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On isoperimetric problem in a complex plane

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Abstract

Classical isoperimetric inequality is shown in a complex plane. In a complex plane we can use effectively the complex Fourier expansion in the computations.

0 Introduction: isoperimetric problem and isoperimetric inequality in a plane

Let C be a simply closed curve in a plane and D be the domain enclosed by C. Let l be the length of C and A be the area of D. Then the isoperimetric inequality is

$$A \leq \frac{l^2}{4\pi}.$$

The classical isoperimtric problem claims that for every simply closed curve C in a plane the isoperimetric inequality holds and that its equality holds if and only if C is a circle of radius $l/2\pi$.

Since the radius of a circle which has the length l is $r = l/2\pi$ the circle has area $\pi(l/2\pi)^2 = l^2/4\pi$. The isoperimetric inequality thus shows that among all simply closed curves of length l, circles of radius $l/2\pi$ have the largest area $l^2/4\pi$ and the equality condition shows that the largest area is attained only by those circles.

We show the claim of isoperimetric problem in a complex plane $\mathbb{C}$. The proof gets through along the classical line [1, 4]. The use of the complex Fourier series in a complex plane makes the reasoning a little straightforward.

1 A closed curve in $\mathbb{C}$ and its Fourier expansion

By similitude it suffices to consider curves of length $l = 2\pi$ and to show the isoperimetric inequality: $A \leq \pi$. Let C be a simply closed curve of length 2π in a complex plane $\mathbb{C}$. We assume that C is piecewise smooth and is parametrized by its arc length. Let

$$C: \ z(s) = x(s) + iy(s), \quad 0 \leq s \leq 2\pi, \quad z(0) = z(2\pi)$$

be the parametrization of a closed curve $z : [0, 2\pi] \longrightarrow \mathbb{C}$. Then the tangent vector at $z(s)$ is $z'(s) = x'(s) + iy'(s)$. When the curve is parametrized by its arc length s, the length of the tangent vector is one: $|z'(s)| = 1$(except finite points). And the total length of C is

$$2\pi = \oint_C |z'(s)|ds = \int_0^{2\pi} |z'(s)|ds.$$

Expand $z(s)$ into the complex Fourier series:

$$z(s) = \sum_{n\in\mathbb{Z}} c_n e^{ins}, \quad c_n = \int_0^{2\pi} z(s)e^{-ins}\frac{ds}{2\pi}.$$

By the term-by-term differentiation

$$z'(s) = \sum_{n=-\infty}^{\infty} ic_n n e^{ins}.$$

The condition: $1 = |z'(s)|^2 = z'(s)\overline{z'(s)}$ thereby becomes

$$1 = \sum_{n=-\infty}^{\infty} ic_n ne^{ins} \sum_{m=-\infty}^{\infty} -i\overline{c_m} me^{-ims} = \sum_{n,m=-\infty}^{\infty} c_n\overline{c_m} nme^{i(n-m)s}.$$

Integrating $\int_0^{2\pi} * \, ds/2\pi$ term by term, the only terms: $n = m$ remain,

$$1 = \sum_{n=-\infty}^{\infty} c_n\overline{c_n} n^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 n^2 \tag{1}$$

since $\int_0^{2\pi} e^{i(n-m)s} ds/2\pi = \delta_{nm}$. This is the condition of the curve length $l = 2\pi$.

2 The Green formula and isoperimetric inequality

Let D be a bounded domian in a plane with piecewise smooth boundary ∂D. Let $P(x,y)$ and $Q(x,y)$ be C^1-functions near $\overline{D}$. Then the Green formula is:

$$\iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dxdy = \oint_{\partial D} Pdy + Qdx.$$

The formula states a basic relation between integration in a region and integration over its boundary in a plane. So put $P = x, Q = -y$. Then if $A = $area$(D)$,

$$2A = \oint_{\partial D} xdy - ydx.$$

In $\mathbb{C}$ we have $xdy - ydx = (\bar{z}dz - zd\bar{z})/2i = \mathrm{Im}(\bar{z}dz)$ $(dz = dx + idy,\ d\bar{z} = dx - idy)$.

