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ON THE UPPER AND LOWER SEMICONTINUITY OF THE AUMANN INTEGRAL

by

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Abstract: Let (T,τ,μ) be a finite measure space, X be a Banach space, P be a metric space and let $L_1(\mu,X)$ denote the space of equivalence classes of X-valued Bochner integral functions on (T,τ,μ). We show that if $\phi : T\times P \to 2^X$ is a correspondence such that for each fixed $p\in P$, $\phi(\cdot,p)$ has a measurable graph and for each fixed $t\in T$, $\phi(t,\cdot)$ is either upper or lower semicontinuous then the Aumann integral of ϕ, i.e., $\int_T\phi(t,p)d\mu(t) = \{\int_T x(t)d\mu(t) : x\in S_\phi(p)\}$, where $S_\phi(p) = \{y\in L_1(\mu,X) : y(t)\in\phi(t,p)\ \mu\text{-a.e.}\}$, is either upper or lower semicontinuous in the variable p as well. Our results extend those of Aumann (1965, 1976) who has considered the above problem for $X = R^n$, and they have applications in general equilibrium and game theory.

Key words: Integral of a correspondence, upper semicontinuity, lower semicontinuity, quasi upper semicontinuity, measurable selection, Fatou's Lemma in infinite dimensions.

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1. INTRODUCTION

Let T be a measure space, P be a metric space, X be a Banach space and ϕ be a correspondence from $T \times P$ to 2^X (where 2^X denotes the set of all nonempty subsets of X), such that for each fixed $p \in P$, $\phi(\cdot, p)$ has a measurable graph and for each fixed $t \in T$, $\phi(t, \cdot)$ is either upper or lower semicontinuous. We would like to know whether the integral of the correspondence ϕ is either upper or lower semicontinuous as well. It is the purpose of this paper to provide an answer to the above question.[1] Specifically, we show (Theorem 3.1) that integration preserves upper semicontinuity (u.s.c.) and that, (Theorem 3.2) integration preserves lower semicontinuity (l.s.c.).

It should be noted that the problem of whether integration preserves u.s.c. or l.s.c. was first examined in a path breaking paper by Aumann (1965, 1976), [see also Schmeidler (1970)]. However, Aumann considered correspondences taking values in a finite dimensional Euclidean space. It turns out, that the finite dimensional arguments of Aumann cannot be readily adopted to cover Banach-valued correspondences. In particular, Aumann's method of proof of the fact that integration preserves u.s.c. is based heavily on the Lyapunov Theorem, a result which is false in infinite dimensional spaces. Nevertheless, for strong forms of u.s.c. correspondences (i.e., weakly u.s.c. correspondences) a result analogous of that of Aumann has been obtained in Yannelis (1988a) by means of the "approximate version of the Lyapunov Theorem." However, the arguments in Yannelis (1988a) cannot be adopted here to prove Theorem 3.1, since the correspondences we consider are u.s.c. in a much weaker sense than that in the above paper and furthermore they are not convex or compact valued. Hence, our arguments are of necessity quite different than those in Yannelis (1988a).

Finally, we would like to note that as the work of Aumann (1965, 1976) was motivated by the problem of the existence of an equilibrium in economies with a continuum of agents and finitely many commodities, our work was also motivated by the same problem but it allows for a continuum of commodities in addition to the continuum of agents.[2]

The paper proceeds as follows: Section 2 contains notation and definitions. Our main results are stated in Section 3 and their proofs are collected in Section 4. Finally, in Section 5 we show that integration preserves the weak closed graph property.

2. NOTATION AND DEFINITIONS

2.1 Notation

R^n denotes the n-fold Cartesian product of the set of real numbers R.

$\overline{\text{con}}A$ denotes the closed convex hull of the set A.

2^A denotes the set of all nonempty subsets of the set A.

$\emptyset$ denotes the empty set.

/ denotes the set theoretic subtraction.

dist denotes distance.

proj denotes projection.

If $A \subset X$, where X is a Banach space, $c\ell A$ denotes the norm closure of A.

If F_n, $(n=1,2,\ldots)$ is a sequence of nonempty subsets of a Banach space X, we will denote by LsF_n and LiF_n the set of its (strong) limit superior and (strong) limit inferior points respectively, i.e.,

$$LsF_n = \{x\varepsilon X : x = \lim_{k\to\infty} x_{n_k}, \ x_{n_k} \varepsilon F_{n_k}, \ k=1,2,\ldots\}, \text{ and}$$

$$LiF_n = \{x\varepsilon X : x = \lim_{n\to\infty} x_n, \ x_n \varepsilon F_n, \ n=1,2,\ldots\}.$$

