A family of non class-regular symmetric transversal designs of spread type

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Abstract. Many classes of symmetric transversal designs have been constructed from generalized Hadamard matrices and they are necessarily class regular. In [2] we constructed symmetric transversal designs using spreads of $\mathbb{Z}_p^{2n}$ with p a prime. In this article we show that most of them admit no class regular automorphism groups. This implies that they are never obtained from generalized Hadamard matrices. As far as we know, this is the first infinite family of non class-regular symmetric transversal designs.

Keywords: transversal design, generalized Hadamard matrix, class regular, spread

1 Introduction

A *transversal design* $\mathrm{TD}_\lambda(k,u)$ $(u > 1, k = u\lambda)$ is an incidence structure $\mathcal{D} = (\mathbb{P}, \mathbb{B})$, where

(i) $\mathbb{P}$ is a set of uk points partitioned into k classes (called *point classes*), each of size u,

(ii) $\mathbb{B}$ is a collection of k-subsets of $\mathbb{P}$ (called blocks),

(iii) Any two distinct points in the same point class are incident with no block and any two points in distinct point classes are incident with exactly λ blocks.

A transversal design $\mathrm{TD}_\lambda(k,u)$ is said to be *symmetric* and denoted by $\mathrm{STD}_\lambda(k,u)$ if its dual structure is also a transversal design with the same parameters as $\mathrm{TD}_\lambda(k,u)$. In this case the point classes of the dual structure are called *block classes*.

A transversal design $\mathcal{D}$ is called *class regular* with respect to U if U is an automorphism group of $\mathcal{D}$ acting regularly on each point class, which is necessarily symmetric by a result of Jungnickel (Corollary 6.9 of [4]).

Let $\mathcal{D}$ be an $\mathrm{STD}_\lambda(k,u)$. If $\mathcal{D}$ is class regular with respect to U, then there exists a *generalized Hadamard matrix* $[d_{i,j}]$ of order k with entries from U (for short GH(u,λ)) such that whenever $i \neq \ell$ the set of differences $\{d_{ij}d_{\ell j}^{-1} \mid 1 \leq j \leq k\}$ contains each element of U exactly λ times. Conversely, from a generalized Hadamard matrix GH(u,λ) over a group U of order u, one can construct an STD$_\lambda(k,u)$ which admits U as a class regular automorphism group (Theorem 3.6 of [1]).

In [2] we give a modification of generalized Hadamard matrices. Let H be a group of order su. For subsets D_{ij} $(1 \leq i,j \leq t, st = u\lambda)$ of H, we call a $t \times t$ matrix $[D_{ij}]$ a modified generalized Hadamard matrix with respect to subgroups $U_1, \cdots, U_t$ of H of order u if the following two conditions are satisfied.

$$|D_{ij}| = s,\ 1 \leq i,j \leq t \tag{1}$$

$$\sum_{1 \leq j \leq t} D_{ij}D_{\ell j}^{(-1)} = \begin{cases} k + \lambda(H - U_i) & \text{if } i = \ell, \\ \lambda H & \text{otherwise.} \end{cases} \tag{2}$$

For short, we say $[D_{ij}]$ is a *GH(s,u,λ) matrix with respect to subgroups* $U_1, \cdots, U_t$ and the subgroups are called *forbidden subgroups*. Clearly any GH$(1,u,\lambda)$ matrix is an ordinary generalized Hadamard matrix GH(u,λ) and a GH$(u\lambda,u,\lambda)$ matrix with respect to U is a $(u\lambda,u,u\lambda,\lambda)$-difference set relative to U (see [1]).

Given a GH(s,u,λ) matrix, we can construct a transversal design TD$_\lambda(k,u)$. For a $t \times t$ GH(s,u,λ) matrix $[D_{ij}]$ $(st = u\lambda)$, we define a set of points $\mathbb{P}$ and a set of blocks $\mathbb{B}$ in the following way.

$$\mathbb{P} = \{1,2,\cdots,t\} \times H,\ \ \mathbb{B} = \{B_{jh} : 1 \leq j \leq t,\ h \in H\}, \tag{3}$$
$$\text{where } B_{jh} = \bigcup_{1 \leq i \leq t} (i, D_{ij}h)\ (= \bigcup_{1 \leq i \leq t} \{(i,dh) : 1 \leq i \leq t,\ d \in D_{ij}\}).$$

Then we have

Result 1.1. ([2]) Let $[D_{ij}]$ be a $t \times t$ GH(s,u,λ) matrix over a group H of order su with respect to subgroups U_i $(1 \leq i \leq t)$, where $t = u\lambda/s$. If we define $\mathbb{P}$ and $\mathbb{B}$ by (3), then the following holds.

(i) $(\mathbb{P},\mathbb{B})$ is a transversal design TD$_\lambda(k,u)$ $(k = u\lambda)$.

(ii) For each i $(1 \leq i \leq t)$ and $x \in H$, set $\mathbb{P}_{i,U_ix} = \{(i,wx) : w \in U_i\}$ $(1 \leq i \leq t, x \in H)$. Then $\mathbb{P}_{i,U_ix}$ is a point class of $(\mathbb{P},\mathbb{B})$.

