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On Rotation Matrices of given Axes and Angles and the Group Structure on $SO(3)$

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Abstract

We treat rotation matrices of given axes and angles in the space $\mathbb{R}^3 = \mathrm{Im}\mathbb{H}$ of pure imaginary quaternions. We give a product formula of rotation matrices of given axes vectors and so explain the group structure on $SO(3) \simeq \mathbb{R}P^3$ from the view point of axes and angles.

1 Introduction

We give the matrix expression $g(\theta; u) \in SO(3)$ of rotation in $\mathbb{R}^3$ of given axis $u \in \mathbb{R}^3$, $|u| = 1$ and angle θ by using the adjoint representation Ad: $S^3 = Sp(1) \longrightarrow SO(3)$, as the following form:

$$g(\theta; u) = g(\theta u) = \mathrm{Ad}\left(\exp \frac{\theta}{2} u\right)$$

where $u \in \mathbb{R}^3$ is identified with a quaternion in $\mathrm{Im}\mathbb{H}$ and $\theta u \in \mathbb{R}^3$ is called the axis vector of the rotation. $g(\theta; u)$ is to rotate clockwise around the axis u with angle θ. The description is classically known as the Cayley-Klein parameter, and is equivalent to that given by the adjoint representation of $SU(2)$. We next give the product formula:

$$g(\theta_1; u_1) g(\theta_2; u_2) = g(\theta_3; u_3)$$

and so look closely at the group structure in $SO(3) = \mathbb{R}P^3$ which is a closed ball of radius π in $\mathbb{R}^3$ whose antipodal points in the boundary are identified.

2 Description of Rotational Transformation by Quaternions

We identify the set Im$\mathbb{H}$ of all pure imaginary quaternions with the real 3-dimensional space $\mathbb{R}^3$ by a linear isomorphism over $\mathbb{R}$:

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{\sim} & \mathrm{Im}\mathbb{H} \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} & \longmapsto & ai+bj+ck \end{array} \tag{1}$$

Let $x = x_1 i + x_2 j + x_3 k$, $y = y_1 i + y_2 j + y_3 k \in \mathrm{Im}\mathbb{H}$. Define an inner product in Im$\mathbb{H}$ by

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3.$$

Then identification (1) is an isomorphism of Euclidean spaces.

Let $S^3 = Sp(1) = \{\rho \in \mathbb{H} |\ |\rho| = 1\}$. For $\rho \in S^3$, we denote the adjoint representation of $S^3 = Sp(1)$ by F_ρ:

$$F_\rho = \mathrm{Ad}\rho : x \mapsto \rho x \rho^{-1}, \quad \mathrm{Im}\mathbb{H} \to \mathrm{Im}\mathbb{H}. \tag{2}$$

For any $u \in \mathrm{Im}\mathbb{H}$, $|u| = 1$, we have $u^2 = -1$. Hence the exponential is given by

$$e^{\theta u} = \cos\theta + u\sin\theta, \quad \theta \in \mathbb{R}.$$

The exponential map $\exp : \mathrm{Im}\mathbb{H} \to S^3$ is then surjective. We show that

1. The sequence: $1 \to \{\pm 1\} \to S^3 \xrightarrow{F} SO(3) \to 1$ is exact,
2. If $\rho = e^{\frac{\theta}{2}u} (u \in \mathrm{Im}\mathbb{H}, |u| = 1)$ then F_ρ has u as axis and θ as angle.

2.1 $F(S^3) = SO(3)$ and Ker $F = \{\pm 1\}$

Ker $F = \{\pm 1\}$ is a consequence of center($\mathbb{H}$)=$\mathbb{R}$ because $\mathbb{R} \cap S^3 = \{\pm 1\}$. The formula

$$\langle x, y\rangle = -\frac{1}{2}(xy + yx) \tag{3}$$

shows not changing an inner product by F_ρ, i.e.,

$$\langle F_\rho(x), F_\rho(y)\rangle = \langle x, y\rangle.$$

So $F(S^3) \subset O(3)$. The map $\rho \mapsto \det F_\rho$ is a continous map from a connected S^3 to $\{\pm 1\}$, we have $\det F_\rho = +1$ and so $F(S^3) \subset SO(3)$. Since dim S^3 = dim $SO(3)$ = 3 and F is a continuous homomorphism between connected groups with discrete kernel, we know that $F(S^3) = SO(3)$.

