On a non-linear hyperbolic equation

By

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§ I. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. In this note we consider the initial-boundary value problem of the form

(1. 1) $\quad \partial^2 u / \partial t^2 - \Delta u + G(u) = 0, \quad x \in \Omega, \quad t > 0,$

(1. 2) $\quad u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geqq 0,$

(1. 3) $\quad \begin{cases} u(x, 0) = u_0(x), & x \in \Omega, \\ \partial u / \partial t(x, 0) = u_1(x), & x \in \Omega. \end{cases}$

When the non-linear term G is Hölder continuous, J. C. Saut gave existence of global solutions of the above problem in [3].

The purpose of the present note is to extend a part of the Saut's results. We make on a continous function G the following condition

(1. 4) $\quad |G(u)| \leqq k_0|u|^\alpha + k_1, \quad 0 < \alpha \leqq 1,$

Where k_0, k_1 are positive constants.

Remark 1.1. If G is Hölder continuous, then G satisfies (1.4).

We shall deal with real valued functions and use the notation in the book of J. L. Lions [2] and prove the following

Theorem. *Suppose that* $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. *Then there exists a function u such that*

(1.5) $\quad u \in L^\infty(0, T; H_0^1(\Omega)), \quad T > 0,$

(1.6) $\quad \partial u / \partial t \in L^\infty(0, T; L^2(\Omega)), \quad T > 0,$

and which satisfies (1.1) in a generalized sense and (1.3).

Remark 1.2. The boundary condition (1.2) is implied by (1.5).

Remark 1.3. It follows from (1.5) and (1.6) that u is continuous from $[0, T]$ to $L^2(\Omega)$, possibly after a modification on a set of measure zero and from (1.5), (1.6) and (1.1) that $\partial u/\partial t$ is continuous from $[0, T]$ to $H^{-1}(\Omega)$.

§ 2. Existence of an approximate solution

Suppose that $0 < \alpha < 1$. We shall use the Galerkin's method. Let $\{w_i\}_{i=1,2\cdots}$ be a complete system in $H_0^1(\Omega)$. We look for an approximate solution $u_m(x, t)$ of the form

$$(2.1) \qquad u_m(t) = \sum_{i=1}^{m} g_{im}(t)\, w_i, \quad g_{im} \in C^2([0, T]).$$

The unknown functions g_{im} are determined by the system of ordinary different al equations

$$(2.2) \qquad \begin{cases} (u_m''(t), w_j) + a(u_m(t), w_j) + (G(u_m(t)), w_j) = 0, \\ 1 \leqq j \leqq m, \end{cases}$$

with initial conditions

$$(2.3) \qquad u_m(0) = u_{0m}, \quad u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_i \to u \quad \text{in } H_0^1(\Omega) \text{ as } m \to \infty,$$

$$(2.4) \qquad u_m'(0) = u_{1m}, \quad u_{1m} = \sum_{i=1}^{m} \beta_{im} w_i \to u_1 \quad \text{in } L^2(\Omega) \text{ as } m \to \infty.$$

Here $\quad u' = \partial u/\partial t, \quad u'' = \partial^2 u/\partial t^2, \quad (u, v) = \int_\Omega uv\,dx,$

$$(2.5) \qquad a(u, v) = \sum_{i=1}^{n} \int_\Omega (\partial u/\partial x_i)(\partial v/\partial x_i)\,dx.$$

By general theory on the system of ordinary differential equations, there exists a solution of (2.2), (2.3) and (2.4) in an int erval $[0, t_m]$, $t_m > 0$.

A priori es timate in §3 assure that $t_m = T$.

§ 3. A priori estimate

Multiplying (2.2) by $g_{im}'(t)$ and summing over j from l to m, we get

$$(3.1) \qquad (1/2)(d/dt)(|u_m'(t)|^2 + \|u_m(t)\|^2) + (G(u_m(t)), u_m'(t)) = 0,$$

Where $|u|^2 = (u, u)$, $\|u\|^2 = a(u, u)$.