(iii) If we define the action of H on $(\mathbb{P},\mathbb{B})$ by $(i,c)^x = (i,cx), (B_{j,d})^x = B_{j,dx}$, then H is an automorphism group of $(\mathbb{P},\mathbb{B})$ acting semiregularly both on $\mathbb{P}$ and on $\mathbb{B}$ and each (i,H) is an H-orbit on $\mathbb{P}$ for each i $(1 \leq i \leq t)$.

(iv) For every $x \in H$, $x^{-1}U_ix$ acts regularly on a point class $\mathbb{P}_{i,U_ix}$ $(1 \leq i \leq t)$.

A 2-$(n^2,n,1)$ design $\pi = (\mathbb{P},\mathbb{B})$ with $n > 1$ is called an affine plane of order n and satisfies $|\mathbb{P}| = n^2,\ |\mathbb{B}| = n(n+1)$. Moreover, $\mathbb{B}$ is divided into $n+1$ parallel classes ([3]). An automorphism g of an affine plane π is called a *translation* if g

leaves each parallel class invariant and has no fixed points in $\mathbb{P}$. It is well known that if $g \neq 1$, then the set of fixed lines of g forms a unique parallel class of π ([3],[5]).

Let H be a group of order $q^2(> 1)$. A set of $q+1$ subgroups $\mathcal{S} = \{H_1, \cdots, H_{q+1}\}$ of H is called a *spread* of H if $|H_1| = \cdots = |H_{q+1}| = q$ and $H_i \cap H_j = 1$ for all distinct i and j with $1 \leq i, j \leq q+1$. Then H is an elementary abelian p-group for a prime p. From $\mathcal{S}$ one can construct an affine plane $\pi_{\mathcal{S}} = (\mathbb{P}, \mathbb{B})$ of order q as follows (See [3],[5]) :

$$\mathbb{P} = H, \qquad \mathbb{B} = \bigcup_{1 \leq i \leq q+1} \{H_i g \mid g \in H\}$$

For each $g \in H$, a map from $\mathbb{P}$ to $\mathbb{P}$ defined by $\sigma_g(x) = gx$ induces a translation of $\pi_{\mathcal{S}}$. Hence H can be regarded as a subgroup of $\mathrm{Aut}(\pi_{\mathcal{S}})$ acting regularly on $\mathbb{P}$. Conversely, An affine plane π of order n admitting a group T of translations acting regularly on the set of points of π is called a translation plane and T is called the group of translations of π. Let $\mathcal{C}_1, \cdots, \mathcal{C}_{n+1}$ be the parallel classes of π. Then the set of stabilizers $\{T_{\mathcal{C}_1}, \cdots, T_{\mathcal{C}_{n+1}}\}$ forms a spread of T ([3],[5]).

From each spread we obtain many $\mathrm{GH}(q,q,q)$ matrices ([2]) and we say that the resulting transversal designs are of *spread type*.

Result 1.2. ([2]) Let q be a power of a prime p and Let $\mathcal{S} = \{H_1, \cdots, H_{q+1}\}$ be a spread of an elementary abelian p-group H of order q^2. Let $A = [n_{ij}]$ be a $q \times q$ matrix with entries from $I = \{1, 2, \cdots, q+1\}$ satisfying

$$I = \{n_{i1}, n_{i2}, \cdots, n_{iq}, m_i\} = \{n_{1i}, n_{2i}, \cdots, n_{qi}, \ell_j\} \quad (1 \leq i \leq q) \tag{4}$$

for some $m_1, \cdots, m_q \in I$ and $\ell_1, \cdots, \ell_q \in I$. Set $D_{ij} = H_{n_{ij}}$ for each i, j with $1 \leq i, j \leq q$ and set $M_{\mathcal{S},A} = [D_{ij}]$. Then $M_{\mathcal{S},A}$ is a $\mathrm{GH}(q,q,q)$ matrix with respect to subgroups $H_{m_1}, \ldots, H_{m_q}$ and the transversal design $\mathrm{TD}_q(q^2, q)$ corresponding to $M_{\mathcal{S},A}$ is symmetric. We note that $H_{m_1}, \ldots,$ and H_{m_q} are not always distinct.

Let $A = [n_{ij}]$ be a $q \times q$ matrix in Result 1.2. Exchanging columns of A if necessary we may assume that

$$\{n_{i1}, \cdots, n_{iq}\} = \{n_{1i}, \cdots, n_{qi}\}, \quad m_i = \ell_i \tag{5}$$

for each i $(1 \leq i \leq q)$.

Let Γ_q be the set of matrices satisfying (5) and let $M_{\mathcal{S},A}$ be a $\mathrm{GH}(q,q,q)$ matrix over an elementary abelian p-group H of order q^2 corresponding to a spread $\mathcal{S} = \{H_1, \cdots, H_{q+1}\}$ of H and a $q \times q$ matrix $A = [n_{ij}] \in \Gamma_q$. A symmetric transversal design obtained from $M_{\mathcal{S},A}$ is called *an* $STD_q(q^2, q)$ *of spread type corresponding to* $M_{\mathcal{S},A}$.

