A construction for modified generalized Hadamard matrices using QGH matrices

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Abstract. Let G be a group of order mu and U a normal subgroup of G of order u. Let $G/U = \{U_1, U_2, \cdots, U_m\}$ be the set of cosets of U in G. We say a matrix $H = [h_{ij}]$ order k with entries from G is a quasi-generalized Hadamard matrix with respect to the cosets G/U if $\sum_{1 \le t \le k} h_{it} h_{jt}^{-1} = \lambda_{ij1} U_1 + \cdots + \lambda_{ijm} U_m$ $(\exists \lambda_{ij1}, \cdots, \exists \lambda_{ijm} \in \mathbb{Z})$ for any $i \neq j$. On the other hand, in our previous article we defined a modified generalized Hadamard matrix $\mathrm{GH}(s, u, \lambda)$ over a group G, from which a $\mathrm{TD}_\lambda(u\lambda, u)$ admitting G as a semiregular automorphism group is obtained. In this article, we present a method for combining quasi-generalized Hadamard matrices and semiregular relative difference sets to produce modified generalized Hadamard matrices.

Keywords: transversal design, generalized Hadamard matrix, semiregular relative difference set

1 Introduction

A *transversal design* $\mathrm{TD}_\lambda(k, u)$ $(u > 1, k = u\lambda)$ is an incidence structure $(\mathbb{P}, \mathbb{B})$, where

(i) $\mathbb{P}$ is a set of uk points partitioned into k classes (called *point classes*), each of size u,

(ii) $\mathbb{B}$ is a collection of k-subsets of $\mathbb{P}$ (called blocks),

(iii) Any two distinct points in the same point class are incident with no block and any two points in distinct point classes are incident with exactly λ blocks.

A transversal design $\mathcal{D} = (\mathbb{P}, \mathbb{B})$ is called *symmetric* (and often denoted by $\mathrm{STD}_\lambda(k, u)$) if the dual structure $\mathcal{D}^*$ of $\mathcal{D}$ is also a transversal design with the same parameters as $\mathcal{D}$. If $\mathcal{D}$ is symmetric, the point classes of $\mathcal{D}^*$ are said to be the *block classes* of $\mathcal{D}$. A transversal design $\mathcal{D}$ is called *class regular* with respect to U if U is an automorphism group of $\mathcal{D}$ acting regularly on each point class.

Throughout the article all groups are assumed to be finite. Let G be a group. A subset S of G is identified with a group ring element $\sum_{x\in S} x \in \mathbb{Z}[G]$ and $S^{(-1)}$ denotes the set of inverses of the elements of S. A matrix $M = [g_{ij}]$ of order $k(= u\lambda)$ with entries from G is called a generalized Hadamard matrix over G if it satisfies $\sum_{1\le t\le k} g_{it}g_{\ell t}^{-1} = \lambda G$ for any $i \ne \ell$, where $\lambda = k/|G|$. From a generalized Hadamard matrix we obtain a symmetric transversal design admitting G as a class regular automorphism group ([3]). On the other hand a modified generalized Hadamard matrix $\mathrm{GH}(s, u, \lambda)$ over a group is defined in [6] and from this one can construct a transversal design $\mathrm{TD}_\lambda(u\lambda, u)$ admitting G as a automorphism group (see Result 2.2).

Let G be a group of order mu and U a normal subgroup of G of order u. Let $\mathcal{S} = \{U_1, \cdots, U_m\}$ be the set of cosets of U in G. We say that a matrix $M = [d_{ij}]$ of order k with entries from G is a quasi-generalized Hadamard matrix with respect to $\mathcal{S}$ if $\sum_{1\le t\le k} d_{it}d_{\ell t}^{-1} = \sum_{1\le s\le m} \lambda_{i\ell s}U_s (\lambda_{i\ell s} \in \mathbb{Z})$ for any $i \ne \ell$. In this article, we present a method for combining such matrices and semiregular relative difference sets to produce modified generalized Hadamard matrices (Theorem 4.1, Theorem 4.9).

2 Preliminaries

In [6] we introduced the notion of a modified generalized Hadamard matrix over a group. We first give a summary of the related results, which we will use in the later sections.