2.2 Definitions

Let X and Y be sets. The graph of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \ \varepsilon \ X\times Y : y \ \varepsilon \ \phi(x)\}$. Let (T,τ,μ) be a complete, finite measure space, and X be a separable Banach space. The correspondence $\phi : T \to 2^X$ is said to have a measurable graph if $G_\phi \ \varepsilon \ \tau \otimes \beta(X)$, where $\beta(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes product σ-algebra. The correspondence $\phi : T \to 2^X$ is said to be lower measurable if for every open subset V of X, the set $\{t\varepsilon T : \phi(t) \cap V \neq \emptyset\}$ is an element of τ. Recall [see for instance Himmelberg (1975), p. 47 or Debreu (1966), p. 359] that if $\phi : T \to 2^X$ has a measurable graph, then ϕ is lower measurable. Furthermore, if $\phi(\cdot)$ is closed valued and lower measurable then $\phi : T \to 2^X$ has a measurable graph. A well-known result of Aumann (1967) which will be of fundamental importance in this paper, [see also Himmelberg (1975), Theorem 5.2, p. 60] says that if (T,τ,μ) is a complete, finite measure space, X is a separable metric space and $\phi : T \to 2^X$ is a nonempty valued correspondence having a measurable graph, then $\phi(\cdot)$ admits a measurable selection, i.e., there exists a measurable function $f : T \to X$ such that $f(t)\varepsilon\phi(t)$ μ-a.e.

We now define the notion of a Bochner integrable function. We will follow closely Diestel-Uhl (1977). Let (T,τ,μ) be a finite measure space and X be a Banach space. A function $f : T \to X$ is called simple if there exist $x_1, x_2, \ldots, x_n$ in X and $\alpha_1, \alpha_2, \ldots, \alpha_n$ in τ such that $f = \sum_{i=1}^{n} x_i \chi_{\alpha_i}$, where $\chi_{\alpha_i}(t) = 1$ if $t \in \alpha_i$ and $\chi_{\alpha_i}(t) = 0$ if $t \notin \alpha_i$. A function $f : T \to X$ is said to be μ-measurable if there exists a sequence of simple functions $f_n : T \to X$ such that $\lim_{n\to\infty} \| f_n(t) - f(t) \| = 0$ for almost all $t \in T$. A μ-measurable function $f : T \to X$ is said to be Bochner integrable if there exists a sequence of simple functions $\{f_n : n=1,2,\ldots\}$ such that

$$\lim_{n\to\infty} \int_T \| f_n(t) - f(t) \| d\mu(t) = 0.$$

In this case we define for each $E \in \tau$ the integral to be $\int_E f(t)d\mu(t) = \lim_{n\to\infty} \int_E f_n(t)d\mu(t)$. It can be shown [see Diestel-Uhl (1977), Theorem 2, p. 45] that, if $f : T \to X$ is a μ-measurable function then f is Bochner integrable if and only if $\int_T \| f(t) \| d\mu(t) < \infty$. We denote by $L_1(\mu,X)$ the space of equivalence classes of X-valued Bochner integrable functions $x : T \to X$ normed by

$$\| x \| = \int_T \| x(t) \| d\mu(t).$$

It is a standard result that normed by the functional $\| \cdot \|$ above, $L_1(\mu,X)$ becomes a Banach space, [see Diestel-Uhl (1977), p. 50].

We denote by S_ϕ the set of all X-valued Bochner integrable selections from $\phi : T \to 2^X$, i.e.,

$$S_\phi = \{x \in L_1(\mu,X) : x(t) \in \phi(t) \ \mu\text{-a.e.}\}.$$

Moreover, as in Aumann (1965) the integral of the correspondence $\phi : T \to 2^X$ is defined as follows:

$$\int_T \phi(t)d\mu(t) = \{\int_T x(t)d\mu(t) : x \in S_\phi\}.$$

In the sequel we will denote the above integral by $\int\phi$. Recall that the correspondence $\phi : T \to 2^X$ is said to be integrably bounded if there exists a map $h \in L_1(\mu, R)$ such that $\sup\{\|x\| : x \in \phi(t)\} \leq h(t)$ μ-a.e. Moreover, note that if T is a complete measure space, X is a separable Banach space and $\phi : T \to 2^X$ is an integrably bounded, nonempty valued correspondence having a measurable graph, then by the Aumann measurable selection theorem we can conclude that S_ϕ is nonempty and therefore $\int_T \phi(t)d\mu(t)$ is nonempty as well.

Let A_n, (n=1,2,...) be a sequence of nonempty subsets of a Banach space. Following Kuratowski (1966, p. 339) we say that A_n converges in A (written as $A_n \to A$) if and only if $LiA_n = LsA_n = A$. It may be useful to remind the reader that LiA_n and LsA_n are both closed sets and $LiA_n \subset LsA_n$ [see Kuratowski (1966), pp. 336-338].