In this article we show that most of transversal designs of spread type admit no class regular automorphism groups though they are symmetric. This implies that they are never obtained from generalized Hadamard matrices. As far as we know, this is the first infinite family of non class-regular symmetric transversal designs.

Theorem 1.3. *Let $M_{\mathcal{S},A}$ be a $GH(q,q,q)$ matrix over an elementary abelian p-group H of order q^2 corresponding to a spread $\mathcal{S}$ of H and a $q \times q$ matrix $A \in \Gamma_q$. Let $U_1, \cdots, U_{q-1}$ and U_q be forbidden subgroups corresponding to $M_{\mathcal{S},A}$ and $(\mathbb{P}, \mathbb{B})$ the symmetric transversal design $STD_q(q^2, q)$ of spread type obtained form $M_{\mathcal{S},A}$. Then one of the following occurs.*

(i) $U_1 = \cdots = U_q (= U)$ *and U is a class regular automorphism group of $(\mathbb{P}, \mathbb{B})$.*

(ii) $U_1, \cdots, U_{q-1}$ *and U_q are all distinct.*

(iii) $(\mathbb{P}, \mathbb{B})$ *admits no class regular automorphism groups.*

The above theorem indicates that most of $\mathrm{STD}_q(q^2, q)$s of spread type admit no class regular automorphism groups.

2 Preliminaries

Result 2.1. ([4]) Let $(\mathbb{P}, \mathbb{B})$ be a symmetric transversal design admitting U as a class regular automorphism group. Then U acts regularly on the blocks of each block class of $(\mathbb{P}, \mathbb{B})$.

Proof. In Lemma 6.2 of [4] $B \cap B^x = \emptyset$ is shown for each element $x \in U \setminus \{1\}$ and each block $B \in \mathbb{B}$. Thus the result holds. $\square$

Let $\pi = (\mathbb{P}, \mathbb{L})$ be an affine plane of order n and let ℓ_∞ be the line at infinity ([3]). Then ℓ_∞ can be identified with the set of $n+1$ parallel classes of π. Clearly $\mathrm{Aut}(\pi)$ induces a permutation group on ℓ_∞. We say an automorphism x of π has a center $\mathcal{D} \in \ell_\infty$ if x leaves each line of the parallel class $\mathcal{D}$ invariant.

The following is a well known fact on translation planes.

Result 2.2. ([3], [5]) Let π be a translation plane of order p^e with p a prime and ℓ_∞ the line at infinity. Let T be the translation group of π of order p^{2e}. Then the following holds.

(i) Set $G = \mathrm{Aut}(\pi)$. Then $G \triangleright T$ and every translation of π is contained in T (see Theorem 1.10 of [5]).

(ii) Each translation $x \neq 1$ of π has a unique center $\mathcal{D} \in \ell_\infty$ and x fixes every line of $\mathcal{D}$.

(iii) Let $\mathcal{D} \in \ell_\infty$ and let $T_{\mathcal{D}}$ be the subgroup of T consisting of the translations with center $\mathcal{D}$. Then $T_{\mathcal{D}} \simeq E_{p^e}$.

3 Symmetric transversal designs of spread type

In this section we prove Theorem 1.3. To show the theorem we need the following.

Proposition 3.1. *Let $[D_{ij}]$ be a $t \times t$ GH(s, u, λ) matrix over a group H of order su with respect to forbidden subgroups U_i $(1 \le i \le t)$, where $t = u\lambda/s$. Let $(\mathbb{P}, \mathbb{B})$ the transversal design corresponding to $[D_{ij}]$ defined by (3). For any blocks B_{jx} and $B_{\ell y}$ of $\mathbb{B}$, we have $B_{jx} \cap B_{\ell y} \cap (i, H) = (i, D_{ij}x \cap D_{i\ell}y)$ and $|B_{jx} \cap B_{\ell y} \cap (i, H)| = m_{i,j,\ell,xy^{-1}}$, where $D_{ij}^{(-1)} D_{i\ell} = \sum_{g \in H} m_{i,j,\ell,g}\, g$ $(m_{i,j,\ell,g} \ge 0)$.*

Proof. Since $B_{jx} = \bigcup_{1 \le i \le t} (i, D_{ij}x)$ and $B_{\ell y} = \bigcup_{1 \le i \le t} (i, D_{i\ell}y)$, we have $B_{jx} \cap B_{\ell y} \cap (i, H) = (i, D_{ij}x \cap D_{i\ell}y)$. Hence the proposition holds. □

In the rest of the article we assume the following.