Definition 2.1. ([6]) Let G be a group of order su, where s is a divisor of $u\lambda$, and u and λ are positive integers. For subsets D_{ij} $(1 \le i, j \le t, t = u\lambda/s)$ of G, we call a matrix

$$[D_{ij}] = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1t} \\ D_{21} & D_{22} & \cdots & D_{2t} \\ \vdots & \cdots & \cdots & \vdots \\ D_{t1} & D_{t2} & \cdots & D_{tt} \end{bmatrix}$$

a modified generalized Hadamard matrix with respect to subgroups U_i $(1 \le i \le t)$ of G of order u if the following conditions are satisfied :

$|D_{ij}| = s$ for all $i, j,\ 1 \le i, j \le t$, and

$$\sum_{1\le j\le t} D_{ij}D_{\ell j}^{(-1)} = \begin{cases} u\lambda + \lambda(G - U_i) & \text{if } i = \ell, \\ \lambda G & \text{otherwise.} \end{cases} \tag{1}$$

For short, we say $[D_{ij}]$ is a *$GH(s, u, \lambda)$ matrix with respect to U_i*, $1 \le i \le t$. If $U_1 = \cdots = U_t = U$ for a subgroup U of G, we simply say that $[D_{ij}]$ is a $\mathrm{GH}(s, u, \lambda)$ matrix with respect to U. In this case, if U is normal in G, then a $\mathrm{GH}(u, \lambda)$ matrix over U is obtained from the $\mathrm{GH}(s, u, \lambda)$ matrix (see Proposition 6.3 of [6]).

We denote by $\mathrm{M}_t(\mathbb{Z}[G])$ the set of matrices of order t over the group ring $\mathbb{Z}[G]$. An incidence structure $(\mathbb{P}, \mathbb{B})$ is obtained from a $\mathrm{GH}(s, u, \lambda)$ matrix $[D_{ij}] \in \mathrm{M}_t(\mathbb{Z}[G])$ in the following way :

$$\mathbb{P} = \{1, 2, \cdots, t\} \times G, \qquad \mathbb{B} = \{B_{jh} : 1 \leq j \leq t,\ h \in G\}, \tag{2}$$
$$\text{where } B_{jh} = \bigcup_{1 \leq i \leq t} (i, D_{ij}h) \ (= \bigcup_{1 \leq i \leq t} \{(i, dh) : 1 \leq i \leq t,\ d \in D_{ij}\}).$$

Moreover, the action of G on $(\mathbb{P}, \mathbb{B})$ is defined by $(i, c)^x = (i, cx), (B_{j,d})^x = B_{j,dx}$. Then, by [6] we have

Result 2.2. ([6]) Let $[D_{ij}] \in \mathrm{M}_t(\mathbb{Z}[G])$ be a $\mathrm{GH}(s, u, \lambda)$ matrix over a group G of order su with respect to subgroups U_i $(1 \leq i \leq t)$, where $t = u\lambda/s$. If we define $\mathbb{P}$ and $\mathbb{B}$ by (2), then the following holds.

(i) $(\mathbb{P}, \mathbb{B})$ is a transversal design $\mathrm{TD}_\lambda(k, u)$, where $k = u\lambda$.

(ii) G is an automorphism group of $(\mathbb{P}, \mathbb{B})$ acting semiregularly both on $\mathbb{P}$ and on $\mathbb{B}$.

(iii) For any $i(1 \leq i \leq t)$ and $x \in G$, $\mathbb{P}_{i,U_ix}$ is a point class of $(\mathbb{P}, \mathbb{B})$, on which $x^{-1}U_ix$ acts regularly.

Using Result 2.2 we can obtain transversal designs by constructing modified generalized Hadamard matrices. Transversal designs obtained from $GH(s, u, \lambda)$ matrices are not always symmetric (see Example 5.3 of [6]) and do not always admit class regular automorphism groups even if they are symmetric (see [7]). The following gives a criterion for the resulting transversal design to be symmetric.

Result 2.3. (Theorem 3.10 and Corollary 3.11 of [6]) Let $[D_{ij}]$ be a $\mathrm{GH}(s, u, \lambda)$ matrix over a group G with respect to subgroups U_i of G, $1 \leq i \leq t = u\lambda/s$. Then the transversal design $\mathrm{TD}_\lambda(k, u)$, $k = u\lambda$, corresponding to $[D_{ij}]$ is symmetric if and only if the matrix

$$[{D_{ij}}^{(-1)}]^T = \begin{bmatrix} {D_{11}}^{(-1)} & {D_{21}}^{(-1)} & \cdots & {D_{t1}}^{(-1)} \\ {D_{12}}^{(-1)} & {D_{22}}^{(-1)} & \cdots & {D_{t2}}^{(-1)} \\ \vdots & \cdots & \cdots & \vdots \\ {D_{1t}}^{(-1)} & {D_{2t}}^{(-1)} & \cdots & {D_{tt}}^{(-1)} \end{bmatrix}$$

is a $\mathrm{GH}(s, u, \lambda)$ matrix over G with respect to suitable subgroups V_i of G, $1 \leq i \leq t$, of order u. In particular, if $G \rhd U_1 = \cdots = U_t$, then $[{D_{ij}}^{(-1)}]^T$ is also a $\mathrm{GH}(s, u, \lambda)$ matrix over G.

