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What is Perfect Competition?

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What is Perfect Competition?

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ABSTRACT

We provide a mathematical formulation of the idea of perfect competition for any economy with infinitely many agents and commodities. We conclude that in the presence of infinitely many commodities the Aumann (1964,1966) measure space of agents, i.e., the interval [0,1] endowed with Lebesgue measure, is not appropriate to model the idea of perfect competition and we provide a characterization of the "appropriate" measure space of agents in an infinite dimensional commodity space setting. The latter is achieved by modeling precisely the idea of an economy with "many more" agents than commodities. For such an economy the existence of a competitive equilibrium is proved. The convexity assumption on preferences is not needed in the existence proof.

What is Perfect Competition?

Aldo Rustichini and Nicholas C. Yannelis

Abstract. We provide a mathematical formulation of the idea of perfect competition for an economy with infinitely many agents and commodities. We conclude that in the presence of infinitely many commodities the Aumann (1964, 1966) measure space of agents, i.e., the interval $[0,1]$ endowed with Lebesgue measure, is not appropriate to model the idea of perfect competition and we provide a characterization of the "appropriate" measure space of agents in an infinite dimensional commodity space setting. The latter is achieved by modeling precisely the idea of an economy with "many more" agents than commodities. For such an economy the existence of a competitive equilibrium is proved. The convexity assumption on preferences is not needed in the existence proof.

1. Introduction

Perfect competition prevails in an economy if no individual can influence the price at which goods are bought and sold.

In order to model rigorously the idea of perfect competition, Aumann (1964, 1965, 1966) assumed that the set of agents in the economy is an atomless measure space. As a consequence of the non-atomicity assumption, each agent in the economy is negligible and therefore will take prices as given.

A special feature of the Aumann model is that the number of commodities in the economy is finite. In particular the commodity space in his model is the positive cone of the Euclidean space $\mathbb{R}^n$. This is quite important because given the fact that the measure space of agents is atomless and the dimension of the commodity space is finite, it turns out that the convexity assumption on preferences is not needed to prove the existence of a competitive equilibrium. In particular, the Lyapunov theorem is used to convexify the aggregate demand set and make the standard fixed point argument applicable.

However, the situation is quite different when the commodity space is not finite dimensional. Indeed, the Lyapunov theorem fails in infinite dimensional spaces [see for instance Diestel-Uhl (1977)] and one loses the nice convexifying effect on the aggregate demand set. (Here, it is important to note that all the basic results of Aumann (1965) which constitute the main technical tools to model the idea of perfect competition fail in infinite dimensional spaces [see for instance Rustichini (1989) or Yannelis (1990)].)

Moreover, as it is known from the work of Aumann (1964), in the presence of finitely many commodities, core allocation characterize competitive equilibrium allocations. However, contrary to the Aumann core equivalence theorem, in infinite dimensional commodity spaces in general, core allocations do not characterize competitive equilibrium allocations. In particular, Rustichini-Yannelis (1991) showed that even if the measure space of agents is atomless, preferences are (weakly) continuous, strictly convex, monotone, and initial endowments strictly positive, the core equivalence theorem ceases to be true. Does this then suggest that the nonatomicity assumption may not be enough in infinite dimensional commodity spaces in order to model the idea of perfect competition?

It may be useful, before we proceed, to put aside for a moment the strictly mathematical nature of the problem, and look more closely at the economic significance of the nonatomicity condition, and its implications in the case where the commodity space is finite dimensional. By definition, any subset of the space of agents, that is any coalition, must have positive measure in order not to be insignificant. We may think of this as a "critical mass" condition on any coalition.

On the other hand, thanks to the nonatomicity condition, any coalition of nonzero measure will have a set of possible subcoalitions (still of positive measure) which is so large that it makes any collusive behavior arduous.

The previous discussion suggests that one should not look for a characterization of perfect competition (or more generally of a strategic situation where each single player has an insignificant influence) in the space of agents or players by itself, but in the relation of the dimension of the measure space of agents and the dimension of the commodity (or strategy) space. The concept of dimension, of course, has to be given a rigorous formulation. Simply and informally stated, the characterization of perfect competition that we introduce in this paper does not just specify how many agents there are, but how many agents (or players) deal with how many commodities (or strategy choices). To put it differently, the assumption which is introduced in this paper is that the economy needs to have "many more" agents than commodities. Such an assumption (see Sections 3 and 6 for a rigorous definition) is stronger than the nonatomicity condition of the measure space of agents in an infinite dimensional commodity space setting and it is equivalent to the nonatomicity assumption in finite dimensional commodity spaces. In essence, this is the hidden assumption in the Aumann model which drives his results.[1]