Hypothesis 3.2. Let $\mathcal{S} = \{H_1, H_2, \cdots, H_{q+1}\}$ be a spread of an elementary abelian p-group $H (\simeq E_{q^2})$ of order q^2 and $A = [n_{ij}] \in \Gamma_q$. Let $M_{\mathcal{S},A} = [H_{n_{ij}}]$ be a $q \times q$ GH(q, q, q) matrix with respect to $(\mathcal{S}, A)$ and let $H_{m_1}, \cdots, H_{m_q}$ be the forbidden subgroups of $M_{\mathcal{S},A}$, where $\{m_i\} = \{1, 2, \cdots, q+1\} \setminus \{n_{i1}, n_{i2}, \cdots, n_{iq}\}$ $(1 \le i \le q)$(cf. (4)). Let $(\mathbb{P}, \mathbb{B})$ be an STD$_q(q^2, q)$ obtained from $M_{\mathcal{S},A}$.

Example 3.3. Set $q = 3$ and $H = \langle a, b \rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. Then, we can check that each element of Γ_q is equivalent to one of the following after exchanging rows or columns.

$$\begin{bmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{bmatrix}, \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \delta \end{bmatrix}, \begin{bmatrix} \alpha & \beta & \gamma \\ \delta & \alpha & \beta \\ \gamma & \delta & \alpha \end{bmatrix}, \quad (\{\alpha, \beta, \gamma, \delta\} = \{1, 2, 3, 4\})$$

For example, set $[n_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix}$. Then the corresponding GH$(3, 3, 3)$ matrix over H is

$\begin{bmatrix} \langle a \rangle & \langle ab \rangle & \langle a^2 b \rangle \\ \langle ab \rangle & \langle a^2 b \rangle & \langle a \rangle \\ \langle a^2 b \rangle & \langle a \rangle & \langle b \rangle \end{bmatrix}$. By a suitable numbering of $\mathbb{P}$, the resulting SDT$_3(9, 3)$ is represented in the following way :

$\mathbb{P} = \{i \mid 1 \le i \le 27\}$,
$\mathbb{B} = \{1,4,7,10,14,18,19,24,26\}, \{2,5,8,11,15,16,20,22,27\}, \{3,6,9,12,13,17,21,23,25\},$
$\{1,4,7,12,13,17,20,22,27\}, \{2,5,8,10,14,18,21,23,25\}, \{3,6,9,11,15,16,19,24,26\},$
$\{1,4,7,11,15,16,21,23,25\}, \{2,5,8,12,13,17,19,24,26\}, \{3,6,9,10,14,18,20,22,27\},$
$\{1,5,9,10,15,17,19,22,25\}, \{2,6,7,11,13,18,20,23,26\}, \{3,4,8,12,14,16,21,24,27\},$
$\{3,4,8,11,13,18,19,22,25\}, \{1,5,9,12,14,16,20,23,26\}, \{2,6,7,10,15,17,21,24,27\},$
$\{2,6,7,12,14,16,19,22,25\}, \{3,4,8,10,15,17,20,23,26\}, \{1,5,9,11,13,18,21,24,27\},$
$\{1,6,8,10,13,16,19,20,21\}, \{2,4,9,11,14,17,19,20,21\}, \{3,5,7,12,15,18,19,20,21\},$
$\{2,4,9,10,13,16,22,23,24\}, \{3,5,7,11,14,17,22,23,24\}, \{1,6,8,12,15,18,22,23,24\},$
$\{3,5,7,10,13,16,25,26,27\}, \{1,6,8,11,14,17,25,26,27\}, \{2,4,9,12,15,18,25,26,27\}\}$,
$\mathcal{C} = \{\{3i+1, 3i+2, 3i+3\} \mid 0 \le i \le 5\} \cup \{\{19,23,27\}\} \cup \{\{20,24,25\}\} \cup \{\{21,22,26\}\}$.

By computer search, $\mathrm{Aut}(\mathbb{P}, \mathbb{B})_{\mathcal{C}} = 1$. Therefore $(\mathbb{P}, \mathbb{B})$ admits no class regular automorphism groups.

Lemma 3.4. *Let notations be as in Hypothesis 3.2. Then the following holds.*

(i) *If $g \neq h \in H$ and $gh^{-1} \in H_{m_j}$, then $B_{j,g} \cap B_{j,h} = \emptyset$.*

(ii) *If $g \neq h \in H$ and $gh^{-1} \notin H_{m_j}$, then there exists a unique i $(1 \leq i \leq q)$ such that $gh^{-1} \in H_{n_{ij}}$ and $B_{j,g} \cap B_{j,h} = (i, H_{n_{ij}}g)$ $(= (i, H_{n_{ij}}h))$.*

(iii) *If $j \neq k$, then $|B_{j,g} \cap B_{k,h}| = q$ and $|(B_{j,g} \cap B_{k,h}) \cap (i, H)| = 1$ for any i $(1 \leq i \leq q)$.*

Proof. By Proposition 3.1, $B_{j,g} \cap B_{k,h} = \bigcup_{1 \leq i \leq q}(i, D_{ij}g \cap D_{ik}h)$ and $D_{i\ell} = H_{n_{i\ell}}$ for each ℓ. Hence

$$B_{j,g} \cap B_{k,h} = \bigcup_{1 \leq i \leq q} (i, (H_{n_{ij}}gh^{-1} \cap H_{n_{ik}})h). \tag{6}$$

