IMPLICIT ENUMERATION METHOD FOR THE INTEGER PROGRAMMING PROBLEM

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1. Consider the bounded variable integer programming problem

$$\text{minimize} \qquad z=\sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \qquad \sum_{j=1}^{n} a_{ij} x_j\ (\geq, =, \leq)\ b_i \qquad (1\leq i\leq m)$$

$$0\leq x_j\leq l_j,\ x_j\in Z \qquad (1\leq j\leq n),$$

where $c_j\in R$, $a_{ij}\in R$, $b_i\in R$, $l_j\in Z (1\leq i\leq m,\ 1\leq j\leq n)$. R and Z denote the set of all real numbers and the set of all integers, respectively

The above problem can be written in the form

$$\text{minimize} \qquad z=\sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \qquad \sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad (1\leq i\leq m)$$

$$0\leq x_j\leq l_j,\ x_j\in Z \qquad (1\leq j\leq \text{n}),$$

where $c_j\geq 0\ (1\leq i\leq n)$.

If there exists r such that $c_r<0$, replace the variable x_r with another variable y_r:

$$y_r=l_r-x_r,$$

then we have

$$z=\sum_{j\neq r} c_j x_j+c_r x_r=\sum_{j\neq r} c_j x_j+(-c_r)y_r+c_r l_r$$

and

$$0\leq y_r\leq l_r,\ y_r\in Z.$$

The inequality $\sum_{j=1}^{n} a_{ij} x_j\leq b_i$ is equivalent to $\sum_{j=1}^{n} (-a_{ij}) x_j\geq -b_i$, and the equality $\sum_{j=1}^{n} a_{ij}x_j=b_j$ is equivalent to two inequalities $\sum_{j=1}^{n} a_{ij} x_j\geq b_j$ and $\sum_{j=1}^{n} a_{ij} x_j\leq b_i$. If there are u equalities

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (1 \leq i \leq u),$$

we can replace the equivalent (u+1) inequalities :

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad (1 \leq i \leq u),$$

$$\sum_{i=1}^{u} \sum_{j=1}^{n} a_{ij} x_j \leq \sum_{i=1}^{u} b_i .$$

If x is a bounded variable with $0 \leq x \leq l$, substitute

$$x = \sum_{t=0}^{k-1} 2^t y_t + (l - \sum_{t=0}^{k-1} 2^t) y_k ,$$

where k is the smallest integer such that $l \leq 2^{k+1} - 1$, and y_t is a binary variable $(0 \leq t \leq k)$. Hence any bounded variable integer programming problem can be reduced to a 0−1 integer programming problem, so it can be solved by the implicit enumeration for 0−1 variables. In this paper we apply the implicit enumeration method to the bounded variable integer programming problem without transforming to 0−1 variables.

2. Consider the 0−1−2 integer programming model

minimize $\quad z = 5x_1 + 4x_2 + 3x_3$

subject to $\quad Q_1 = 5 - 2x_1 + 5x_2 - 3x_3 \geq 0$

$Q_2 = -3 + 4x_1 + x_2 + 3x_3 \geq 0$

$Q_3 = -1 + x_2 + x_3 \geq 0$

$0 \leq x_j \leq 2 , \ x_j \in Z \quad (1 \leq j \leq 3).$

Step 1

If we put $x_1 = x_2 = x_3 = 0$ then

$z = 0$

$Q_1 = 5 \geq 0$

$Q_2 = -3 \ngeq 0$

$Q_3 = -1 \ngeq 0$,

so constraints Q_2 and Q_3 are violated. Consequently the solution $x_1 = x_2 = x_3 = 0$, which corresponds to node 1 in Figure 1, is not feasible. We determine if further branching can be done from node 1.

Step 2

Let T_1 be the set of all variables with positive coefficients in some violated constraint : $T_1 = \{ x_1 , x_2 , x_3 \}$. We choose a variable in T_1 that would minimize the total distance from feasibility as a partitioning variable.