The main contribution of this paper is to provide a rigorous formulation of the idea of perfect competition, and derive the analogous results to those of Aumann for separable Banach spaces. Moreover, we characterize the measure space of agents which satisfy the condition that the dimensionality of the measure space of agents is bigger than that of the commodity space. As it may be expected, the Aumann (1964, 1966) measure space of agents, i.e., the $[0,1]$ interval endowed with Lebesgue measure is not "large enough" to model the idea of perfect competition in the presence of infinitely many commodities.

The rest of the paper is organized as follows: Section 2 contains definitions and some mathematical preliminaries. Section 3 contains the main result, i.e., the integral of a Banach-valued correspondence is convex provided that the dimensionality of the measure space is greater than the dimensionality of the Banach space. This result is interpreted as the convexifying effect on the aggregate demand set. Section 4 characterizes the measure space whose dimensionality is bigger than that of the Banach space of commodities. Some important corollaries of the main result are collected in Section 5. Section 6 uses the main result of this paper to prove the existence of the competitive equilibrium. Finally, Section 7 contains an application of our main theorem to the problem of the existence of a pure strategy Nash equilibrium in games with a continuum of players and with an infinite dimensional strategy space.

---

[1] It should be noted that several authors have already made remarks of this nature [see for example, Mertens (1990), Mas-Colell (1975), Gretsky-Ostroy (1985), Ostroy-Zame (1988)], but they did not provide a precise mathematical modeling of the idea of "many more" agents than commodities. The only exemption is the work of Lewis (1977) which uses nonstandard analysis. In particular, she makes use of a nonstandard analog of the Lyapunov theorem proved by Loeb (1971).

2. Definitions and Preliminaries

The space X is a Banach space over the field $\mathbb{R}$ of real numbers; it is assumed to be separable. The norm of an element $x \in X$ will be denoted by $||x||$. Unless otherwise specified, the topology on X will be the norm topology.

A special importance will be assumed by a fixed weak compact subset K of X. We recall that (Dunford-Schwartz [1958], V. 6.3) the weak topology of K is a metric topology. As usual for a set $A \subset X$ we denote by co A and $\overline{\text{co}}A$ the convex hull and the closed convex hull of A, respectively. For a subset $A \subseteq X$, we denote by $P(A)$ or 2^A the set of all nonempty subsets of A, and $P_f(A)$ the set of all closed subsets of A.

The Banach space topology on X induces a natural structure of measure space on it if we denote by $\beta(X)$ the set of norm Borel subsets of X (and for any $A \in \beta(X)$, $\beta(A)$ are the Borel subsets of A). If $\beta_w(X)$ denotes the Borel σ-algebra generated by the weak topology of X, then from Masani [1978], Theorem 2.5(b), we have $\beta(X) = \beta_w(X)$.

We now proceed to describe our measure space. T will be a measurable space, with a σ-algebra τ, and a measure μ; we shall always assume (T, τ, μ) to be complete and finite. We do not assume nonatomicity, since such condition will be contained in an assumption which we will introduce later.

$L_\infty(\mu)$ is the space of real valued, measurable, essentially bounded functions defined on T. For any $E \in \tau$, the measure space (E, τ_E, μ_E) is naturally defined; and so is the space $L_{\infty,E}(\mu) \equiv \{f : E \to \mathbb{R}$, f is τ_E-measurable and μ_E-essentially bounded$\}$.

In this paper a *set-valued function* (or *correspondence*) F is defined to be a map from T to a set of all the nonempty subsets of X, $P(X)$. We denote the *graph* of F by $G_F = \{(t, x) \in T \times X : x \in F(t)\}$. For a given correspondence F, the new correspondence co F and $\overline{\text{co}}F$ are defined by $(\text{co}\, F)(t) \equiv \text{co}(F(t))$, and analogously for $\overline{\text{co}}F$.