2.2 Axes and Angles

We show that F_ρ ($\rho = e^{\theta u/2}$) has u as axis and θ as clockwise angle of rotation. We use the formula

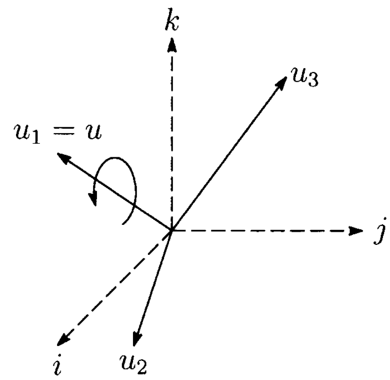
$$F_\rho(x) = x\cos\theta + (u \times x)\sin\theta + \langle u, x\rangle u(1-\cos\theta) \tag{4}$$

where $u \times x$ is an outer product given by

$$x \times y = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} i + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k. \tag{5}$$

F_ρ has u as axis because by (4),

$$\begin{aligned} F_\rho(u) &= u\cos\theta + (u\times u)\sin\theta + \langle u, u\rangle u(1-\cos\theta) \\ &= u\cos\theta + u(1-\cos\theta) \\ &= u. \end{aligned}$$



Changing basis from i, j, k to $u_1 = u, u_2, u_3$ which is orthonormal basis of right hand system, we get F_ρ from (4) as,

$$\begin{cases} F_\rho(u_1) = u_1 \\ F_\rho(u_2) = u_2\cos\theta + u_3\sin\theta \\ F_\rho(u_3) = -u_2\sin\theta + u_3\cos\theta \end{cases}.$$

Hence

$$F_\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

with respect to basis u_1, u_2, u_3. It follows that F_ρ has θ as angle of rotation. Computing $F_\rho(i), F_\rho(j), F_\rho(k)$ with standard basis, we summarize as:

Theorem 1 *The rotation $g(\theta; u) \in SO(3)$ of $\mathbb{R}^3 = \mathrm{Im}\mathbb{H}$ with axis $u \in \mathrm{Im}\mathbb{H}$, $|u| = 1$ and angle θ, is given by*

$$g(\theta; \boldsymbol{u}) = \mathrm{Ad}\left(\exp \frac{\theta}{2} u\right)$$

$$= \begin{pmatrix} (1-a^2)\cos\theta + a^2 & ab - c\sin\theta - ab\cos\theta & ca + b\sin\theta - ca\cos\theta \\ ab + c\sin\theta - ab\cos\theta & (1-b^2)\cos\theta + b^2 & bc - a\sin\theta - bc\cos\theta \\ ca - b\sin\theta - ca\cos\theta & bc + a\sin\theta - bc\cos\theta & (1-c^2)\cos\theta + c^2 \end{pmatrix}.$$

And every rotation $g \in SO(3)$ can be written as the form: $g = g(\theta; u)$ for some axis u and angle θ.

3 Product of Rotations

Let $\rho = e^{\theta u/2} = \cos\frac{\theta}{2} + u\sin\frac{\theta}{2}$, $\rho_1 = e^{\theta_1 u_1/2} = \cos\frac{\theta_1}{2} + u_1\sin\frac{\theta_1}{2}$ and $\rho_2 = e^{\theta_2 u_2/2} = \cos\frac{\theta_2}{2} + u_2\sin\frac{\theta_2}{2}$. Consider the product of rotations:

$$\begin{aligned} g(\theta; u) &= g(\theta_2; u_2) g(\theta_1; u_1), \quad \text{i.e.,} \\ F_\rho &= F_{\rho_2} F_{\rho_1} = F_{\rho_2\rho_1}. \end{aligned}$$

Then since kernel of $\rho \mapsto F_\rho$ is $\{\pm 1\}$,

$$\rho = \varepsilon\rho_2\rho_1 \quad (\varepsilon = \pm 1).$$

From the formula

$$xy = -\langle x, y\rangle + x \times y, \ \ x, y \in \mathrm{Im}\mathbb{H}, \tag{6}$$

we get

$$\begin{aligned} \rho_2\rho_1 &= \left(\cos\frac{\theta_2}{2} + u_2\sin\frac{\theta_2}{2}\right)\left(\cos\frac{\theta_1}{2} + u_1\sin\frac{\theta_1}{2}\right) \\ &= \cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + u_2\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + u_1\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + u_2u_1\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \\ &= \cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - \langle u_2, u_1\rangle\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \\ &\quad\quad + u_2\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + u_1\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + (u_2 \times u_1)\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}. \end{aligned}$$