Let X be a metric space and Y be a Banach space. The correspondence $\phi : X \to 2^Y$ is said to be u.s.c. at $x_0 \in X$, if for any neighborhood $N(\phi(x_0))$ of $\phi(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that for all $x \in N(x_0)$, $\phi(x) \subset N(\phi(x_0))$. We say that ϕ is u.s.c. if ϕ is u.s.c. at every point $x \in X$. Recall that this definition is equivalent to the fact that the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every open subset V of Y, [see for instance Kuratowski (1966), Theorem 3, p. 176].

Let ν be a small positive number and let B be the open unit ball in Y. The correspondence $\phi : X \to 2^Y$ is said to be <u>quasi upper-semicontinuous</u> (q.u.s.c.) at $x \in X$, if whenever the sequence x_n, $(n=1,2,...)$ in X converges to x, then for some n_0, $\phi(x_n) \subset \phi(x) + \nu B$ for all $n \geq n_0$. We say that ϕ is q.u.s.c. if ϕ is q.u.s.c. at every point $x \in X$. It can be easily checked that if ϕ is compact valued, quasi upper-semicontinuity implies upper-semicontintuity and vice-versa.

Let now P and X be metric spaces. The correspondence $F : P \to 2^X$ is said to be l.s.c. if the sequence p_n, $(n=1,2,...)$ in P converges to $p \in P$, then $F(p) \subset \mathrm{Li}F(p_n)$. Finally recall that the correspondence $F : P \to 2^X$ is said to be <u>continuous</u> if and only if it is u.s.c. and l.s.c.

With all these preliminaries out of the way we can now turn to the statements of the main theorems.

3. THE MAIN THEOREMS

We now state our main results:

<u>Theorem 3.1</u>: Let (T,τ,μ) be a complete, finite measure space, P be a metric space and X be a separable Banach space. Let $\psi : T \times P \to 2^X$ be a nonempty valued, integrably bounded correspondence, such that for each fixed $t \in T$, $\psi(t,\cdot)$ is q.u.s.c. and for each fixed $p \in P$, $\psi(\cdot,p)$ has a measurable graph. Then

$$\int \psi(t,\cdot) \text{ is q.u.s.c.}$$

<u>Theorem 3.2</u>: Let (T,τ,μ) be a complete, finite measure space, X be a separable Banach space and P be a metric space. Let $\phi : T\times P \to 2^X$ be an integrably bounded correspondence such that for each fixed $t\in T$, $\phi(t,\cdot)$ is l.s.c. and for each fixed $p\in P$, $\phi(\cdot,p)$ has a measurable graph. Then

$$\int\phi(t,\cdot) \text{ is l.s.c.}$$

<u>Remark 3.1</u>: If in addition to the assumptions of Theorem 3.1, it is assumed that $\int\psi(t,\cdot)$ is compact valued, then we can conclude that $\int\psi(t,\cdot)$ is u.s.c.

<u>Remark 3.2</u>: If in Theorem 3.1 we add the assumption that $\psi(\cdot,\cdot)$ is convex valued and that for all $(t,p)\in T\times P$, $\psi(t,p) \subset K$, where K is a weakly compact, convex, nonempty subset of X, then it follows from Lemma 4.1 (see next section) that $\int_T\psi(t,\cdot)d\mu(t)$ is weakly compact valued and we can conclude that $\int_T\psi(t,\cdot)d\mu(t)$ is weakly u.s.c., i.e., the set $\{p\in P : \int_T\psi(t,p)d\mu(t) \subset V\}$ is open in P for every weakly open subset V of X. Hence, from Theorem 3.1 we can obtain a version of Theorem 4.1 in Yannelis (1988a) which does not require (T,τ,μ) to be atomless.

The Corollaries below follow directly from Theorems 3.1, 3.2 and Remark 3.1. They extend some results of Aumann (1965, Theorem 5, and Corollary 5.2) to separable Banach spaces.

<u>Corollary 3.1</u>: Let (T,τ,μ) be a complete, finite measure space, P be a metric space and X be a separable Banach space. Let $\psi : T\times P \to 2^X$ be an integrably bounded, nonempty valued correspondence such that for

each fixed $p \varepsilon P$, $\psi(\cdot,p)$ has a measurable graph and for each fixed $t \varepsilon T$, $\psi(t,\cdot)$ is continuous. Moreover, suppose that $\int_T \psi(t,\cdot)d\mu(t)$ is compact valued. Then

$$\int_T \psi(t,\cdot)d\mu(t) \text{ is continuous.}$$

Corollary 3.2:[3] Let (T,τ,μ) be a complete, finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$, $(n=1,2,\ldots)$ be a sequence of integrably bounded, nonempty valued correspondence having a measurable graph, such that:

(i) For all n, $(n=1,2,\ldots)$, $\phi_n(t) \subset K$ μ-a.e., where K is a compact, nonempty subset of X, and

(ii) $\phi_n(t) \to \phi(t)$ μ-a.e.