Using (1.4), Hölder's inequality and Sobolev's inequality, we have

$$(3.2)\quad \begin{cases} (1/2)(d/dt)(|u_m'(t)|^2+\|u_m(t)\|^2) \leqq k_2|u_m'|\,\|u_m\|^r + k_3|u_m'| \\ \qquad \leqq k_4(|u_m'|^{1+\alpha}+\|u_m\|^{1+\alpha}) + k_5 \quad \text{(by Young's inequality)} \\ \qquad \leqq k_6\{(1/2)(|u_m'|^2+\|u_m\|^2\}^{\frac{1+\alpha}{2}} + k_7 \end{cases}$$

Where $k_2, \cdots, k_7$ are positive constants.

We define $e_m(t)$ by

$$(3.3)\qquad e_m(t) = (1/2)(|u_m'(t)|^2+\|u_m(t)\|^2),$$

then by integrating with respect to t and using (2.3) and (2.4), we have an integral inequality

$$(3.4)\qquad e_m(t) \leqq c_0 + c_1 t + c_2\int_0^t e_m^{\frac{1+\alpha}{2}}(s)\,ds,$$

Where c_0, c_1, c_2 are positive constants independent of m.

From the results of integral inequalities (see [1]),
we obtain

$$(3.5)\qquad e_m(t) \leqq c_0 + c_1 t + M(t).$$

Here M(t) is the maximal solution of ordinary differential equation of the form

$$(3.6)\qquad y'(t) = c_2(c_0 + c_1 t + y)^{\frac{1+\alpha}{2}},$$

with initial condition

$$(3.7)\qquad y(0) = 0.$$

Since $0 < \frac{1+\alpha}{2} < 1$, the above maximal solution M(t) exists in the large. Hence

$$(3.8)\qquad (1/2)(|u_m'(t)|^2+\|u_m(t)\|^2) \leqq c(T),$$

and c(T) is a constant independent of m.

This a priori estimate shows $t_m = T$.

§4. Existence of global solutions

From a priori estimate (3.8) and Rellich's theorem, we can find a function u and a subsequence $\{u_\mu\}$ of $\{u_m\}$ such that

$$(4.1)\qquad u_\mu \rightarrow u \quad \text{in } L^\infty(0,T;H_0^1(\Omega)) \text{ weakly star,}$$

$$(4.2)\qquad u_\mu' \rightarrow u' \quad \text{in } L^\infty(0,T;L^2(\Omega)) \text{ weakly star,}$$

(4.3) $u_\mu \to u$ in $L^2(0, T; L^2(\Omega))$ strongly and a. e. in $\Omega \times [0, T]$.

By (4.3),(1.4) and a well-known lemma (see Lemma 1.3 [2]), we have

(4.4) $G(u_\mu) \to G(u)$ in $L^{2/\alpha}(0, T; L^{2/\alpha}(\Omega))$ weakly,

Which implies the function u satisfies (1.1) in a generalized sense.

We next prove that u satisfies (1.3). From (4.1),(4.2) and Lemma 1.2 [2], in particular

(4.5) $u_\mu(0) \to u(0)$ in $L^2(\Omega)$ weakly

and since $u_\mu(0) = u_{0\mu} \to u_0$ in $H_0^1(\Omega)$, we have

(4.6) $u(0) = u_0$.

Using (2.2), (4.1), (4.2) and (4.4), we get

(4.7) $(u_\mu'', w_j) \to (u'', w_j)$ in $L^{2/\alpha}(0, T)$ weakly.

It follows from (4.2) (4.7) that

(4.8) $(u_\mu'(0), w_j) \to (u', w_j)|_{t=0} = (u'(0), w_j) \quad \forall j$.

Since $(u_\mu'(0), w_j) = (u_{1\mu}, w_j) \to (u_1, w_j) \quad \forall j$ we have

(4.9) $u'(0) = u_1$.

This completes the proof of theorem when $0 < \alpha < 1$.

Remark 4.1. When $\alpha = 1$, we can more easily prove the existence of global solutions.

References

[1] V. Lakshmikantham and S. Leela, Differential and Integral Inequalities Vol. 1, Academic Press, New York and London, 1969.

[2] J. L. Lions, Quelques M'ethodes de R'esolution des Probl'emes aux Limites non Lin'eaire, Dunod, Paris, 1969.

[3] J. C. Saut, Sur un probleme hyperbolique non lineaire, C. R. Acad. Sc. Paris, t. **273** (1971), 671-673.