Let G be a group of order $u^2\lambda$ and U a subgroup of G of order u. A $u\lambda$-subset D of G is called a $(u\lambda, u, u\lambda, \lambda)$-*difference set relative to* U if the list of quotients

$d_1 d_2^{-1}$ with distinct elements $d_1, d_2 \in D$ contains each element of $G - U$ exactly λ times and no elements of U :

$$DD^{(-1)} = u\lambda + \lambda(G - U) \tag{3}$$

We note that if D is a $(u\lambda, u, u\lambda, \lambda)$-difference set relative to U, then $[D]$ is a GH$(u\lambda, u, \lambda)$ matrix of order 1 and the corresponding transversal design is not always symmetric (see Proposition 4.4 of [5]). A $(u\lambda, u, u\lambda, \lambda)$-difference set is often called *a semiregular relative difference set.*

For an abelian group G, we denote by G^* the set of (linear) characters of G. Let χ_0 be the principal character of G. The following is a well known result on G^*.

Result 2.4. ([12]) Let G be an abelian group and let $z \in \mathbb{Z}[G]$. If $\chi(z) = 0$ for any character $\chi \in G^*, \chi \neq \chi_0$, then $z = cG$ for an integer c.

The following is a slight modification of Result 2.4.

Lemma 2.5. *Let U be a subgroup of an abelian group G and let $z \in \mathbb{Z}[G]$. If $\chi(z) = 0$ for every character $\chi \in G^*$ such that $\chi_{|U} \neq \chi_0$, then $z = Uf$ for some $f \in \mathbb{Z}[G]$.*

Proof. It suffices to show that $zg = z$ for every $g \in U$. On the other hand, for any $\chi \in G^*$ we have $\chi(g - 1) = 0$ or $\chi(z) = 0$ according as $\chi_{|U} = \chi_0$ or $\chi_{|U} \neq \chi_0$. Hence $\chi(z(g - 1)) = 0$. By Result 2.4 the lemma holds. □

3 Quasi-Generalized Hadamard Matrices with respect to cosets

In this section we give a modification of generalized Hadamard matrices from a different point of view to construct GH(s, u, λ) matrices that we have given in Definition 2.1.

Definition 3.1. Let N be a group of order mu and U a normal subgroup of N of order u. Let $N/U = \{U_1(= U), U_2, \cdots, U_m\}$ be the set of cosets of U in N. We say a matrix $H = [h_{ij}]$ of order $k(= u\lambda)$ with entries from N is a *quasi-generalized Hadamard matrix with respect to the cosets N/U* (a QGH(u, λ) matrix with respect to N/U for brevity) if there exist integers $\lambda_{ijt} \geq 0$ such that

$$\sum_{1 \leq t \leq k} h_{it} h_{jt}^{-1} = \lambda_{ij1} U_1 + \cdots + \lambda_{ijm} U_m, \tag{4}$$

for any i, j $(1 \leq i \neq j \leq k)$.

We note that the condition (4) is equivalent to the following :

$$H(H^{(-1)})^T = \begin{bmatrix} k & Uz_{12} & \cdots & Uz_{1k} \\ Uz_{21} & k & & Uz_{2k} \\ \vdots & & \ddots & \vdots \\ Uz_{k1} & Uz_{k2} & \cdots & k \end{bmatrix}$$

where $z_{ij} \in \mathbb{Z}[N]$ $(i \neq j)$ and each coefficient of z_{ij} is a non-negative integer and satisfies $\chi_0(z_{ij}) = \lambda$ for the principal character χ_0 of N.

Remark 3.2. (i) An ordinary GH(u, λ) matrix over U is a QGH(u, λ) matrix with respect to U/U.

(ii) If $H = [h_{ij}]$ is a generalized Hadamard matrix over a group U, then H is also a quasi-generalized Hadamard matrix with respect to the cosets U/V for any normal subgroup V of U. Hence, there always exists a QGH(p^s, p^m) matrix of order p^{s+m} over $(\mathbb{Z}_p)^s$ with respect to the cosets $(\mathbb{Z}_p)^s/(\mathbb{Z}_p)^t$ for any non-negative integers m, s and $t(\leq s)$ (see Table 5.10 of [2]).

(iii) Let U be a normal subgroup of a group G and N a subgroup of G such that $N \geq U$. If H is a QGH(u, λ) matrix with respect to N/U, then H can be regarded as a QGH(u, λ) matrix with respect to G/U.

(iv) Since $u\lambda = (\lambda_{ij1} + \cdots + \lambda_{ijm})|U|$ by (4), we have

$$\lambda = \lambda_{ij1} + \cdots + \lambda_{ijm}$$

for any i, j $(i \neq j)$.

