FREE VIBRATION ANALYSIS OF RECTANGULAR PLATES WITH VARIABLE THICKNESS

by

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[1]**Summary** An approximate method for analyzing the free vibration of rectangular plates with variable thickness is proposed. The approximate method is based on the Green function of a rectangular plate. The Green function of a rectangular plate with arbitrary variable thickness is obtained as a discrete form solution for deflection of the plate with a concentrated load. The discrete form solution is obtained at each discrete point equally distributed on the plate. It is shown that the numerical solution for the Green function has the good convergency and accuracy. By applying the Green function, the free vibration problem of plate is translated into the eigen-value problem of matrix. The convergency and accuracy of the numerical solutions for the natural frequency parameter calculated by the proposed method are investigated, and the freqency parameters and their modes of free vibration are shown for some rectangular plates.

Key words free vibration , rectangular plate , variable thickness , discrete Green function

1 INTRODUCTION

The fundamental differential equations of free vibration of plates with variable thickness have variable coefficients concerning the flexural rigidity and thickness of plate, and it seems almost impossible to get generally the analytical solution.

For some special cases of variable thickness of rectangular plate, investigations have been made and solutions have been obtained. Apple and byers [1] investigated the case when the thickness varied only in one direction, and calculated the fundamental frequency of simply supported rectangular plate having a linear thickness variation. Plunkett [2] investigated the free vibration of linearly tapered rectangular cantilever plates.

In this paper an approximate method for generally analyzing the free vibration of rectangular plates with variable thickness is proposed. At first the approximate solutions of rectangulr plate with variable thickness for a concentrated load are obtained at the discrete points equally distributed on the plate. The solution for deflection gives the discrete-type Green function of the plate. It is shown that the numerical solution for the Green function has the good convergency and accuracy.

By applying the Green function, the free vibration problem of plate is translated into the eigen-value problem of matrix. For some rectangular plates with various boundary conditions and variable thickness, the convergency and accuracy of the numerical solutions for the natural frequency parameter calculated by the proposed method are investigated and the lowest twenty one frequency parameters and their modes of free vibration are shown.

2 DISCRETE GREEN FUNCTION OF PLATE WITH VARIABLE THICKNESS

The Green function of plate bending problem is

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given by the displacement function of the plate with a unit concentrated load, so the Green function $w(x, y, x_q, y_r)/\overline{P}$ of plates with variable thickness can be obtained from the fundamental differential equations of the plate with a concentrated load $\overline{P}$ at a point (x_q, y_r), which are given by following equations.

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \overline{P}\delta(x-x_q)\delta(y-y_r) = 0 \tag{1-a}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{1-b}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{1-c}$$

$$\frac{\partial \theta_x}{\partial x} + \nu \frac{\partial \theta_y}{\partial y} = \frac{M_x}{D} \tag{1-d}$$

$$\frac{\partial \theta_y}{\partial y} + \nu \frac{\partial \theta_x}{\partial x} = \frac{M_y}{D} \tag{1-e}$$

$$\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} = \frac{2}{(1-\nu)} \frac{M_{xy}}{D} \tag{1-f}$$

$$\frac{\partial w}{\partial x} + \theta_x = \frac{Q_x}{Gt_s} \tag{1-g}$$

$$\frac{\partial w}{\partial y} + \theta_y = \frac{Q_y}{Gt_s} \tag{1-h}$$

where Q_x, Q_y the shearing forces, M_{xy} the twisting moment, M_x, M_y the bending moments, θ_x, θ_y the slopes, w the deflection, $D = Eh^3/12(1-\nu^2)$: the bending rigidity, E, G: modulus, shear modulus of elasticity, ν: Poisson's ratio, $h=h(x, y)$: the thickness of plate, $t_s = h/1.2$, $\delta(x-x_q)$, $\delta(x-x_r)$: Dirac's delta functions.

By introducing the following non-dimensional expressions,

$$[X_1, X_2] = \frac{a^2}{D_0(1-\nu^2)}[Q_y, Q_x], \ [X_3, X_4, X_5] = \frac{a}{D_0(1-\nu^2)}[M_{xy}, M_y, M_x]$$

$$[X_6, X_7, X_8] = [\theta_y, \theta_x, w/a],$$

the differential equations (1-a)~(1-h) can be rewritten as follows.

$$\sum_{e=1}^{8}[F_{1te}\frac{\partial X_e}{\partial \zeta} + F_{2te}\frac{\partial X_e}{\partial \eta} + F_{3te}X_e] + P\delta(\eta-\eta_q)\delta(\zeta-\zeta_r)\delta_{1t} = 0 \tag{2}$$

where $t=1\sim8$, $\mu = b/a, \eta = x/a, \zeta = y/b$, $D_0 = Eh_0^3/12(1-\nu^2)$: standard bending rigidity, h_0: standard thickness of a plate, a, b: breadth, length of a rectangular plate, $P = \overline{P}a/D_0(1-\nu^2)$, δ_{ft}: Kronecker's delta, F_{1te}, F_{2te}, F_{3te} : Appendix I

3 DISCRETE SOLUTION OF FUNDAMENTAL DIFFERENTIAL EQUATION

With a rectangular plate divided vertically into m equal-length parts and horizontally into n equal-length parts as shown in Fig.1, the plate can be considered as a group of discrete points which are the intersections of the $(m+1)$-vertical and $(n+1)$-horizontal dividing lines. In this paper, the rectangular area, $0 \le \eta \le \eta_i$, $0 \le \zeta \le \zeta_j$, corresponding to the arbitrary intersection (i, j) as shown in Fig.1 is denoted as the area $[i, j]$, the intersection (i, j) denoted by ◯ is called the main point of the area $[i, j]$, the intersections denoted by ○ are called the inner dependent points of the area, and the intersections denoted by ● are called the boundary dependent points of the area.

By integrating the equation (2) over the area $[i, j]$, the following integral equation is obtained.