Then

$$\int_T \phi_n(t)d\mu(t) \to c\ell \int_T \phi(t)d\mu(t).$$

Moreover, if $\phi(\cdot)$ is convex valued then

$$\int_T \phi_n(t)d\mu(t) \to \int_T \phi(t)d\mu(t).$$

4. PROOF OF THE MAIN THEOREMS

4.1 Lemmata

For the proof of our main results we will need some preparatory Lemmata.

Lemma 4.1: Let (T,τ,μ) be a finite measure space and X be a Banach space. Let $\phi : T \to 2^X$ be a correspondence satisfying the following condition:

(i) $\phi(t) \subset K$ μ-a.e., where K is a compact, nonempty subset of X.

Then

$$c\ell\int_T \overline{\text{con}\phi}(t)d\mu(t) = \int_T \overline{\text{con}\phi}(t)d\mu(t).$$

Proof: Let $\tilde{K} = \overline{\text{con}K}$. Note that $\tilde{K}$ is compact, [see Dunford-Schwartz (1958), Theorem 6, p. 416] nonempty and convex. Hence, from Diestel's theorem [Diestel (1977), Theorem 2] we have that $S_{\tilde{K}}$ is weakly compact in $L_1(\mu,X)$. Since $\overline{\text{con}\phi}(\cdot)$ is norm closed and convex valued so is $S_{\overline{\text{con}\phi}}$. It is a consequence of the Separation Theorem that the weak and norm topologies coincide on closed convex sets. Hence, $S_{\overline{\text{con}\phi}}$ is weakly closed. Since $S_{\overline{\text{con}\phi}} \subset S_{\tilde{K}}$ and the latter set is weakly compact we can conclude that $S_{\overline{\text{con}\phi}}$ is weakly compact. Define the mapping $\gamma : L_1(\mu,X) \to X$ by $\gamma(x) = \int_T x(t)d\mu(t)$. Certainly γ is linear and norm continuous. It follows from Theorem 15 in Dunford-Schwartz (1958, p. 422) that γ is also weakly continuous. Therefore, $\gamma(S_{\overline{\text{con}\phi}}) = \{\gamma(x) : x \varepsilon S_{\overline{\text{con}\phi}}\} = \int_T \overline{\text{con}\phi}(t)d\mu(t)$ is weakly compact, and we can conclude that $c\ell\int_T \overline{\text{con}\phi}(t)d\mu(t) = \int_T \overline{\text{con}\phi}(t)$. This completes the proof of the Lemma.

Notice that the above proof of the Lemma showed that $\int_T \overline{\text{con}\phi}(t)d\mu(t)$ is weakly compact. Hence, the above Lemma may be seen as the infinite dimensional extension of Theorem 4 of Aumann (1965).

The result below is an infinite dimensional Li version of the Fatou Lemma for the set of all integrable selections from a correspondence.

Lemma 4.2: Let (T,τ,μ) be a complete, finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$, $(n=1,2,\ldots)$ be a sequence of integrably bounded correspondences having a measurable graph, i.e., $G_{\phi_n} \varepsilon \tau \otimes \beta(X)$. Then

$$S_{Li\phi_n} \subset LiS_{\phi_n} .$$

Proof: See Yannelis (1988b, Lemma 5.3).

4.2 Proof of Theorem 3.1

Without loss of generality we may assume throughout the argument that $\int_T d\mu(t) = 1$. Let B be the open unit ball in X, and ν be a small positive number. We must show that if $\{p_n : n=1,2,\ldots\}$ is a sequence in P converging to $p \varepsilon P$, then for a suitable n_0,

$$\int_T \psi(t,p_n)d\mu(t) \subset \int_T \psi(t,p)d\mu(t) + \nu B \text{ for all } n \geq n_0.$$

Define the mapping $S_\psi : P \to 2^{L_1(\mu,X)}$ by $S_\psi(p) = \{x \varepsilon L_1(\mu,X) : x(t) \varepsilon \psi(t,p)\ \mu\text{-a.e.}\}$. Let B and $\tilde{B}$ be the open unit balls in X and $L_1(\mu,X)$ respectively. We first show that for a suitable n_0, $S_\psi(p_n) \subset S_\psi(p) + \nu\tilde{B}$ for all $n \geq n_0$.

We begin by finding the suitable n_0. Since for each fixed $t \varepsilon T$, $\psi(t,\cdot)$ is q.u.s.c. we can find a minimal M_t such that

$$(4.1) \qquad \psi(t,p_n) \subset \psi(t,p) + \delta B \text{ for all } n \geq M_t,$$

where $\delta = \frac{\nu}{3\mu(T)}$.

