Then we have, for an integer $r \geq 0$,*

$$arith\text{-}deg_r(I) \leq \Delta^r H(S/I, \ell)$$

for all $\ell \geq m + r - t - 1$ if $r - t$ is even, and for all $\ell \geq m + r - t$ if $r - t$ is odd.

Before embarking on the proof of Theorem 3.1 and the corollaries we need two lemmas. The first one follows from [11] Nr.79 (see also [12], Proof of Lemma I.4.3).

Lemma 3.4: *Let I be a homogeneous ideal of S and $t = depth\ S/I$. Then we have*

(a) $P(S/I, \ell) = H(S/I, \ell) - \sum_{i=0}^{d}(-1)^i \dim_K = [H^i_{\mathsf{m}}(S/I)]_\ell$ *for all integers ℓ.*

(b) $P(S/I, \ell) = H(S/I, \ell)$ *for all $\ell \geq m - t$.*

Lemma 3.5: *Let I be a homogeneous ideal of S. Then we have*

$$\Delta^r P(I, \ell) \geq 0$$

for all $\ell \geq m - 1$ and $r \geq 0$.

Proof. For a generic hyperplane H given by $h = 0$, we can take an exact sequence

$$0 \to [0 : h]_I(-1) \to I(-1) \xrightarrow{h} I \to I_H \to 0$$

with $\mathrm{Supp}[0 : h]_I \subseteq \{\mathsf{m}\}$, where $I_H = (I, h)/h$. ¿From the exact sequence, we have

$$\Delta P(I, \ell) = P(I_H, \ell)$$

for all ℓ and I_H is m-regular. Repeating this step, we see that

$$\Delta^r P(I, \ell) = P(I_{H_1\cap\cdots\cap H_r}, \ell)$$

for all ℓ and for generic hyperplanes $H_1, \cdots, H_r$ defined by $h_1 = 0, \cdots, h_r = 0$, resp., where

$$I_{H_1\cap\cdots\cap H_r} = (I, h_1, \cdots, h_r)/(h_1, \cdots, h_r),$$

and that $I_{H_1\cap\cdots\cap H_r}$ is m-regular. So $(I_{H_1\cap\cdots\cap H_r})_{\geq 0}$ is also m-regular and $\mathrm{depth}_S S/(I_{H_1\cap\cdots\cap H_r})_{\geq 0} \geq 1$. Therefore we have

$$\begin{aligned}
\Delta^r P(I, \ell) &= P(I_{H_1\cap\cdots\cap H_r}, \ell) \\
&= P(S, \ell) - P(S/I_{H_1\cap\cdots\cap H_r}, \ell) \\
&= P(S, \ell) - P(S/(I_{H_1\cap\cdots\cap H_r})_{\geq 0}, \ell) \\
&= H(S, \ell) - H(S/(I_{H_1\cap\cdots\cap H_r})_{\geq 0}, \ell) \\
&= H((I_{H_1\cap\cdots\cap H_r})_{\geq 0}, \ell) \geq 0
\end{aligned}$$

for $\ell \geq m - 1$, by (3.4), (b).

Proof of Theorem 3.1. Without the loss of generality, we may assume that I is a saturated ideal. First we prove the case $r = 0$. By Lemma 2.5, we have

$$(*) \qquad \text{arith-deg}_0(I) = P(S/I, \ell) - P(S/I_{\geq 1}, \ell).$$

Now we want to show that $I_{\geq 1}$ is m-regular. From the short exact sequence

$$0 \to I_{\geq 1}/I \to S/I \to S/I_{\geq 1} \to 0$$

and the fact that $I_{\geq 1}/I$ has at most 1-dimensional support, we have

$$0 \to H^1_{\mathsf{m}}(I_{\geq 1}/I) \to H^1_{\mathsf{m}}(S/I) \to H^1_{\mathsf{m}}(S/I_{\geq 1}) \to 0$$

and $H^i_{\mathsf{m}}(S/I) \cong H^i_{\mathsf{m}}(S/I_{\geq 1})$ for $i \geq 2$. Thus we have $I_{\geq 1}$ is m-regular. Hence

$$P(S/I_{\geq 1}, \ell) = H(S/I_{\geq 1}, \ell) \geq 0$$

for all $\ell \geq m-1$, by (3.4), (b). Therefore we have from $(*)$

$$\text{arith-deg}_0(I) \leq P(S/I, \ell)$$

for all $\ell \geq m-1$.

