Sci, Bull, Fac, Educ., Nagasaki Univ., No.58, PP. 9~13 (1998)

Note on the $\bar{\partial}$-problem on the complex ellipsoid

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(Received Oct. 31, 1997)

abstract

Let D be a complex ellipsoid in $\mathbf{C}^n$. In this paper we study Hölder estimates for solutions of the $\bar{\partial}$-problem in D.

1. Introduction.

Let D be a complex ellipsoid in $\mathbf{C}^n$. Then D can be written in the following form.

$$D = \{z : r(z) < 0\}, \qquad r(z) = \sum_{i=1}^{n} |z_i|^{2m_i} - 1,$$

where $m_i (i = 1, \cdots, n)$ are positive integers. We denote by $C_{(0,q)}(\overline{D})$ the space of all C^1 $(0, q)$-forms on $\overline{D}$. We also denote by $\Lambda_{\alpha,(0,q)}(D)$ the space of all $(0, q)$-forms in D whose coefficients are Lipschitz functions of order α. Let $M = \max\{2m_i\}$. Let f be a $C^1(0,1)$-form in $\overline{D}$ with $\bar{\partial} f = 0$. Then Range[2] proved that there exists a Lipschitz function u of order $\alpha (\alpha < 1/M)$ in D such that $\bar{\partial} u = f$. On the other hand, Diederich-Fornaess-Wiegerinck[1] obtained Lipschitz solutions of the $\bar{\partial}$-problem in real ellipsoids. In their paper they pointed out that Range's result is still valid in the case where $\alpha = 1/M$. In the present paper we shall prove the following:

THEOREM. *Let D be the complex ellipsoid defined as above. For $f \in C^1_{(0,q)}(\overline{D}), 1 \leq q \leq n$, with $\bar{\partial} f = 0$, there exists $u \in \Lambda_{1/M,(0,q-1)}(D)$ such that $\bar{\partial} u = f$.*

2. Some lemmas.

Define

$$r_j(z) = \frac{\partial r}{\partial z_j}(z), \qquad \Phi(\zeta, z) = \sum_{j=1}^{n} r_j(\zeta)(\zeta_j - z_j).$$

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Further we set

$$\beta = |\zeta - z|^2, \qquad W = \sum_{j=1}^{n} \frac{r_j(\zeta)}{\Phi(\zeta, z)} d\zeta_j, \qquad B = \frac{\partial\beta}{\beta} = \sum_{j=1}^{n} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2} d\zeta_j.$$

For $\hat{W} = \lambda W + (1-\lambda)B$, we define

$$\Omega_q(\hat{W}) = c_{n,q}\hat{W} \wedge (\bar{\partial}_{\zeta,\lambda}\hat{W})^{n-q-1} \wedge (\bar{\partial}_z \hat{W})^q,$$

where $c_{q,n}$ are numerical constants. Now we define for a continuous $(0,q)$-form $f (1 \leq q \leq n)$ on $\overline{D}$

$$T_q^W f = \int_{\partial D \times I} f \wedge \Omega_{q-1}(\hat{W}) - \int_D f \wedge \Omega_{q-1}(B).$$

Then $T_q^W f$ satisfies $\bar{\partial} T_q^W f = f$.

If we set $\Omega_{q-1}(\hat{W}) = d\lambda \wedge \Omega^{(1)} + \Omega^{(0)}$, then after integrating with respect to $d\lambda$ we have

$$\int_{\partial D \times I} f \wedge \Omega_{q-1}(\hat{W}) = \int_{\partial D \times I} f \wedge d\lambda \wedge \Omega^{(1)} = \int_{\partial D} \Omega^{(2)},$$

where $\Omega^{(2)}$ is written by using a symbol $P = \sum_{j=1}^{n} r_j(\zeta) d\zeta_j$, $Q = \sum_{k=1}^{n} d\bar{\zeta}_k \wedge d\zeta_k$,

$$\Omega^{(2)} = \sum_{j=1}^{n-q-1} b_{j,k} \frac{\partial_\zeta \beta \wedge P \wedge (\bar{\partial}_\zeta P)^j \wedge Q^{n-q-1-j} \wedge (\sum_{j=1}^{n} d\bar{z}_j \wedge d\zeta_j)^{q-1}}{\Phi^{j+1}\beta^{n-j-1}}.$$

Range[2] proved the following:

