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The Risk-Free Rate In Heterogeneous-Agent, Incomplete-Insurance Economies

Mark Huggett

College of Commerce and Business Administration
Bureau of Economic and Business Research
University of Illinois Urbana-Champaign

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The Risk-Free Rate in
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Mark Huggett

Department of Economics
University of Illinois

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ABSTRACT:

A heterogeneous-agent, incomplete-insurance economy is constructed to address why the average, real, risk-free, interest rate has been so low. The economy is calibrated and equilibria are characterized by computational methods. The risk-free interest rate generated by the calibrated economy is below those of comparable representative-agent economies.

1. Introduction

Why has the average, real, risk-free, interest rate been less than one percent?[1] The question is motivated by the work of Mehra and Prescott (1985). They argue that a class of calibrated, representative-agent models does not match the average, real return to equity and risk-free debt.[2] They suggest that the rate of return observations may be understood by explaining why the risk-free rate has been so low.

The conjecture that market imperfections are important for determining the risk-free rate is investigated. One approach for considering the conjecture is to describe an environment and arrangement that represent key features of actual economies. Then one can examine if the data are generated. This approach follows the modeling rules of Lucas (1987) by completely describing the game being played. Another approach, discussed by Townsend (1987), is to describe an environment and hypothesize that agents act to achieve pareto optimal allocations. Then one can check if the data result from some arrangement that achieves a pareto optimal allocation.

The first approach is explored here. One consideration motivating the choice is purely technical. Pareto optimal allocations are sometimes difficult to characterize.[3] Another consideration is that the second approach is likely to be more helpful in interpreting a collection of interest rates rather than the specific one considered here. Lastly, we would like to know if features of observed arrangements are or are not important determinants of rates of return, regardless of whether we have underlying explanations for them. I view the two approaches as being largely complimentary.

This paper examines the importance of idiosyncratic shocks and incomplete insurance for determining the risk-free rate. A pure exchange economy where agents experience idiosyncratic, endowment shocks and smooth consumption by holding credit balances is constructed. Many elements that may be important determinants of the risk-free rate (eg. discrete asset levels, other assets and shocks, production and government policy)

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[1] Mehra and Prescott (1985) state that from 1889 to 1978 the average, real return on short term relatively riskless debt has been .8%.

[2] The representative-agent models in this class predict a risk-free rate that is too high and an equity premium that is too low.

[3] This is especially the case for environments with private information. See Green (1987), Spear and Srivastava (1987), Atkeson (1987) and Phelan and Townsend (1989) for recent advances in characterizing pareto optimal allocations in private information environments.

are abstracted from to concentrate on the effects of idiosyncratic shocks and incomplete insurance on the risk-free rate. At this stage a relatively simple explanation is given for why this structure may generate a low risk-free rate. With a borrowing limit, agents must be persuaded from accumulating large credit balances so that the credit market clears. A low risk-free rate does this. To examine the risk-free rate generated by this structure, the economy is calibrated and equilibria are characterized using computational methods.

There has been a considerable amount of work on heterogeneous-agent, incomplete-insurance models of asset pricing. In monetary economics, work by Bewley (1980, 1983), Imrohoroglu (1989), Lucas (1980) and Taub (1988) employ a similar structure to that used here. In financial economics, similar structures include Manuelli (1986), Diaz and Prescott (1989), Taub (1989) and Aiyagari and Gertler (1989). Manuelli studies international debt markets in an economy with taste shocks, traded and nontraded goods. Diaz and Prescott study movements in the return to money and T-Bills in response to monetary and fiscal policies. Taub is primarily concerned with the efficiency properties of money and credit in an environment with taste shocks. Aiyagari and Gertler concentrate on the effect of transaction costs on asset returns. The above work builds on the work on consumption smoothing problems by Schechtman and Escudero (1977), Mendelson and Amihud (1982), Clarida (1984) and others. Models with a different structure that address similar questions include Mankiw (1986) and Kahn (1988).

The paper is organized in six sections.The next section, section two, describes the environment and arrangement in more detail. Section three describes the equilibrium concept and some theorems that will be useful in computing equilibria. Section four describes model calibration and computation. Section five presents the results. Section six concludes.

2. Environment and Arrangement

This paper considers an exchange economy with a continuum of agents of total mass equal to one. Each period each agent receives an endowment $e \in E$ of the one perishable consumption good in the economy. Each agent's endowment follows a Markov process with transition probability $\pi(e' \mid e) = \text{Prob}(e_{t+1} = e' \mid e_t = e)$ for $e, e' \in E$ that is independent of all other agents current and past endowments. Each agent has preferences defined over stochastic processes for consumption given by a utility function.

(2.1) $$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \qquad \text{where } \beta \in (0,1).$$

(2.2) $$u(c) = \frac{c^{(1-\sigma)}}{(1-\sigma)}, \qquad \text{where } \sigma > 1.$$

The particular arrangement considered allows each agent to smooth consumption by holding a single asset. The asset can be interpreted as a credit balance with a central credit authority or as a one period ahead sure claim on consumption goods. I will use the credit balance interpretation. A credit balance of $a \in A$ entitles an agent to a goods this period. To obtain a credit balance of $a' \in A$ next period, an agent must pay $a'q$ goods this period, where q is the price of next period credit balances. Credit balances are restricted to never fall below a credit limit $\underline{a}$, $\underline{a} < 0$. An agent's decision problem will be described at a more technical level after setting down some notation.