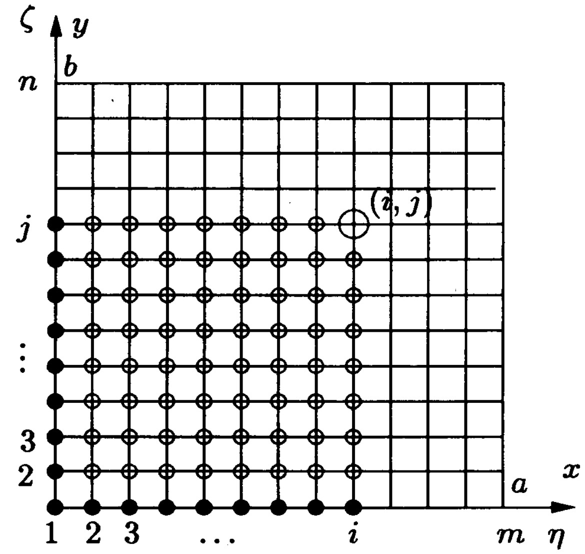


Fig.1 Discrete points on plate

$$\sum_{e=1}^{8}\Big\{F_{1te}\int_0^{\eta_i}[X_e(\eta,\zeta_j)-X_e(\eta,0)]d\eta + F_{2te}\int_0^{\zeta_j}[X_e(\eta_i,\zeta)-X_e(0,\zeta)]\,d\zeta + F_{3te}\int_0^{\eta_i}\int_0^{\zeta_j}X_e(\eta,\zeta)d\eta d\zeta\Big\} + Pu(\eta-\eta_q)u(\zeta-\zeta_r)\delta_{1t} = 0 \tag{3}$$

where $u(\eta-\eta_q)$, $u(\zeta-\zeta_r)$: unit step function

Next, by applying the numerical integration method the simultaneous equation for the unknown quantities $X_{eij} = X_e(\eta_i, \zeta_j)$ at the main point (i, j) is obtained as follows.

$$\sum_{e=1}^{8}\left\{F_{1te}\sum_{k=0}^{i}\beta_{ik}(X_{ekj}-X_{ek0})+F_{2te}\sum_{l=0}^{j}\beta_{jl}(X_{eil}-X_{e0l})\right.$$
$$\left.+F_{3te}\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\,\beta_{jl}\,X_{ekl}\right\}+Pu_{iq}\,u_{jr}\,\delta_{1t}=0 \qquad (4)$$

where

$$u_{iq}=\begin{cases}0 & \text{if } i<q\\ 0.5 & \text{if } i=q\\ 1 & \text{if } i>q\end{cases},\quad u_{jr}=\begin{cases}0 & \text{if } j<r\\ 0.5 & \text{if } j=r\\ 1 & \text{if } j>r\end{cases},$$

$$\alpha_{ik}=\begin{cases}0.5 & \text{if } k=0,i\\ 1 & \text{if } k\neq 0,i\end{cases},\quad \alpha_{jl}=\begin{cases}0.5 & \text{if } l=0,j\\ 1 & \text{if } l\neq 0,j\end{cases},$$

$$\beta_{ik}=\alpha_{ik}/m,\ \beta_{jl}=\alpha_{jl}/n$$

The solution X_{pij} of the simultaneous equation (4) is obtained as follows.

$$X_{pij}\sum_{e=1}^{8}\left\{\sum_{k=0}^{i}\beta_{ik}\,A_{pe}[X_{ek0}-X_{ekj}(1-\delta_{ki})]\right.$$
$$+\sum_{l=0}^{j}\beta_{jl}\,B_{pe}\,[X_{e0l}-X_{eil}(1-\delta_{lj})]$$
$$\left.+\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\,\beta_{jl}\,C_{pekl}\,X_{eki}(1-\delta_{ik}\,\delta_{lj})\right\}$$
$$-\gamma_{p1}\,Pu_{iq}\,u_{jr} \qquad (5)$$

where $p=1,2,\cdots,8,\ i=1,2,\cdots,m,\ j=1,2,n,\ A_{pe},\ B_{pe},\ C_{pekl},\ \gamma_{pf}$: Appendix Ⅱ

In the equation (5), the quantity X_{pij} at the main point (i, j) of the area$[i, j]$ is related to the quantities X_{ek0} and X_{e0l} at the boundary dependent points of the area and the quantities X_{ekj}, X_{eil} *and* X_{ekl} at the inner dependent points of the area. With the spreading of the area $[i, j]$ according to the regular order as $[1,1]$, $[1,2]$, $\cdots$, $[1,n]$, $[2,1]$, $[2,2]$, $\cdots$, $[2,n]$, $\cdots$, $[m,1]$, $[m,2]$, $\cdots$, $[m,n]$, a main point of a smaller area becomes one of the inner dependent points of the following larger areas. Whenever the quantity X_{pij} *at the main point* (i, j) is obtained by using the equation (5) in above mentioned order, the quantities X_{ekj}, X_{eil} and X_{ekl} at the inner dependent points of the following larger areas can be eliminated by substituting the obtained results into the corresponding terms of the right side of equation (5). By repeating this process, the equation X_{pij} at the main point is related to only the quantities X_{vk0}, $(v=1,3,4,6,7,8)$ and X_{s0l}, $(s=2,3,5,6,7,8)$ which are six independent quantities at the each boundary dependent points along the horizontal axis and the vertical axis in Fig.1 respectively. The result is as follows.

$$X_{pij}=\sum_{k=0}^{i}\left\{a_{1pijk1}(Q_y)_{k0}+a_{1pijk2}(M_{xy})_{k0}+a_{1pijk3}(M_y)_{k0}\right.$$
$$\left.+a_{1pijk4}(\theta_y)_{k0}+a_{1pijk5}(\theta_x)_{k0}+a_{1pijk6}(w)_{k0}\right\}$$
$$+\sum_{l=0}^{j}\left\{a_{2pijl1}(Q_x)_{0l}+a_{2pijl2}(M_{xy})_{0l}+a_{2pijl3}(M_x)_{0l}\right.$$
$$\left.+a_{2pijl4}(\theta_y)_{0l}+a_{2pijl5}(\theta_x)_{0l}+a_{2pijl6}(w)_{0l}\right\}+\bar{q}_{pij}\,P \qquad (6)$$

where $(Q_y)=X_1$, $(Q_x)=X_2$, $(M_{xy})=X_3$, $(M_y)=X_4$, $(M_x)=X_5$, $(\theta_y)=X_6$, $(\theta_x)=X_7$, $(w)=X_8$, $\bar{q}_{pij}$: Appendix Ⅲ

the equation (6) gives the discrete solution (3) of the fundamental differential equation (2) of plate bending problem, and the discrete Green function of plate is obtained from $X_{8ij}=G(x_i,y_j,x_q,y_r)[\bar{P}a/D_0(1-\nu^2)]$ which is the displacement at a point (x_i, y_j) of a plate with a concentrated load $\bar{P}$ at a piont (x_q, y_r).