Various notions of measurability are discussed in Himmelberg [1957]. In the case where the measure space T is complete, X is Souslin (conditions which are satisfied in our case) and F is closed valued they turn out to be equivalent [Himmelberg [1957], Theorem 3.5]. We adopt the following:

Definition. *The correspondence $F : T \to P(X)$ is said to be measurable if the graph of F, $\{(t,x) \in T \times X : x \in F(t)\}$ is an element of the product measure space $\tau \otimes \beta(X)$. The $\overline{\text{co}}$ operation preserve measurability [Himmelberg [1975], Theorem 9.1].*

K is now a compact Hausdorff space (in the weak topology). Let $C(K)$ be the space of continuous functions on K, and $C(K)^*$ its dual space. It is well known [Dunford-Schwartz [1957] Theorem IV.6.3] that this last space is (isometrically isomorphic to) the space $\text{rca}(K)$, the space of all regular, countably additive real valued set functions, defined on the σ-field of the Borel (norm) subsets of K; $\text{rca}^+(K)$ are the nonnegative elements of $\text{rca}(K)$. For any Borel subset A, $\text{rca}(A)$ and $\text{rca}^+(A)$ are defined in the natural way. For any Borel subset A of K, we denote M_A the unit ball of $\text{rca}^+(A)$; that is:

$$M_A \equiv \{\mu \in \text{rca}^+(A) : \mu(A) \leq 1\}.$$

If $\mu(A) = 1$, we refer to μ as usual as a *probability measure.* Among the probability measures over a Borel set we single out those having unit mass concentrated at one point:

Definition. *For any Borel subset A, the set of Dirac measures on A, denoted D_A, is the subset of M_A of elements δ_a, with $a \in A$, such that*

$$(\delta_a, \phi) = \phi(a) \quad \textit{for every} \quad \phi \in C(K).$$

For any $m \in \text{rca}(K)$, its norm $|m|$ is the total variation. We shall consider on the following measurable functions defined on T and with values in the space M_K. More precisely we introduce $L_\infty^{w^*}(\mu, M_K)$, the set of M_K-valued functions f which satisfy: (i) $\|f\| \in L^\infty(\mu)$; (ii) $t \mapsto (f(t), \phi)$ is a measurable (real valued) function for every $\phi \in C(K)$; [see Ionescu-Tulcea, A. and C., (1969), p. 99]. Also we let

$$L_\infty^{w^*}(\mu, M_F) \equiv \{\gamma \in L_\infty^{w^*}(\mu, M_K) : \nu(t) \in M_{F(t)} \ \mu\text{-a.e.}\}$$
$$L_\infty^{w^*}(\mu, D) \equiv \{\gamma \in L_\infty^{w^*}(\mu, M_K) : \nu(t) \in D \ \mu\text{-a.e.}\}$$
$$L_\infty^{w^*}(\mu, D_F) \equiv \{\gamma \in L_\infty^{w^*}(\mu, D) : \nu(t) \in D_{F(t)} \ \mu\text{-a.e.}\}.$$

Also for $\nu \in M$, F a closed subset of K, we define $(\nu, F) \equiv \int_K x\chi_F(x)d\nu(x)$ where χ_F is the indicator function of the set F. If X is a Banach space, we will denote by $L_1(\mu, X)$ the space of equivalence classes of X-valued Bochner integrable functions $f : T \to X$ normed by

$$\|f\| = \int_T \|f(t)\| d\mu(t).$$

The *integral of the set-valued function* $F : T \to 2^X$ is as usual:

$$\int_T F(t) d\mu(t) = \left\{ \int_T x(t) d\mu(t) : x(t) \in F(t) \ \mu\text{-a.e.} \right\}.$$

We will denote the above integral by $\int F$ or $\int F d\mu$.

3. The Main Theorem
(Convexifying Effect on Aggregation)

The following simple lemmata will be used later.

Lemma 1. *Let $\nu \in M_A \setminus D_A$. Then there exist two measures $\nu_\pm$ which satisfy:*

$$\nu_\pm = \nu \pm \nu_2 \in M_A \quad \textit{with:} \quad \nu_2 \neq 0, \ \nu_2 \in \mathrm{rca}(A).$$

Remark. Note ν_2 is *not* an element of M_A.

Proof. Since ν is not Dirac, there exist two disjoint Borel subsets of A, S and B say, such that

$$0 < \nu(S) \leq \nu(B) < 1. \tag{1}$$

We denote with $\nu(S)$, ν_B the restriction of ν to the two set S, B. Define now $\theta \equiv \nu(S)/\nu(B)$: from (1) above follows that $\theta \in (0, 1]$; also define: $\nu_2 \equiv -\nu_S + \theta\nu_B$ (clearly $\nu_2 \neq 0$); and $\nu_\pm \equiv \nu \pm \nu_2$. It is now an easy computation to check that $\nu_\pm \in M_A$.