Notation:

*An agent's position at a point in time is described by an individual state vector $x \in X$. $x = (a,e)$ indicates credit balance a and endowment e. $X = A \times E$, $A = [\underline{a}, \infty)$ and $E = \{e_1, e_2\}$, $e_1 > e_2$.

*Let $q > 0$ be the constant price of credit balances each period.

*Let $v(x;q)$ be the expected utility for an agent who starts in state x faces price q and behaves optimally. $v: X \times R_{++} \rightarrow R$.

*Let ψ be a probability measure on $(S,\mathcal{S})$, where $S = [\underline{a}, \bar{a}] \times E$ and $\mathcal{S}$ is the Borel σ-algebra . For $B \in \mathcal{S}$, $\psi(B)$ indicates the mass of agents whose individual state vector lies in B.

An agent's decision problem is then:

(2.3) $$v(x;q) = \max \left[u(c) + \beta \sum_{e'} v(a',e';q)\, \pi(e' \mid e)\right]$$

$$(c,a') \in \Gamma(x;q) = \{(c,a'): c + a'q \leq a + e;\ c \geq 0;\ a' \geq \underline{a}\}$$

If (2.3) has a solution, then measurable functions $c: X \times R_{++} \rightarrow R_+$ and $a: X \times R_{++} \rightarrow A$ are optimal decision rules provided $c(x;q)$ and $a(x;q)$ are feasible and

$$(2.4) \qquad v(x;q) = u(c(x;q)) + \beta \sum_{e'} v(a(x;q),e';q)\, \pi(e' \mid e)$$

The decision rule $a(x;q)$ and the transition probabilities help define a transition function P, $P: S \times S \times R_{++} \rightarrow [0,1]$. $P(x,B;q)$ indicates the probability of being in B next period given that an agent's current state is x and the price is q. In the appendix such a transition function is constructed. In the paper the dependence of the decision rules and the transition function on q will often be suppressed for notational convenience.

3. Equilibrium

The equilibrium concept and some theorems that will be useful in computing equilibria are described. The stationary, recursive, equilibrium structure described in Lucas (1980) is employed.

Definition: An equilibrium to this economy is $(c(x), a(x), q, \psi)$ satisfying:

1) $c(x)$ and $a(x)$ are optimal decision rules, given q.

2) Markets Clear:

$$\text{i) } \int_S c(x)d\psi = \int_S e\,d\psi \qquad \text{ii) } \int_S a(x)d\psi = 0$$

3) ψ is a stationary probability measure

$$\psi(B) = \int_S P(x,B)d\psi \quad \text{for all } B \in S.$$

Some discussion of the equilibrium concept is in order. The first condition says that agents optimize. The second condition says that consumption and endowment averaged over the population are equal and that credit balances averaged over the population are zero. The third condition says that the distribution of agents over states is unchanging.[4] Note that the measure ψ is defined over subsets of S instead of X. Subsequent arguments will show that this is legitimate.

The following theorems will be useful in computing equilibria. Theorem 1 states conditions under which for given q there exists a unique solution to (2.3) and gives a method for computing optimal decision rules. Theorem 2 lists properties of decision rules

[4]An equilibrium concept that allows for changing probability measures is not difficult to state. However, general methods for characterizing equilibria to that equilibrium concept have not been developed. Therefore, this paper considers stationary equilibria.

that are used in the proof of Theorem 3. Theorem 3 states conditions under which for given q there exists a unique stationary probability measure ψ on (S,S) and gives a method for computing excess demand in the credit market. Some additional notation is provided.

$$(3.1) \qquad (Tv)(x;q) = \max_{(c,a') \in \Gamma(x;q)} [u(c) + \beta \sum_{e'} v(a',e';q)\, \pi(e' \mid e)]$$

The functions v on which the mapping T is defined are in C(X) the space of continuous, bounded, real-valued functions on X.

<u>Theorem 1</u>: For $q > 0$ and $\underline{a} + e_2 - \underline{a}q > 0$, there exists a unique solution $v(x;q) \in C(X)$ to (2.3) and $T^n v_0$ converges uniformly to v as $n \to \infty$ from any $v_0 \in C(X)$. Furthermore, v(x;q) is strictly increasing, strictly concave and continuously differentiable in a.

<u>Theorem 2</u>: Under the conditions of theorem 1, there exist continuous, optimal decision rules c(x;q) and a(x;q). a(x;q) is nondecreasing in a and strictly increasing in a for (x;q) such that $a(x;q) > \underline{a}$.

A theorem is presented for the existence of a unique, stationary, probability measure. The theorem is used to prove Theorem 3. The structure assumed by the theorem (here specialized to the case of probability measures) is now described:

*$(S,\geq)$ is an ordered space. $\geq$ is a closed order.

*S is a compact metric space.

*(S,S) is a measurable space and S is the Borel σ-algebra.

*P is a transition function, $P: S\times S \to [0,1]$.