We give some examples of quasi-generalized Hadamard matrices with respect to cosets.

Let p^n be any prime power and r a positive integer. We denote by GR(p^n, r) the Galois ring over $\mathbb{Z}_{p^n}$ (see [10]).

Proposition 3.3. *Let $R =GR(p^n, r)$ be the Galois ring over $\mathbb{Z}_{p^n}$. We define a matrix $M = [m_{ij}]$ of degree p^{nr} over the additive group $(R, +)$ by $m_{ij} = ij$ for $i, j \in R$. Then M is a $QGH(p^r, p^{(n-1)r})$ matrix with respect to the cosets R/I, where $I = (p^{n-1})$ is the smallest non-zero ideal of R.*

Proof. As $\bigcup_{j\in R}(m_{ij} - m_{\ell j}) = (i - \ell)\bigcup_{j\in R} j$. Assume $i \neq \ell$. Then, as a mapping $f(j) = (i - \ell)j$ from R to the ideal $(i - \ell)R$ of R is an epimorphism, $(i - \ell)\bigcup_{j\in R} j = dJ$, where d is the order of the kernel of f and $J = (i - \ell)R$. We note that any nonzero ideal of R is of the form $(p^s)(\supset (p^{n-1}))$ for some s $(0 \leq s \leq n - 1)$ (see [10] p.308). Set $J = (p^s)$ and $I = (p^{n-1})$. Then $\bigcup_{j\in R}(m_{ij} - m_{\ell j}) = dU_1 \cup dU_2 \cup \cdots \cup dU_t$, where $J/I = \{U_1(= I), U_2, \cdots, U_t\}$ and $t = p^{n-s-1}$. Thus the proposition holds. □

Example 3.4. (i) In Proposition 3.3, set $n = 2$ and $r = 1$. Then $R = \mathbb{Z}_{p^2}$. Hence there exists a QGH(p, p) matrix over $\langle a\rangle \simeq \mathbb{Z}_{p^2}$ with respect to the cosets $\langle a\rangle/\langle a^p\rangle$ for any prime p.

(ii) Set $N = \langle a, b\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, $U = \langle b\rangle \simeq \mathbb{Z}_3$. Then $[\ell_{ij}]$ below is a QGH$(3, 3)$ matrix with respect to N/U.

$$[\ell_{ij}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & b & b & b & b^2 & b^2 & b^2 \\ 1 & 1 & 1 & b^2 & b^2 & b^2 & b & b & b \\ 1 & b & b^2 & 1 & b^2 & b & a & ab & ab^2 \\ 1 & b & b^2 & b & 1 & b^2 & ab^2 & a & ab \\ 1 & b & b^2 & b^2 & b & 1 & ab & ab^2 & a \\ 1 & b^2 & b & 1 & b & b^2 & a & ab^2 & ab \\ 1 & b^2 & b & b^2 & 1 & b & ab & a & ab^2 \\ 1 & b^2 & b & b & b^2 & 1 & ab^2 & ab & a \end{bmatrix}$$

We can verify that $\sum_{1 \le t \le 9} \ \ell_{it}\ell_{jt}^{-1} \in \{3U, 2U + Ua, 2U + Ua^2\}$ $(i \neq j)$.

Example 3.5. Let $N = \langle a \rangle \simeq \mathbb{Z}_6$ and $U = \langle a^2 \rangle \simeq \mathbb{Z}_3$. Then the following matrix $[h_{ij}]$ of degree 12 is a QGH$(3, 4)$ matrix with respect to N/U. We note that $\sum_{1 \le t \le 12} h_{it}{h_{jt}}^{-1} \in \{4U, 3U + Ua, 2U + 2Ua\}$ $(i \neq j)$.

$$[h_{ij}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & a & a^2 & a^4 & a^4 & a^5 & 1 & a^2 & a^2 & a^3 & a^4 \\ 1 & 1 & a^4 & a^2 & 1 & a^2 & a^4 & a^2 & 1 & a^4 & a^4 & a^2 \\ 1 & 1 & a^2 & a^4 & 1 & a^2 & a^2 & a^4 & a^4 & 1 & a^2 & a^4 \\ 1 & a^4 & a & 1 & a^2 & a^3 & a & a^2 & a^5 & a^3 & a^5 & a^4 \\ 1 & a^4 & a^3 & a^4 & a^2 & a^5 & a & 1 & a & a^5 & a^3 & a^2 \\ 1 & a^4 & a^3 & a^2 & a^2 & a & a^5 & a^4 & a^3 & a & a^5 & 1 \\ 1 & a^4 & a^5 & a^4 & 1 & a^4 & a^3 & a^2 & a^2 & a^2 & a & 1 \\ 1 & a^2 & a^5 & a^2 & a^2 & 1 & a^3 & 1 & a^4 & a^4 & a & a^4 \\ 1 & a^2 & a^3 & 1 & a^4 & a^4 & a^5 & a^2 & a^4 & 1 & a & a^2 \\ 1 & a^2 & a & a^4 & a^4 & 1 & a^3 & a^4 & 1 & a^2 & a^5 & a^2 \\ 1 & a^2 & a^5 & 1 & a^4 & a^2 & a & a^4 & a^2 & a^4 & a^3 & 1 \end{bmatrix}$$

By the definition of the Kronecker product the following holds.