Now let us assume $r > 0$. By Corollary 2.4 we see

$$\text{arith-deg}_r(I) = \text{arith-deg}_0(I, h_1, \cdots, h_r)$$

for generic hyperplanes $h_1, \cdots, h_r$. Thus we have

$$\text{arith-deg}_r(I) \leq P(S/(I, h_1, \cdots, h_r), \ell)$$

for all $\ell \geq m-1$. On the other hand, we see

$$\begin{aligned} P(S/(I, h_1, \cdots, h_r), \ell) &= \Delta P(S/(I, h_1, \cdots, h_{r-1}), \ell) \\ &\vdots \\ &= \Delta^r P(S/I, \ell) \end{aligned}$$

for all ℓ. Hence the assertion is proved.

Proof of Corollary 3.2. By Lemma 3.5, we have

$$\Delta^r P(S/I, \ell) \leq \Delta^r P(S, \ell)$$

for all $\ell \geq m-1$. On the other hand, $\Delta^r P(S, \ell) = \begin{pmatrix} n+\ell-r \\ n-r \end{pmatrix}$. Hence the assertion is proved.

Proof of Corollary 3.3. By Lemma 3.4, we see that

$$\Delta^r P(S/I, \ell) = \Delta^r H(S/I, \ell)$$

for all $\ell \geq m+r-t$, and that

$$\Delta^r P(S/I, m+r-t-1) = \Delta^r H(S/I, m+r-t-1) - (-1)^{r-t} \dim_K \left[H^t_m(S/I)\right]_{m-t-1}.$$

Hence the assertion follows from Theorem 3.1.

4 Bezout-type results

The aim of this section is to state properties of arithmetic degree under iterated hyperplane sections, and Bezout-type results. Our Theorem 4.1 describes a Bezout's theorem in terms of the arithmetic degree.

Theorem 4.1: *Let I be a homogeneous ideal of $S := K[x_0, x_1, \cdots, x_n]$. Let $r \geq 0$ and $s \geq 1$ be integers with $r+1 \geq s$. Let $F_1, \cdots, F_s$ be homogeneous polynomials of S such that F_i does not belong to any associated prime ideal p of $(I, F_1, \cdots, F_{i-1})$ with $\dim \mathsf{p} \geq r-i+1$, for all $i = 1, \cdots, s$. Then we have*

(i) arith-deg$_{r-s}(I, F_1, \cdots, F_s) \geq [\prod_{i=1}^{s}$ degree$(F_i)] \cdot$ arith-deg$_r(I)$;

(ii) We have equality in (i) if and only if $((I, F_1, \cdots, F_{i-1})_{\geq r-i+2}, F_i)$ has no $(r-i)$-dimensional primes and F_i does not belong to any associated prime ideal p of $(I, F_1, \cdots, F_{i-1})$ with $\dim \mathsf{p} = r-i$, for all $i = 1, \cdots, s$;

(iii) Assume that there is an integer t with $-1 \leq t \leq r+1$ such that F_i does not belong to any associated prime ideal p of $(I_{\geq t}, F_1, \cdots, F_{i-1})$ with $\dim \mathsf{p} \geq r-i+1$, for all $i = 1, \cdots, s$, and $(I_{\geq t}, F_1, \cdots, F_s)$ has no $(r-s)$-dimensional associated prime ideals. Then we have equality in (i) if and only if F_i does not belong to any associated prime ideal p of $(I, F_1, \cdots, F_{i-1})$ with $\dim \mathsf{p} = r-i$, for all $i = 1, \cdots, s$.

Proof. (i) and (ii) follow from (2.2). In order to prove (iii) we need Lemma 4.2 and Lemma 4.3 below. First we replace the ideal I of (4.2) by the ideal $I_{\geq t}$ of (iii). Then Lemma 4.3 shows that we can apply (ii) of (4.1). This provides our result (iii).

We note that special cases of (4.1) describe generalizations of classical results in the degree theory (see, e.g., [6], [14]).

We prove the two lemmas.