*P(S) is the space of probability measures on (S,S).

*Define $(W\psi)(B) = \int_S P(s,B)\,d\psi$ for $B \in S$.

<u>Theorem 2</u>: (Hopenhayn and Prescott (1987)) Suppose P is an increasing transition function, S has a greatest (d) and a least (c) element in S and the following condition is satisfied:

Monotone Mixing Condition: There exists $s^* \in S$, $\varepsilon > 0$ and N such that
$P^N(d,\{s: s \leq s^*\}) > \varepsilon$ and $P^N(c,\{s: s \geq s^*\}) > \varepsilon$

, then there exists a unique stationary probability measure ψ and, for any $\psi_0 \in P(S)$, $W^n \psi_0$ converges weakly to ψ as $n \to \infty$.

Theorem 3 below is an application of the theorem of Hopenhayn and Prescott.

Theorem 3: If the conditions to theorem 1 hold, $\beta/q < 1$ and $\pi(e_1 \mid e_1) \geq \pi(e_1 \mid e_2)$, then there exists a unique stationary probability measure ψ (given q) on (S,S) and, for any $\psi_0 \in P(S)$, $W^n \psi_0$ converges weakly to ψ as $n \to \infty$.

4. Calibration and Computation

The economy is calibrated following the procedures described in Lucas (1981).[5] This involves using microeconomic and macroeconomic observations to set values of the parameters $\{e_1, e_2; \pi(e_1 \mid e_1), \pi(e_2 \mid e_2); \beta; \sigma; \underline{a}\}$ and the period length. I follow Imrohoroglu (1989) in interpreting e_1 and e_2 as earnings when employed and not employed. Consider the following observations:

1) Kydland (1984) calculates the standard deviation of annual hours worked for individual prime-age males from 1970-1980. He groups males by education levels. He calculates the average of group members' standard deviation as a percentage of group members' average annual hours. The statistic varies from 16% to 32% for the groups with the highest and lowest education levels.

2) The average duration of unemployment spells for men from 1948-1988 is 12.3 weeks (Handbook of Labor Statistics).

When $e_1 = 1.0$, $e_2 = 0.1$, $\pi(e_1 \mid e_1) = .925$, $\pi(e_2 \mid e_2) = .5$ and there are six model periods in one year, the standard deviation of annual earnings as a percentage of mean for an agent is 20% and the average duration of the low endowment shock is 17 weeks. The duration of the low endowment shock is higher than the data cited above. However, Clark and Summers (1979) calculate that in 1974 26% of unemployment spells for men age 20

[5] An earlier version of this paper used a different calibration. The calibration described here uses evidence on hours variability cited in Aiyagari and Gertler (1989). The results obtained with the previous and current calibration are similar.

and over ended in withdrawal from the labor force. They argue that duration understates the average time to reemployment.

The discount factor β is set to .993. This gives an annual discount factor of .96. The microeconomic studies reviewed by Mehra and Prescott (1985) estimate the risk-aversion coefficient, σ, to be about 1.5. A range of values for the credit limit are selected, $\underline{a} \in \{-2, -4, -6, -8\}$, to examine the sensitivity of the results to different credit limits.

The procedures used to compute equilibria to the calibrated model economies are described next. The computation method consists of four steps:

1) Given price q, compute $a(x;q)$ using theorem 1.

2) Given $a(x;q)$, iterate on $\psi_{t+1}(B) = \int_S P(x,B)d\psi_t$ from arbitrary $\psi_0 \in P(S)$ for sets B in a certain class.

3) Given the results from steps 1 and 2, compute $\int_S a(x;q)d\psi$.

4) Update q and repeat the steps until market clearing is approximately obtained.

These steps are now discussed in more detail. Step one is to iterate on on (3.1) from arbitrary bounded, concave, differentiable time 0 value function, v_0. This is a concave programming problem. First order conditions to the time 1 problem reduce to

$$(4.1) \qquad u'(a + e - a'q)q \geq \beta \sum_{e'} v_0'(a', e')\, \pi(e' \mid e) \qquad ; \text{with equality if } a' > \underline{a}.$$

Values of $a_1(x;q)$ are given by solutions to (4.1). The first order conditions to the time 2 problem also reduce to (4.1) with $v_1'(x) = u'(a + e - a_1(x;q)q)$. This result follows from Lucas (1978) Proposition 2. Values of $a_2(x;q)$ are determined in the same manner. The iterations are repeated until convergence of decision rules is approximately obtained. To implement this procedure on a computer some changes need to be made. First, compute $u'(a + e - a'q)$ and $v_0'(a,e)$ on finite grids on $X \times A$ and X respectively. Between gridpoints let the values of the functions be given by linear interpolation. Next, solve for $a_1(a,e)$ on gridpoints using (4.1). Iterate until convergence is approximately obtained.[6] See figure 1 for a plot of $a(a,e)$.

[6]This computation procedure is similar to Coleman's (1988) methods for computing equilibria to representative-agent models.