Lemma 2. *Let $E \in \tau$ with $\mu(E) > 0$, and $\nu \in L_\infty^{w^*}(\mu, M_F)$ with $\nu(t) \in M_{F(t)} \setminus D_{F(t)}$ for every $t \in E$. Then there exists a measurable function $\nu_2(t) \in \mathrm{rca}(F(t))$, $\nu_2 \neq 0$ and $\nu \pm \nu_2 \in M_{F(t)}$, μ-a.e.*

Proof. The function ν_2 will result as a measurable selection from a correspondence. We recall $M_{F(t)}$ is a w^*-compact, convex subset of M_K. Define now: $\hat{\theta} = \max\{\theta \geq 0 : B(\nu(t), \theta) \subset M_{F(t)}\}$. From Lemma 1, $\hat{\theta} > 0$ for every $t \in E$; also $\hat{\theta}$ is a measurable function.

Now the correspondence $N(t) \equiv B(\gamma(t)), \hat{\theta}(t))$ is measurable [see Castaing and Valadier, (1977), III-41] with complete values, and takes values in M_K, a separable space (because this space is compact and metric). The result follows from the previous lemma and the Kuratowski and Ryll-Nardzewski (1962) measurable selection theorem.

Recall now that for any vector space over the real field an algebraic (Hamel) basis exists. The cardinality of any Hamel basis is the same, and we denote, for any vector space X, $\dim X$ the cardinality of any of its bases.

We introduce the following condition.

(A1) For the pair $((T, \tau, \mu), X)$, if E belongs to τ, $\mu(E) > 0$, then $\dim L_{E,\infty}(\mu) > \dim X$.

Remark 1. This is the condition that there are "many more" agents than commodities. In section 4 we shall characterize the measure spaces that satisfy the above condition.

Remark 2. The assumption implies that for any linear map T from $L_{E,\infty}(\mu)$ to X there exists a function f such that $f \neq 0$, $T(f) = 0 \in X$. This is the condition which will be actually used in the proof of the main theorem and its corollaries.

Remark 3. When X is finite dimensional, the assumption that the measure space (T, τ, μ) is non-atomic implies the assumption (A1) above. Of course, the reverse is also true. Hence the Main Theorem below, as well as the corollaries of the section 5 imply the analogous results in Aumann (1965).

Remark 4. We recall here that a Banach space is of second category and so (since every finite dimensional proper subspace is closed and with empty interior) it cannot have, if it is infinite dimensional, a countable Hamel basis. On the other hand such a space, as being a separable metric space, has cardinality at most $\aleph_0$. We conclude (under the continuum hypothesis) $\dim X = \aleph_0$.

Remark 5. Readers familiar with the proof by Lindenstrauss (1966) of Lyapunov's convexity theorem, or Knowles' theorem [see Knowles (1974) or Kluvanek-Knowles (1975)] will recognize the main idea of the proof.

Main Theorem. *Let K be a weakly compact, nonempty subset of X and let $F : T \to 2^K$ be a measurable, closed valued correspondence. Assume that the pair $((T, \tau, \mu), X)$ satisfy (A1). Then*

$$\int F = \int \overline{\text{co}} F.$$

Proof. Fix $t \in T$. Define

$$I(M_{F(t)}) \equiv \{x \in X : x = \int_k a d\mu(a), \quad \text{with} \quad \mu \in M_{F(t)}\}.$$

Clearly

$$\text{co}\, F(t) \subset I(M_{F(t)}).$$

Also, since $M_{F(t)}$ is w^*-compact convex, and the map $\mu \to \int_k a d\mu(a)$ is linear, and w^* to w-continuous, $I(M_F)$ is closed and convex, and therefore

$$\overline{\text{co}} F \subset I(M_F).$$

Clearly

$$\int_T F d\mu = \{x \in X : x = \int_T (\delta(t), F(t)) d\mu(t), \quad \delta \in D_F\}.$$

Obviously, $\int_T F d\mu \subset \int_T \overline{\text{co}} F d\mu$. We shall now prove the converse inclusion. From $\overline{\text{co}} F(t) \subset I(M_{F(t)})$ it follows that

$$\int_T \overline{\text{co}} F d\mu \subset \int I(M_{F(t)}) d\mu(t).$$

We shall now prove that

$$\int_T I(M_{F(t)}) d\mu(t) \subset \int_T F d\mu,$$

and this will complete the proof of the theorem.

