# Geometrically Nonlinear Analysis of Nonuniform Arches of Any Shape

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## Introduction

In this paper a semianalytical method is developed to study the in-plane geometrical nonlinearity of nonprismatic arches with any shape. The incremental differential equations of arch are first translated into the integral equations and by applying the approximate solution of integral equation, semianalytical solutions of the original differential equations are obtained. The solutions have discrete type expressions concerning with the equally spaced points on arch axis. The method is applied to the analysis of stability limit loads and elastic bifurcation buckling loads of parabolic, catenary, circular and cycloidal arches.

## Basic Equation

The incremental differential equations which analyze the geometrical nonlinearity of nonprismatic arches with any shape can be derived from the equilibrium equations of the deformed arch element. In the case that the load retains its original direction like dead load of rib arch bridge, the basic equations for the incremental loads $\Delta p$, $\Delta q$ and $\Delta m$ are derived as follows.

$$d\Delta Q/ds+\Delta N/R+N\cdot d\Delta\theta/ds -q\cdot\Delta\theta+\Delta p+\Delta p_0=0 \quad \cdots\cdots(1.a)$$

$$d\Delta N/ds-\Delta Q/R-Q\cdot d\Delta\theta/ds +p\cdot\Delta\theta+\Delta q+\Delta q_0=0 \quad \cdots\cdots(1.b)$$

$$d\Delta M/ds-\Delta Q-Q\cdot\Delta\varepsilon-\Delta m-\Delta m_0=0 \quad \cdots\cdots(1.c)$$

$$\Delta M=-EI\cdot d\Delta\theta/ds \quad \cdots\cdots(1.d)$$

$$\Delta N=EA\cdot\Delta\varepsilon=EA\cdot(d\Delta w/ds-\Delta u/R) \quad \cdots\cdots(1.e)$$

$$\Delta\theta=d\Delta u/ds+\Delta w/R \quad \cdots\cdots(1.f)$$

The load correction factors $\Delta p_0$, $\Delta q_0$ and $\Delta m_0$ are given by following equations.

$$\Delta p_0=\Delta N\cdot d\Delta\theta/ds-\Delta q\cdot\Delta\theta+(p+\Delta p) \cdot(\cos\Delta\theta-1)-(q+\Delta q)(\sin\Delta\theta-\Delta\theta)$$

$$\Delta q_0=-\Delta Q\cdot d\Delta Q/ds+\Delta p\cdot\Delta\theta+(p+\Delta p) \cdot(\sin\Delta\theta-\Delta\theta)+(q+\Delta q)(\cos\Delta\theta-1)$$

$$\Delta m_0=\Delta Q\cdot\Delta\varepsilon$$

In the Eqs. (1.a)~(1.f), $Q$ and $N$ are the initial stresses for the initial loads $p$, $q$ and $m$, and $\Delta Q$, $\Delta N$, $\Delta M$, $\Delta\theta$, $\Delta w$, $\Delta u$ and $\Delta\varepsilon$ are the increments of shearing force $Q$, normal force $N$, bending moment $M$, deflection angle $\theta$, tangential displacement $w$, normal displacement $u$ and axial strain $\varepsilon$.

## Discrete General Solution

For the convenience of numerical works, following nondimensional variables are introduced.

$$X_1=-L^2\Delta Q/EI_0,\quad X_2=-L^2\Delta N/EI_0,$$
$$X_3=-L\Delta M/EI_0,\quad X_4=\Delta\theta,\quad X_5=\Delta W/L,$$
$$X_6=\Delta u/L,\quad \eta=s/l.$$

where $l$, $L$, $EI_0$ and $EA_0$ are the length of arch axis, span of arch, standard flexural and axial rigidities.

By using the nondimensional variables $X_t$, the differential Eqs. (1.a)~(1.f) are rearranged together as follows.

$$dX_t/d\eta=\sum_{n=1}^{7}G_{tk}\cdot X_k,\quad (t=1\sim6,\ X_7=1) \quad \cdots\cdots(2)$$

The semianalytical solutions of differential Eq. (2) concerning with the discrete point $i$ which is one of the equally spaced points on arch axis shown in Fig. 1 are given by

$$X_{ti}=\sum_{n=1}^{7}d_{tni}\cdot X_{n0},\quad (i=1\sim m,\ X_{70}=1) \quad \cdots(3)$$

where

$$d_{tni}=\delta_{nt}+\sum_{j=0}^{i}\sum_{k=1}^{7}\beta_{ij}\cdot G_{tkj}\cdot d_{knj},$$

$\beta_{ij}=\alpha_{ij}/24m$, $\delta_{nt}$: Kronecker delta.