Step two involves iterations on $\psi_{t+1}(B) = \int_S P(x,B)d\psi_t$ from arbitrary initial $\psi_0 \in$ P(S) for sets of the form $B = \{x \in S: x_1 \leq a, x_2 = e\}$, where $(a,e) \in S$ and $S = [\underline{a}, \bar{a}] \times E$. To implement this procedure on a computer, define the function $F_0(a,e) = \psi_0(\{x: x_1 \leq a, x_2 = e\})$ on gridpoints. Between gridpoints let values of the function be given by linear interpolation. Then iterate on

$$(4.2) \qquad F_{t+1}(a',e') = \sum_{e} \pi(e' \mid e)\, F_t\,(a^{-1}(\bullet,e)(a'), e)$$

on gridpoints (a', e'). Since a(x) may not be invertible in its first argument when $\underline{a}$ is chosen, define $a^{-1}(\bullet,e)(\underline{a})$ as the maximum a such that $\underline{a}$ is chosen when the state is (a,e). See figure 2 for a plot of F(a,e).

Step three approximates the excess demand for credit using the results from steps one and two. Theorem 3 provides the justification for this approximation.

In step four the initial value of q is selected to be the midpoint of some interval of candidate q's. New values are increased if there is an excess demand and decreased if there is an excess supply of credit balances at the previous price. This process is stopped after approximate market clearing is obtained.

5. Results

This section presents the results and investigates the sensitivity of the results to changes in parameter values and computational methods. Table 1 presents the results. Table 2 describes the sensitivity of the results to changes in the coefficient of relative risk-aversion. The interest rates (r) are annual rates, whereas prices are for model periods. For comparison, a credit limit of - 5.3 is equal to one years average endowment.

Two caveats are mentioned. First, the results presented in the tables are not upper or lower bounds to true equilibrium prices and interest rates. The error involved in the computation is unknown and will be a topic of future research. Second, the issue of multiplicity of stationary equilibria has yet to be resolved. However, for all the examples considered excess demand is a monotone function of the price of credit balances.

The sensitivity of excess demand to grid size, the number of grid points and the criteria for approximate market clearing has been examined. For Tables 1 and 2 the grid size is between .03 and 0.1, the number of grid points is between 150 and 350 and the criteria for market clearing is $0 \pm .005$. To see the range of prices that are approximately market clearing by this criteria, excess demand was computed at many prices for the second entry in Table 1, other things equal. Prices in the range .998045 to .998074 are approximately market clearing. The corresponding annual interest rates are between 1.18% and 1.16%. Excess demand is not sensitive to the number of grid points, other things equal. However, excess demand is somewhat sensitive to changes in grid size, other things equal. For the second entry in Table 1 when the grid size changes from .05 to 0.1 excess demand changes from -.001 to .065.

6. Conclusion

The paper addresses the question why the average, real, risk-free, interest rate has been so low. The paper examines the importance of idiosyncratic, endowment shocks and incomplete insurance for determining the risk-free rate. The experiments listed in Table 1 show that the risk-free rate is negative for sufficiently restrictive credit limits and increases as the credit limit is relaxed. For a similar result in a different context see Taub (1989). The experiments listed in Table 2 show the sensitivity of the results to changes in the coefficient of relative risk-aversion, σ. The higher value of σ reduces the risk-free rate for all credit levels examined.

Are the results likely to change under variations in calibration or model structure? An improved calibration of the endowment process may change the results somewhat. However, Theorem 3 provides conditions under which the interest rate is likely to remain below the time preference rate. Adding capital would be an interesting extension. As the capital-output ratio for the U.S. economy is about 2.5, physical capital would appear to be an important consideration in individual consumption smoothing problems. It remains to be seen whether capital can be added in a reasonable way without adding aggregate shocks and, hence, without abandoning the stationary, recursive, equilibrium concept used here. Another interesting extension would be to examine alternative preference structures. Structures that separate risk-aversion and intertemporal substitution as in Epstein and Zin (1989) and Weil (1988) are prime candidates.

The economy studied here can be compared to a similar representative-agent economy where the representative-agent receives the average endowment. In that economy the risk-free rate is equal to the annual time preference rate, approximately 4%. So, in all the experiments considered, the economy differs from the representative-agent economy by having a lower risk-free rate. Note, however, that the risk-free rate does not differ dramatically from the representative-agent model for all feasible values of the credit limit. In light of this fact, it would be nice to have a theory of endogenous credit constraints.

There are several theoretical motivations for credit constraints. Private information on effort or output is one. Green (1987) provides an example. Limited commitment is another motivation. Kehoe and Levine (1990) provide an analysis. These studies among many others represent a promising start at understanding credit constraints.

Appendix:

A transition function on the state space S is constructed.

Let (S,S) be a state space and corresponding Borel σ-algebra. Let z be a random variable defined on the probability measure space (Z,Z,λ). Let g be a function mapping S×Z into S. Define a mapping $P{:}S \times S \rightarrow [0,1]$ by

(A.1) $P(s,B) = \lambda(\{z{:}g(s,z) \in B\})$ for $B \in S$

The following lemma gives conditions under which P is a transition function.

Lemma 5 (Hopenhayn and Prescott (1987)) If g is measurable in S×Z (with the product σ-algebra), then P described in (A.1) is a transition function for a Markov process.