For any $x \in \int_T F d\mu$, define

$$H_x \equiv \{\nu \in L_\infty^{w^*}(\mu, M_F) : \int_T (\nu, F) d\mu = x\}.$$

The set H_x is convex, and w^*-compact [see Castaing and Valadier, (1977), V-2]. We shall now prove that the set of extreme points of H_x (denoted by extH_x) are contained in $L_\infty^{w^*}(\mu, D_F)$. From the representation formula for $\int_T F d\mu$, this will prove our claim. Let $\nu \in \text{ext}H$; we assume, arguing by contradiction that $\nu \notin L_\infty^{w^*}(\mu, D_F)$, i.e., that for a set E of μ-measure nonzero: $\nu(t) \notin D_{F(t)}$, $t \in E$.

From Lemma 2 we derive the existence of a rca(K) valued τ-measurable function ν_2 such that $\nu(t) \pm \nu_2(t) \in M_{F(t)}$ and $\nu_2 \neq 0$. Now from (A1) we deduce the existence of an $f \in L_{E,\infty}(\mu)$ such that

$$\int_T (f\nu_2, F) d\mu(t) = 0; \quad ||f||_{L_{E,\infty}(\mu)} > 0.$$

Now extend f to a function $\bar{f} \in L_{\infty(\mu)}$ setting $\bar{f}(t) = 0$, for every $t \notin E$. We conclude:

$$\nu_1 \pm f\nu_2 \in H, \quad \nu_1 + f\nu_2 \neq \nu_1 - f\nu_2$$

so that ν_1 is *not* an extreme points of H, a contradiction.

The above result tells us the following: Suppose that (T, τ, μ) is interpreted as the measure space of agents and X is the commodity space. Let P be a separable metric space denoting the price space. Denote by $D(t,p)$ the demand set of agent t at prices p, i.e., D is a set-valued function from $T \times P$ to K. If (A1) is satisfied (i.e., if there are "many more" agents than commodities), then it follows from our Main Theorem that the aggregate demand set $\int_{t \in T} D(t,p) d\mu(t)$ is convex (see also Corollary 1 in Section 5). In other words there is a convexifying effect on aggregation despite the fact that we have infinitely many commodities [compare with Khan-Yannelis (1990) or Ostroy-Zame (1988)].

4. Models of Agents' Spaces

We now take a closer look at the possible variety of measure spaces of agents. Our main purpose is to describe those spaces for which the assumption (A1) holds, when X is an (infinite dimensional) separable Banach space.

In the following we shall say that two measure spaces (T_i, τ_i, μ_i), $i = 1, 2$, are *isomorphic* if there exists a *measure preserving set isomorphism* between them. A set isomorphism is a one to one mapping $\phi : \tau_1 \to \tau_2$ such that $\phi(T_1 \setminus E) = T_2 \setminus \phi(E)$, $\phi(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} \phi(E_i)$ if $E_i \notin \tau_1$ for $i = 1, 2, \ldots$. ϕ is said to be measure preserving if $\mu_2(\phi(E)) = \mu_1(E)$ for every $E \in \tau_1$; we consider here the equivalence classes of measurable sets modulo the ideal of sets of measure zero.

Consider now the unit interval $I = [0,1]$, with β the Borel subsets of I, and λ the Lebesgue measure. For every ordinal number α we now define that product

$$T^{\alpha} = \prod_{\gamma > \alpha} I_{\alpha}, \quad \text{with } I_{\alpha} \equiv I \text{ for every } \alpha.$$

The measure theoretic product space, a complete finite measure space, is denoted $(T^{\alpha}, \tau^{\alpha}, \mu^{\alpha})$. Recall that a measurable rectangle is a set of the form $\prod_{\gamma<\alpha} A_{\gamma}$, where $A_{\gamma} = I$ for all but a finite number of indices, and $A_{\gamma} \in \beta$ otherwise. The σ-algebra τ^{α} is the σ-algebra generated by the clan of all measurable rectangles. The measure μ^{α} satisfies

$$\mu^{\alpha}\left(\prod_{\gamma<\alpha} A_{\gamma}\right) = \prod_{i=1}^{m} \lambda(A_{\gamma i})$$

for every measurable rectangle where the sides $\{A_{\gamma i} \mid i = 1, \ldots, m\}$ are different from I (for details, see for instance von Neumann (1950)).