Proposition 3.6. *Let N be a group and U a normal subgroup of N. If $H_i (i = 1, 2)$ is a QGH(u, λ_i) matrix with respect to N/U for $i \in \{1, 2\}$, then $H_1 \otimes H_2$ is a QGH$(u, \lambda_1\lambda_2 u)$ matrix with respect to N/U.*

We note that when $N = U$ the assertion of the proposition coincides with that of Theorem 5.11 in [2].

4 Semiregular relative difference sets and QGH(u, λ) matrices with respect to cosets

In this section we present a construction method for transversal designs by combining quasi-generalized Hadamard matrices with respect to cosets and semiregular relative difference sets.

Theorem 4.1. *Let G be a group of order $u^2\mu$ and let U and N be subgroups of G such that $N_G(U) \geq N \geq U$ and $|U| = u$. Let $H = [h_{ij}]$ be a QGH(u, λ) matrix with respect to N/U and let $\mathcal{D} = (D_1, D_2, \cdots, D_k)$ $(k = u\lambda)$ be a k-tuple of $(u\mu, u, u\mu, \mu)$-difference sets in G relative to U. Then the following is a GH$(u\mu, u, u\lambda\mu)$ matrix of order k with respect to U and the resulting $TD_{u\mu\lambda}(u^2\mu\lambda, u)$ admits G as a semiregular automorphism group.*

$$M_{H,\mathcal{D}} = \begin{bmatrix} h_{11}D_1 & h_{12}D_2 & \cdots & h_{1k}D_k \\ h_{21}D_1 & h_{22}D_2 & \cdots & h_{2k}D_k \\ \vdots & \vdots & & \vdots \\ h_{k1}D_1 & h_{k2}D_2 & \cdots & h_{kk}D_k \end{bmatrix} \tag{5}$$

Proof. Set $N/U = \{U_1(= U), U_2, \cdots, U_m\}$, where $m = [N : U]$. By assumption, for any i, j $(1 \leq i \neq j \leq k)$ there exist $\lambda_{ijs} \geq 0$ $(1 \leq s \leq m)$ satisfying

$$\sum_{1\leq t\leq k} h_{it}{h_{jt}}^{-1} = \lambda_{ij1}U_1 + \lambda_{ij2}U_2 + \cdots + \lambda_{ijm}U_m \tag{6}$$

$$\text{and} \quad \lambda = \lambda_{ij1} + \cdots + \lambda_{ijm} \tag{7}$$

by Remark 3.2(iv). Moreover, by assumption,

$$D_t D_t^{(-1)} = u\mu + \mu(G - U) \quad (1 \leq t \leq k) \tag{8}$$

Set $M_{H,\mathcal{D}} = [D_{ij}]$, where $D_{ij} = h_{ij}D_j$.

Assume $i \neq j$. Then we have

$$\begin{aligned} & \sum_{1\leq t\leq k} D_{it}{D_{jt}}^{(-1)} \\ = & \sum_{1\leq t\leq k} h_{it}(u\mu + \mu(G-U)){h_{jt}}^{-1} && \text{(by (8))} \\ = & \sum_{1\leq t\leq k} h_{it}{h_{jt}}^{-1}(u\mu + \mu(G-U)) && \text{(as } N \triangleright U) \\ = & \sum_{1\leq s\leq m} \lambda_{ijs}U_s(u\mu + \mu(G-U)) && \text{(by (6))} \\ = & \ u\mu \sum_{1\leq s\leq m} \lambda_{ijs}U_s + \mu(\sum_{1\leq s\leq m} \lambda_{ijs}|U_s|)G \\ & -\mu \sum_{1\leq s\leq m} \lambda_{ijs}|U|U_s \\ = & \ \mu(\sum_{1\leq s\leq m} \lambda_{ijs}u)G = \mu\lambda uG && \text{(by (7))} \end{aligned}$$

Assume $i = j$. Then, similarly we have

$$\begin{aligned} \sum_{1\leq t\leq k} D_{it}{D_{it}}^{(-1)} &= \sum_{1\leq t\leq k} h_{it}(u\mu + \mu(G-U)){h_{it}}^{-1} \\ &= ku\mu + k\mu(G-U) \end{aligned}$$