Let (Z,Z) and λ be Lesbegue measure on the unit interval. Let $g(s,z) = (g_1(s,z), g_2(s,z)$, where $g_1(s,z) = a(s)$ and

$$g_2(s,z) = e_1 \text{ if } (s_2 = e_1 \text{ and } z \in (0,\pi(e_1 \mid e_1)]) \text{ or } (s_2 = e_2 \text{ and } z \in (0,\pi(e_1 \mid e_2])$$
$$e_2 \text{ if } (s_2 = e_1 \text{ and } z \in (\pi(e_1 \mid e_1),1]) \text{ or } (s_2 = e_2 \text{ and } z \in (\pi(e_1 \mid e_2),1])$$

Note that g is measurable with respect to the product σ-algebra because g_2 is measurable by construction and g_1 is measurable (S,S).[7]

Theorem 1: For $q > 0$ and $\underline{a} + e_2 - \underline{a}q > 0$, there exists a unique solution $v(x;q) \in C(X)$ to (2.3) and $T^n v_0$ converges uniformly to v as $n \rightarrow \infty$ from any $v_0 \in C(X)$. Furthermore, $v(x;q)$ is strictly increasing, strictly concave and continuously differentiable in a.

proof: Consider the mapping T defined in (3.1). Show that T: $C(X) \rightarrow C(X)$. First, note that the maximum of the objective in the definition of T is not obtained for c too close to zero. If $|v(x;q)| < M$ for all x, then $c \in [0,c^*]$, where $c^* = u^{-1}(u(\underline{a} + e_2 - \underline{a}q) - 3\beta M)$, will never be selected. Define $H(x;q,M) = \{(c,a') \in \Gamma(x;q){:}\ c \geq c^*\}$. H(x;q,M) is a continuous

[7]Conditions under which g maps SxZ into S are given in theorem 3. The function g_1 is the optimal decision rule a(x). Theorem 2 states conditions under which a(x) is continuous, hence a(x) defined on S will be measurable (S,S).

correspondence and, for fixed x, H(x;q,M) is a nonempty, compact set. Apply the Theorem of the Maximum (Lucas and Stockey (1989) p. 62) to get that

$$h(x;q,M) = \max_{(c,a') \in H(x;q,M)} [u(c) + \beta \sum_{e'} v(a',e';q)\ \pi(e' \mid e)]$$

is a continuous function. h(x;q,M) is bounded above because the objective is bounded above. It is also bounded below because the objective on H(x;q,M) is bounded below. So h(x;q,M) is in C(X). To show T: C(X) $\rightarrow$ C(X), note h(x;q,M) = (Tv)(x;q) for any v such that $|v(x;q)| < M$ for all x.

Next note that T is a contraction because C(X) with the sup norm defines a complete metric space and Blackwell's sufficient conditions for T to be a contraction are satisfied. The contraction mapping theorem yields a unique v in C(X) solving (2.3) and guarantees that $T^n v_0$ converges uniformly to v as $n \rightarrow \infty$ from any $v_0 \in C(X)$.

v(x;q) is strictly increasing in a because u(c) is strictly increasing and, for increases in a , it's always possible to increase c holding a' constant.

v(x;q) can be shown to be strictly concave in a by standard arguments.

v(x;q) can be shown to be continuously differentiable in a, v'(x;q) = u'(c(x;q)), by applying Proposition 2 Lucas (1978).◊

<u>Theorem 2</u>: Under the conditions of theorem 1, there exist continuous, optimal decision rules c(x;q) and a(x;q). a(x;q) is nondecreasing in a and strictly increasing in a for (x, q) such that $a(x;q) > \underline{a}$.

proof: An application of the theorem of the maximum in theorem 1 when v is the solution to (2.3) gives an u.h.c correspondence g: $X \rightarrow A \times R_+$. The continuity of g(x;q) = (a(x;q),c(x;q)) follows because the program involves maximizing a strictly concave function over a convex set. To show a(x) is nondecreasing in a, note that the first order conditions are

(A.2) $\quad u'(a + e - a(a,e)q)q \geq \beta \sum_{e'} v'(a(a,e), e')\ \pi(e' \mid e) \qquad$; with equality if $a(a,e) > \underline{a}$.

For $a_1 > a_2$, assume $a(a_1,e) < a(a_2,e)$.

(A.3) $$\beta \sum_{e'} v'(a(a_1,e), e')\, \pi(e'|\, e) \; > \; \beta \sum_{e'} v'(a(a_2,e), e')\, \pi(e'|\, e)$$

(A.4) $$u'(a_1 + e - a(a_1,e)q)q \; > \; u'(a_2 + e - a(a_2,e)q)q$$

(A.3) holds by strict concavity of v. (A.4) holds by (A.3) and (A.2). Finally, (A.4) and the strict concavity of u implies that $(a_1 - a_2) < (a(a_1,e) - a(a_2,e))\, q$. Contradiction. So $a(a,e)$ is nondecreasing in a.