4 INTEGRAL CONSTANT AND BOUNDARY CONDITION OF RECTANGUYLAR PLATE

The integral constants $(Q_y)_{k0}$, $(M_{xy})_{k0}$, $\cdots$, $(w)_{k0}$, $(Q_x)_{0l}$, $(M_{xy})_{0l}$, $\cdots$, $(w)_{0l}$ being involved in the discrete solution (6) are to be evaluated by the boundary conditions of a rectangular plate. The combinations of the integral constants and the boundary conditions for some cases are shown in Fig.2 ~ Fig.4, in which the integral constants and the boundary conditions at the four corners are shown in the boxes. The integral constants and the boundary conditions along

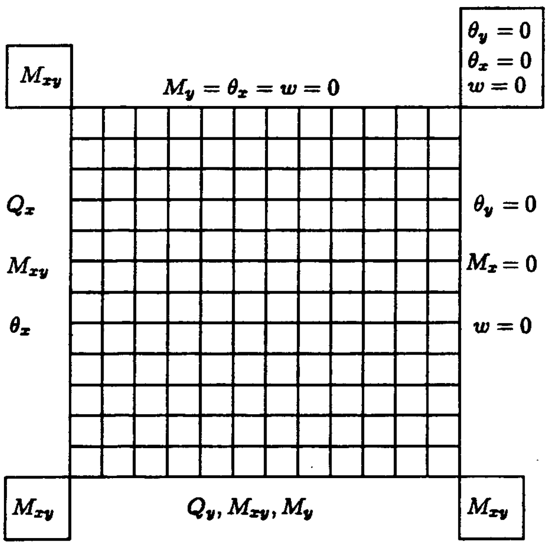


Fig.2 Simply supported plate

the four edges are given at the each equally-spaced discrete point. In this paper simply supported, fixed and free edges are denoted by solid line ———, thick solid line ▬▬▬ and dotted line - - - -.

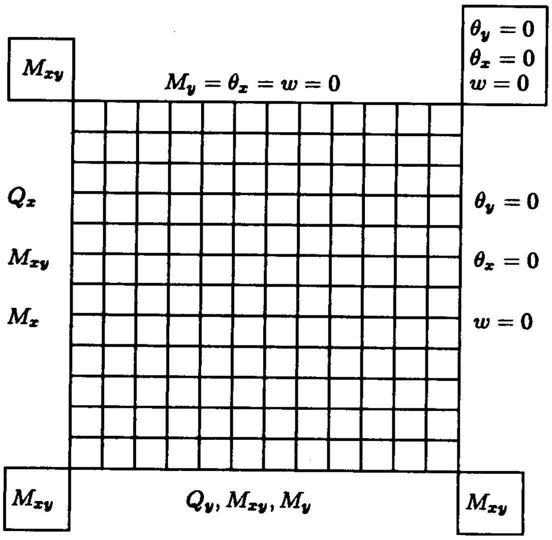


Fig.3 Fixed plate

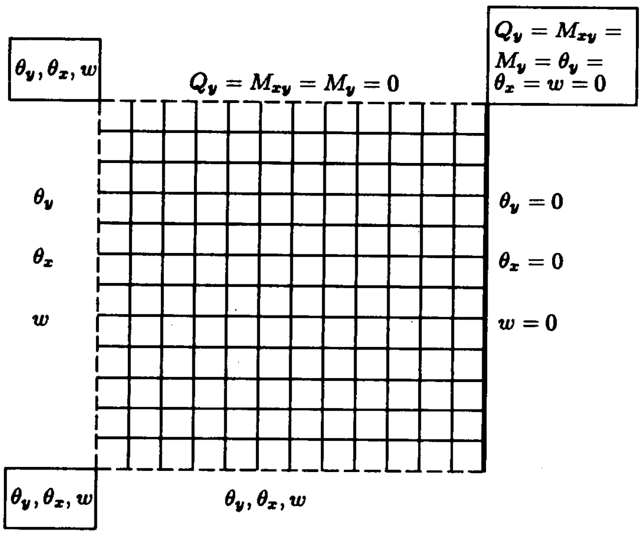


Fig.4 Cantilever plate

5 CHARACTERISTIC EQUATION OF FREE VIBRATION OF RECTANGULAR PLATE WITH VARIABLE THICKNESS

By applying the Green function $\hat{w}(x_0, y_0, x, y)/\bar{P}$ which is the displacement at a point (x_0, y_0) of a plate with a concentrated load $\bar{P}$ at a point (x, y), the displacement amplitude $\hat{w}(x_0, y_0)$ at a point (x_0, y_0) of the rectangular plate during the free vibration is given as follows

$$\hat{w}(x_0, y_0)=\int_0^b\int_0^a \rho h\omega^2\hat{w}(x,y)[w(x_0,y_0,x,y)/\bar{P}]dxdy \quad (7)$$

where ρ is the mass density of the plate material.