Assume $k = j$. We note that $\{H_{n_{1j}}, H_{n_{2j}}, \cdots, H_{n_{qj}}, H_{m_j}\}$ is a spread of H. Hence, if $gh^{-1} \in H_{m_j}$, then $H_{n_{ij}}gh^{-1} \cap H_{n_{ij}} = \emptyset$ for each i and so (i) holds. On the other hand, if $gh^{-1} \notin H_{m_j}$, there exists a unique i such that $gh^{-1} \in H_{n_{ij}}$ and $gh^{-1} \notin H_{n_{\ell j}}$ for any $\ell \neq i$. Thus (ii) holds.

Assume $k \neq j$. Then, as $H = H_{n_{ij}}H_{n_{ik}}$ and $H_{n_{ij}} \cap H_{n_{ik}} = 1$, we have $|H_{n_{ij}}g \cap H_{n_{ik}}| = 1$ for any i $(1 \leq i \leq q)$ and $g \in H$. Thus (iii) holds. □

As a corollary we have

Corollary 3.5. *Let $\mathcal{S} = \{H_1, H_2, \cdots, H_{q+1}\}$ be a spread of a p-group H of order q^2 and $A = [n_{ij}] \in \Gamma_q$. Let $(\mathbb{P}, \mathbb{B})$ be an $STD_q(q^2, q)$ of spread type with respect to $(\mathcal{S}, A)$. Then, for each H-orbit Ω on $\mathbb{P}$ and for any distinct $B, C \in \mathbb{B}$, we have $|\Omega \cap B \cap C| \in \{0, 1, q\}$.*

Under Hypothesis 3.2, from now on we use the following notations.

Notation 3.6. (i) $U_i = H_{m_i} (\simeq E_q)$, $1 \leq i \leq q$.

(ii) Let $(\mathbb{P}, \mathbb{B})$ be an $\mathrm{STD}_q(q^2, q)$ corresponding to a $\mathrm{GH}(q, q, q)$ matrix $[H_{n_{ij}}]$.

(iii) Let $\Omega_1, \cdots, \Omega_{q-1}$ and Ω_q be the H-orbits on $\mathbb{P}$ $(|\Omega_1| = \cdots = |\Omega_q| = q^2)$.

(iv) Let $\mathcal{S}_i = \{\mathcal{C}_{i1}, \cdots, \mathcal{C}_{iq}\}$ be the set of point classes contained in Ω_i $(1 \leq i \leq q)$ and denote by $\mathcal{C} = \bigcup \mathcal{S}_i$ the set of point classes of $(\mathbb{P}, \mathbb{B})$.

In the rest of the article, we assume the following and prove Theorem 1.3 by way of a contradiction.

Hypothesis 3.7. (i) There exist distinct r, s and t $(1 \leq r, s, t \leq q)$ such that $U_r \neq U_s = U_t$.

(ii) $(\mathbb{P}, \mathbb{B})$ admits a class regular automorphism group U of order q.

Let X be a group acting on a set Δ. We define the kernel of the action of X on Δ by $\mathrm{Ker}(X, \Delta) = \{x \in X \mid \alpha^x = \alpha, \ \forall \alpha \in \Delta\}$. Clearly $\mathrm{Ker}(X, \Delta)$ is a normal subgroup of X.

Lemma 3.8. *The following holds.*
(i) $U \le \mathrm{Ker}(\mathrm{Aut}(\mathbb{P},\mathbb{B}),\mathcal{S}_i)$ *and* $U_i = \mathrm{Ker}(H,\mathcal{S}_i)$ *for any* i.
(ii) $H \cap U = 1$.

Proof. Clearly $U \le \mathrm{Ker}(\mathrm{Aut}(\mathbb{P},\mathbb{B}),\mathcal{S}_i)$ for each i. Since H is abelian, by Result 1.1(iii)(iv), we have $U_i = \mathrm{Ker}(H,\mathcal{S}_i)$.

By (i), $H \cap U \le U_i$ for any i $(1 \le i \le q)$. Applying Hypothesis 3.7(i), $H \cap U \le U_r \cap U_s = 1$. Thus the lemma holds. □

Lemma 3.9. *Set* $G = \langle H, U\rangle$ $(\le \mathrm{Aut}(\mathbb{P},\mathbb{B}))$. *Then we have the following.*

(i) *We may assume that* G *is a* p*-group. In particular* $q^3 \mid |G|$.

(ii) Ω_i *is a* G*-orbit on* $\mathbb{P}$ *for each* i $(1 \le i \le q)$.