Hence,

$$\begin{aligned} \cos\frac{\theta}{2} + u\sin\frac{\theta}{2} &= \varepsilon\left\{\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - \langle u_2, u_1\rangle\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\right. \\ &\quad\quad \left. + u_2\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + u_1\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + (u_2 \times u_1)\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\right\}. \end{aligned}$$

Comparing real and imaginary parts we get the product formula:

$$\begin{aligned} \cos\frac{\theta}{2} &= \varepsilon\left(\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - \langle u_2, u_1\rangle \sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\right) \\ u\sin\frac{\theta}{2} &= \varepsilon\left(u_2\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + u_1\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + (u_2\times u_1)\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\right) \end{aligned} \tag{7}$$

The axis u and angle θ of product rotation is determined by this formula.

Consider the easy case $u_1 = u_2 = u'$. Then rotations in 3-space is in a plane. Since $\langle u', u'\rangle = 1, u'\times u' = 0$,

$$\begin{aligned} \cos\frac{\theta}{2} &= \varepsilon\left(\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - \sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\right) = \varepsilon\cos\frac{\theta_2+\theta_1}{2} \\ u\sin\frac{\theta}{2} &= \varepsilon u'\left(\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + \cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\right) = \varepsilon u'\sin\frac{\theta_2+\theta_1}{2}. \end{aligned}$$

It is addition formula of sine and cosine.

4 Group Structure on $SO(3) \simeq \mathbb{R}P^3$

We have several relations among $g(\theta; u)$'s:

$$g(0; u) = g(\theta; 0) = I,$$

$$g(\theta + 2\pi; u) = g(\theta; u), \quad g(\theta; u)^{-1} = g(-\theta; u) = g(\theta; -u),$$

for any $u \in \mathbb{R}^3$, $|u| = 1$, $\theta \in \mathbb{R}$ and hence,

$$g(\theta + \pi; u) = g(\theta - \pi; u) = g(\pi - \theta; -u).$$

Therefore we can strengthen theorem 1 in part: every rotation $g \in SO(3)$ is of the form: $g = g(\theta; u)$ with $0 \leq \theta \leq \pi$. For any $v \in \mathrm{Im}\mathbb{H}$, $v \neq 0$, let $v = \theta u$, $\theta = |v|$, $u = v/|v|$ be its polar decomposition. Define $g(v) \in SO(3)$ by

$$g(v) = g(\theta; u) = \mathrm{Ad}\left(\exp\frac{v}{2}\right)$$

and call $v \in \mathrm{Im}\mathbb{H}$ the axis vector of $g(v) \in SO(3)$. An axis vector indicates the axis and angle of a rotation by its direction and length. We then have a surjection

$$g : \mathrm{Im}\mathbb{H} \xrightarrow{\exp} S^3 \xrightarrow{F} SO(3).$$

We know $g(D^3) = SO(3)$ where $D^3 = \{v \in \mathrm{Im}\mathbb{H} \mid |v| \leq \pi\}$. Since $g(\pi; u) = g(\pi; -u)$, $g|D^3$ induces a homeomorphism of topological spaces:

$$g : D^3/(v \sim -v, \ |v| = \pi) \xrightarrow{\sim} S^3/(x \sim -x) \xrightarrow{\sim} SO(3).$$

$\mathbb{R}P^3 = S^3/(x \sim -x)$ is the 3-dimensional real projective space. Since $D^3/(v \sim -v, \ |v| = \pi) = \mathrm{Im}\mathbb{H}/\sim$ where $v \sim w \Leftrightarrow g(v) = g(w)$, we here look on $\mathbb{R}P^3$ as the set of all the axes vectors modulo some equivalence. The rotation group $SO(3)$ induces a group structure on this $\mathbb{R}P^3$ as:

Theorem 2 *Let* $\mathbb{R}P^3 = D^3/(v \sim -v,\ |v| = \pi)$ = *the set of all the axes vectors of rotations modulo equivalence. Then the above* g *induces a group structure on* $\mathbb{R}P^3 = SO(3)$. *In the group,*

1. *the unit element is zero vector.*
2. *the inverse of* v *is* $-v$.
3. *the product of 2 axes vectors is computed by the product formula* (7) *modulo equivalence.*

5 Proof of Formulas

We give proofs of some facts and formulas. Refer to [2].