			Distance from feasibility
For variable 1	$(x_1=1)$:	$Q_1=3$	0
		$Q_2=1$	0
		$Q_3=-1$	1
	$(x_1=2)$:	$Q_1=1$	0
		$Q_2=5$	0
		$Q_3=-1$	1
			Total = 2
For variable 2	$(x_1=1)$:	$Q_1=10$	0
		$Q_2=-2$	2
		$Q_3=0$	0
	$(x_2=2)$:	$Q_1=15$	0
		$Q_2=-1$	1
		$Q_3=1$	0
			Total = 3
For variable 3	$(x_3=1)$:	$Q_1=2$	0
		$Q_2=0$	0
		$Q_3=0$	0
	$(x_3=2)$:	$Q_1=-1$	1
		$Q_2=3$	0
		$Q_3=1$	0
			Total = 1

Therefore we choose x_3.

Step 3

x_3 is specified to be 1, and x_1 and x_2 are still free to take on 0 or 1 or 2, which corresponds to node 2 in Figure 1.

Step 4

Put $x_1=x_2=0$ (and $x_3=1$), then we have

$$z=3$$

$$Q_1 = 2 \geq 0$$
$$Q_2 = 0 \geq 0$$
$$Q_3 = 0 \geq 0$$

Thus $x_1 = x_2 = 0$, $x_3 = 1$ is a feasible solution with $z = 3$. Let $z_{min} = 3$. This is the best value to date. Since all coefficients c_j of the objective function are non-negative, $z = 3$ is the minimal value among all the solutions with $x_3 = 1$. It is not necessary to examine any solution from node 2.

Step 5

Go back to node 1 and specify $x_3 = 2$, and x_1 and x_2 are still free variables, which corresponds to node 3 in Figure 1.

Step 6

Put $x_1 = x_2 = 0$ (and $x_3 = 2$), then

$$z = 6$$
$$Q_1 = -1 \ngeq 0$$
$$Q_2 = 3 \geq 0$$
$$Q_3 = 1 \geq 0,$$

so Q_1 is a violated constraint.

Step 7

Let T be the set of all free variables which have an objective coefficient less than $z_{min} - z$, and a positive coefficient in some violated constraint. In this case we have $T = \emptyset$, so there is no feasible solution with $x_3 = 2$ and $z < 3$, Hence node 3 is fathomed.

Step 8

Go back to node 1 and specify $x_3 = 0$, which corresponds to node 4 in Figure 1.

Step 9

Put $x_1 = x_2 = 0$ (and $x_3 = 0$), then

$$z = 0$$
$$Q_1 = 5 \geq 0$$
$$Q_2 = -3 \ngeq 0$$
$$Q_3 = -1 \ngeq 0,$$

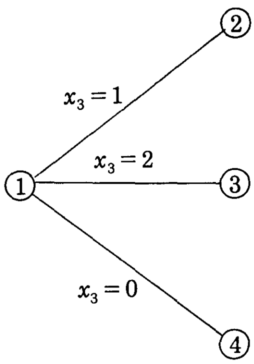
so constraints Q_2 and Q_3 are violated.

Step 10

We have $T = \emptyset$, so there is no feasible solution with $x_3 = 0$ and $z < 3$. Hence node 4 is fathomed.

Step 11

Thus $x_1 = x_2 = 0$ and $x_3 = 1$ is an optimal solution with $z = 3$.



$F = \{x_1, x_2\}$
Feasible solution
$z = 3$

$F = \{x_1, x_2\}$
No feasible solution with $z < 3$

$F = \{x_1, x_2\}$
No feasible solution with $z < 3$

Figure 1

3. Consider the bounded variable integer programming problem

$$\text{minimize} \qquad z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \qquad Q_i = -b_i + \sum_{j=1}^{n} a_{ij} x_j \geq 0 \qquad (1 \leq i \leq m)$$

$$0 \leq x_j \leq l_j, \; x_j \in Z \qquad (1 \leq j \leq n),$$

where $c_j \geq 0$ $(1 \leq j \leq n)$.

We denote by F the set of variables that have not been specified, by NF the set of variables whose value has been specified, and by z_{min} the value of the objective function corresponding to the best feasible solution to date.

