On the equation $u_t - \Delta u + u^3 = f$

Kazuo Okamoto

(昭和52年9月17日受理)

Introduction

Let Ω be a bounded domain of R^3 with sufficiently smooth boundary Γ. This note is concerned with the boundary value problem for the equation

(1) $\quad -\Delta b + b^3 = f \qquad (x \in \Omega)$

under the boundary condition.

(2) $\quad b|_\Gamma = b_o(x)$

and also with the initial-boundary value problem for the equation

(3) $\quad u_t - \Delta u + u^3 = f \qquad (x \in \Omega,\ t \geqq o)$

under the initial condition

(4) $\quad u|_{t=o} = u_o(x)$

and the boundary condition

(5) $\quad u|_\Gamma = b_o(x)$

We study the equation (3) when $u(x)$ is sufficiently closed to a solution $b(x)$ of the equation (1) with the boundary condition (2), and prove that if $b(x)$ satisfies certain conditions, a solution u converges to b as $t \to \infty$. The problem for the Navier-Stokes equations has been treated by Heywood in [1], [2].

Preliminaries

We denote by $L^p(\Omega)$, $1 \leqq p < \infty$, the Banach space of all real functions on Ω, with norm

$$|u|_p = \{ \int_\Omega |u(x)|^p dx \}^{1/p}$$

For $p=2$, the space $L^2(\Omega)$ is a Hilbert space for the scalar product

$$(u, v) = \int_\Omega u(x)v(x)dx \quad ,$$

and we set

$$|u| = (u, u)^{1/2} \quad ,$$

$H^1(\Omega)$ is the space of functions of $L^2(\Omega)$ whose first derivatives (in the sense of distributions) are in $L^2(\Omega)$. $H^1(\Omega)$ is a Hilbert space with the scalar product

$$((u, v)) = (u, v) + \sum_{i=1}^{3} (D_i u, D_i v) \ , \ D_i = \frac{\partial}{\partial x_i}$$

$H^1_0(\Omega)$ is the closure in $H^1(\Omega)$ of $C^\infty_0(\Omega)$, the space of infinitely differentiable functions with compact support contained in Ω. We write

$$(\nabla u, \nabla v) = \sum_{i=1}^{3} (D_i u, D_i v), \quad (\nabla u, \nabla u) = |\nabla u|^2.$$

Lemma (Sobolev) *If $u \in H^1_0(\Omega)$, then*

$$|u|_p \leqq C_\Omega |\nabla u|, \quad 1 \leqq p \leqq 6$$

Where C_Ω is a constant depending only on Ω.

Generalized solution

We assume the functions f, b_o are time independent and b_o has an extension b(x) into Ω satisfying

$$(6) \quad \begin{cases} b \in L^4(\Omega) \\ -\Delta b + b^3 - f \in L^2(\Omega) \\ v_0 = u_0 - b \in H^1_0(\Omega) \end{cases}$$

Definition We call $u(x, t) = v(x, t) + b(x)$ a generalized solution of (3), (4), (5) in $\Omega \times (0, \infty)$ if b satisfies (6) and if for all $T > 0$:

(i) $v \in L^2(0, T; H^1_0(\Omega) \cap L^4(\Omega))$, $v_t \in L^2(0, T; L^2(\Omega))$

(ii) $|v(x, t) - v_o(x)| \to 0$ as $t \to \infty$

(iii) $\int_0^T \{(v_t, \phi) + (\nabla v, \nabla\phi) + (v^3 + 3bv^2 + 3b^2v, \phi) + (b^3 - \Delta b - f, \phi)\} dt = 0$ for all $\phi \in C^\infty_0(\Omega \times (0, T))$.

Theorem *Let f, u_o, b_o be given. Suppose a solution b of equations (1), (2) satisfies the condition (6), and*

$$(i) \quad 1 - \frac{3}{2} C_\Omega |b|_4 = \mu > 0$$

$$(ii) \quad |v_o| \cdot |\Delta u_o + f - u_0^3| \leqq \frac{\mu}{36\, C^6_\Omega |b|^2_4}$$

Then the initial-boundary problem (3), (4), (5) has a generalized solution u in $\Omega \times (0, \infty)$, and

$$|u(t) - b| \leqq |v_o| \exp(-\mu C^{-2}_\Omega t).$$

A Priori estimates

We shall employ Galerkin's method to prove the existence of generalized solutions.

Let $\{w_j(x)\}$ be a complete system of functions in $H^1_o(\Omega)$.

We suppose that $u_o = |v_o| w_1 + b$ Let

$$v^m(x, t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x) \quad , m = 1, 2, \ldots\ldots$$

be the solution of the system (j= 1,....m) of ordinary differential equations,

(7) $(v_t^m, w_j) + (\nabla v^m, \nabla w_j) + ((v^m)^3 + 3b(v^m)^2 + 3b^2 v^m, w_j) = 0$

which satisfy the initial conditions $g_{1m}(0) = |v_0|$ and $g_{jm}(0) = 0$ for j= 1,..., m. There exists v^m in $[0, t_m]$, $t_m > 0$.

By multiplying each equation (7) by g_{jm}, summing $\sum_{j=1}^{m}$, noting the Sobolev's lemma, inequality (8) is obtained.

(8) $\frac{1}{2}\frac{d}{dt}|v^m|^2 + \mu(|\nabla v^m|^2 + |v^m|_4^4) \leqq 0.$

This shows that $t_m = T$. According to the Sobolev's lemma, we have

(9) $|v^m(t)| \leqq |v_0| \exp(-\frac{\mu t}{C_\Omega^2})$

An applications of the Schwarz inequality to (8) yields

(10) $\mu|\nabla v^m|^2 \leqq |v_0| \cdot |v_t|$

By differentiating each equation (7) with respect to t, multiplying by $\frac{d}{dt} g_{jm}(t)$, summing $\sum_{j=1}^{m}$, and using (10), we obtain

$$\frac{1}{2}\frac{d}{dt}|v_t^m|^2 + (1 - 6\mu^{1/2}C_\Omega^3|b|_4|v_0|^{1/2}|v_t|^{1/2})|v_t^m|^2 \leqq 0.$$

From the assumption of the theorem, it follows that $|v_t^m|$ and hence $|\nabla v^m|$ are bounded :

(11) $|v_t^m| \leqq |\Delta u_0 + f - u_0^3|$

(12) $|\nabla v^m| \leqq \mu^{1/2}|v_0|^{1/2}|\Delta u_0 + f - u_0^3|^{1/2}.$

(the proof of the theorem)

By the estimates (8), (11), (12) and the Rellich theorem, a subsequence $\{v^k\}$ can be selected from $\{v^m\}$ such that

$v^k \to v$ weakly in $L^2(0, T; H_0^1(\Omega))$

$v_t^k \to v_t$ weakly in $L^2(0, T; L^2(\Omega))$

$v^k \to v$ strongly and a.e. in $L^2(0, T; L^2(\Omega))$,

$(v^k)^3 \to v'$ weakly in $L^{4/3}(0, T; L^{4/3}(\Omega))$.

According to well known results, it follows that $v' = v^3$ and v is a generalized solutions of the equations (3), (4), (5).

By (9), each $|v^m(t)|$ decays exponentially, uniformly in m.

Thus this estimate must hold for $|v(t)|$ also.

Refferences

〔1〕 J. Heywood, On stationary solutions of the Navier-Stokes equations as limits of nonstationary solutions,
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〔2〕 J. Heywood, On nonstationary problems for the Navier-Stokes equations, and the stability of stationary flows.
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