Proof. Let S be a Sylow p-subgroup of G containing H. Then there exists a subgroup U' of S such that $U' = U^g$ for some $g \in G$. Set $K = \mathrm{Ker}(G,\mathcal{C})$. Then $G \rhd K \ge U$. Hence $U' \le K$. Therefore U' is also a class regular automorphism group of $(\mathbb{P},\mathbb{B})$. Exchanging U for U' if necessary, we may assume that $U \le S$ and G is a p-group. Hence, by Lemma 3.8(ii), $|G| \ge q^3$. Thus (i) holds. On the other hand, since $U \le K$ and each H-orbit is a union of point classes of $(\mathbb{P},\mathbb{B})$, we have (ii). □

Lemma 3.10. *For each* i $(1 \le i \le q)$, *set* $\mathfrak{A}_i = \{B \cap \Omega_i \mid B \in \mathbb{B}\}$ *and* $\widehat{\mathfrak{A}_i} = \mathfrak{A}_i \cup \mathcal{S}_i$. *Then the incidence structure* π_i *defined by* $\pi_i = (\Omega_i, \widehat{\mathfrak{A}_i})$ *is an affine plane of order* q.

Proof. Fix i $(1 \le i \le q)$. We call each element of $\widehat{\mathfrak{A}_i}$ a line of π_i. Since any block and any point class of $(\mathbb{P},\mathbb{B})$ intersect in a unique point and Ω_i contains exactly q point classes of $\mathcal{C}$, we have $|B \cap \Omega_i| = q$ for any $B \in \mathbb{B}$. Hence each line of π_i contains exactly q points of Ω_i. Let $P, Q \in \Omega_i$ $(P \ne Q)$. If P and Q are contained in the same point class of $\mathcal{S}_i$, say $\mathcal{C}_{ij} (\in \mathcal{S}_i)$, there is no block of $\mathbb{B}$ containing P and Q and so $\mathcal{C}_{ij}$ is a unique line of $\widehat{\mathfrak{A}_i}$ through P and Q. If P and Q are not in the same point class of $\mathcal{S}_i$, there exists a block B of $\mathbb{B}$ containing P and Q. Hence $B \cap \Omega_i$ is a line of $\widehat{\mathfrak{A}_i}$ through P and Q and Corollary 3.5 implies that this is the only line of $\widehat{\mathfrak{A}_i}$ through P and Q. Therefore π_i is a 2-$(q^2, q, 1)$ design. Hence π_i is an affine plane of order q. Thus the lemma holds. □

Let π_i be the affine plane defined in Lemma 3.10. Let $\ell_\infty^{(i)}$ be the set of $q+1$ parallel classes of π_i. Clearly $\mathrm{Aut}(\pi_i)$ induces a permutation group on the $q+1$ parallel classes of $\ell_\infty^{(i)}$. We note that $\mathcal{S}_i \in \ell_\infty^{(i)}$.

Lemma 3.11. *Let notations be as in Lemma 3.10. Then the following holds.*

(i) $U|_{\pi_i}$ *is the group of translations of order* q *with center* $\mathcal{S}_i$.

(ii) $H|_{\pi_i}$ *is the full translation group of* π_i *order* q^2.

(iii) $G|_{\pi_i} = H|_{\pi_i} \leq \mathrm{Aut}(\pi_i)$ *and* $G|_{\pi_i}$ *is the translation group of* π_i *of order* q^2 *isomorphic to* E_{q^2}*. In particular,* π_i *is a translation plane admitting the translation group* $G|_{\pi_i}$*.*

Proof. By Result 2.1, $B \cap B^g = \emptyset$ for any $B \in \mathbb{B}$ and any $g \in U \setminus \{1\}$. Hence U acts on π_i as a group of translations of π_i.

Let $\ell \in \mathfrak{A}_i$ be a line. Then $\ell = B_{j,g} \cap (i, H) = (i, H_{n_{ij}} g)$ for some $j \in \{1, \cdots, q\}$ and $g \in H$. For any $x \in H$,

$$\ell \cap \ell^x = (B_{j,g} \cap B_{j,gx}) \cap (i, H) = (i, H_{n_{ij}} g \cap H_{n_{ij}} gx) = \begin{cases} \ell & \text{if } x \in H_{n_{ij}}, \\ \emptyset & \text{if } x \notin H_{n_{ij}}. \end{cases}$$

From this, we have $\ell = \ell^x$ or $\ell \cap \ell^x = \emptyset$ for any $\ell \in \mathfrak{A}_i$ and $x \in H$. Hence H is the full translation group of order q^2 of π_i. Thus (ii) holds.

By (i)(ii) and by Result 2.2(i)(ii), $G|_{\pi_i} = H|_{\pi_i} \leq \mathrm{Aut}(\pi_i)$ and (iii) holds. $\square$

By Results 2.2(ii) and 3.11, for each element x in G, $x|_{\pi_i}$ is a translation with a unique center unless $x|_{\pi_i} = 1$.

