PERFECT ONE-FACTORIZATIONS OF THE COMPLETE GRAPH

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1. Introduction

We denote by $K_{2n}=(V, E)$ the complete graph with 2n vertices, where V is the set of $2n$ vertices and E is the set of $n(2n-1)$ edges. A 1-factor of K_{2n} is a set of pairwise disjoint edges that partition the set of vertices V. A 1-factorization of K_{2n} is a set of 1-factors that partition the set of edges E. A 1-factorization is called perfect if the union of every pair of distinct 1-factors is a Hamiltonian circuit. Two 1-factorizations F and F' are isomorphic if there exists a permutation of V which sends each member of F into a member of F'.

The existence of a perfect 1-factorization of the complete graph K_{2n} for all $n \geqq 2$ is conjectured, and the problem is settled only for $2n=p+1, 2p$ (p is prime), and $2n=$ 16, 28, 244, 344. In this paper, these perfect 1-factorizations are explicitly shown.

Perfect 1-factorizations of K_{36}, K_{1332} and K_{6860} have recently been found ([4, 5]). The papers are in submission.

2. Perfect 1-factorization of K_{p+1} (p is an odd prime)

$GF(p)$ denotes the Galois field with p elements. We put

$$V=GF(p) \cup \{\infty\}$$

and

$$F_0=\left\{\{i, j\} \,\middle|\, i+j=0,\ i, j \in GF(p)-\{0\}\right\} \cup \left\{(0, \infty)\right\}$$

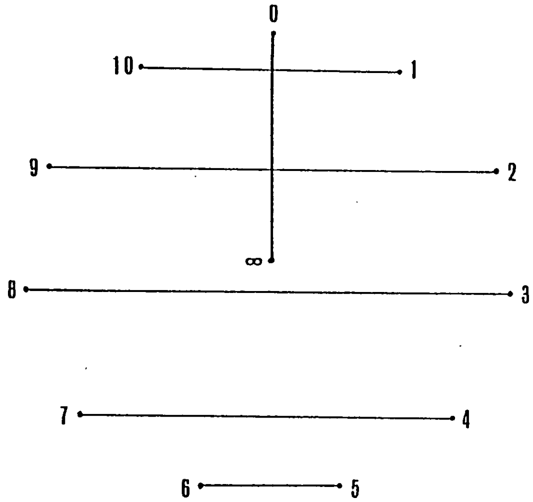
F_0 is called a starter 1-factor and

$$\begin{aligned} F_g &= F_0+g \\ &= \left\{\{i+g, j+g\} \,\middle|\, i+j=0,\ i, j \in GF(p)-\{0\}\right\} \cup \left\{(g, \infty)\right\}. \end{aligned}$$

is an induced 1-factor, where g is an element of $GF(p)$. We obtain a perfect 1-factorization GK_{p+1}:

$$GK_{p+1} = \left\{ F_g \,\middle|\, g \in GF(p) \right\}.$$

For example, a starter of GK_{12} is shown in FIGURE 1.



a starter of GK_{12}

FIGURE 1

3. Perfect 1-factorization of K_{2p} (p is an odd prime)

Let

$$V = \left\{ w_0, w_1, \cdots, w_{p+1}, w_0^*, w_1^*, \cdots, w_{p-1}^* \right\}.$$

For mathematical simplicity, we use w_{i+kp} and w_{i+kp}^* instead of w_i and w_i^*, respectively, where k is an integer.

For an integer s with $0 \leqq s \leqq p-1$, we put

$$OG_S = \left\{ \{w_i, w_j\} \,\middle|\, i+j \equiv s,\ i \not\equiv j (\text{mod } p) \right\}$$

$$\cup \left\{ \{w_i^*, w_j^*\} \,\middle|\, i+j \equiv p-2-s,\ i \not\equiv j (\text{mod } p) \right\}$$