LEMMA 1. *Let $M = \max_i(2m_i)$. Then it holds that for $(\zeta, z) \in \partial D \times D$,*

$$(2.1) \qquad |\Phi(\zeta, z)| \gtrsim |\mathrm{Im}\Phi(\zeta, z)| + |r(z)| + \sum_{i=1}^{n} |\zeta_i|^{2m_i - 2}|z_i - \zeta_i|^2 + |z - \zeta|^M.$$

Let $\zeta \in \partial D$. Then $r_i(\zeta) \neq 0$ for some i. We may assume without loss of generality that $i = n$. Then we can choose a small ball $\tilde{U}$ with center ζ. We denote by U a ball with center ζ such that $U \subset\subset \tilde{U}$. By using the partition of unity argument it is sufficient to estimate $\int_{\partial D \cap U} f \wedge \Omega^{(2)}$. Now we have the following.

LEMMA 2. *For $z, \zeta \in U$, we define $x_{2j-1}(\zeta) = \mathrm{Re}(\zeta_j - z_j)$, $x_{2j}(\zeta) = \mathrm{Im}(\zeta_j - z_j), j = 1, \cdots, n-1$, $y(\zeta) = \mathrm{Im}\Phi(\zeta, z)$, $t(\zeta) = r(\zeta) + |r(z)|$, then $t, y, x_1, \cdots, x_{2n-2}$ constitute coordinates system in U.*

PROOF. In view of the equality

$$\frac{\partial y}{\partial x_{2j}}(z) = -\frac{1}{2}\frac{\partial r}{\partial x_{2j-1}}(z), \qquad \frac{\partial y}{\partial x_{2j-1}}(z) = \frac{1}{2}\frac{\partial r}{\partial x_{2j}}(z),$$

we have

$$\frac{\partial(x_1, \cdots, x_{2n-2}, y, t)}{\partial(x_1, \cdots, x_{2n})} = -2\left|\frac{\partial r}{\partial \zeta_n}\right|^2 \neq 0.$$

This completes the proof of Lemma 2.

We need the following(cf. [1]):

LEMMA 3. *Let R be a positive constant and j a non-negative integer. For $A > 0, q \geq 1$ and $z = x + iy$ it holds that*

$$\int_{|z|<R} \frac{|z+w|^j dxdy}{(A + |z+w|^j |z|^2)^q} = \begin{cases} O(A^{1-q}) & (q > 1) \\ O(\log A) & (q = 1). \end{cases}$$

PROOF. We divide the domain of integration into three parts.

$$\begin{aligned} \{z : |z| < R\} &= \{z : |z| < R, |z| < \tfrac{1}{2}|w|\} \\ &\quad \cup\{z : |z| < R, |z| \geq \tfrac{1}{2}|w|, |z+w| < \tfrac{1}{2}|w|\} \\ &\quad \cup\{z : |z| < R, |z| \geq \tfrac{1}{2}|w|, |z+w| \geq \tfrac{1}{2}|w|\}. \end{aligned}$$

We only estimate

$$I_1 = \int_{|z|<R, |z|<\frac{1}{2}|w|} \frac{|z+w|^j}{(A + |z+w|^j |z|^2)^q} dxdy.$$

Using polar coordinates we have

$$I_1 \lesssim \int_{|z|<R} \frac{(\frac{3}{2}|w|)^j}{(A + (\frac{1}{2}|w|)^j |z|^2)^q} dxdy = 2\pi \int_0^R \frac{(\frac{3}{2}|w|)^j}{(A + (\frac{1}{2}|w|)^j r^2)^q} dr.$$

Thus we have

$$I_1 = \begin{cases} O(A^{1-q}) & (q > 1) \\ O(\log A) & (q = 1). \end{cases}$$

Using similar methods, we can prove the other cases. This completes the proof of Lemma 3.

In order to prove our theorem we use the following Hardy-Littlewood argument.