We now show that M_t is a measurable function of t. However, first we make a few observations. By assumption for each fixed p and n, $G_{\psi(\cdot,p_n)+\delta B} \varepsilon \tau \otimes \beta(X)$ and so does $(G_{\psi(\cdot,p_n)+\delta B})^c$, (where S^c denotes the complement of the set S). It is easy to see that

$G_{\psi(\cdot,p)} \cap (G_{\psi(\cdot,p_n)}+\delta B)^c \in \tau \otimes \beta(X)$. Therefore, the set

$$U = \{(t,x)\in T\times X : (t,x)\in G_{\psi(\cdot,p)} \cap (G_{\psi(\cdot,p_n)}+\delta B)^c\}$$

belongs to $\tau \otimes \beta(X)$.

It follows from the projection theorem [see for instance Hildenbrand (1974), p. 44] that

$$\mathrm{proj}_T(U)\in\tau.$$

Notice that,

$$\mathrm{proj}_T(U) = \{t\in T : \psi(t,p) \not\subset \psi(t,p_n) + \delta B\}$$

$$= \{t\in T : \psi(t,p)/(\psi(t,p_n) + \delta B) \neq \emptyset\}.$$

By virtue of the measurability of the above set we can now conclude that M_t is a measurable function of t. In particular, simply notice that,

$$\{t\in T: M_t=m\} = \bigcap_{n\geq m} \{t\in T: \psi(t,p_n) \subset \psi(t,p)+\delta B\} \cap \{t\in T: \psi(t,p_{m-1}) \not\subset \psi(t,p)+\delta B\}.$$

We are now in a position to choose the desired n_0. Since $\psi(\cdot,\cdot)$ is integrably bounded there exists $h\in L_1(\mu,R)$ such that for almost all $t\in T$, $\sup\{\|x\| : x\in\psi(t,p)\} \leq h(t)$ for each $p\in P$.

Choose δ_1 such that if $\mu(S) < \delta_1$, $(S \subset T)$, then $\int_S h(t)d\mu(t) < \frac{\nu}{3}$. Since M_t is a measurable function of t, we can choose n_0 such that $\mu(\{t\in T : M_t \geq n_0\}) < \delta_1$. This is the desired n_0. Let $n \geq n_0$ and $y\in S_\psi(p_n)$. We must show that $y\in S_\psi(p) + \nu\tilde{B}$.

By assumption, for each fixed $p \varepsilon P$, $\psi(\cdot,p)$ has a measurable graph and $\psi(\cdot,\cdot)$ is nonempty valued. Hence, by the Aumann measurable selection theorem there exists a measurable function $f_1 : T \to X$ such that $f_1(t) \varepsilon \psi(t,p)$ μ-a.e. Define the correspondence $\theta : T \to 2^X$ by $\theta(t) = (\{y(t)\} + \delta B) \cap \psi(t,p)$. It follows from (4.1) that for all $t \varepsilon T_0 = \{t : M_t \leq n_0\}$, $\theta(t) \neq \emptyset$. Moreover, $\theta(\cdot)$ has a measurable graph. Another application of the Aumann measurable selection theorem allows us to guarantee the existence of a measurable function $f_2 : T \to X$ such that $f_2(t) \varepsilon \theta(t)$ μ-a.e. Define $f : T \to X$ by

$$f(t) = \begin{cases} f_1(t) \text{ for } t \notin T_0 \\ f_2(t) \text{ for } t \varepsilon T_0. \end{cases}$$

Then $f(t) \varepsilon \psi(t,p)$ μ-a.e. and since $\psi(\cdot,\cdot)$ is integrably bounded we can conclude that $f \varepsilon S_\psi(p)$. If we show that $\|f-y\| < \nu$ then $y \varepsilon S_\psi(p) + \nu\tilde{B}$ and we will be done. But this is easy to see. We have

$$\begin{aligned} \|f-y\| &= \int_{T/T_0} \|f_1(t) - y(t)\| d\mu(t) + \int_{T_0} \|f_2(t)-y(t)\| d\mu(t) \\ &< 2\int_{T/T_0} h(t) d\mu(t) + \int_{T_0} \delta d\mu(t) \\ &< \frac{2\nu}{3} + \delta\mu(T) = \frac{2\nu}{3} + \frac{\nu}{3\mu(T)} \cdot \mu(T) = \nu. \end{aligned}$$

This completes the proof of the fact that, if the sequence $\{p_n : n=1,2,\ldots\}$ in P converges to $p \varepsilon P$, then for a suitable n_0

(4.2) $\quad S_\psi(p_n) \subset S_\psi(p) + \nu\tilde{B}$ for all $n \geq n_0$.