Let x_s be any variable in F. x_s is a bounded variable with $0 \leq x_s \leq l_s$. Let d be any integer with $0 < d \leq l_s$. Evaluate each constraint Q_i $(1 \leq i \leq m)$ using the variables in NF with their specified values, x_s=d, and the remaining variables in F each set equal to 0. Let denote by $Q_i(d)$ the value of Q_i:

$$Q_i(d) = -b_i + \sum_{x_j \in NF} a_{ij} x_j + a_{is} d \, .$$

Define

$$TD(x_s)=\sum_{d=1}^{l_s}\sum_{\substack{i=1\\ Q_i(d)<0}}^{m}|Q_i(d)|$$

to be the total distance from feasibility of x_s.

The implicit enumeration algorthm is as follows :

Step 1 (Initialization)

At node 1, set $F=\{x_1, x_2, \dots, x_n\}$, $NF=\emptyset$ and $z_{min}=\infty$. Go to Step 2.

Step 2 (Calculating bounds)

At node k, $z=\sum_{x_j\in NF} c_j x_j$. If $NF=\emptyset$, then we put $z=0$. Go to Step 3.

Step 3 (Fathoming)

Evaluate the constraints $Q_i\,(1\leq i\leq m)$ putting the variables in NF with their specified values and the variables in F with value 0 : $Q_i=-b_i+\sum_{x_j\in NF} a_{ij}x_j$. Let VC be the set of violated constraints. If $VC=\emptyset$ and $z\geq z_{min}$, node k is fathomed, so go to Step 4. If $VC=\emptyset$ and $z<z_{min}$, we put $z_{min}=z$ and node k is fathomed, so go to Step 4. If $VC\neq\emptyset$, go to Step 5.

Step 4 (Backtracing)

If no live node exists, go to Step 6. Otherwise branch to the live node and go to Step 2.

Step 5 (Partitioning and branching)

Put $B=z_{min}-z$. Let T be the set of the free variables that have a positive coefficient in some violated constraint and an objective function coefficient less than B:

$$T=\{x_j\in F:\ \ ^{\exists}Q_i\in VC,\ a_{ij}>0,\ \text{and } c_j<B\}.$$

If $T=\emptyset$, there is no feasible solution with $z<z_{min}$, so it is fathomed and go to Step 4. If $T\neq\emptyset$, evaluate each constraint Q_i in VC using the variables in NF with their specified values, the variables x_s in T with value l_s, and the remaining variables in F with value 0 :

$$Q_i=-b_i+\sum_{x_j\in NF} a_{ij}x_j+\sum_{x_s\in T} a_{is}l_s\,.$$

If any of the constraints are still violated, it is fathomed, so go to Step 4. Otherwise, we choose a variable x_p in T that would minimize the total distance from feasibility as a partitioning variable. Branch to x_p with a specified value. Go to Step 2.

Step 6 (Termination)

If $z_{min}=\infty$, there is no feasible solution. If $z_{min}<\infty$, that feasible solution which yielded z_{min} is opital.

4. Example

minimize $z=6x_1+3x_2+x_3+5x_4$

subject to $Q=-6+5x_1+2x_2+x_3+3x_4\geq 0$

$x_1, x_2, x_3, x_4=0$ or 1 or 2.

Step 1

At node 1, $F=\{x_1, x_2, x_3, x_4\}$, $NF=\emptyset$, $z_{min}=\infty$.

Step 2

At node 1, $z=0$.

Step 3

$Q=-6 \ngeq 0$, $VC=\{Q\}$.

Step 5

$B=\infty$, $T=\{x_1, x_2, x_3, x_4\}$. Choose x_1.

Step 2A

At node 2, $F=\{x_2, x_3, x_4\}$, $NF=\{x_1\}$, $x_1=2$, $z=12$.

Step 3A

$Q=4\geq 0$, $VC=\emptyset$, $x_{min}=12$, and node 2 is fathomed.

Step 4A

Branch to node 3.

Step 2B

At node 3, $F=\{x_2, x_3, x_4\}$, $NF=\{x_1\}$, $x_1=1$, $z=6$.

Step 3B

$Q=-1 \ngeq 0$, $VC=\{Q\}$.

Step 5A

$B=12-6=6$, $T=\{x_2, x_3, x_4\}$. Choose x_3.

Step 2C

At node 4, $F=\{x_2, x_4\}$, $NF=\{x_1, x_3\}$, $x_1=1$, $x_3=2$, $z=8$.