Now argue that $a_1 > a_2$ and $a(a_2,e) > \underline{a}$ imply that $a(a_1,e) > a(a_2,e)$. Suppose that $a(a_1,e) = a(a_2,e)$, then (A.2) implies

$$u'(a_1 + e - a(a_1,e)q) = u'(a_2 + e - a(a_2,e)q)$$

This contradicts the fact that u is strictly increasing and strictly concave. So, $a(a_1,e) > a(a_2,e)$.◊

Theorem 3: If the conditions to theorem 1 hold, $\beta/q < 1$ and $\pi(e_1|\, e_1) \geq \pi(e_1 \,|\, e_2)$, then there exists a unique stationary probability measure ψ (given q) on (S,S) and, for any $\psi_0 \in P(S)$, $W^n \psi_0$ converges weakly to ψ as $n \rightarrow \infty$.

proof: The strategy is to first justify restricting attention to a compact set $S = [\underline{a}, \bar{a}]\times E$ and then to show that the conditions to Theorem 2 (Hopenhayn and Prescott) hold. For the first step consider the following lemmas:

Lemma 1: Under the conditions of Theorem 3, $a(a,e_2) < a$ for $a > \underline{a}$.

proof: Define the functions v_t for t = 0, 1, 2,... by iterating on (3.1) starting with $v_0(a,e) = 0$. Using first order conditions (4.1) and $\pi(e_1 \,|\, e_1) \geq \pi(e_1 \,|\, e_2)$, induction yields $v'_t(a,e_1) \leq v'_t(a,e_2)$ for all t. Show that $v'_t(a,e)$ converges pointwise to $v'(a,e)$. Since $v'_t(a,e) = u'(a + e - a_t(a,e)q)$, $v'(a,e) = u'(a + e - a(a,e)q)$ and u' is continuous, it is sufficient to show that $a_t(a,e)$ converges pointwise to $a(a,e)$. It is straight forward to show that the argument in Lemma 3.7 Lucas and Stockey (1989) can be applied to obtain this result. Pointwise convergence of v'_t to v' establishes that $v'(a,e_1) \le v'(a,e_2)$. The conclusion follows because $\beta/q < 1$ and $v'(a,e_1) \le v'(a,e_2)$ imply that the hypothesis to Lemma 2 below holds for $e = e_2$ and any $a^* > \underline{a}$.◊

Lemma 2: If $v'(a,e) > (\beta/q)\, E[\, v'(a,e') \mid e]$ for $a \ge a^* > \underline{a}$, then $a(a,e) < a$ for $a \ge a^*$.

proof: An agent's first order condition is

$$\text{(A.5)} \quad u'(a + e - a(a,e)q)q \ \ge \beta \sum_{e'} v'(a(a,e), e')\, \pi(e' \mid e) \qquad ; \text{with equality if } a(a,e) > \underline{a}.$$

For $a \ge a^*$, either $a(a,e) = \underline{a}$ or $a(a,e) > \underline{a}$. If the first occurs then $a(a,e) < a$. If the second occurs, then (A.5), the hypothesis, $v'(a,e) = u'(a + e - a(a,e)q)$ and v' decreasing in a imply that $a(a,e) < a$. v concave and differentiable implies that v' is decreasing in a.◊

Lemma 3: Under the conditions of Theorem 3, there exists a such that $a(a,e_1) = a$.

proof: Suppose not. Then $a(a,e_1) > a$ for all a. Lemma 1 then implies that $a(a,e_1) \ge a(a,e_2)$ for all a. Three inequalities follow:

$$a + e_2 - a(a,e_1)q \ \le a + e_2 - a(a,e_2)q$$

$$c(a,e_1) - (e_1 - e_2) \le c(a,e_2)$$

$$c(a,e_2)/c(a,e_1) \ \ge 1 - (e_1 - e_2)/c(a,e_1)$$

Note that v is bounded, increasing, v' decreasing and $v'(a,e) = u'(c(a,e))$ imply that $c(a,e_1) \to \infty$ as $a \to \infty$. So for all sufficiently large a,

$$v'(a,e_1)/v'(a,e_2) = (c(a,e_2)/c(a,e_1))^{\sigma} \ge (1 - (e_1 - e_2)/c(a,e_1))^{\sigma}$$

Since $\beta/q < 1$, there is an a^* such that $v'(a,e_1)/v'(a,e_2) > \beta/q$ for $a \ge a^*$. This fact and $v'(a,e_1) \le v'(a,e_2)$ from Lemma 1 imply that the hypothesis of Lemma 2 holds for $e = e_1$. Contradiction.◊

The previous Lemmas imply that there is $S = [\underline{a}, \bar{a}] \times E$ such that if an agent starts with state x in S then the agent stays in S. Choose $\bar{a}$ to be the smallest fixed point to $a(a,e_1) = a$. Now show that the conditions to Theorem 2 (Hopenhayn and Prescott) hold. First, define an order $\geq$ on S. For $x,x' \in S$, where $x = (x_1,x_2)$

$$x \geq x' \text{ iff } [\ (x_1 \geq x_1' \text{ and } x_2 = x_2') \text{ or } (x' = c = (\underline{a},e_2)) \text{ or } (x = d = (\bar{a},e_1))]$$

This is a closed order with minimum (c) and maximum (d) elements.