$$\cup \left\{ \{w_{s/2}, w_{(p-2-s)/2}^*\} \right\},$$

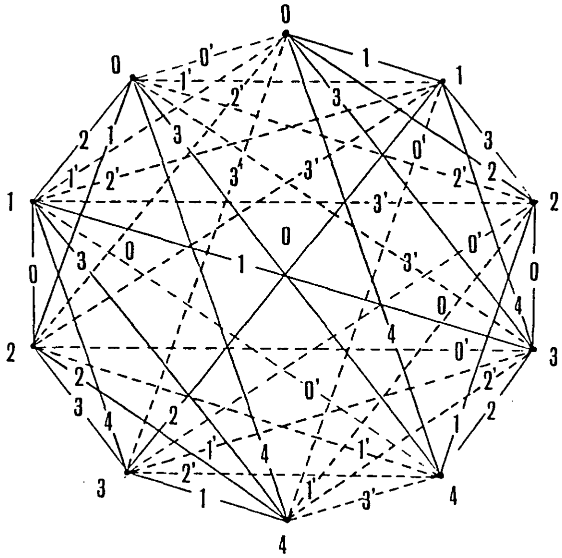
where $1/2$ means 2^{-1} (mod p). For an integer s with $0 \leqq s \leqq p-2$, we put

$$IG_s = \left\{ \{w_i, w_j^*\} \,\middle|\, i+j \equiv s (\text{mod } p) \right\}.$$

Then

$$GA_{2p} = \left\{ OG_s \,\middle|\, s = 0, 1, \cdots, p-1 \right\} \cup \left\{ IG_s \,\middle|\, s = 0, 1, \cdots, p-2 \right\}$$

is a perfect 1-factorization of $K_{2p}([2])$. For example, GA_{10} is shown in FIGURE 2.



GA_{10}

FIGURE 2

Let

$$V' = \left\{ v_0,\ v_1,\ \cdots,\ v_{2p-1} \right\},$$

$$E' = \left\{ \{v_i,\ v_j\} \,\middle|\, 0 \leqq i \leqq 2p-1,\ 0 \leqq j \leqq 2p-1,\ i \neq j \right\}.$$

For mathematical simplicity, we use v_{i+2pk} instead of v_i, where k is an integer.

For any integer s with $0 \leq s \leq 2p-1$ and $s \neq p$, we define $G_s (\subset E)$ as follows:

If s is even, then

$$G_S = \left\{ \{v_i,\ v_j\} \,\middle|\, i + j \equiv s,\ i \not\equiv j \ (\mathrm{mod}\ 2p) \right\} \cup \left\{ \{ v_{S/2},\ v_{S/2+p} \} \right\}$$

If s is odd and $s \neq p$, then

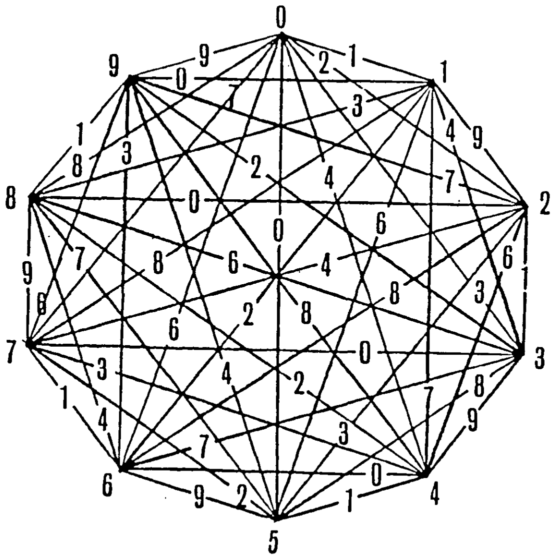
$$G_S = \left\{ \{v_i,\ v_j\} \,\middle|\, i: \mathrm{odd},\ i - j \equiv s \ (\mathrm{mod}\ 2p) \right\}.$$

G_S is a 1-factor of K_{2p} and the set of G_S denoted by

$$GN_{2p} = \left\{ G_S \,\middle|\, 0 \leqq s \leqq 2p-1,\ s \neq p \right\}$$

is a perfect 1-factorization of $K_{2p}([7])$. For example, GN_{10} is shown in FIGURE 3.

GA_{2p} and GN_{2p} are isomorphic perfect 1-factorizations ([3]).