Lemma 4.2: *Let I be a homogeneous ideal of S. Let r and s be integers with $1 \leq s \leq r+1$. Let $F_1, \cdots, F_s$ be homogeneous polynomials of S with degree $(F_i) \geq 1$, $i = 1, \cdots, s$, such that F_i does not belong to any associated prime ideal p of $(I, F_1, \cdots, F_{i-1})$ with $\dim \mathsf{p} \geq r-i+1$, for all $i = 1, \cdots, s$. If the ideal $(I, F_1, \cdots, F_s)$ has no $(r-s)$-dimensional associated prime ideals, then the ideal $((I, F_1, \cdots, F_{i-1})_{\geq r-i+2}, F_i)$ has no $(r-i)$-dimensional associated prime ideals, for all $i = 1, \cdots, s$.*

Proof. It is easy to see that the ideal $(I, F_1, \cdots, F_i)$ has no $(r-i)$-dimensional associated prime ideals. By Theorem 2.1, we have

$$\text{arith-deg}_{r-i}(I, F_1, \cdots, F_i)_{\geq r-i} - \text{arith-deg}_{r-i}((I, F_1, \cdots, F_{i-1})_{\geq r-i+2}, F_i) \geq 0.$$

This shows that arith-deg$_{r-i}((I, F_1, \cdots, F_{i-1})_{\geq r-i+2}, F_i) = 0$. Thus the assertion is proved.

Lemma 4.3: *Let I be a homogeneous ideal of S. Let r and s be integers with $1 \leq s \leq r+1$. Let t be an integer with $-1 \leq t \leq r+1$. Let $F_1, \cdots, F_s$ be homogeneous polynomials of S with degree $(F_i) \geq 1$, $i = 1, \cdots, s$, such that F_i does not belong to any associated prime ideal $\mathfrak{p}$ of $(I, F_1, \cdots, F_{i-1})$ with $\dim \mathfrak{p} \geq r-i+1$, for all $i = 1, \cdots, s$. Then we have*

$$(I_{\geq t}, F_1, \cdots, F_{i-1})_{\geq r-i+2} = (I, F_1, \cdots, F_{i-1})_{\geq r-i+2}$$

for $i = 1, \cdots, s$.

Proof. By Lemma 2.7, we have

$$\begin{aligned} (I, F_1, \cdots, F_{i-1})_{\geq r-i+2} &= ((I, F_1, \cdots, F_{i-2})_{\geq r-i+3}, F_{i-1})_{\geq r-i+2} \\ &= \cdots = (\cdots((I, F_1)_{\geq r}, F_2)_{\geq r-1}, \cdots, F_{i-1})_{\geq r-i+2} \\ &= (\cdots((I_{\geq t}, F_1)_{\geq r}, F_2)_{\geq r-1}, \cdots, F_{i-1})_{\geq r-i+2}. \end{aligned}$$

On the other hand, we see

$$(I_{\geq t}, F_1, F_2, \cdots, F_{i-1})_{\geq r-i+2} \subset (\cdots((I_{\geq t}, F_1)_{\geq r}, F_2)_{\geq r-1}, \cdots, F_{i-1})_{\geq r-i+2}$$

and

$$(I_{\geq t}, F_1, F_2, \cdots, F_{i-1})_{\geq r-i+2} \supset (I, F_1, \cdots, F_{i-2})_{\geq r-i+2}.$$

Therefore we have $(I_{\geq t}, F_1, \cdots, F_{i-1})_{\geq r-i+2} = (I, F_1, \cdots, F_{i-1})_{\geq r-i+2}$ for all $i = 1, \cdots, s$.

5 Some examples

The first example sheds some light on Theorem 2.1 and Corollary 2.2 in case that F has degree one and is a non-zero-divisor on S/I. It shows that we have no equality in Corollary 2.2 even under these assumptions.

Example 1: Let $S = K[x_0, x_1, x_2, x_3, y_1, y_2, \cdots, y_r]$ be a polynomial ring, where r is a non-negative integer. Take $\mathfrak{q} = (x_0x_3 - x_1x_2, x_0^2, x_1^2, x_0x_1) \subset S$, which is a primary ideal belonging to (x_0, x_1) (cf. [12], Claim 1 on page 182). We set $I = \mathfrak{q} \cap (x_0^2, x_1, x_2)$ and $F(x_0, x_1, x_2, x_3) = x_3 + G(x_0, x_1, x_2)$, where $G(x_0, x_1, x_2)$ is a linear form. Then we have by (2.1)

$$\text{arith-deg}_{r-1}(I, F) - \text{arith-deg}_{r-1}(I_{\geq r+1}, F) = \text{arith-deg}_r(I).$$