Step 3C

$Q=1\geq 0$, $VC=\emptyset$, $z<z_{min}$, put $z_{min}=8$.

Step 4B

Branch to node 5.

Step 2D

At node 5, $F=\{x_2, x_4\}$, $NF=\{x_1, x_3\}$, $x_1=1$, $x_3=1$, $z=7$.

Step 3D

$Q=0$, $VC=\varnothing$, $z<z_{min}$, put $z_{min}=7$.

Step 4C

Branch to node 6.

Step 2E

At node 6, $F=\{x_2, x_4\}$, $NF=\{x_1, x_3\}$, $x_1=1$, $x_3=0$, $z=6$.

Step 3E

$Q=-1 \ngeq 0$, $VC=\{Q\}$.

Step 5B

$B=7-6=1$, $T=\varnothing$, so node 6 is fathomed.

Step 4D

Branch to node 7.

Step 2F

At node 7, $F=\{x_2, x_3, x_4\}$, $NF=\{x_1\}$, $x_1=0$, $z=0$.

Step 3F

$Q=-6 \ngeq 0$, $VC=\{Q\}$.

Step 5C

$B=7-0=7$, $T=\{x_2, x_3, x_4\}$. Choose x_4.

Step 2G

At node 8, $F=\{x_2, x_3\}$, $NF=\{x_1, x_4\}$, $x_1=0$, $x_4=2$, $z=10$.

Step 3G

$Q=0$, $VC=\varnothing$, $z=z_{min}$, so node 8 is fathomed.

Step 4E

Branch to node 9.

Step 2H

At node 9, $F=\{x_2, x_3\}$, $NF=\{x_1, x_4\}$, $x_1=0$, $x_4=1$, $z=5$.

Step 3H

$Q=-3$, $VC=\{Q\}$.

Step 5D

$B=7-5=2$, $T=\{x_3\}$, $Q=-6+2+3=-1 \ngeq 0$, node 9 is fathomed.

Step 4F

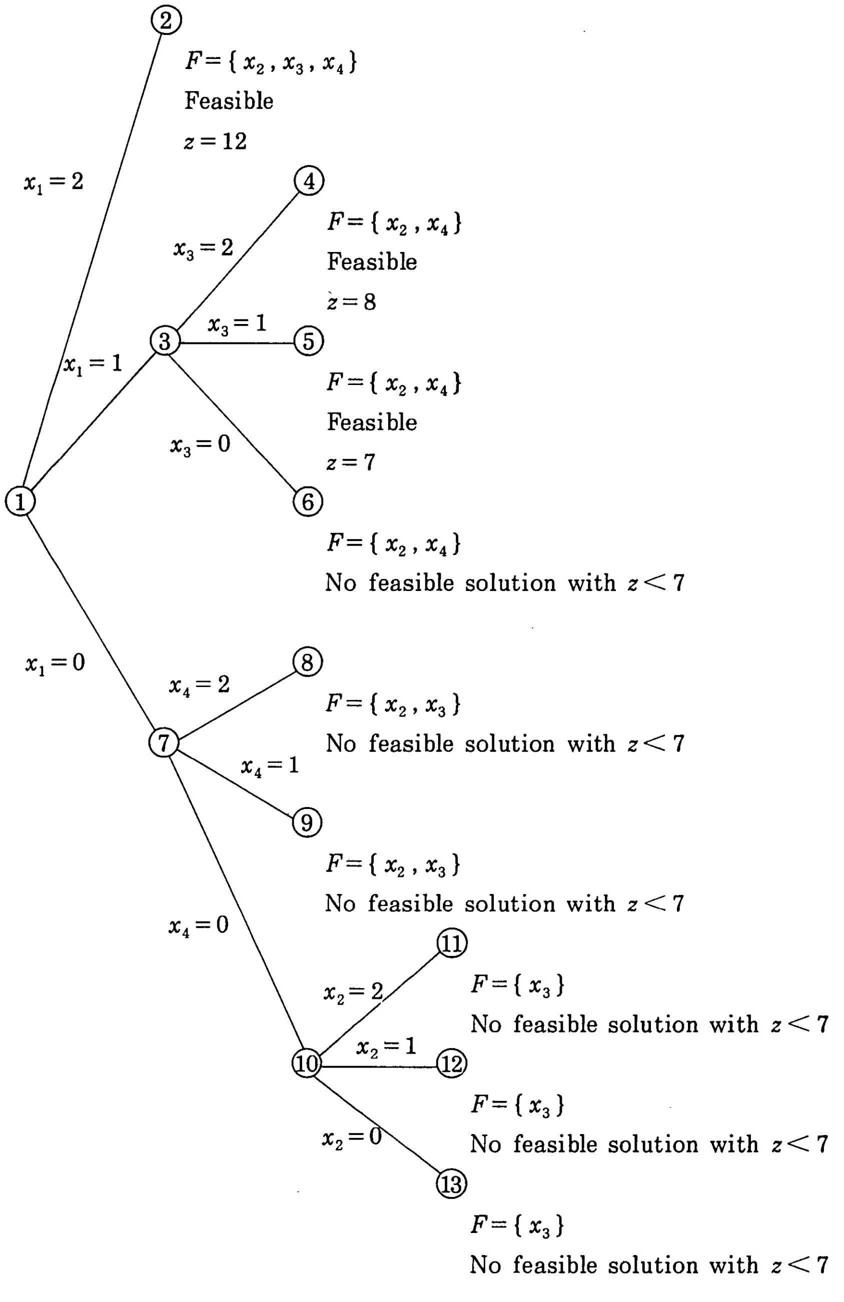
Branch to node 10.

