On certain classes of analytic functions in the unit disk*

By

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Abstract

In this note we shall consider the problem of finding the radius of convexity in some univalent region for functions $f(z)=z+a_2z^2+\cdots\cdots$ which are analytic and satisfy
$\mathrm{Re}\ \{f'(z)/(\lambda f'(z)+(1-\lambda)\phi'(z))\}>\beta$ for $|z|<1$, $0\leqq\beta<1$, $0\leqq\lambda<1$
where $\phi(z)=z+b_2z^2+\cdots\cdots$ is analytic, univalent and convex of order α, $0\leqq\alpha<1$.
Finally a distortion theorem for $f(z)$ is shown.

1. Introduction

Let $k(\alpha)$ denote the class of functions analytic in $|z|<1$ and of the form

$$\phi(z)=z+b_2z^2+\cdots\cdots,$$

such that $\mathrm{Re}\ \{z\phi''(z)/\phi'(z)+1\}>\alpha$ for $|z|<1$ and $0\leqq\alpha<1$.
Then $\phi(z)$ is said to be convex of order α.
we say that an analytie function $f(z)=z+a_2z^2+\cdots\cdots$ is in the class $C(\alpha,\beta)$ if there exists a function $\phi(z)\in k(\alpha)$ such that

$$\mathrm{Re}\ \{f'(z)/\phi'(z)\}>\beta,\ 0\leqq\beta<1.$$

Kaplan [2] proved that C(o, o), the class of close-to-convex functions, is univalent in $|z|<1$.
In this paper we shall consider the radius of convexity of $f(z)$ under the conditions which are pointed out in abstract. And finally we state on the distortion theorems for $f(z)$.

2. Proof of the theorems

In proving the theorems we will make use of the following lemmas.

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Lemma 1. [5]. If $h(z)=1+d_1z+\cdots\cdots$ is analytic for $|z|<1$ and Re $h(z)>\alpha$, $0\leqq\alpha<1$,then

$$\frac{1-(1-2\alpha)\ |z|}{1+|z|}\leqq \text{Re } h(z)\leqq |h(z)|\leqq\frac{1+(1-2\alpha)|z|}{1-|z|}.$$

Lemma 2. [4]. Let $p(z)$ be analytic for $|z|<1$, $p(o)=1$.
Then Re $p(z)>\beta$, $0\leqq\beta<1$, if and only if

$$p(z)=\frac{1+(1-2\beta)\,\omega(z)}{1-\omega(z)},$$

where $\omega(z)$ is analyitc, $\omega(o)=0$ and $|\omega(z)|<1$ for $|z|<1$.

Lemma 3. Let $p(z)=1+c_1z+\cdots\cdots$ is analytie and Re $p(z)>\beta$, then $0\leqq\beta<1$, for $|z|<1$.

$$(1-\lambda\,|p(z)|)^{-1}\leqq(1-|z|)\Big/(1-\lambda-(1+(1-2\beta)\lambda)|z|)$$

for $|z|<\dfrac{1-\lambda}{1+(1-2\beta)\lambda}<1$, where $0\leqq\lambda<1$.

Proof. Using Lemma 1,

$$\frac{1}{1-\lambda|p(z)|}\leqq\frac{1}{1-\lambda\dfrac{1+(1-2\beta)|z|}{1-|z|}}=\frac{1-|z|}{1-\lambda-(1+(1-2\beta)\lambda)|z|}$$

for $|z|<\dfrac{1-\lambda}{1+(1-2\beta)\lambda}<1$.

Lemma 4. [3]. With the same hypothesis as in Lemma 3, we have

$$\left|\frac{zp'(z)}{p(z)}\right|\leqq\frac{2\gamma}{(1-\gamma)\left\{1+\gamma+\dfrac{\beta}{1-\beta}(1-\gamma)\right\}}$$

for $|z|=\gamma<1$.

Lemma 5. [1]. Let $\phi(z)\in k(\alpha)$. Then

$$|\phi'(z)|\leqq\frac{1}{(1-\gamma)^{2(1-\alpha)}},\quad |\phi'(z)|\geqq\frac{1}{(1+\gamma)^{2(1-\alpha)}}$$

for $|z|=\gamma<1$.

Theorem 1. Let $f(z)=z+a_2z^2+\cdots\cdots$ be analytic for $|z|<1$ and $\phi(z)\in k(\alpha)$.
If Re$\{f'(z)\big/(\lambda f'(z)+(1-\lambda)\phi'(z))\}>\beta$,
$0\leqq\beta<1$, $0\leqq\lambda<1$ for $|z|<1$, then Re $\dfrac{f'(z)}{\phi'(z)}>0$
for $|z|<R^*=\dfrac{1-\lambda}{(1+(1-2\beta)\lambda)}$.

Proof. Let

(1) $p(z)=f'(z)\big/(\lambda f'(z)+(1-\lambda)\phi'(z))=1+c_1z+\cdots\cdots,$

then $p(z)$ is analytic and Re $p(z)<\beta$ for $|z|<1$.