Now we will show that

$$\text{arith-deg}_{r-1}(I, F) > \text{arith-deg}_r(I).$$

For simplicity we assume that $G = 0$, that is, $F = x_3$. Clearly, $\text{arith-deg}_r(I) = 1$. On the other hand,

$$\begin{aligned} (I, x_3) &= (x_0^2, x_1^2, x_0x_1, x_0x_2x_3 - x_1x_2^2) + (x_3) \\ &= (x_0^2, x_1, x_3) \cap (x_0^2, x_1^2, x_2^2, x_0x_1, x_3). \end{aligned}$$

Hence arith-deg$_{r-1}(I, x_3) = 2$. Also, we have

$$\begin{aligned}(I_{\geq r+1}, x_3) &= (\mathfrak{q}, x_3) \\ &= (x_0^2, x_1^2, x_0x_1, x_1x_2, x_3) \\ &= (x_0^2, x_1, x_3) \cap (x_0^2, x_1^2, x_2, x_3, x_0x_1).\end{aligned}$$

Hence arith-deg$_{r-1}(I_{\geq r+1}, x_3) = 1$.

We note that

$$\text{arith-deg}_{r-1}(I, x_3) > \text{arith-deg}_r(I)$$

even in the case that x_3 is a non-zero-divisor on S/I.

The second example shows that the bound of Theorem 3.1 is sharp and improves the result of [1], Proposition 3.6 (see Corollary 3.2).

Example 2: Take $I = (x_0^2x_1, x_0x_2^2, x_1^2, x_2, x_2^3) = (x_0^2, x_2) \cap (x_1, x_2^2) \cap (x_0^2, x_1^2, x_0x_2^2, x_2^3)$ $\subset S := K[x_0, x_1, x_2]$. We get $m = 5$, $P(S/I, \ell) = 4$ for all $\ell \geq 4$. We consider the case $r = 0$ in (3.1) and (3.2). Then we have

$$4 = \deg I = \text{arith-deg}_0(I) \leq P(S/I, m-1) = 4 < \binom{5+2-0-1}{2-0} = 15.$$

References

[1] Bayer, D. and Mumford, D.: What can be computed in algebraic geometry? In: Eisenbud, D. and Robbiano, L. (eds): "Computational Algebraic Geometry and Commutative Algebra", pp. 1-48. Cambridge University Press, 1993.

[2] Bingener, J. and Flenner, H.: On the Fibers of Analytic Mappings. In: Ancona, V. and Silva, A. (eds): "Complex Analysis and Geometry", pp. 45-101. Plenum Press, New York, 1993.

[3] Eisenbud, D. and Goto, S.: Linear free resolutions and minimal multiplicity. J. Algebra 88 (1984), 89-133.

[4] Eisenbud, D. and Harris, J.: Schemes: The Language of Modern Algebraic Geometry. Belmont, Wadsworth, 1992.

[5] Flenner, H.: Die Sätze von Bertini für lokale Ringe. Math. Ann. 229 (1977), 97-111.

[6] Flenner, H. and Vogel, W.: On multiplicities of local rings. Manuscripta math. 78 (1993), 85-97.

[7] Hartshorne, R.: Connectedness of the Hilbert scheme. Publ. I.H.E.S. 29, 261-304, Paris, 1966.

[8] Kalkbrener, M. and Sturmfels, B.: Initial complexes of prime ideals. Advances in Mathematics, (to appear).

[9] Kollár, J.: Sharp effective Nullstellensatz. J. Amer. Math. Soc. 1 (1988), 963-975.

[10] Mumford, D.: Lectures on curves on an algebraic surface. Princeton University Press, Princeton, New Jersey, 1966.

[11] Serre, J.-P.: Faisceaux algébriques cohérents. Ann. of Math. 61 (1955), 197-278.

[12] Stückrad, J. and Vogel, W.: Buchsbaum rings and applications. Springer-Verlag, Berlin, 1986.

[13] Sturmfels, B., Trung, N.G., and Vogel, W.: Bounds on degrees of projective schemes. Math. Ann. (to appear).

[14] Vogel, W.: Lectures on results on Bezout's theorem. Springer-Verlag, Berlin, 1984.