When $\alpha \leq \omega$ the measure space $(T^{\alpha}, \tau^{\alpha}, \mu^{\alpha})$ is (isomorphic to) the unit interval with Borel subsets and Lebesgue measure. Note that by the isomorphism theorem of Halmos and von Neumann (1948) every *separable* and nonatomic measure space of total measure one is isomorphic to it.

For cardinals of cardinality higher than ω a complete classification of the measure spaces is possible thanks to Maharam's Theorem [Maharam (1942)]. For a σ-algebra τ, let $\bar{\tau}$ be the least cardinal number which is the power of a basis of τ. The σ-algebra is called homogeneous if $\bar{\lambda} = \bar{\tau}$ for any principal ideal $\lambda \leq \tau$ which is not the null ideal. Maharam's theorem now states that every homogeneous measure algebra of total one is isomorphic to a $(T^\alpha, \tau^\alpha, \mu^\alpha)$, as previously described for some cardinal α. We may therefore restrict our attention to such product spaces.

For any ordinal number $\bar{\gamma}$, $\bar{\gamma} < \alpha$, and every set $A \in \beta$ we define the function:

$$f_{A,\bar{\gamma}}(x) = \begin{cases} 1 & \text{if } x_{\bar{\gamma}} \in A; \\ 0 & \text{if } x_{\bar{\gamma}} \notin A, \end{cases}$$

where $x_{\bar{\gamma}}$ is the coordinate of the point x in the measure space which corresponds to the ordinal $\bar{\gamma}$. Clearly, every such function is measurable (note $\{f_{\bar{\gamma}} > 1/2\} = \prod_{\gamma<\bar{\gamma}} D_\gamma \times A \times \prod_{\bar{\gamma}<\gamma<\alpha} D_\gamma$) and essentially bounded. The set of functions $\{f_\gamma\}_{\gamma<\alpha}$ is a set of linearly independent functions, and has cardinality $\aleph(\alpha)$. Setting $\alpha = \omega_1$ is clearly enough to satisfy (A1). We have proved the following result:

Theorem 4.1. *For every separable Banach space X, there exists a measure space (T_1, τ_1, μ_1) such that the assumption (A1) is satisfied. This measure space can be taken to be the product measure space with factors (I, β, λ) and $\alpha = \omega_1$.*

The conclusion to be drawn from the above result is that the $[0,1]$ interval endowed with Lebesgue measure [as Aumann (1964, 1966) formulated his model] is not "large enough" to model perfect competition in infinite dimensional commodity spaces.

5. Some Corollaries

Suppose that the correspondence $F : T \to 2^K$ satisfies all the assumptions of the Main Theorem. Then we can deduce the following Corollaries.

Corollary 1. *$\int_T F d\mu$ is convex.*

Proof. Directly from the fact that $\int_T F d\mu = \int_T \overline{\text{co}} F d\mu$, and that $\int_T \overline{\text{co}} F d\mu$ is convex because it is the linear image of the convex set $S_{\overline{\text{co}}F} = \{y \in L_1(\mu, X) : y(t) \in \overline{\text{co}} F(t)\ \mu\text{-a.e.}\}$.

Corollary 2 (Fatou's Lemma, exact version). *If $F_n(\cdot) : T \to 2^K$ $(n = 1, 2, \ldots)$ are measurable correspondences, then*

$$\text{w-Ls} \int_T F_n(t) d\mu \subset \int_T \text{w-Ls}\, F_n(t) d\mu.$$

Proof. The proof is the following chain of inclusions:

$$\begin{aligned}\text{w-Ls}\int_T F_n(t)d\mu &\subset \text{cl}\int_T \text{w-Ls}\, F_n(t)d\mu \\ &= \int_T \overline{\text{co}}\, \text{w-Ls}\, F_n(t)d\mu \\ &= \int_T \text{w-Ls}\, F_n(t)d\mu.\end{aligned}$$

The first inclusion is the approximate version of Fatou's lemma in infinite dimensional spaces [see Yannelis (1988)], the second is the Datko-Khan theorem [Khan (1985)]. Finally the third inclusion follows from our Main Theorem.