Next, define P as described in the appendix. To show that P is increasing, Hopenhayn and Prescott (1987) prove that it is sufficient to show

$$x,x' \in S \qquad x \geq x' \text{ imply } \int_S f\, P(x,dx) \geq \int_S f\, P(x',dx)$$

$$\text{where } f = \chi_B\ ,\ B = \{y \in S\text{: } y \geq x \text{ for some x in } B\} \in S.$$

Let $B_x = \{z \in Z\text{: } g(x,z) \in B\}$ and $B_{x'} = \{z \in Z\text{: } g(x',z) \in B\}$. Show $B_{x'} \subseteq B_x$. This is obvious if g(x,z) is monotone in x for fixed z. g(x,z) can be shown to be monotone by considering all possible cases. Therefore, $P(x,B) \geq P(x',B)$ as was to be shown.

Lastly, show that the mixing condition holds. Choose $s^* = (a(\underline{a},e_1) + \bar{a})/2,\ e_1)$. Define a sequence $x_1 = \underline{a}$, $x_2 = a(x_1,e_1)$, $x_3 = a(x_2,e_1)$, ... and a sequence $y_1 = \bar{a}$, $y_2 = a(y_1,e_2)$, $y_3 = a(y_2,e_2)$, ... Note that $\{x_n\} \rightarrow \bar{a}$ monotonically and $\{y_n\} \rightarrow \underline{a}$ monotonically. Therefore, there is an N_1 such that an agent goes from c to $\{x \in S\text{: } x \geq s^*\}$ with positive probability in N_1 or greater steps and there is an N_2 such that an agent goes from d to $\{x \in S\text{: } x \leq s^*\}$ with positive probability in N_2 or greater steps. Choose $N = \max\{N_1,N_2\}$ in the mixing condition. The conclusion to theorem 3 follows by theorem 2 (Hopenhayn and Prescott).◊

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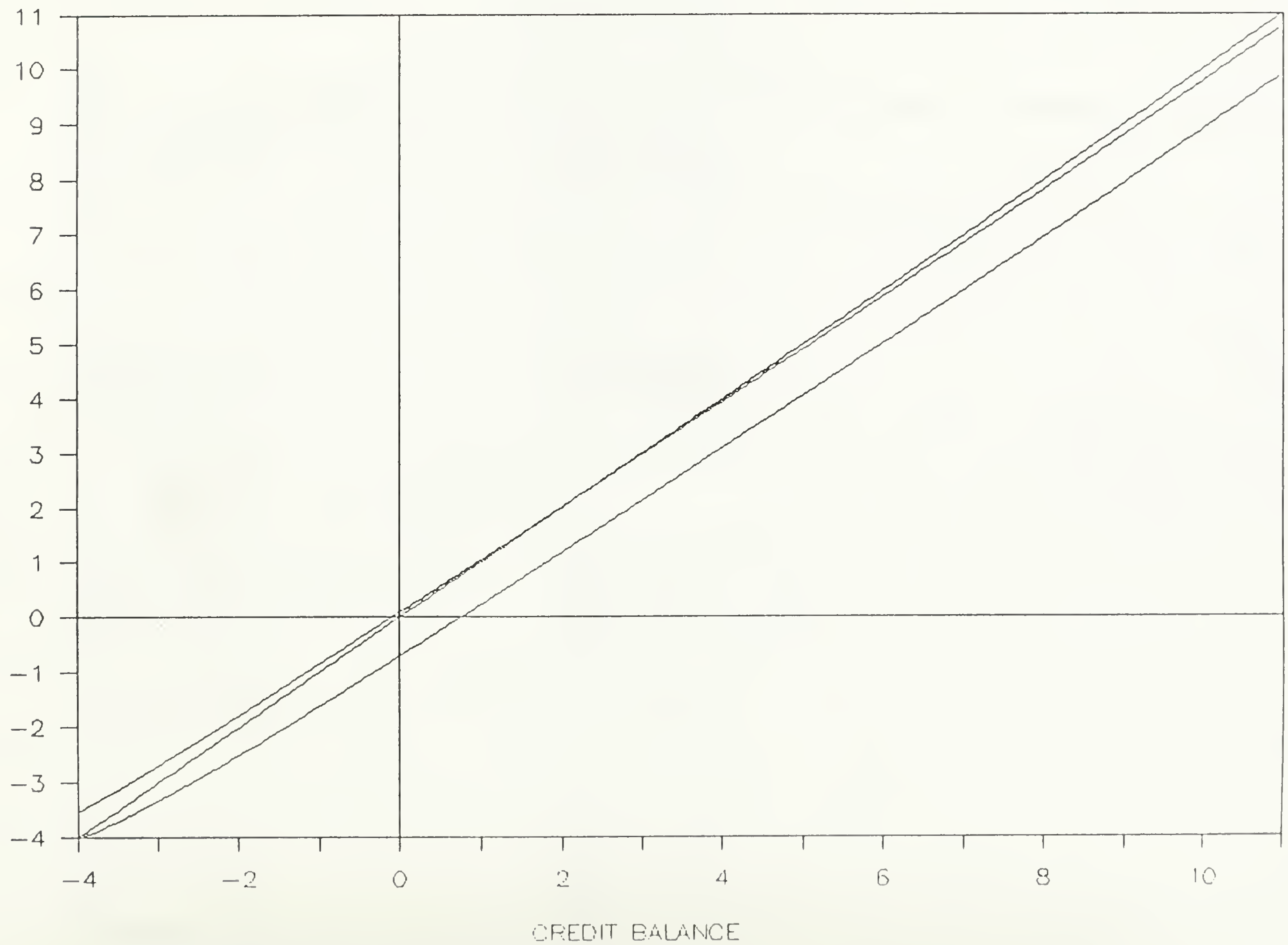
DECISION RULE