The exponential map $\exp : \mathrm{Im}\mathbb{H} \to S^3$ is surjective.

Proof. Let $\rho = a + bu \in S^3$, $a, b \in \mathbb{R}, u \in \mathrm{Im}\mathbb{H}$, $|u| = 1$. From $|\rho|^2 = a^2 + b^2 = 1$, we get $a = \cos\theta$, $b = \sin\theta$ for some θ. So $\exp(\theta u) = e^{\theta u} = \cos\theta + u\sin\theta = \rho$. □

$$(3)\quad xy = -\langle x, y\rangle + x \times y, \quad (6)\quad \langle x, y\rangle = -\frac{1}{2}(xy + yx)$$

Proof. Let $x = x_1 i + x_2 j + x_3 k, y = y_1 i + y_2 j + y_3 k \in \mathrm{Im}\mathbb{H}$,

$$\begin{aligned} xy &= (x_1 i + x_2 j + x_3 k)(y_1 i + y_2 j + y_3 k) \\ &= -(x_1y_1 + x_2y_2 + x_3y_3) + (x_2y_3 - y_2x_3)i + (x_3y_1 - y_3x_1)j + (x_1y_2 - y_1x_2)k \\ &= -(x_1y_1 + x_2y_2 + x_3y_3) + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} i + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k. \end{aligned}$$

For

$$\begin{aligned} \langle x, y\rangle &= x_1y_1 + x_2y_2 + x_3y_3, \\ x \times y &= \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} i + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k, \end{aligned}$$

we have

$$xy = -\langle x, y\rangle + x \times y.$$

It follows that immediately,

$$\begin{aligned} \langle x, y\rangle &= -\frac{1}{2}(xy + yx) \\ x \times y &= \frac{1}{2}(xy - yx). \end{aligned} \qquad (8)$$

This completes the proof. □

$$(4)\quad F_\rho(x) = x\cos\theta + (u \times x)\sin\theta + (1-\cos\theta)\langle u, x\rangle u$$

Proof. Let $\rho = e^{\theta u/2} = \cos\frac{1}{2}\theta + u\sin\frac{1}{2}\theta \in Sp(1)$, $x \in \mathrm{Im}\mathbb{H}$.

$$\begin{aligned} F_\rho(x) &= \rho x \rho^{-1} \\ &= \left(\cos\frac{1}{2}\theta + u\sin\frac{1}{2}\theta\right) x \left(\cos\frac{1}{2}\theta - u\sin\frac{1}{2}\theta\right) \\ &= x\cos^2\frac{1}{2}\theta - uxu\sin^2\frac{1}{2}\theta + ux\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta - xu\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta \\ &= x\cos^2\frac{1}{2}\theta - uxu\sin^2\frac{1}{2}\theta + (u\times x)2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta. \end{aligned}$$

Here $uxu = x - 2\langle u, x\rangle$ because by (6),

$$uxu = \{-\langle u, x\rangle + (u\times x)\}u = -\langle u, x\rangle u + (u\times x)u.$$

And by (8),

$$uxu = -\langle u, x\rangle u + \frac{1}{2}(uxu + x) \Rightarrow uxu = x - 2\langle u, x\rangle u.$$

Therefore

$$\begin{aligned} F_\rho(x) &= x\cos^2\frac{1}{2}\theta + (2\langle u, x\rangle u - x)\sin^2\frac{1}{2}\theta + (u\times x)\sin\theta \\ &= x\cos\theta + (u\times x)\sin\theta - 2\sin^2\frac{1}{2}\theta\langle u, x\rangle u \\ &= x\cos\theta + (u\times x)\sin\theta + (1-\cos\theta)\langle u, x\rangle u. \end{aligned}$$

This completes the proof. □

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