By using the non-dimensional expressions,

$$\lambda^4=\frac{\rho_0 h_0\omega^2 a^4}{D_0(1-\nu^2)},\ H(\eta,\zeta)=\frac{\rho(x,y)}{\rho_0}\frac{h(x,y)}{h_0},$$

$$W(\eta,\zeta)=\frac{\hat{w}(x,y)}{a},\ G(\eta_0,\zeta_0,\eta,\zeta)$$

$$=\frac{w(x_0,y_0,x,y)}{a}\frac{D_0}{}\frac{(1-\nu^2)}{Pa}$$

ρ_0 : standard mass desity

the integral equation (7) can be rewritten as follows,

$$W(\eta_0,\zeta_0)=\int_0^1\int_0^1 \mu\lambda^4 H(\eta,\zeta)G(\eta_0,\zeta_0,\eta,\zeta)W(\eta,\zeta)\,d\eta d\zeta \quad (8)$$

By applying the numerical integration method mentioned at the third-section, equation (8) is discretely expressed as

$$kW_{kl}=\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}\beta_{mi}\beta_{nj}H_{ij}G_{klij}W_{ij},\ k=1/(\mu\lambda^4) \quad (9)$$

From equation (9) homogeneous linear equations in $(m+1)\times(n+1)$ unknowns W_{11}, W_{12}, $\cdots$, W_{1n}, W_{21}, W_{22}, $\cdots$ W_{2n}, $\cdots$ W_{m1}, W_{m2}, $\cdots$, W_{mn}, are obtained as follows

$$\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}(\beta_{mi}\beta_{nj}H_{ij}G_{klif}-\kappa\delta_{ik}\delta_{jl})W_{ij}$$
$$=0,\ (\kappa=1,2,\cdots,m+1,\ l=1,2,\cdots,n+1) \quad (10)$$

The characteristic equation of the free vibration of a rectangular plate with variable thickness is obtained from the equation (10) as follows.

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & \mathbf{K}_{1(n+1)} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_{(m+1)1} & \mathbf{K}_{(m+1)2} & \cdots & \mathbf{K}_{(m+1)(n+1)} \end{bmatrix}=0 \quad (11)$$

where

$$\mathbf{K}_{ij}=\beta_{mj}\begin{bmatrix} \beta_{n1}H_{j1}G_{i1j1}-\kappa\delta_{ij} & \beta_{n2}H_{j2}G_{i1j2} & \cdots & \beta_{n(n+1)}H_{j(n+1)}G_{i1j(n+1)} \\ \beta_{n1}H_{j1}G_{i2j1} & \beta_{n2}H_{j2}G_{i2j2}-\kappa\delta_{ij} & \cdots & \beta_{n(n+1)}H_{j(n+1)}G_{i2j(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1}H_{j1}G_{i(n+1)j1} & \beta_{n2}H_{j2}G_{i(n+1)j2} & \cdots & \beta_{n(n+1)}H_{j(n+1)}G_{i(n+1)j(n+1)}-\kappa\delta_{ij} \end{bmatrix}$$

6 NUMERICAL RESULTS

The convergency and accuracy of numerical solutions have been investigated for the free vibration problem of some rectangular plates with uniform thickness or with variable thickness.

The convergent values of numerical solutions of frequency parameter for these plates have been obtained by using Richardson's extrapolation formula for two cases of combinations of divisionl numbers m and n.

6.1 Simply supported square plate and rectangular plate with uniform thickness

Numerical solutions for the lowest twenty one natural frequency parameters λ of a simply supported square plate and a rectangular plate of aspect ratio $b/a=2$ are shown in Table 1. The convergent values of numerical solutions were obtained by using Richardson's extrapolation formula for the two cases of division numbers $m(=n)$ of 12 and 16 for Ref.[4] by Leissa, and it shows the good convergency and satisfiable accuracy of the numerical solutions by present method. The nodal lines of twenty one modes of free vibration of the two plates are shown in Figure 5

6.2 Fixed square plate and rectangular plate with uniform thickness

Numerical solutions for the lowest twenty one natural frequency parameters λ of a fixed square plate and a rectangular plate of aspect ratio b/a=2 are obtained for the two cases of divisional numbers $m(=n)$ of 12 and 16 for the whole part of the plate. Table 2 involves the other theoretical values by Claassen and Thorne[5]. The numerical solutions by the present method have the good convergency and satisfiable accuracy. The nodal lines of twenty one modes of free vibration of the two plates are shown in Figure 6.

Table 1 Natural frequency parameter λ for simple rectangular plate; $\nu=0.3$

mode	b/a=1				b/a=2			
	m		Extra-polation	Ref.[4]	m		Extra-polation	Ref.[4]
	12	16			12	16		
1	4.574	4.563	4.548	4.549	3.617	3.607	3.596	3.596
2	7.333	7.270	7.188	7.192	4.615	4.585	4.547	4.549
3	7.333	7.270	7.188	7.192	6.029	5.924	5.789	5.799
4	9.306	9.211	9.089	9.098	6.778	6.712	6.627	6.631
5	10.672	10.442	10.146	–	7.793	7.511	7.148	7.192
6	10.672	10.442	10.146	–	7.359	7.284	7.187	7.192
7	12.110	11.873	11.569	11.597	8.318	8.192	8.030	8.041
8	12.110	11.873	11.569	11.597	9.960	9.326	8.511	9.098
9	14.530	13.931	13.161	13.262	9.672	9.403	9.056	8.661
10	14.530	13.931	13.161	13.262	12.691	11.402	9.745	10.172
11	14.374	14.037	13.603	13.647	10.299	10.062	9.757	9.782
12	15.614	15.032	14.282	–	10.689	10.451	10.146	10.172
13	15.614	15.032	14.282	–	11.489	10.906	10.156	10.298
14	17.424	16.789	15.972	16.083	16.306	13.808	10.596	–
15	17.424	16.789	15.972	16.083	11.369	11.102	10.757	10.789
16	19.083	17.769	16.079	–	13.855	12.725	11.272	–
17	19.083	17.769	16.079	–	12.393	12.022	11.545	11.597
18	19.918	18.642	17.000	17.322	13.921	13.230	12.341	–
19	19.918	18.642	17.000	17.322	14.259	13.650	12.865	–
20	20.009	19.145	18.033	18.196	14.543	13.938	13.160	–
21	21.360	20.081	18.436	–	15.049	14.432	13.639	