Define now the mapping $\gamma : L_1(\mu,X) \to X$ by $\gamma(x) = \int_T x(t) d\mu(t)$. It follows from (4.2) that for all $n \geq n_0$,

$$\gamma(S_\psi(p_n)) = \{\gamma(x) : x \in S_\psi(p_n)\}$$

$$= \int_T \psi(t,p_n)d\mu(t) \subset \gamma(S_\psi(p) + \nu\tilde{B}) = \gamma(S_\psi(p)) + \gamma(\nu\tilde{B})$$

$$= \int_T \psi(t,p)d\mu(t) + \nu B.$$

Hence,

$$\int_T \psi(t,p_n)d\mu(t) \subset \int_T \psi(t,p)d\mu(t) + \nu B \text{ for all } n \geq n_0.$$

i.e., $\int_T \psi(t,\cdot)d\mu(t)$ is q.u.s.c. as was to be shown.

4.3 Proof of Theorem 3.2

We first show that the correspondence $S_\phi : P \to 2^{L_1(\mu,X)}$ defined by

$$S_\phi(p) = \{y \in L_1(\mu,X) : y(t) \in \phi(t,p)\ \mu\text{-a.e.}\}$$

is l.s.c.

To see this, let $\{p_n : n=1,2,\ldots\}$ be a sequence in P converging to $p \in P$. We must show that $S_\phi(p) \subset \operatorname{Li} S_\phi(p_n)$. Since by assumption for each fixed $t \in T$, $\phi(t,\cdot)$ is l.s.c. we have that $\phi(t,p) \subset \operatorname{Li}\phi(t,p_n)$ for all $t \in T$, and therefore,

$$(4.3) \qquad S_\phi(p) \subset S_{\operatorname{Li}\phi}(p_n).$$

It follows now from Lemma 4.2 that (4.3) can be written as:

$$S_\phi(p) \subset S_{\operatorname{Li}\phi}(p_n) \subset \operatorname{Li} S_\phi(p_n).$$

Hence,

$$S_\phi(\cdot) \text{ is l.s.c.}$$

Define now the mapping $\gamma : L_1(\mu,X) \to X$ by $\gamma(x) = \int_T x(t)d\mu(t)$. Then γ is linear and norm continuous. Notice that $\gamma(S_\phi(p)) = \{\gamma(x) : x \in S_\phi(p)\} = \int_T \phi(t,p)d\mu(t)$. Since $S_\phi(\cdot)$ is l.s.c. so is $\gamma(S_\phi)$, i.e., $\int_T \phi(t,\cdot)d\mu(t)$ is l.s.c. as was to be shown. This completes the proof of Theorem 3.2.

4.4 Proof of Corollary 3.2

We begin by proving an approximate version of the Fatou Lemma in infinite dimensions [see also Balder (1987), Khan-Majumdar (1986) and Yannelis (1988a) for w-Ls versions of this Lemma], which may be considered as an extension of the finite dimensional Fatou-type Lemmata obtained in Aumann (1965), Artstein (1979), Balder (1984), Hildenbrand-Mertens (1971), Rustichini-Yannelis (1986), and Schmeidler (1970).

Lemma 4.3: Let (T,τ,μ) be a complete, finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$, $(n=1,2,\ldots)$ be a sequence of nonempty valued, graph measurable and integrably bounded correspondences, taking values in a compact, nonempty subset of X. Then

$$\mathrm{Ls}\int_T \phi_n(t)d\mu(t) \subset c\ell \int_T \mathrm{Ls}\,\phi_n(t)d\mu(t).$$

Moreover, if $\mathrm{Ls}\,\phi_n(\cdot)$ is convex valued, then

$$\mathrm{Ls}\int_T \phi_n(t)d\mu(t) \subset \int_T \mathrm{Ls}\,\phi_n(t)d\mu(t).$$

Proof: Denote by P the interval $[0,1)$. Define the correspondence $\psi : T\times P \to 2^X$ by

$$\psi(t,p) = \begin{cases} \phi_n(t) & \text{if } \frac{1}{n+1} < p < \frac{1}{n} \\ \phi_n(t) \cup \phi_{n+1}(t) & \text{if } p = \frac{1}{n+1} \\ Ls\phi_n(t) & \text{if } p = 0. \end{cases}$$

It can be easily checked that for each fixed $t \varepsilon T$, $\psi(t,\cdot)$ is u.s.c. and that for each fixed $p \varepsilon P$, $\psi(\cdot,p)$ has a measurable graph. Moreover, ψ is integrably bounded. Hence, ψ satisfies all the assumptions of Theorem 3.1 and thus, $\int_T \psi(t,\cdot)d\mu(t)$ is q.u.s.c. Let now $x \varepsilon Ls\int_T \phi_n(t)d\mu(t)$, i.e., there exists x_{n_k} such that $\lim_{k\to\infty} x_{n_k} = x$, $x_{n_k} \varepsilon \int_T \phi_{n_k}(t)d\mu(t)$, $(k=1,2,\ldots)$. We wish to show that $x \varepsilon c\ell\int_T Ls\phi_n(t)d\mu(t)$.