For a curve $C : z = z(s)$ $(0 \le s \le 2\pi)$ and its enclosed region D in $\mathbb{C}$ we have

$$2A = \mathrm{Im} \oint_C \bar{z}dz = \mathrm{Im} \int_0^{2\pi} \overline{z(s)} z'(s) ds.$$

We calculate quantity $A/\pi = 2A/2\pi$.

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{z(s)} z'(s) ds = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-ins} \sum_{m=-\infty}^{\infty} i c_m m e^{ims} \frac{ds}{2\pi}$$

$$= i \sum_{n,m=-\infty}^{\infty} \overline{c_n} c_m m \int_0^{2\pi} e^{i(m-n)s} \frac{ds}{2\pi} = i \sum_{n=-\infty}^{\infty} |c_n|^2 n.$$

Hence we get

$$\frac{A}{\pi} = \frac{2A}{2\pi} = \frac{1}{2\pi}\mathrm{Im}\int_0^{2\pi} \overline{z(s)} z'(s) ds = \sum_{n=-\infty}^{\infty} |c_n|^2 n. \tag{2}$$

Subtract (2) from (1) we have

$$1 - \frac{A}{\pi} = \sum_{n=-\infty}^{\infty} |c_n|^2 n^2 - \sum_{n=-\infty}^{\infty} |c_n|^2 n = \sum_{n=-\infty}^{\infty} |c_n|^2 (n^2 - n)$$

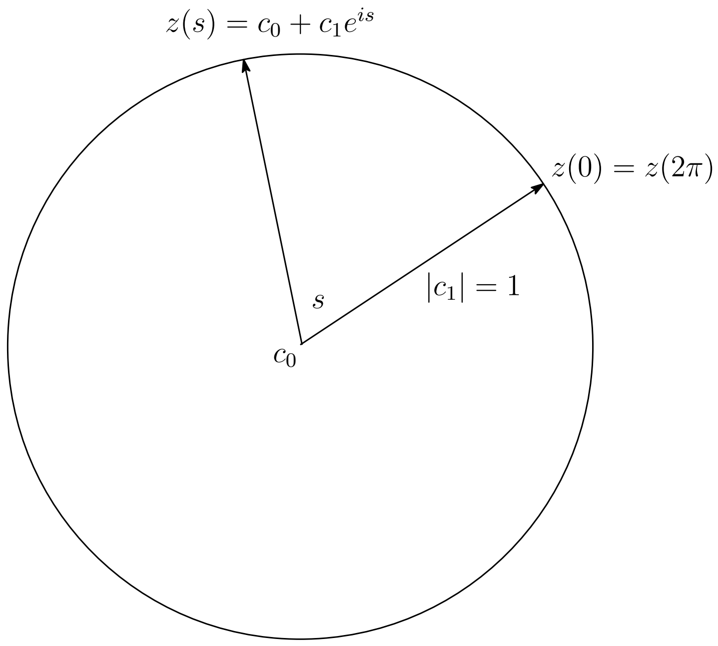
$$= \sum_{n=-\infty}^{\infty} |c_n|^2 \left\{ \left(n - \frac{1}{2}\right)^2 - \frac{1}{4} \right\} \ge 0$$

since $n \in \mathbb{Z}$. This proves the isoperimetric inequality for the curve C.

Because $n^2 - n = n(n-1) = 0$ iff $n = 0, 1$, the equality above holds if and only if all $c_n = 0$ except $n = 0, 1$. In the case in which the equality holds the condition (1) becomes $1 = |c_1|^2$ and the Fourier expansion of $z(s)$ has the only two non-zero terms:

$$z(s) = c_0 + c_1 e^{is}, \quad (0 \leq s \leq 2\pi).$$

Since $|c_1| = 1$ this is exactly the parametrization of a circle of radius one and of center c_0 in the complex plane $\mathbb{C}$.



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