Fig.5 Nodal patterns for simply supportd plate

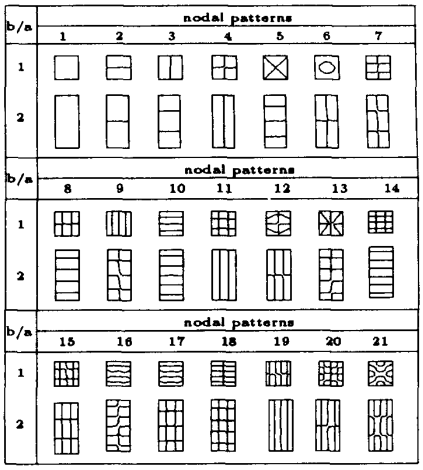
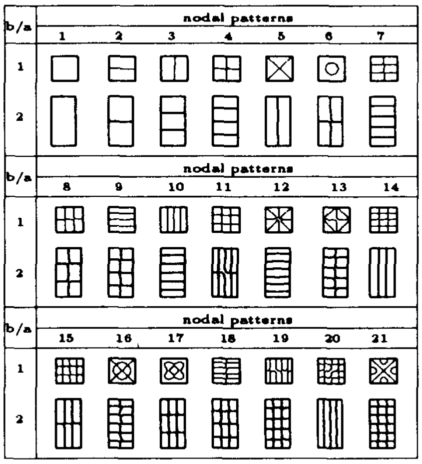


Table 2 Natural frequency parameter λ for fixed rectangular plate; $\nu=0.3$

mode	b/a=1				b/a=2			
	m		Extra-polation	Ref.[5]	m		Extra-polation	Ref.[5]
	12	16			12	16		
1	6.205	6.175	6.138	6.142	5.133	5.107	5.073	5.076
2	9.030	8.911	8.756	8.771	5.883	5.834	5.771	5.776
3	9.030	8.911	8.756	8.771	7.175	7.023	6.829	6.851
4	10.985	10.829	10.629	10.651	8.970	8.573	8.064	8.148
5	12.533	12.162	11.686	11.745	8.475	8.344	8.176	8.19
6	12.563	12.191	11.714	11.772	8.923	8.798	8.617	8.632
7	13.910	13.551	13.091	13.152	11.305	10.430	9.305	9.668
8	13.910	13.551	13.091	13.152	9.742	9.558	9.320	9.343
9	16.690	15.803	14.663	14.856	11.023	10.667	10.209	10.279
10	16.690	15.803	14.663	14.856	14.372	12.611	10.347	–
11	16.194	15.717	15.105	–	16.671	14.087	10.765	–
12	17.655	16.817	15.741	15.933	18.600	15.193	10.813	11.044
13	17.697	16.856	15.744	–	12.892	12.134	11.160	–
14	19.420	18.550	17.432	–	12.196	11.808	11.309	–
15	19.421	18.550	17.432	–	12.505	12.123	11.632	–
16	21.676	19.825	17.446	–	15.565	13.991	11.970	–
17	21.684	19.833	17.454	–	13.074	12.672	12.155	–
18	22.409	20.633	18.351	–	13.995	13.492	12.845	–
19	22.409	20.633	18.351	–	15.422	14.626	13.603	–
20	22.063	20.929	19.471	19.712	16.444	15.536	14.369	–
21	23.707	21.982	19.765	–	17.610	16.193	15.466	–

Fig.6 Nodal patterns for fixed plate



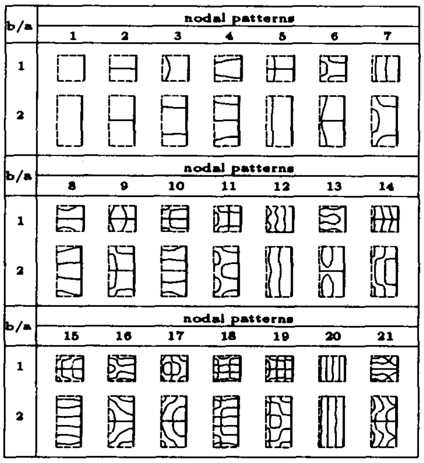
6.3 Cantilever square plate and rectangular plate with uniform thickness

Numerical solutions for the lowest twenty one natural frequency parameters λ of a cantilever square plate and a rectangular plate of aspect ratio b/a=2 are obtained for the two cases of divisional numbers $m(=n)$ of 12 and 16 for the whole part of the plate. Table 3 involves the other theoretical values by Claassen and Thorne[6]. The numerical solutions by the present method have the good convergency and satisfiable accuracy. The nodal lines of twenty one modes of free vibration of the two plates are shown in Figure 7.

Table 3 Natural frequency parameter λ for cantilever rectangular plate; $\nu=0.3$

mode	b/a=1				b/a=2			
	m 12	m 16	Extra-polation	Ref.[4]	m 12	m 16	Extra-polation	Ref.[4]
1	1.909	1.908	1.907	1.908	1.915	1.914	1.913	1.914
2	2.990	2.987	2.982	2.998	2.372	2.370	2.366	2.375
3	4.783	4.756	4.721	4.724	3.294	3.281	3.264	3.273
4	5.399	5.371	5.335	5.340	4.605	4.543	4.465	4.463
5	5.756	5.725	5.685	5.710	4.848	4.820	4.783	4.791
6	7.640	7.588	7.522	7.545	5.155	5.122	5.080	5.099
7	8.305	8.172	8.000	8.016	5.846	5.800	5.741	5.746
8	8.478	8.350	8.185	8.204	6.354	6.173	5.940	5.979
9	8.913	8.777	8.602	8.633	6.907	6.818	6.705	6.729
10	10.179	10.029	9.836	–	8.310	7.912	7.400	–
11	10.389	10.244	10.058	–	8.192	8.011	7.779	–
12	11.999	11.614	11.118	–	8.333	8.190	8.007	–
13	12.259	11.863	11.356	–	8.531	8.345	8.107	–
14	12.404	12.041	11.575	–	8.943	8.803	8.623	–
15	12.606	12.349	12.019	–	9.704	9.444	9.111	–
16	13.337	12.958	12.470	–	9.950	9.596	9.142	–
17	13.671	13.284	12.786	–	10.764	9.938	8.692	–
18	16.339	15.426	14.252	–	10.907	10.534	10.238	–
19	15.327	14.886	14.319	–	12.022	11.203	10.150	–
20	16.376	15.476	14.319	–	12.148	11.691	11.103	–
21	15.483	15.027	14.441	–	12.139	11.730	11.204	–