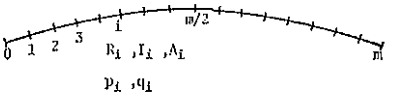


Fig. 1 Discrete points along arch axis.

Table 1 Weight coefficient $\alpha_{ij}$.

| $i$ | $j$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ··· |
| 1 | 12 | 12 | | | | | | | | |
| 2 | 8 | 32 | 8 | | | | | | | |
| 3 | 9 | 27 | 27 | 9 | | | | | | |
| 4 | 8 | 32 | 16 | 32 | 8 | | | | | |
| 5 | 9 | 27 | 27 | 17 | 32 | 8 | | | | |
| 6 | 8 | 32 | 16 | 32 | 16 | 32 | 8 | | | |
| 7 | 9 | 27 | 27 | 17 | 32 | 16 | 32 | 8 | | |
| 8 | 8 | 32 | 16 | 32 | 16 | 32 | 16 | 32 | 8 | |
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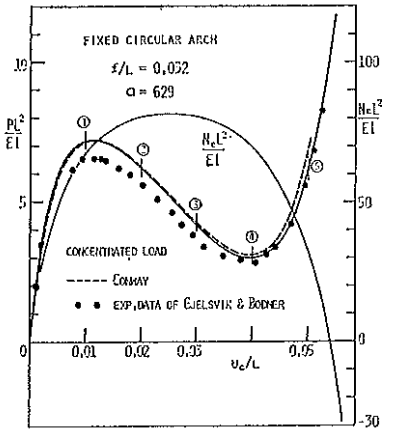


Fig. 2 Fixed shallow arch: Central load-deflection.

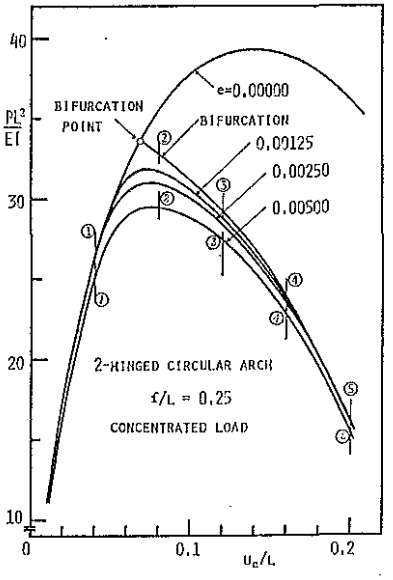


Fig. 3 Load-deflection relation of a circular arch with initial inperfection.

The weight coefficient $\alpha_{ij}$ of numerical integration are given in Table 1 for the case of using Simpson's rule.

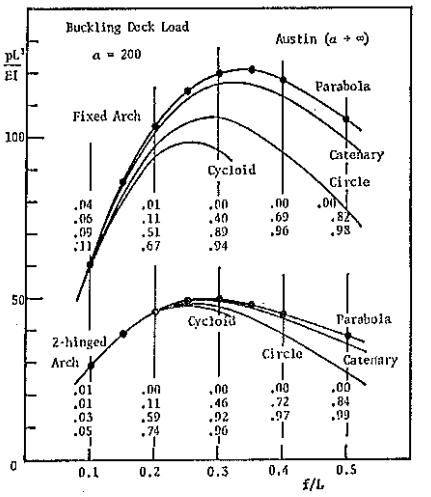


Fig. 4 Bifurcation buckling load of 2-hinged and fixed arches deck load.

## Numerical Results

**(1) Comparison with Existing Analytical and Experimental Investigations**

Fig. 2 shows the load-deflection relation of a fixed shallow arch which accepts a concentrated load $P$ at the arch crown. The calculated values agree well with Conway's analytical results and Gjelsvik's experimental results. $N_0$ means the normal force at the arch crown.

**(2) Effects of Initial Displacements to Load-Deflection Relations**

The load-deflection relations of circular arches with initial displacements are shown in Fig. 3. The initial normal displacements are supposed to be sinusoidal as $u/L=e\sin 2\pi\eta$. The parameter $e$ is the ratio of maximum normal displacement to arch span $L$.

**(3) Bifurcation Buckling Loads of 2-hinged and Fixed Arches**

The elastic bifurcation buckling loads of 2-hinged and fixed arches with parabolic, catenary, circular and cycloidal axis are shown in Fig. 4. The numbers in Fig. 4 are the ratios of strain energy by bending moment to the whole strain energy. These show that the effects of bending moment can not be neglected except for the case of pure axial compression.