Lemma 3.12. *Set* $N_i = \langle U, U_i \rangle$ *and* $K_i = \mathrm{Ker}(N_i, \pi_i)$ $(1 \leq i \leq q)$*. Then the following holds.*

(i) $U_i|_{\pi_i} = U|_{\pi_i}$*, and* $U_i|_{\pi_i}$ *is the group of translations of* π_i *of order* q *with center* $\mathcal{S}_i$*. Moreover,* $N_i = UK_i = U_i K_i \rhd K_i$ *and* $U \simeq E_q$*.*

(ii) $K_i \cap K_j = 1$ *for any distinct* i *and* j $(1 \leq i, j \leq q)$*.*

(iii) *Let* r, s *and* t *be as defined in Hypothesis 3.7. Then* $N_s = N_t = K_s \times K_t \simeq E_q \times E_q$ *and* K_s *acts faithfully and semiregularly on* Ω_i *for any* $i \neq s$*.*

Proof. By Lemma 3.11(i), $U|_{\pi_i}$ is a group of translations of π_i with center $\mathcal{S}_i \in \ell_\infty^{(i)}$. By Lemma 3.8(i), U_i is also a group of translations of π_i with center $\mathcal{S}_i \in \ell_\infty^{(i)}$. By Lemma 3.11(iii), $U|_{\pi_i} = N_i|_{\pi_i} = U_i|_{\pi_i} \simeq E_q$ because U and U_i are groups of translations of order q of the same center. Thus $U \cap K_i = U_i \cap K_i = 1$. It follows that $N_i/K_i \simeq E_q$, $N_i = UK_i = U_i K_i$. Therefore (i) holds.

Let $x \in K_i \cap K_j$ $(i \neq j)$. For each $B \in \mathbb{B}$, $\mathrm{F}(x) \supset (B \cap \Omega_i) \cup (B \cap \Omega_j)$. Hence $|B \cap B^x| \geq |(B \cap \Omega_i)| + |B \cap \Omega_j| = 2q$. However, as $(\mathbb{P}, \mathbb{B})$ is a symmetric transversal design, this implies that $B = B^x$. Thus $x = 1$. Therefore $K_i \cap K_j = 1$ and (ii) holds.

By assumption, $U_s = U_t$ and so $N_s = N_t \rhd K_s, K_t$. On the other hand, $K_t \cap K_s = 1$ by (ii). Therefore, by (i) and (ii), $E_q \simeq N_s/K_s \geq K_t K_s/K_s \simeq K_t/(K_t \cap K_s) \simeq K_t$. Hence $|K_t| \leq q$. As $N_s = N_t$, by (i) $|K_s| = |K_t|$. Hence $|K_t| = |K_s| = |N_s|/|U| \geq |UU_s|/|U| = q$ and so $|K_t| = |K_s| = q$. It follows that $N_s = K_s \times K_t$ and N_s is an elementary abelian p-group of order q^2.

Assume that an element $x \in K_s$, $x \neq 1$, has a fixed point on Ω_i $(i \neq s)$, say Q. Let $\mathcal{C}_{ik}$ be the point class containing Q. As $U \leq N_s$ and N_s is abelian, $[U, x] = 1$. Hence x fixes all points on $\mathcal{C}_{ik}$. From this, $\mathrm{F}(x) \supset (B \cap \Omega_s) \cup (B \cap \mathcal{C}_{ik})$ for any block $B \in \mathbb{B}$. Therefore $|B \cap B^x| \geq |B \cap \Omega_s| + |B \cap \mathcal{C}_{ik}| = q + 1$, which implies $B = B^x$ for any $B \in \mathbb{B}$. Thus $x = 1$, a contradiction. $\square$

Lemma 3.13. *Set $K = \mathrm{Ker}(G, \pi_r)$ and let L be the stabilizer of a point class contained in π_r. Then $[G : K] = q^2, |K| = q, |L| = q^2$ and $L = UU_r$.*

Proof. By Lemma 3.11(iii), $|G/K| = q^2$ and by Lemmas 3.9(i) $|G| \geq q^3$. Hence $|K| \geq q$. As $H \cap K = 1$, we have $|K| = q$ and so $|G| = q^3$. By Lemma 3.9(ii), $[G : L] = q$. Hence $|L| = q^2$. On the other hand, clearly $UU_r \leq L$ and $U \cap U_r = 1$. Thus $L = UU_r$. □

Lemma 3.14. *K_s acts faithfully and regularly on each point class of $\mathcal{S}_t$. In particular, K_s acts on π_t as the translation group of order q with center $\mathcal{S}_t$ on $\ell_\infty^{(t)}$.*

Proof. Since $K_s = \mathrm{Ker}(\langle U_s, U\rangle, \pi_s) = \mathrm{Ker}(\langle U_t, U\rangle, \pi_s)$, it follows from Lemma 3.12(iii) that K_s acts faithfully and semiregularly on any point class $\mathcal{C}_{tk}$ of $\mathcal{S}_t$. As $|K_s| = q$, K_s acts regularly on $\mathcal{C}_{tk}$. Thus the lemma follows from Lemma 3.11(iii). □