Fig.7 Nodal patterns for cantilever plate



6.4 Simply supported square plate and rectangular plate with variable thickness

Numerical solutions for the lowest twenty one natural frequency parameters λ of a simply supported square plate and a rectangular plate of aspect ratio b/a=2 with a linear thickness variation in the η-direction given by $h(\eta,\zeta)=h_0(1+\alpha\eta)$ are shown in Table 4 and 5 for two cases of $\alpha=0.1$ and 0.8. The convergent values of numerical solution were obtained for the two cases of divisional numbers $m(n)$ of 12 and 16 for the whole part of the plate. Table 4 and 5 involves the other theoretical values of the fundamental frequency by Apple and Byers[1]. The numerical solutions by present method have the good convergency and satisfiable accuracy of fundamental frequency. The nodal lines of sixteen modes of free vibration of the four plates of $b/a=1, 2$ and $\alpha=0.1, 0.8$ are shown in Figure 8 and 9.

Table 4 Natural frequency parameter λ for simple square plate with variable thickness; $\nu=0.3$

mode	$\alpha=0.1$				$\alpha=0.8$			
	m 12	m 16	Extra-polation	Ref.[1]	m 12	m 16	Extra-polation	Ref.[1]
1	4.687	4.675	4.660	4.661	5.386	5.372	5.354	5.335
2	7.512	7.446	7.362	–	8.576	8.501	8.404	–
3	7.513	7.447	7.363	–	8.611	8.535	8.437	
4	9.534	9.436	9.311	–	10.944	10.829	10.680	–
5	10.927	10.692	10.389	–	12.342	12.080	11.742	–
6	10.932	10.696	10.393	–	12.512	12.238	11.886	–
7	12.405	12.162	11.850	–	14.210	13.926	13.560	–
8	12.407	12.164	11.852	–	14.265	13.977	13.607	–
9	14.870	14.269	13.496	–	16.581	15.918	15.066	–
10	14.723	14.258	13.659	–	16.889	16.308	15.561	–
11	14.883	14.378	13.730	–	17.018	16.481	15.791	–
12	15.993	15.396	14.628	–	18.294	17.602	16.712	–
13	15.997	15.400	14.631	–	18.402	17.701	16.799	–
14	17.846	17.195	16.358	–	21.467	20.047	18.222	–
15	17.848	17.197	16.360	–	20.513	19.742	18.752	–
16	19.514	18.174	16.451	–	20.428	19.670	18.695	–
17	19.543	18.198	16.468	–	22.320	20.775	18.789	–
18	20.397	19.091	17.411	–	23.299	21.798	19.867	–
19	20.407	19.099	17.416	–	23.396	21.895	19.964	–
20	20.492	19.607	18.469	–	23.481	22.446	21.116	–
21	21.874	20.564	18.880	–	23.992	23.484	21.546	–

Fig.8 Nodal patterns for simply supportd square plate with variable thickness

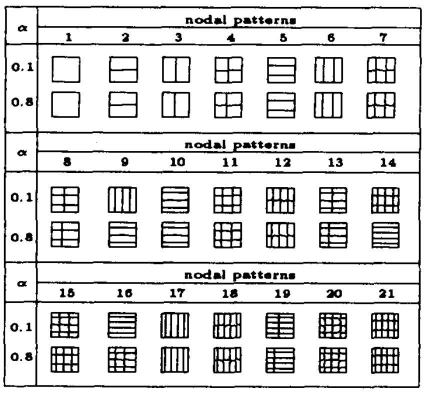
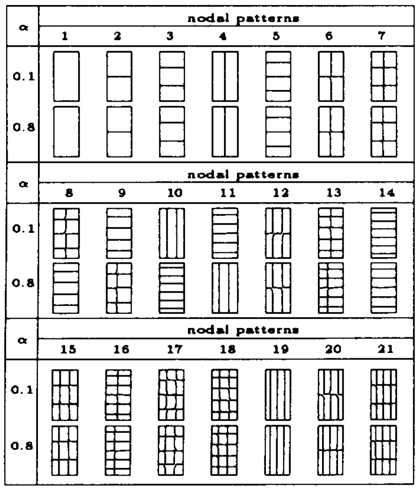


Table 5 Natural frequency parameter λ for simple rectangular plate with variable thickness; $b/a=2\nu=0.3$

mode	$\alpha=0.1$				$\alpha=0.8$			
	m		Extra-polation	Ref.[1]	m		Extra-polation	Ref.[1]
	12	16			12	16		
1	3.705	3.696	3.684	3.684	4.2446	4.234	4.220	4.221
2	4.728	4.698	4.659	–	5.433	5.398	5.352	–
3	6.176	6.069	5.930	–	7.078	6.955	6.797	–
4	6.943	6.876	6.789	–	7.955	7.876	7.775	–
5	7.982	7.693	7.322	–	9.101	8.776	8.359	–
6	7.539	7.462	7.362	–	8.641	8.552	8.436	–
7	8.521	8.392	8.226	–	9.775	9.624	9.430	–
8	9.909	9.633	8.903	–	11.547	10.831	9.910	–
9	10.200	9.551	9.091	–	11.377	11.055	10.641	–
10	10.550	10.307	9.995	–	14.574	13.145	11.308	–
11	12.992	11.673	9.977	–	12.072	11.791	11.430	–
12	10.949	10.706	10.393	–	12.533	12.250	11.886	–
13	11.770	11.173	10.405	–	13.529	12.833	11.938	–
14	16.682	14.132	10.853	–	18.498	15.784	12.295	–
15	11.647	11.373	11.021	–	13.336	13.017	12.813	–
16	14.262	13.036	11.460	–	16.4.7	14.984	13.154	–
17	12.695	112.315	11.826	–	14.545	14.102	13.532	–
18	14.262	13.552	12.728	–	16.275	15.527	14.565	–
19	14.605	13.980	13.177	–	16.698	15.977	15.050	–
20	14.896	14.276	13.479	–	17.033	16.317	15.396	–
21	15.414	14.781	13.967	–	17.628	16.897	15.957	–

Fig.9 Nodal patterns for simply supported rectangular plate with variable thickess