LEMMA 4. *Let D be a bounded domain in $\mathbf{R}^n$ with smooth boundary. Then there exists a positive constant C with the following property: If g is a C^1 function in D such that for some $K > 0$ and $0 < \alpha < 1$*

$$||dg(x)|| \leq K|dist(x, \partial D)|^{-\alpha}(x \in D),$$

then it holds that

$$|g(x) - g(y)| \leq CK|x - y|^{1-\alpha}(x, y \in D).$$

3. Proof of the theorem.

We set

$$g = \int_{\partial D} f \wedge \Omega^{(2)}.$$

Then we have

$$dg = \int_{\partial D} f \wedge d\Omega^{(2)}.$$

Thus it is sufficient to estimate the following two integrals:

$$I_1 = \int_{\partial D} \left| \frac{\partial_\zeta \beta \wedge P \wedge (\bar{\partial}_\zeta P)^j}{\Phi^{j+2}\beta^{n-j-1}} \right|, \qquad I_2 = \int_{\partial D} \left| \frac{P \wedge (\bar{\partial}_\zeta P)^j}{\Phi^{j+1}\beta^{n-j-1}} \right|.$$

We set $x = (t, y, x_1, \cdots, x_{2n-2})$ and $x' = (x_{2j+1}, \cdots, x_{2n-2})$. Then we have by using (2.1)

$$\begin{aligned}
I_1 &\lesssim \int_{|x|<c} \frac{|\zeta_1|^{2m_1-2} \cdots |\zeta_j|^{2m_j-2} dy dx_1 \cdots dx_{2n-2}}{(|y| + t + \sum_{i=1}^n |\zeta_i|^{2m_i-2}|z_i - \zeta_i|^2 + |z - \zeta|^M)^{j+2}|\zeta - z|^{2n-2j-3}} \\
&\lesssim \int_{|x|<c} \frac{|\zeta_1|^{2m_1-2} \cdots |\zeta_j|^{2m_j-2} dy dx_1 \cdots dx_{2n-2}}{(|y| + t + \sum_{i=1}^j |\zeta_i|^{2m_i-2}|z_i - \zeta_i|^2 + |x'|^M)^{j+2}|x'|^{2n-2j-3}} \\
&\lesssim \int_{|x|<c} \frac{|\zeta_1|^{2m_1-2} \cdots |\zeta_j|^{2m_j-2} dx_1 \cdots dx_{2n-2}}{(t + \sum_{i=1}^j |\zeta_i|^{2m_i-2}|z_i - \zeta_i|^2 + |x'|^M)^{j+1}|x'|^{2n-2j-3}}.
\end{aligned}$$

We set $\zeta_i - z_i = w_i$. Using Lemma 3 we have

$$\begin{aligned}
I_1 &\lesssim \int_{|x|<c} \frac{|z_1 + w_1|^{2m_1-2} \cdots |z_j + w_j|^{2m_j-2} dx_1 \cdots dx_{2n-2}}{(t + \sum_{i=1}^j |z_i + w_i|^{2m_i-2}|w_i|^2 + |x'|^m)^{j+1}|x'|^{2n-2j-3}} \\
&\lesssim \int_{|x|<c} \frac{|z_2 + w_2|^{2m_2-2} \cdots |z_j + w_j|^{2m_j-2} dx_3 \cdots dx_{2n-2}}{(t + \sum_{i=2}^j |z_i + w_i|^{2m_i-2}|w_i|^2 + |x'|^M)^{j+1}|x'|^{2n-2j-3}} \\
&\lesssim \int_{|x'|<c} \frac{dx_{2j+1} \cdots dx_{2n-2}}{(t + |x'|^M)|x'|^{2n-2j-3}} \\
&\lesssim \int_0^c \frac{dr}{(t + r^M)}.
\end{aligned}$$

We set $t^{-1/M}r = u$. Then we have

$$I_1 \lesssim \int_0^\infty \frac{t^{1/M-1}}{1+u^M}du \lesssim (dist(z,\partial D))^{1/M-1}.$$

Next we estimate I_2. Following the estimate of I_1, we obtain

$$\begin{aligned} I_2 &\lesssim \int_{|x'|<c} \frac{|\log(t+|x'|^M)|dx_{2j+1}\cdots dx_{2n-2}}{(t+|x'|)^{2n-2j-2}} \lesssim \int_0^c \frac{|\log(t+r^M)|}{r+t}dr \\ &\lesssim \int_0^c \frac{|\log t|}{r+t}dr \lesssim (\log t)^2. \end{aligned}$$

This completes the proof of the theorem.

References

[1] Diederich, Fornaess and Wiegerinck, *Sharp Hölder estimates for $\bar{\partial}$ on ellipsoids*, Manuscripta Math., **56**(1986), 399-417.

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