Proof of Theorem 1.3

By Lemmas 3.11(iii) and 3.12(iii), K_s acts on π_r as a group of translations and $|N_s| = q^2$. Hence, as $|U_s| = |U| = q$ and $U_s \cap U = 1$, we have $N_s = U_sU$. On the other hand, $N_r (= \langle U_r, U\rangle)$ leaves each point class of $\mathcal{S}_r$ invariant. Hence, as $U_s \neq U_r$, $N_r \cap U_s = 1$. Therefore $N_r \cap U_sU = (N_r \cap U_s)U = U$. If a nontrivial element x of K_s fixes an element of $\mathcal{S}_r (\subset \pi_r)$, then x fixes every element of $\mathcal{S}_r$ because $x|_{\pi_r}$ is a translation of π_r by Result 2.2(ii). This, together with Lemma 3.13, implies that $x \in K_s \cap N_r \leq U_sU \cap N_r = U$, a contradiction. Therefore every nontrivial element x of K_s fixes each line of some parallel class $\mathcal{D} (\neq \mathcal{S}_r)$ of $\mathfrak{A}_r$ as $x|_{\pi_r}$ must be a translation. Let $\ell (\in \mathcal{D})$ be such a line and let B be a block containing ℓ. Since $B \cap B^x \supset \ell \cup (B \cap \Omega_s)$ and $|\ell \cup (B \cap \Omega_s)| > q$, we have $B = B^x$ and so x fixes B. As $U_s = U_t$, $x \in \langle U, U_t\rangle$ and so x fixes $B \cap \mathcal{C}_{ti}$ for every i $(1 \leq i \leq q)$. This is contrary to Lemma 3.14.

Remark 3.15. We note that the $\mathrm{STD}_3(9, 3)$ of the case Theorem 1.3(ii) is class regular. Let M be the $\mathrm{GH}(3, 3, 3)$ matrix in the third case of Example 3.3 and $H = \langle a, b\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $\mathrm{Aut}(H)$ acts transitively on the set subgroups $\{\langle a\rangle, \langle ab\rangle, \langle ab^2\rangle, \langle b\rangle\}$, we may assume that M has the following form :

$$M = \begin{bmatrix} N_3 & N_0 & N_1 \\ N_2 & N_3 & N_0 \\ N_1 & N_2 & N_3 \end{bmatrix} \text{ where } N_i = \langle ab^i\rangle \ (0 \leq i \leq 2) \text{ and } N_3 = \langle b\rangle.$$

Let $(\mathbb{P}, \mathbb{B})$ be a symmetric transversal design obtained from M. By Result 1.1, $\mathbb{P} = \{1, 2, 3\} \times H$ and the 9 point classes are

$$\{1\} \times R \ (R \in H/N_2), \quad \{2\} \times S \ (S \in H/N_1), \quad \{3\} \times T \ (T \in H/N_0),$$

where H/N_i denotes the three cosets of N_i in H $(0 \leq i \leq 2)$. We define a permutation σ of $\mathbb{P}$ by

$$(1, x)\sigma = (1, xab^2), \quad (2, x)\sigma = (1, xa^2b^2), \quad (3, x)\sigma = (1, xa)$$

for each $x \in H$. Since $ab^2 \in N_2$, $a^2b^2 \in N_1$ and $a \in N_0$, σ acts regularly on each point class of $(\mathbb{P}, \mathbb{B})$. We show that σ induces a permutation of $\mathbb{B}$. We can check the following.

$$\begin{aligned} N_3ab^2 \cap N_2a^2b^2 \cap N_1a &= \{a\} \\ N_0ab^2 \cap N_3a^2b^2 \cap N_2a &= \{a^2b^2\} \\ N_1ab^2 \cap N_0a^2b^2 \cap N_3a &= \{ab^2\} \end{aligned}$$

Hence the action of σ on the base blocks $B_{1,1}, B_{2,1}, B_{3,1}$ is

$$(B_{1,1})\sigma = B_{1,a}, \quad (B_{2,1})\sigma = B_{1,a^2b^2}, \quad (B_{3,1})\sigma = B_{1,ab^2}.$$

It follows that

$$(B_{1,h})\sigma = B_{1,ah}, \quad (B_{2,h})\sigma = B_{1,a^2b^2h}, \quad (B_{3,h})\sigma = B_{1,ab^2h}$$

for any $h \in H$. Therefore σ is an automorphism of $(\mathbb{P}, \mathbb{B})$ and $\langle\sigma\rangle$ is a class regular automorphism group of $(\mathbb{P}, \mathbb{B})$.

Concerning the above remark we would like to raise the following question.

Question 3.16. Let M be a $\mathrm{GH}(q, q, q)$ matrix over an elementary abelian p-group H of order q^2 corresponding to a spread $\mathcal{S} = \{U_1, \cdots, U_{q+1}\}$ of H. Assume $U_1, \cdots, U_{q-1}$ and U_q are forbidden subgroups corresponding to M. What is the condition of the spread $\mathcal{S}$ under which the $\mathrm{STD}_q(q^2, q)$ obtained form M is class regular.

It is conceivable that whether or not the $\mathrm{STD}_q(q^2, q)$ in Question3.16 is class regular depends on the choice of the corresponding spread.

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