6.5 Fixed square plate with variable thickness

Numerical solutions for the lowest twenty one natural frequency parameters λ of a fixed square with a sinusoidal thickness variation in the η, ζ–directons given by

$$h(\eta,\zeta)=h_0(1-\alpha\sin\pi\eta)(1-\alpha\sin\pi\zeta)$$

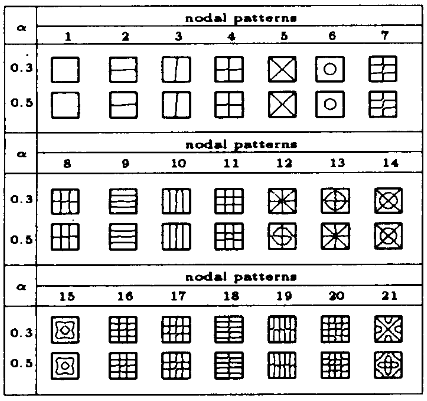
are shown in Table 6 for two cases of $\alpha=0.3$ and 0.5.

The convergent values of numerical solutions were obtained for the two cases of divisional numbers $m(=n)$ of 12 and 16 for the whole part of the plate. The nodal lines of twenty one modes of free vibration of the two plates of $\alpha=0.3, 0.5$ are shown in Figure 10. There are some changes of mode order in 12th, 13th and 21th modes.

Table 6 Natural frequency parameter λ for fixed square plate with variable thickness; $\nu=0.3$

mode	$\alpha=0.3$			$\alpha=0.5$		
	m		Extra-polation	m		Extra-polation
	12	16		12	16	
1	5.097	5.071	5.038	4.315	4.292	4.262
2	7.225	7.128	7.003	5.944	5.863	5.758
3	7.225	7.128	7.003	5.944	5.863	5.758
4	8.866	8.736	8.570	7.360	7.243	7.093
5	9.858	9.563	9.185	7.965	7.724	7.415
6	9.894	9.599	9.220	7.965	7.726	7.420
7	11.172	10.878	10.500	9.218	8.961	8.631
8	11.172	10.878	10.500	9.218	8.961	8.631
9	13.027	12.329	11.431	10.381	9.824	9.108
10	13.027	12.329	11.431	10.381	9.824	9.108
11	13.033	12.640	12.135	10.811	10.459	10.006
12	14.119	13.438	12.561	11.560	10.986	10.248
13	14.167	13.481	12.599	11.601	11.023	10.280
14	16.844	15.393	13.528	13.318	12.174	10.703
15	16.846	15.397	13.534	13.315	12.174	10.707
16	15.630	14.913	13.990	12.961	12.327	11.512
17	15.630	14.913	13.990	12.961	12.327	11.512
18	17.885	16.447	14.599	14.535	13.357	11.842
19	17.885	16.447	14.599	14.535	13.357	11.842
20	17.885	16.842	15.642	14.795	13.962	12.890
21	19.095	17.670	15.837	15.780	14.545	12.957

Fig.10 Nodal patterns for fixed square plate with variable thickess



6.6 Cantilever square plate with variable thickness

Numerical solutions for the lowest twenty one natural parameters λ of a cantilever square plate with a linear thickness variation in the η–directions given by

$$h(\eta,\zeta)=h_0\,[\alpha+(1-\alpha)\eta]$$

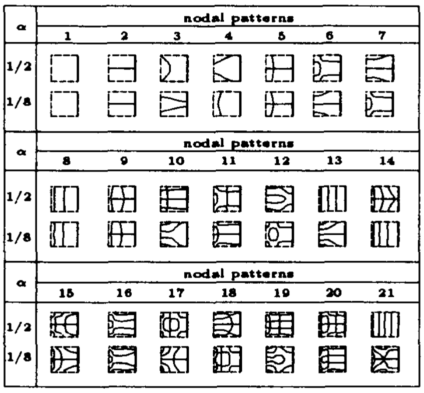
are shown in Table 7 for two cases of $\alpha=1/2$ and $1/8$.

The convergent values of numerical solutions were obtained for the two cases of divisional numbers $m(=n)$ of 12 and 16 for the whole part of the plate.The nodal lines of twenty one modes of free vibration of the two plates of $\alpha=1/2, 1/8$ are shown in Figure 11. There are some differences of mode shape and mode order between the two cantilever plates.

Table 7 Natural frequency parameter λ for cantilever square plate with variable thickness; $\nu=0.3$

mode	$\alpha=1/2$			$\alpha=1/8$		
	m		Extra-polation	m		Extra-polation
	12	16		12	16	
1	1.990	1.989	1.987	2.166	2.164	2.162
2	2.799	2.794	2.789	2.675	2.670	2.664
3	4.324	4.300	4.270	3.745	3.720	3.687
4	4.604	4.576	4.541	4.033	4.006	3.971
5	5.015	4.986	4.949	4.337	4.308	4.269
6	6.524	6.477	6.416	5.148	5.046	4.914
7	6.861	6.747	6.601	5.378	5.324	5.253
8	7.274	7.153	6.997	6.178	6.063	5.914
9	7.695	7.575	7.421	6.447	6.332	6.183
10	8.677	8.540	8.363	6.671	6.423	6.103
11	8.869	8.736	8.564	6.993	6.843	6.650
12	9.592	9.282	8.885	7.328	7.189	7.009
13	10.466	10.116	9.666	8.434	7.905	7.225
14	10.665	10.380	10.014	8.701	8.379	7.964
15	10.793	10.523	10.176	8.766	8.522	8.208
16	11.375	11.034	10.595	8.942	8.597	8.155
17	11.716	11.373	10.932	8.963	8.637	8.219
18	12.697	11.987	11.074	9.633	9.282	8.831
19	13.114	12.716	12.204	10.582	9.585	8.304
20	13.260	12.849	12.321	10.832	10.370	9.776
21	14.113	13.305	12.265	10.869	10.424	9.852

Fig.11 Nodal patterns for cantilever plate with variable thickess



7 CONCLUTION

Under the concept that the behaviour of a rectangular plate can be analyzed from the geometrical, material and mechanical properties at the discrete points uniformly distributed on the plate,an approximate method was proposed for analyzing the free vibration problem of various types of rectangular plates with uniform or non-uniform thickness. As a result of numerical works, it was shown that the numerical solutions by the proposed method had the good convergency and satisfiable accuracy for various type of rectangular plates with uniform or non-uniform thickness.