It follows that

$$\sum_{1\le t\le k} D_{it}{D_{jt}}^{(-1)} = \begin{cases} ku\mu + k\mu(G-U) & \text{if } i=j, \\ k\mu G & \text{otherwise.} \end{cases}$$

Therefore the theorem holds. $\square$

Remark 4.2. (i) In Theorem 4.1, if there exists a $(u\mu, u, u\mu, \mu)$-difference set D in G relative to U, then we may choose a k-tuple $\mathcal{D} = (Dg_1, Dg_2, \cdots, Dg_k)$, where $g_1, \cdots, g_k \in G$.
(ii) We note that U is not always a normal subgroup of G in Theorem 4.1 and so the transversal design corresponding to D_i might not admit a class regular automorphism group.

Corollary 4.3. *Let G be a group of order $u^2\mu$ and U a normal subgroup of G of order u. Let $H = [h_{ij}]$ be a $QGH(u, \lambda)$ matrix with respect to G/U and $\mathcal{D} = (D_1, D_2, \cdots, D_k)(k = u\lambda)$ an n-tuple of $(u\mu, u, u\mu, \mu)$-difference sets in G relative to U. Then the matrix of order k defined by (5) is a $GH(u\mu, u, u\lambda\mu)$ matrix with respect to U and gives an $STD_{u\mu\lambda}(u^2\mu\lambda, u)$.*

Proof. The corollary immediately follows from Result 2.3 and Theorem 4.1. $\square$

Lemma 4.4. *Assume the existence of a $(p\mu, p, p\mu, \mu)$-difference set in a group G relative to a subgroup $U \simeq \mathbb{Z}_p$ of G for a prime p. If $p^2 \mid |C_G(U)|$, then there exists a $TD_{p^2\mu}(p^3\mu, p)$ admitting G as a semiregular automorphism group.*

Proof. By assumption, there exists a subgroup N of G such that $U \le N \simeq \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. Let $\mathcal{D} = (D_1, \cdots, D_{p^2})$ be a p^2-tuple of $(p\mu, p, p\mu, \mu)$-difference sets in G relative to U. It follows from Example 3.4(i) or Remark 3.2(ii) that there is a QGH(p, p) matrix with respect to N/U, say H. Applying Theorem 4.1, $M_{H,\mathcal{D}}$ is a GH$(p\mu, p, p^2\mu)$ matrix with respect to U and we obtain a TD$_{p^2\mu}(p^3\mu, p)$ from $M_{H,\mathcal{D}}$, which admits G as a semiregular automorphism group. Thus the lemma holds. $\square$

Example 4.5. (i) Set $G = \langle a, b, c \mid a^7 = b^3 = c^3 = 1, ac = ca, bc = cb, b^{-1}ab = a^2\rangle$ and let D be a $(21, 3, 21, 7)$-difference set relative to $U = \langle c\rangle \simeq \mathbb{Z}_3$ ([1]) . By Lemma 4.4, there exists a TD$_{3^27}(3^37, 3)$ admitting $G \simeq (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times \mathbb{Z}_3$ as a semiregular automorphism group.
(ii) Set $G = \langle r, s\rangle \times \langle t\rangle \simeq Sym(3) \times \mathbb{Z}_6$, where $r^2 = s^3 = t^6 = 1, [r, t] = [s, t] = 1$ and $rsr = s^{-1}$ and let D be a $(12, 3, 12, 4)$-difference set in G relative to a non-normal subgroup $U = \langle st^2\rangle$ ([5]). By Lemma 4.4, there exists a TD$_{36}(108, 3)$ admitting $G \simeq Sym(3) \times \mathbb{Z}_6$ as a semiregular automorphism group.

Example 4.6. Assume that there exists a $(3\mu, 3, 3\mu, \mu)$-difference set in a group G relative to a subgroup $U \simeq \mathbb{Z}_3$ of G and that $2 \mid |C_G(U)|$. Let $\mathcal{D} = (D_1, \cdots, D_{12})$ be a 12-tuple of $(3\mu, 3, 3\mu, \mu)$-difference sets in G relative to U. By assumption, there exists a subgroup N of G such that $U \le N \simeq \mathbb{Z}_6$. It follows from Example 3.5 that there is a QGH$(3, 4)$ matrix with respect to N/U, say H. Applying Theorem 4.1, $M_{H,\mathcal{D}}$ is a GH$(3\mu, 3, 12\mu)$ matrix with respect to U and we obtain a TD$_{12\mu}(36\mu, 3)$ from $M_{H,\mathcal{D}}$, which admits G as a semiregular automorphism group. For example, let G be the group of Example 4.5(ii). Then we obtain a TD$_{48}(144, 3)$ admitting $G \simeq Sym(3) \times \mathbb{Z}_6$ as a semiregular automorphism group.