Since $\int_T \psi(t,\cdot)d\mu(t)$ is q.u.s.c. it follows that if p_{n_k} converges to 0 then $\int_T \psi(t,p_{n_k})d\mu(t) \subset \int_T \psi(t,0)d\mu(t) + \nu B$ for all sufficiently large k. Consequently, $x_{n_k} \varepsilon \int_T \psi(t,0)d\mu(t) + \nu B$ for all sufficiently large k and therefore, $x \varepsilon c\ell\int_T \psi(t,0)d\mu(t) \equiv c\ell\int_T Ls\phi_n(t)d\mu(t)$ as was to be shown. If now $Ls\phi_n(\cdot)$, is convex valued (recall that $Ls\phi_n(\cdot)$ is closed valued as well) it follows from Lemma 4.1 and the first conclusion of Lemma 4.3 that

$$Ls\int_T \phi_n d\mu(t) \subset c\ell\int_T Ls\phi_n(t)d\mu(t) = \int_T Ls\phi_n(t)d\mu(t).$$

The proof of the Lemma is now complete.

We are now ready to complete the proof of Corollary 3.2. Notice first that it follows from Lemma 4.2 that

(4.4) $\quad \int Li\phi_n \subset Li\int \phi_n.$

To see this define the linear mapping $\gamma : L_1(\mu,X) \to X$ by $\gamma(x) = \int_T x(t)d\mu(t)$. Note that $\gamma(S_{Li\phi_n}) = \{\gamma(x) : x \varepsilon S_{Li\phi_n}\} = \int Li\phi_n$ and hence by virtue of Lemma 4.2 we can conclude that $\gamma(S_{Li\phi_n}) \subset \gamma(LiS_{\phi_n}) = \{\gamma(x) : x \varepsilon LiS_{\phi_n}\} = Li\int\phi_n$. This completes the proof of (4.4). Since by assumption $\phi_n(t) \to \phi(t)$ μ-a.e., i.e., $\phi(t) = Li\phi_n(t) = Ls\phi_n(t)$ μ-a.e., it follows from Lemma 4.3 and the expression (4.4) above that:

$$(4.5) \qquad \int\phi = \int Li\phi_n \subset Li\int\phi_n \subset Ls\int\phi_n \subset c\ell\int Ls\phi_n = c\ell\int\phi.$$

Therefore,

$$c\ell\int_T\phi(t)d\mu(t) = Li\int_T\phi_n(t)d\mu(t) = Ls\int_T\phi_n(t)d\mu(t),$$

i.e.,

$$\int_T\phi_n(t)d\mu(t) \to c\ell\int_T\phi(t)d\mu(t).$$

If now $\phi(\cdot)$ is convex valued, (4.5) can be written (recall the second conclusion of Lemma 4.3) as:

$$\int\phi = \int Li\phi_n \subset Li\int\phi_n \subset Ls\int\phi_n \subset \int Ls\phi_n = \int\phi.$$

Thus,

$$\int_T\phi(t)d\mu(t) = Li\int_T\phi_n(t)d\mu(t) = Ls\int_T\phi_n(t)d\mu(t),$$

i.e.,

$$\int_T\phi_n(t)d\mu(t) \to \int_T\phi(t)d\mu(t),$$

and this completes the proof of Corollary 3.2.

5. ON THE WEAK CLOSED GRAPH PROPERTY OF THE AUMANN INTEGRAL

Let $\{A_n : n=1,2,\ldots\}$ be a sequence of nonempty subsets of a Banach space X, and denote by w-LsA_n the set of its weak limit superior points, i.e.,

$$\text{w-Ls}A_n = \{x\varepsilon X : x = \text{w-}\lim_{k\to\infty} x_{n_k},\ x_{n_k} \varepsilon A_{n_k},\ k=1,2,\ldots\}.$$

Let (T,τ,μ) be a complete finite measure space, P be a metric space and X be a separable Banach space. The correspondence $\psi : T\times P \to 2^X$ is said to have a <u>weakly closed graph</u> if w-Ls$\psi(t,p_n) \subset \psi(t,p)$ μ-a.e., whenever the sequence $\{p_n : n=1,2,\ldots\}$ in P converges to $p\varepsilon P$.

The following result in Yannelis (1988b) will be used to prove that if for each fixed $t\varepsilon T$, $\psi(t,\cdot)$ has a weakly closed graph then so does the integral of $\psi(t,\cdot)$.

<u>Lemma 5.1</u>: Let (T,τ,μ) be a finite measure space and X be a separable Banach space. Let $\{f_n : n=1,2,\ldots\}$ be a sequence of functions in $L_p(\mu,X)$, $1 \leq p < \infty$ such that f_n converges weakly to $f\varepsilon L_p(\mu,X)$. Suppose that for all n, (n=1,2,...), $f_n(t)\varepsilon F(t)$ μ-a.e., where $F : T \to 2^X$ is a weakly compact, integrably bounded, nonempty valued correspondence. Then

$$f(t)\varepsilon\overline{\text{con}}\text{w-Ls}\{f_n(t)\}\ \mu\text{-a.e.}$$

<u>Proof</u>: See Yannelis (1988b, Corollary 3.1).