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Appendix I

$F_{111}=F_{123}=F_{134}=F_{146}=F_{167}=F_{178}=F_{188}=1$, $F_{212}=F_{225}=F_{233}=F_{257}=F_{266}=\mu$, $F_{156}=\nu$, $F_{247}=\nu\mu$, $F_{322}=F_{331}=-\mu$, $F_{344}=F_{355}=-I$, $F_{363}=-J$, $F_{372}=-k$, $F_{377}=1$, $F_{381}=-\mu k$, $F_{386}=\mu$, other $F_{1te}=F_{2te}=F_{3te}=0$, $I=\mu(1-\nu^2)(h_0/h)^3$, $J=2\mu(1+\nu)(h_0/h)^3$, $k=(1/10)(E/G)(h_0/a)^2\ (h_0/\mathrm{h})$

Appendix II

$A_{p1}=\gamma_{p1}$, $A_{p2}=0$, $A_{p3}=\gamma_{p2}$, $A_{p4}=\gamma_{p3}$, $A_{p5}=0$, $A_{p6}=\gamma_{p4}+\nu\gamma_{p5}$, $A_{p7}=\gamma_{p6}$, $A_{p8}=\gamma_{p7}$, $B_{p1}=0$, $B_{p2}=\mu\gamma_{p1}$, $B_{p3}=\mu\gamma_{p3}$, $B_{p4}=0$, $B_{p5}=\mu\gamma_{p2}$, $B_{p6}=\mu\gamma_{p6}$, $B_{p7}=\mu\ (\nu\gamma_{p1}+\gamma_{p5})$, $B_{p8}=\gamma_{p8}$, $C_{p1kl}=\mu(\gamma_{p3}+k_{kl}\gamma_{p7})$, $C_{p2kl}=\mu\gamma_{p2}+k_{kl}\gamma_{p8}$, $C_{p3kl}=J_{kl}\gamma_{p6}$, $C_{p4kl}=I_{kl}\gamma_{p4}$, $C_{p5kl}=I_{kl}\gamma_{p5}$, $C_{p6kl}=-\mu\gamma_{p7}$, $C_{p7kl}=-\gamma_{p8}$, $C_{p8kl}=0$, $[\gamma_{pk}]=[\bar{\gamma}_{pk}]^{-1}$, $\bar{\gamma}_{11}=\beta_{ii}$, $\bar{\gamma}_{12}=\mu\beta_{jj}$, $\bar{\gamma}_{22}=-\mu\beta_{ij}$, $\bar{\gamma}_{23}=\beta_{ii}$, $\bar{\gamma}_{25}=\mu\beta_{jj}$, $\bar{\gamma}_{31}=-\mu\beta_{ij}$, $\bar{\gamma}_{33}=\mu\beta_{jj}$, $\bar{\gamma}_{34}=\beta_{ii}$, $\bar{\gamma}_{44}=-I_{ij}\beta_{ij}$, $\bar{\gamma}_{46}=\beta_{ii}$, $\bar{\gamma}_{47}=\mu\nu\beta_{jj}$, $\bar{\gamma}_{55}=-I_{ij}\beta_{ij}$, $\bar{\gamma}_{56}=\nu\beta_{ii}$, $\bar{\gamma}_{57}=\mu\beta_{jj}$, $\bar{\gamma}_{63}=-J_{ij}\beta_{ii}$, $\bar{\gamma}_{66}=\mu\beta_{jj}$, $\bar{\gamma}_{67}=\beta_{ii}$, $\bar{\gamma}_{71}=-\mu k_{ij}\beta_{ij}$, $\bar{\gamma}_{76}=\mu\beta_{ij}$, $\bar{\gamma}_{78}=\beta_{ii}$, $\bar{\gamma}_{82}=-k_{ij}\beta_{ij}$, $\bar{\gamma}_{87}=\beta_{ij}$, $\bar{\gamma}_{88}=\beta_{jj}$, other $\bar{\gamma}_{pk}=0$, $\beta_{ij}=\beta_{ii}\beta_{jj}$

Aappendix III

$a_{11i0i1}=a_{13i0i2}=a_{14i0i3}=a_{16i0i4}=a_{17i0i5}=a_{18i0i6}=1$, $a_{15i0i3}=\nu$, $a_{220jj1}=a_{230jj2}=a_{250jj3}=a_{260jj4}=a_{270jj5}=a_{280jj6}=1$, $a_{240jj3}=\nu$, $a_{230002}=0$

$$a_{hpijuv} = \sum_{e=1}^{8}\left\{\sum_{k=1}^{i} \beta_{ik}A_{pe}[a_{hek0uv}-a_{hekjuv}(1-\delta_{ki})]\right.$$
$$+\sum_{l=1}^{j} \beta_{jl}B_{pe}[a_{he0luv}-a_{heiluv}(1-\delta_{lj})]$$
$$\left.+\sum_{k=1}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl}\, C_{pekl}\, a_{hekluv}(1-\delta_{ki}\delta_{lj})\right\}$$

where $h=1, 2,\ p=1, 2, \cdots, 8,\ i=1, 2, \cdots, m, j=1, 2, \cdots, n, v=1, 2, \cdots, 6,\ u=0, 1, \cdots, i(h=1),\ 0, 1, \cdots, j(h=2)$

$$\bar{q}_{pij} = \sum_{e=1}^{8}\left\{\sum_{k=1}^{i} \beta_{ik}\, A_{pe}[\bar{q}_{ek0}-\bar{q}_{ekj}(1-\delta_{ki})]\right.$$
$$+\sum_{l=1}^{i} \beta_{jl}\, B_{pe}[\bar{q}_{e0l}-\bar{q}_{eil}(1-\delta_{lj})]$$
$$\left.+\sum_{k=0}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl}\, C_{pekl}\, \bar{q}_{ekl}(1-\delta_{ki}\delta_{lj})\right\}-\gamma_{p1}\, u_{iq}u_{jr}$$