Corollary 3 (integration preserves u.s.c.). *Let Δ be a metric space and $F : \Delta \times T \to 2^K$ be a correspondence such that for each fixed $t \in T$, $F(\cdot\,,t)$ is weakly u.s.c. (w-u.s.c.), i.e., if the sequence p_n converges to p, then* w-Ls $F(p_n,t) \subset F(p,t)$. *Then $\int_T F(\cdot\,,t)d\mu$ is w-u.s.c.*

Proof. Since for each $t \in T$, $F(\cdot\,,t)$ is w-u.s.c., we have that

$$\text{w-Ls}\, F(p,t) \subset F(p_0,t), \quad \text{whenever} \quad p_n \to p_0. \tag{5.1}$$

We show that $\int_T F(\cdot\,,t)d\mu(t)$ is w-u.s.c., i.e., if $p_n \to p_0$ then

$$\text{w-Ls}\int_T F(p_n,t)d\mu(t) \subset \int_T F(p_0,t)d\mu(t).$$

It follows from Corollary 2 that

$$\text{w-Ls}\int_T F(p_n,t)d\mu(t) \subset \int_T \text{w-Ls}\, F(p_n,t)d\mu(t). \tag{5.2}$$

From (5.1) it follows that

$$\int \text{w-Ls}\, F(p_n,t)d\mu(t) \subset \int F(p_0,t)d\mu(t). \tag{5.3}$$

Combining now (5.2) and (5.3) we can conclude that

$$\text{w-Ls}\int F(p_n,t)d\mu(t) \subset \int F(p_0,t)d\mu(t),$$

i.e., $\int F(\cdot\,,t)d\mu(t)$ is w-u.s.c.

Note that Corollary 1 is the infinite dimensional generalization of Theorem 1 in Aumann (1965) or Debreu (1967, p. 369) known as the Lyapunov-Richter theorem. Corollary 2 is an exact version of the Fatou Lemma in infinite dimensional spaces. It is worth pointing out that the sequence of set-valued functions $F_n(\cdot)$ need not be convex valued contrary to the related results obtained in Yannelis (1988) and Rustichini (1989). The latter is also true for Corollary 3, i.e., no convexity assumption is needed to show that integration preserves w-u.s.c. It is worth pointing out that all the above Corollaries are false in infinite dimensional spaces [see for instance Rustichini (1989) or Yannelis (1990) for a counterexample] even if the measure space is atomless. What seems to drive the above results is condition (A1).

6. The Existence of a Competitive Equilibrium

Let X denote the *commodity space* where X is an ordered, separable Banach space whose positive cone X_+ has an interior point u. An economy $\mathcal{E}$ is a quadruple $[(T, \tau, \mu), Y, \succsim, e]$ where

(1) (T, τ, μ) is the measure space of agents;

(2) $Y : T \to 2^{X_+}$ is the *consumption set-valued function* of each agent;

(3) $\succsim_t \subset Y(t) \times Y(t)$ is the *preference relation* of agent t;

(4) $e : T \to X_+$ is the *initial endowment* of agent t, where $e(t) \in Y(t)$ μ-a.e., and for all $t \in T$, $e(t)$ belongs to a norm compact subset of $Y(t)$;

(5) the pair $[(T, \tau, \mu), X]$ satisfies the assumption (A1).

Denote the *budget set* of agent t at prices p by $B(t,p) = \{x \in Y(t) : p \cdot x \leq p \cdot e(t)\}$. The *demand set* of agent t at prices p is $D(t,p) = \{x \in B(t,p) : x \succsim_t y \text{ for all } y \in B(t,p)\}$. A *competitive equilibrium* for $\mathcal{E}$ is a pair (p, f), $p \in X_+ \setminus \{0\}$, $f \in L_1(\mu, X_+)$ such that:

(i) $f(t) \in D(t,p)$ μ-a.e., and

(ii) $\int_T f(t) d\mu(t) \leq \int_T e(t) d\mu(t)$.

We now introduce the following assumptions:

(a.1) (T, τ, μ) is a complete finite separable measure space.

(a.2) The correspondence $Y : T \to 2^{X_+}$ is integrably bounded, closed, convex, nonempty, weakly compact valued, and it has a measurable graph, i.e., $G_Y \in \tau \otimes \beta(X_+)$.

(a.3)(a) For each $t \in T$ and each $x \in Y(t)$ the set $R(t,x) = \{t \in Y(t) : y \succsim_t x\}$ is weakly closed and the set $R^{-1}(t,x) = \{y \in Y(t) : x \succsim_t y\}$ is norm closed,

(b) $\succsim_t$ is measurable in the sense that the set $\{(t,x,y) \in T \times X_+ \times X_+ : y \succsim_t x\}$ belongs to $\tau \otimes \beta(X_+) \otimes \beta(X_+)$,

(c) $\succsim_t$ is transitive, complete, and reflexive.