Lemma 4.7. *Let G be an abelian group of order $u^2\mu$ and let $\mathcal{D} = (D_1, \cdots, D_k)$ be a k-tuple of $(u\mu, u, u\mu, \mu)$-difference sets in G relative to a subgroup U of G of order u, where $k = u\lambda$ for some $\lambda \in \mathbb{Z}$. Let $h_{ij} (1 \leq i, j \leq k)$ be elements of G. Then a matrix $M = [h_{ij}D_j]$ of order k is a GH$(u\mu, u, u\lambda\mu)$ matrix with respect to U if and only if $H = [h_{ij}]$ is a QGH(u, λ) matrix with respect to G/U. If this is the case, the resulting TD$_{u\mu\lambda}(u^2\mu\lambda, u)$ is symmetric.*

Proof. By definition, M is a GH$(u\mu, u, u\lambda\mu)$ matrix if and only if

$$\sum_{1 \leq j \leq k} h_{ij}D_j(h_{\ell j}D_j)^{(-1)} = \begin{cases} ku\mu + k\mu(G - U) & \text{if } j = \ell, \\ \mu kG & \text{otherwise.} \end{cases} \tag{9}$$

Since G is abelian, $\sum_{1 \leq j \leq k} h_{ij}D_j(h_{\ell j}D_j)^{(-1)} = \sum_{1 \leq j \leq k} h_{ij}h_{\ell j}^{-1}D_jD_j^{(-1)} = (u\mu + \mu(G - U)) \sum_{1 \leq j \leq k} h_{ij}h_{\ell j}^{-1}$. Hence, by Result 2.4, (9) is equivalent to

$$\chi(u\mu - \mu\chi(U))\chi(\sum_{1 \leq j \leq k} h_{ij}h_{\ell j}^{-1}) = 0 \quad (i \neq \ell) \tag{10}$$

for any character $\chi (\neq \chi_0)$ of G. Clearly (10) is equivalent to $\chi(\sum_{1 \leq j \leq k} h_{ij}h_{\ell j}^{-1}) = 0$ for any character χ of G such that $\chi_{|U} \neq \chi_0$. Applying Lemma 2.5, this is equivalent to the condition that H is a QGH(u, λ) matrix with respect to the cosets G/U. If this is the case, the resulting TD$_{u\lambda\mu}(u^2\lambda\mu, u)$ is symmetric by Result 2.3. Therefore the proposition holds. $\square$

Example 4.8. Set $G = \langle a \rangle \times \langle b \rangle \simeq \mathbb{Z}_9 \times \mathbb{Z}_3, N = \langle a \rangle \simeq \mathbb{Z}_9$ and $U = \langle a^3 \rangle \simeq \mathbb{Z}_3$. Let $H = [h_{ij}]$ be a QGH$(3, 3)$ matrix with respect to N/U in Example 3.4(i). As G contains $(9, 3, 9, 3)$-difference sets relative to U (see [9]), we can choose a 9-tuple $\mathcal{D} = (D_1, \cdots, D_9)$ of $(9, 3, 9, 3)$-difference sets in G relative to U. Then, by Theorem 4.1, $M_{H,\mathcal{D}}$ is a GH$(9, 3, 27)$ matrix with respect to U. Moreover, by Lemma 4.7, the TD$_{27}(81, 3)$ obtained from $M_{H,\mathcal{D}}$ is symmetric.

When D is a $(u\mu, u, u\mu, \mu)$-difference set in G relative to U, D is a complete set of right coset representatives of U in G by (3), but $D^{(-1)}$ is not so in general. If some $(u\mu, u, u\mu, \mu)$-difference set in G satisfies this condition, then we have the following.

Theorem 4.9. *Let G be a group of order $u^2\mu$ and let U and N be subgroups of G such that $|N| = mu, |U| = u$ and $N_G(U) \geq N \geq U$ and $|U| = u$. Let $H = [h_{ij}]$ $(h_{ij} \in N)$ be a QGH(u, λ) matrix with respect to N/U and let $\mathcal{D} = (D_1, D_2, \cdots, D_k)$ $(k = u\lambda)$ be a k-tuple of $(u\mu, u, u\mu, \mu)$-difference sets in G. Assume at least $k - 1$ of D_i's are complete sets of right and left coset* *representatives of U in G. Then the following matrix $M'_{H,\mathcal{D}}$ of order k is a $GH(u\mu, u, u\lambda\mu)$ matrix with respect to U and the resulting $TD_{u\mu\lambda}(u^2\mu\lambda, u)$ admits G as a semiregular automorphism group.*

$$M'_{H,\mathcal{D}} = \begin{bmatrix} D_1 h_{11} & D_1 h_{12} & \cdots & D_1 h_{1k} \\ D_2 h_{21} & D_2 h_{22} & \cdots & D_2 h_{2k} \\ \vdots & \vdots & & \vdots \\ D_k h_{k1} & D_n h_{k2} & \cdots & D_k h_{kk} \end{bmatrix} \tag{11}$$