We are now ready to state the main result of this section, which generalizes Theorem 4.1 in Yannelis (1988a).

<u>Theorem 5.1</u>: Let (T,τ,μ) be a complete, finite measure space, P be a metric space and X be a separable Banach space. Let $\psi : T\times P \to 2^X$ be a nonempty, closed, convex valued correspondence such that:

(i) for each fixed $t\varepsilon T$, $\psi(t,\cdot)$ has a weakly closed graph,

(ii) for all $(t,p) \varepsilon T\times P$, $\psi(t,p) \subset K(t)$ where $K : T \to 2^X$ is an integrably bounded, weakly compact and nonempty valued correspondence.

Then

$$\int\psi(t,\cdot) \text{ has a weakly closed graph.}$$

<u>Proof</u>: We first show that the set-valued function $S_\psi : P \to 2^{L_1(\mu,X)}$ defined by

$$S_\psi(p) = \{x\varepsilon L_1(\mu,X) : x(t)\varepsilon\psi(t,p) \ \mu\text{-a.e.}\},$$

has a weakly closed graph, i.e., if $\{p_n : n=1,2,\ldots\}$ is a sequence in P converging to $p\varepsilon P$, then

$$\text{(5.1)} \qquad w\text{-}\mathrm{Ls}S_\psi(p_n) \subset S_\psi(p).$$

To this end let $x\varepsilon w\text{-}\mathrm{Ls}S_\psi(p_n)$, i.e., there exists x_k, $(k=1,2,\ldots)$ in $L_1(\mu,X)$ such that x_k converges weakly to $x\varepsilon L_1(\mu,X)$, and $x_k(t)\varepsilon\psi(t,p_{n_k})$ μ-a.e., we must show that $x\varepsilon S_\psi(p)$. It follows from Lemma 5.1 that $x(t)\varepsilon\overline{\mathrm{con}}w\text{-}\mathrm{Ls}\{x_k(t)\}$ μ-a.e. and therefore,

$$\text{(5.2)} \qquad x(t)\varepsilon\overline{\mathrm{con}}w\text{-}\mathrm{Ls}\psi(t,p_n) \ \mu\text{-a.e.}$$

Since for each fixed $t\varepsilon T$, $\psi(t,\cdot)$ has a weakly closed graph we have that:

(5.3) $\quad w\text{-}Ls\psi(t,p_n) \subset \psi(t,p)$ μ-a.e.

Combining (5.2) and (5.3) and taking into account the fact that ψ is convex valued we have that $x(t)\varepsilon\psi(t,p)$ μ-a.e. Since ψ is integrably bounded, we can conclude that $x\varepsilon S_\psi(p)$. This completes the proof of the fact that $S_\psi(\cdot)$ has a weakly closed graph. Define the linear mapping

$$\pi : L_1(\mu,X) \to X \text{ by } \pi(x) = \int x(t)d\mu(t).$$

It follows from (5.1) that if the sequence $\{p_n: n=1,2,\ldots\}$ in P converges to $p\varepsilon P$, then

$$\begin{aligned}\pi(w\text{-}LsS_\psi(p_n)) &= \{\pi(x) : x\varepsilon w\text{-}LsS_\psi(p_n)\} \\ &= w\text{-}Ls\int\psi(t,p_n) \subset \pi(S_\psi(p)) = \{\pi(x) : x\varepsilon S_\psi(p)\} \\ &= \int\psi(t,p),\end{aligned}$$

i.e., $\int\psi(t,\cdot)$ has a weakly closed graph as was to be shown.

Remark 5.1: It can be easily shown by means of the failure of the Lyapunov theorem in infinite dimensional spaces that Theorem 5.1 is false without the convex valueness of the correspondence $\psi : T\times P \to 2^X$ [see Rustichini (1987) for a complete argument].

FOOTNOTES

[1] In general equilibrium theory, T denotes the measure space of agents, X denotes the commodity space, P denotes the price space, $\phi(t,p)$ denotes the demand set of agent t at prices p and the integral of ϕ denotes the aggregate demand set [see for instance Aumann (1966) or Schmeidler (1969) or Hildenbrand (1974) who have considered the above problem for $X = R^{\ell}$, in order to prove the existence of an equilibrium for an economy with an atomless measure space of agents and with finitely many commodities].

[2] See for instance Khan-Yannelis (1987) for the usefulness of our results in general equilibrium theory. Also, applications of our results in game theory are given in Balder-Yannelis (1988).

[3] Compare with Corollary 3.2 in Yannelis (1988b) where a different notion of convergence of sequences of set-valued functions was used.

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