Step 2I

At node 10, $F=\{x_2, x_3\}$, $NF=\{x_1, x_4\}$, $x_1=0$, $x_4=0$, $z=0$.

Step 3I

$Q=-6 \ngeq 0$, $VC=\{Q\}$.



2
F = { x2 , x3 , x4 }
Feasible
z = 12
x1 = 2
4
x3 = 2
F = { x2 , x4 }
Feasible
z = 8
x3 = 1
3
5
x1 = 1
F = { x2 , x4 }
Feasible
x3 = 0
z = 7
1
6
F = { x2 , x4 }
No feasible solution with z < 7
x1 = 0
8
x4 = 2
F = { x2 , x3 }
7
No feasible solution with z < 7
x4 = 1
9
F = { x2 , x3 }
No feasible solution with z < 7
x4 = 0
11
x2 = 2
F = { x3 }
No feasible solution with z < 7
x2 = 1
10
12
F = { x3 }
x2 = 0
No feasible solution with z < 7
13
F = { x3 }
No feasible solution with z < 7

Figure 2

Step 5E

$B=7-0=7$, $T=\{x_2, x_3\}$. Choose x_2.

Step 2J

At node 11, $F=\{x_3\}$, $NF=\{x_1, x_2, x_4\}$, $x_1=0$, $x_2=2$, $x_4=0$, $z=6$.

Step 3J

$Q=-2 \ngeq 0$, $VC=\{Q\}$.

Step 5F

$B=7-6=1$, $T=\varnothing$, node 11 is fathomed.

Step 4G

Branch to node 12.

Step 2K

At node 12, $F=\{x_3\}$, $NF=\{x_1, x_2, x_4\}$, $x_1=0$, $x_2=1$, $x_4=0$, $z=3$.

Step 3K

$Q=-4 \ngeq 0$, $VC=\{Q\}$.

Step 5G

$B=7-3=4$, $T=\{x_3\}$, $Q=-6+2+2=-2 \ngeq 0$, node 12 is fathomed.

Step 4H

Branch to node 13.

Step 2L

At node 13, $F=\{x_3\}$, $NF=\{x_1, x_2, x_4\}$, $x_1=x_2=x_4=0$, $z=0$.

Step 3L

$Q=-6 \ngeq 0$, $VC=\{Q\}$.

Step 5H

$B=7$, $T=\{x_3\}$, $Q=-6+2=-4 \ngeq 0$, node 13 is fathomed.

Step 4I

No live node exists.

Step 6

$x_1=1$, $x_2=0$, $x_3=1$, $x_4=0$ is an optimal solution with $z=7$.

REFERENCES

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[2] B.E.Gillett, Introduction to Operations Research, McGraw-Hill, 1976.