Proof. Set $N/U = Ug_1 \cup \cdots \cup Ug_m$ $(g_1, \cdots, g_m \in N)$, where $m = [N : U]$. By assumption,

$$\sum_{1 \le t \le k} h_{it} h_{jt}^{-1} = \lambda_{ij1} U g_1 + \cdots + \lambda_{ijm} U g_m \tag{12}$$

for some non-negative integers λ_{ijs} $(1 \le i \ne j,\ 1 \le s \le m)$.

Set $M'_{H,\mathcal{D}} = [D_{ij}]$, where $D_{ij} = D_i h_{ij}$. Then Then

$$\begin{aligned} \sum_{1 \le t \le k} D_{it} {D_{jt}}^{(-1)} &= \sum_{1 \le t \le k} D_i h_{it} h_{jt}^{-1} D_j^{(-1)} \\ &= D_i \big(\sum_{1 \le t \le k} h_{it} h_{jt}^{-1} \big) D_j^{(-1)} \end{aligned}$$

Hence, by (12)

$$\sum_{1 \le t \le k} D_{it} {D_{jt}}^{(-1)} = \begin{cases} k(u\mu + \mu(G - U)) & \text{if } i = j, \\ D_i(\lambda_{ij1} U g_1 + \cdots + \lambda_{ijm} U g_m) D_j^{(-1)} & \text{otherwise.} \end{cases}$$

Assume $i \ne j$. By assumption, either D_i or D_j is a complete set of right and left coset representatives of U in G as $i \ne j$. Hence we have either $D_i U = G$ or $U D_j^{(-1)} = G$. In either case, $\sum_{1 \le t \le k} D_{it} {D_{jt}}^{(-1)} = \lambda u \mu G$ as $N \rhd U$. Thus

$$\sum_{1 \le t \le k} D_{it} {D_{jt}}^{(-1)} = \begin{cases} ku\mu + k\mu(G - U) & \text{if } i = j, \\ k\mu G & \text{otherwise.} \end{cases}$$

Therefore the theorem holds. □

Corollary 4.10. *Let G be a group of order $u^2\mu$ and U a normal subgroup of G of order u. Let $H = [h_{ij}]$ be a $QGH(u, \lambda)$ matrix with respect to G/U and $\mathcal{D} = (D_1, D_2, \cdots, D_k)$ $(k = u\lambda)$ a k-tuple of $(u\mu, u, u\mu, \mu)$-difference sets in G relative to U. Then the matrix of order k defined by (11) is a $GH(u\mu, u, u\lambda\mu)$ matrix with respect to U and the resulting $TD_{u\mu\lambda}(u^2\mu\lambda, u)$ admits G as a semiregular automorphism group.*

Example 4.11. Many $(4n^2, 2n^2 - n, n^2 - n)$-difference sets have been constructed in abelian groups of order $4n^2$ and they are called Menon Hadamard difference sets ([8]). Let L be an abelian group of order $4n^2$ containing a Menon Hadamard difference set A. Assume that L is not an elementary abelian 2-group. We define a group $G = L\langle t\rangle$ of order $8n^2$, where an element t of G inverts L. By a similar way as in Proposition 4.14 of [4], we can verify that $D = A + (L - A^{(-1)})t$ is a $(4n^2, 2, 4n^2, 2n^2)$-difference set in G relative to $U = \langle t\rangle$. We choose A so that it satisfies $A = A^{(-1)}$ (see Problem 2 in Chapter 4 of [8]). For $g \in L$, Dg is a $(4n^2, 2, 4n^2, 2n^2)$-difference set in G relative to U. However, as $(Dg)^{(-1)}(Dg) = 4n^2 + 2n^2(G - \langle gt\rangle)$, Dg is not a complete set of left coset representatives of U in G. Clearly $C_G(t)$ contains a subgroup N of the form $N = \langle t\rangle \times \langle s\rangle$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $H = [h_{ij}]$ be the following QGH$(2, 2)$ matrix with respect to N/U :

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & t & s & st \\ 1 & 1 & t & t \\ 1 & t & st & s \end{bmatrix}$$

Set $\mathcal{D} = (D, D, D, Dg)$. Then, applying Theorem 4.9, $M'_{H,\mathcal{D}}$ is a $GH(4n^2, 2, 8n^2)$ matrix with respect to U and the resulting $\mathrm{TD}_{8n^2}(16n^2, 2)$ admits G as a semiregular automorphism group.

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