Now from (1),

(2) $$\frac{f'(z)}{\phi'(z)}=\frac{(1-\lambda)p(z)}{1-\lambda p(z)}.$$

The expression (2) is valid for those z for which $1-\lambda p(z)\neq 0$ for $|z|<1$.

Since $|p(z)| \leqq \frac{1+(1-2\beta)|z|}{1-|z|}$, $1-\lambda p(z)\neq 0$ in particular if $|z| < \frac{1-\lambda}{1+(1-2\beta)\lambda}$.

From Lemma 2, by Schwarz's lemma,

(3) $|\omega(z)| \leqq |z|$.

Using Lemma 2 and (3),

$$\operatorname{Re} \frac{f'(z)}{\phi'(z)} = \operatorname{Re}\left\{\frac{(1-\lambda)}{1+(1-2\beta)\lambda} \cdot \frac{1+(1-2\beta)\omega(z)}{\left(\frac{1-\lambda}{1+(1-2\beta)\lambda} - \omega(z)\right)}\right\}$$

$$(4) \quad \geqq \frac{1-\lambda}{(1+(1-2\beta)\lambda)^2} \cdot \frac{1-\lambda-2(\beta+(1-2\beta)\lambda)|z|-(1-2\beta)(1+(1-2\beta)\lambda)|z|^2}{\left|\frac{1-\lambda}{1+(1-2\beta)\lambda} - \omega(z)\right|^2}.$$

Using the right-hand part of inequality in (4), we obtain that

$$\operatorname{Re} \frac{f'(z)}{\phi'(z)} > 0 \quad \text{for } |z| < R^* < 1.$$

This shows that f(z) is univalent and close-to-convex for $z < R^*$.

Theorem 2. With the same hypothesis as is Theorem 1, f(z) maps the disk $|z|<R$ onto a convex domain, where R is the smallest positive root of the equation $q(\gamma, \alpha, \beta, \lambda)=0$, where

$$\begin{aligned} q(\gamma, \alpha, \beta, \lambda) = (1-\alpha)\Big[&(1-\lambda)+2\Big\{(1-\lambda)\alpha-(1+(1-2\beta)\lambda)\Big\}\gamma \\ &-\Big\{(1-\lambda)(1-2\alpha)+4\alpha(\beta+(1-2\beta)\lambda)+4(1-\beta)+(1-2\beta)(1+(1-2\beta)\lambda)\Big\}\gamma^2 \\ &+2(1-2\alpha)(1-2\beta)(1+(1-2\beta)\lambda)\gamma^4\Big]. \end{aligned}$$

Proof. Now from (1),

$$(5) \quad f'(z) = \Big((1-\lambda)\phi'(z)p(z)\Big)\Big/(1-\lambda p(z)), \quad \text{for } |z| < R^* < 1.$$

From equation (5), we have

$$\frac{zf''(z)}{f'(z)} = \frac{z\phi''(z)}{\phi'(z)} + \frac{zp'(z)}{p(z)} \quad \frac{1}{1-\lambda p(z)}.$$

In [1], it is shown that

$$(6) \quad \operatorname{Re}\left\{\frac{z\phi''(z)}{\phi'(z)}\right\} \geqq -\frac{2(1-\alpha)\gamma}{1+\gamma} \qquad \text{for } |z| = \gamma < 1.$$

From Lemma 4, we obtain

$$(7) \quad \left| z\frac{p'(z)}{p(z)} \right| \leqq \frac{2(1-\beta)\gamma}{(1-\gamma)(1+(1-2\beta)\gamma)}.$$

Using (6), (7) and Lemma 3,

$$(8)\quad \mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\}\geqq 1-\frac{2(1-\alpha)\gamma}{1+\gamma}-\frac{2(1-\beta)\gamma}{(1+(1-2\beta)\gamma)}\cdot\frac{1}{1-\lambda-(1+(1-2\beta)\lambda)\gamma},$$

for $|z|<R^*<1$.

Simplifying the right-hand side of (8), we obtain

$$\frac{q(\gamma,\ \alpha,\ \beta,\ \lambda)}{(1+\gamma)(1+(1-2\beta)\gamma)\{1-\lambda-(1+(1-2\beta)\lambda)\gamma\}}.$$

Since $q(o,\ \alpha,\ \beta,\ \lambda)>0$,

$q(R^*,\ \alpha,\ \beta,\ \lambda)<0$,

and $q(R,\ \alpha,\ \beta,\ \lambda)=0$,

it follows that f(z) maps the disk $|z|<R(<R^*<1)$ onto a convex domain. This completes the proof.

Remark. If $\alpha=\beta=0$, $\lambda=0$, we see that $f(z)\in C(o, o)$. The Koebe function $z(1-z)^{-2}$ is in C(o, o) relative for $\frac{z}{1-z}$ and the least positive root of

$q(\gamma, o, o, o)=\gamma^4-2\gamma^3-6\gamma^2-2\gamma+1$ is $2-\sqrt{3}$, the radius of convexity for the class S. (S is the class of normalized univalent functions analytic in $|z|<1$.)