(a.4) For all $t \in T$, there exists $z(t) \in Y(t)$ such that $e(t) - z(t)$ belongs to the norm interior of X_+.

Theorem 6.1. *Let $\mathcal{E}$ be an economy satisfying (a.1)–(a.4). Then a competitive equilibrium exists in $\mathcal{E}$.*

Proof. It follows by combining the main existence result in Khan-Yannelis (1990) together with the Main Theorem and its corollaries.

Note that Theorem 6.1 above is the counterpart of the Main Existence Theorem of a competitive equilibrium in Khan-Yannelis (1990). It should be emphasized, that contrary to the Khan-Yannelis (1990) setting, the economy that we considered in this paper has "many more" agents than commodities [recall (5) above] and this allows us to drop the convexity assumption on preferences.

It is worth pointing out that for the core equivalence theorem [see for instance Rustichini-Yannelis (1989, 1991)] the convexity assumption on preferences is also not needed regardless as to whether the economy has "many more" agents than commodities. The reason is that for the proof of the core equivalence theorem, one has to separate an open set from the closure of the "aggregate net trade preferred set." By virtue of the approximate Lyapunov theorem we will have that the norm closure of the "aggregate net trade preferred set" is convex [see Rustichini-Yannelis (1989, 1991)] and this will enable us to apply the standard separating hyperplane argument.

However, despite the fact that without convexity of preferences both sets coincide (i.e., the core and the set of competitive equilibrium allocations are the same), they may be empty unless of course we have "many more" agents than commodities, as the above theorem indicates.

Finally it is worth pointing out that the preference relation $\gtrsim$ in Theorem 6.1 need not be complete contrary to the Khan-Yannelis (1990) existence theorem.

7. The Existence of a Pure Strategy Nash Equilibrium

We now indicate how from our main theorem one can obtain a generalization of Schmeidler's (1973) result on the existence of a pure strategy Nash equilibrium for a game with a continuum of players with a finite dimensional strategy space. In particular, in view of the failure of the Lyapunov theorem in infinite dimensional spaces, as it was shown by Khan (1986), one can only obtain an approximate pure strategy Nash equilibrium once the dimensionality of the strategy space is allowed to be infinite. However, by applying our main theorem, we show that one can still obtain an exact pure strategy Nash equilibrium for a game with a continuum of players and with an infinite dimensional strategy space, provided that the game has "many more" players than strategies. We outline the details below:

A *game* G is a quadruple $\langle (T, \tau, \mu), X, Y, u \rangle$ where

(1) (T, τ, μ) is a complete finite *measure space of players*,

(2) X is the *strategy space* which is assumed to be a separable Banach space,

(3) $Y : T \to 2^X$ is a measurable function denoting the *strategy set* of player t,

(4) $u : T \times X \times L_1(\mu, Y) \to \mathbb{R}$ is the *utility function* of player t, where for each fixed x, $u(\cdot, \cdot, x)$ is a Borel measurable function on $\{(t, y) : y \in Y(t)\}$,

(5) the pair $((T, \tau, \mu), X)$ satisfies the condition (A.1).

Note that the above definition of a game is identical with that of Khan (1986) with the addition of (5), i.e., we have replaced the nonatomicity assumption in Khan (1986), with the condition that we have "many more" players than strategies.

Denote by ext$Y(\cdot)$ the extreme points of the set $Y(\cdot)$. We are now ready to state the following result.

Theorem 7.1. *Let $G = \langle (T, \tau, \mu), X, Y, u \rangle$ be a game satisfying the following assumptions:*

(7.1) $Y : T \to 2^X$ *is a nonempty, closed, convex, weakly compact and integrably bounded correspondence;*

(7.2) $u : T \times X \times L_1(\mu, Y) \to \mathbb{R}$ *is linear on* X*;*

(7.3) $u : T \times X \times L_1(\mu, Y) \to \mathbb{R}$ *is weakly continuous on* $X \times L_1(\mu, Y)$.

Then G *has a pure strategy Nash equilibrium, i.e., there exists* $x^* \in L_1(\mu, \text{ext}Y)$ *such that for almost all* t *in* T*,*

$$u(t, x^*(t), x_T^*) = \max_{y \in Y(t)} u(t, y, x_T^*) \quad \textit{where} \quad x_T^* = \int_{t \in T} x^*(t) d\mu(t).$$

Proof. Combine Khan's (1986) theorem together with our Main Theorem.

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