GN_{10}
FIGURE 3

4. Perfect 1-factorization of K_{16}

A 1-factorization F is called factor-1-rotational if F has an automorphism fixing two vertices (and one 1-factor), and permuting the remaining $2n$-2 vertices (and $2n$-2 1-factors) in a single cycle. It has a convenient geometric representation. One takes the vertices of the regular polygon with $2n$-2 vertices and labels them with elements of Z_{2n-2} ; the other two vertices is labeled with ∞_1, ∞_2, where Z_{2n-2} denotes the residue class group modulo $2n$-2. Let F_1 be a starter 1-factor of a factor-1-rotational 1-factorization F. The $2n$-2 1-factors are obtained by rotating the figure successively through an angle $2\pi/(2n$-2). F consists of these $2n$-2 1-factors and the fixed 1-factor F^*:

$$F^* = \left\{ \{i, j\} \,\middle|\, i-j \equiv n-1 \ (\mathrm{mod}\ 2n-2) \right\} \cup \left\{ \{\infty_1, \infty_2\} \right\}$$

A starter 1-factor of a factor-1-rotational, perfect 1-factorization of K_{16} is shown in FIGURE 4.

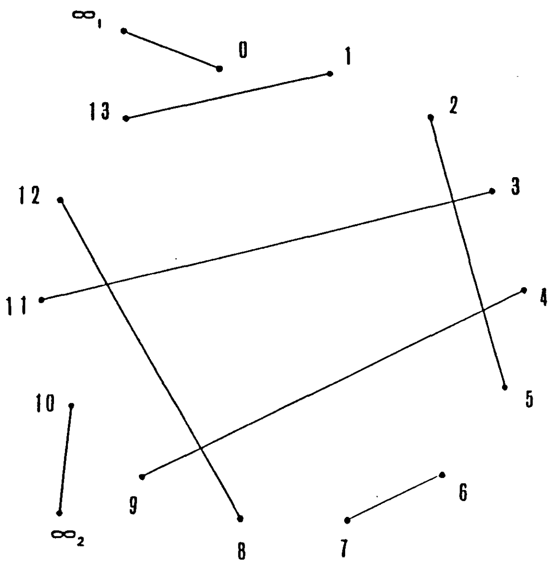


FIGURE 4

5. Perfect 1-factorizations of K_{2n} for $2n$ =28, 244, 344

Let p be a prime number and m be a natural number such that $p^m \equiv 3 \pmod 4$. We put $q = p^m$, $s = (q-1)/2$ and $2n = q+1$. $GF(q)$ denotes the Galois field with q elements. $K_{2n} = (V, E)$ denotes the complete graph with $2n$ vertices, and

$$V = GF(q) \cup \{\infty\}.$$

Let ω be a primitive element of $GF(q)$. We define a starter 1-factor F_0:

$$F_0 = \left\{ \{ \omega^{2i}, \omega^{2i+1} \} \,\middle|\, i = 0, 1, 2, \cdots, s-1 \right\} \cup \left\{ \{0, \infty\} \right\}$$

For any $g \in GF(q)$,

$$F_g = F_0 + g$$

$$= \left\{ \{\omega^{2i} + g, \omega^{2i+1} + g\} \,\middle|\, i = 0, 1, 2, \cdots, s-1 \right\} \cup \left\{ \{g, \infty\} \right\}$$

is a 1-factor which is induced by the starter F_0. Then

$$F(\omega) = \left\{ F_g \,\middle|\, g \in GF(q) \right\}$$

is a 1-factorization of K_{2n}. It is proved that $F(\omega)$ is semi-regular ([1]). By suitable selections of the semi-regulars, we may construct perfect 1-factorizations.

In case $p=3$ and $m=3$, let ω be a primitive element of $GF(3^3)$ with a minimal polynomial x^3+2x^2+1. Then $F(\omega)$ is a perfect 1-factorization of K_{28}.

In case $p=3$ and $m=5$, let ω be a primitive element of $GF(3^5)$ with an minimal polynomial $x^5+x^4+x^2+1$. Then $F(\omega^5)$ is a perfect 1-factorization of K_{244}.

In case $p=7$ and $m=3$, let ω be a primitive element of $GF(7^3)$ with a minimal polynomial x^3+x^2+x+2. Then $F(\omega^{37})$ is a perfect 1-factorization of K_{344}.

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