Theorem 3. Let $f(z)=z+a_2z^2+\ldots\ldots$ be analytic for $|z|<1$ and $\phi(z)\in k(\alpha)$. If $\mathrm{Re}\left\{f'(z)\big/(\lambda f'(z)+(1-\lambda)\phi'(z))\right\}>\beta$, $0\leqq\beta<1$, $0\leqq\lambda<1$ for $|z|<1$, then we have for $|z|<R^*$

$$(9)\quad \left|f'(z)\right|\leqq\frac{(1-\lambda)(1+(1-2\beta)\gamma)}{(1-\lambda)-(1+(1-2\beta)\lambda)\gamma}\cdot\frac{1}{(1-\gamma)^{2(1-\alpha)}},$$

$$(10)\quad \left|f'(z)\right|\geqq\frac{(1-\lambda)(1-(1-2\beta)\gamma)}{(1-\lambda)+(1+(1-2\beta)\lambda)\gamma}\cdot\frac{1}{(1+\gamma)^{2(1-\alpha)}}.$$

Equality holds in (9) for the function

$$f_1(z)=\int_o^z\frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda)-(1+(1-2\beta)\lambda)t}\cdot\frac{1}{(1-t)^{2(1-\alpha)}}\,dt$$

and equality holds in (10) for the function

$$f_2(z)=\int_o^z\frac{(1-\lambda)(1-(1-2\beta)t)}{(1-\lambda)+(1+(1-2\beta)\lambda)t}\cdot\frac{1}{(1+t)^{2(1-\alpha)}}\,dt.$$

Proof. Using Lemma 2, we put

$$(11)\quad \frac{f'(z)}{\lambda f'(z)+(1-\lambda)\phi'(z)}=\frac{1+(1-2\beta)G(z)}{1-G(z)},$$

where $G(o)=0$, $|G(z)|<1$ for $|z|<1$.

Therefore by Schwarz's lemma, (11) yields

$$\frac{1-(1-2\beta)\gamma}{1+\gamma} \leqq \left| \frac{f'(z)}{\lambda f'(z)+(1-\lambda)\phi'(z)} \right| \leqq \frac{1+(1-2\beta)\gamma}{1-\gamma} \quad (|z|=\gamma<1),$$

or

$$(12)\quad \frac{1+\gamma}{1-(1-2\beta)\gamma} \geqq \left| \frac{\lambda f'(z)+(1-\lambda)\phi'(z)}{f'(z)} \right| \geqq \frac{1-\gamma}{1+(1-2\beta)\gamma}.$$

Using (12) and Lemma 5, the result follows.

Now, let

$$f(z)=\int_0^z \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda)-(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^{2(1-\alpha)}}\, dt$$

and

$$\phi(z)=\int_0^z \frac{1}{(1-t)^{2(1-\alpha)}}\, dt .$$

For $\phi(z)$, we can able to show that $\phi(z)\in k(\alpha)$, and for $f(z)$, $\phi(z)$, it is shown that

$$\mathrm{Re}\, \frac{f'(z)}{\lambda f_1(z)+(1-\lambda)\phi'(z)} = \mathrm{Re}\, \frac{1+(1-2\beta)z}{1-z} > \beta.$$

Therefore $f_1(z)$ satisfies the conditions of Theorem 3. The proof that $f_2(z)$ satisfies the conditions of Theorem 3. is similar, with

$$\phi(z)=\int_0^z \frac{1}{(1+t)^{2(1-\alpha)}}\, dt .$$

Theorem 4. With the same hypothesis as in Theorem 3, we have for $|z|<R^*$

$$(13)\quad |f(z)| \leqq \int_0^r \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda)-(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^{2(1-\alpha)}}\, dt.$$

$$(14)\quad |f(z)| \geqq \int_0^r \frac{(1-\lambda)(1-(1-2\beta)t)}{(1-\lambda)+(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1+t)^{2(1-\alpha)}}\, dt .$$

Equality holds in (13) for $f_1(z)$ in Theorem 3. and in (14) for $f_2(z)$ in Theorem 3.

Proof. To prove (13), let $z=\gamma e^{i\theta}$. Then

$$|f(\gamma e^{i\theta})| \leqq \int_0^r |f'(te^{i\theta})|\, dt$$

$$\leqq \int_0^r \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda)-(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^{2(1-\alpha)}}\, dt .$$

To prove (14), let z_0, $|z_0| = \gamma$, be chosen in such a way that $|f(z_0)| \leqq |f(z)|$, for all z, $|z| = \gamma$.

If $L(z_0)$ is the pre-image of the segment o, $f(z_0)$, then

$$|f(z)| \geqq |f(z_0)| \geqq \int_{L(Z_0)} |f'(z)|\, |dz|$$

$$\geqq \int_0^r \frac{(1-\lambda)(1-(1-2\beta)t)}{(1-\lambda)+(1-(1-2\beta)\lambda)t} \cdot \frac{1}{(1+t)^{2(1-\alpha)}}\, dt .$$

This completes the proof.

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