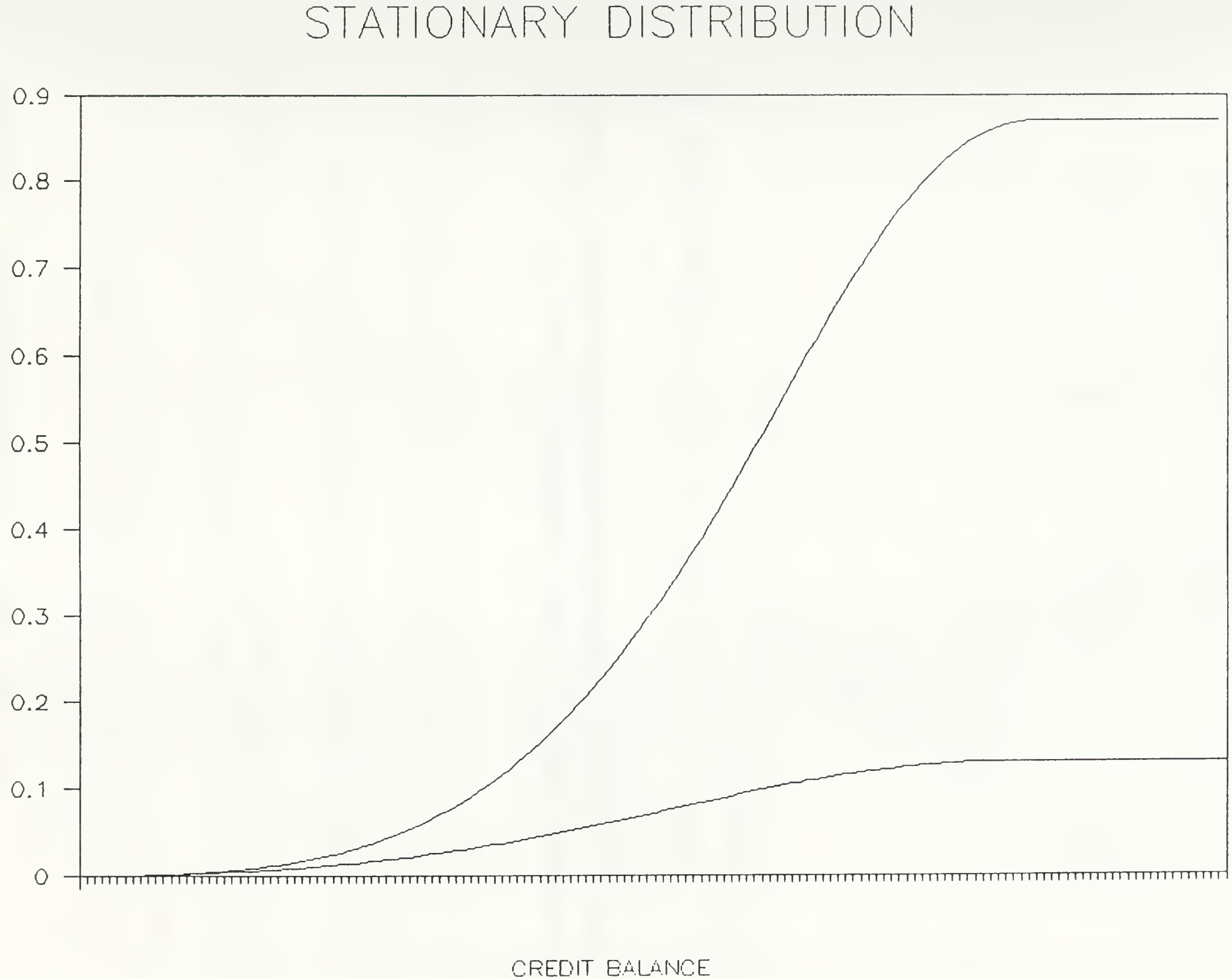

STATIONARY DISTRIBUTION
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
CREDIT BALANCE

Table 1

$\sigma = 1.5$

Credit Limit	Interest Rate	Price	Excess Demand
$\underline{a} = -2$	- 7.4 %	1.0129	- .0025
$\underline{a} = -4$	1.2 %	.9980	- .0010
$\underline{a} = -6$	3.0 %	.9951	- .0008
$\underline{a} = -8$	3.5 %	.9942	.0044

Table 2

$\sigma = 3.0$

Credit Limit	Interest Rate	Price	Excess Demand
$\underline{a} = -2$	- 23 %	1.0461	.0003
$\underline{a} = -4$	- 4.5 %	1.0077	- .0007
$\underline{a} = -6$	0.5 %	.9991	.0004
$\underline{a} = -8$	2.4 %	.9961	.0013