INVERTIBILITY AND OBSERVABILITY OF SWITCHED SYSTEMS WITH INPUTS AND OUTPUTS

BY

ANEEL KUMAR TANWANI

DISSERTATION

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Doctoral Committee:

Associate Professor Daniel Liberzon, Chair
Adjunct Professor P. R. Kumar
Assistant Professor Alejandro Domínguez-García
Assistant Professor Sayan Mitra

Abstract

Hybrid dynamical systems or switched systems can operate in several different modes, with some discrete dynamics governing the mode changes. Each mode of operation is described by a dynamical subsystem having an internal state, an external input (which can be thought of as a disturbance or a control signal), and a measured output. Hybrid/switched systems may arise in practice because of the interaction of digital devices with physical components in order to implement control schemes, or due to integration of small-scale systems to form a large network, or due to transitions occurring in the model of some physical phenomenon. Because of the richness of their application, switched systems have attracted the attention of many researchers over the past decade for the study of analysis and control design problems.

In this thesis, we analyze the properties of invertibility and observability for switched systems and study their related applications in system design. The common facet to both these problems involves the extraction of unknown variables from the knowledge of the output. It is well known that, under certain assumptions, the state trajectory and the output response of any dynamical system are uniquely defined once the initial condition and the input are fixed. Broadly speaking, if the output is assumed to be known, the problems considered in our work deal with: (a) the reconstruction of the input when the initial state is known, or (b) the recovery of the initial state when the inputs are known; the former is called the invertibility problem and the latter is called observability.

Invertibility is an important property in system design and system security analysis, and has only recently been studied for switched systems. Since we treat the switching signal as an exogenous signal, invertibility of switched systems relates to the ability to reconstruct the unknown input and the unknown switching signal from the knowledge of the measured output and the initial state. The thesis addresses the invertibility problem of switched systems where the subsystem dynamics are nonlinear but affine in controls. The novel concept of switch-singular pairs, which arises in the reconstruction of the switching signal, is extended to nonlinear systems and a formula is developed for checking if the given state and output form a switch-singular pair. We give a necessary and sufficient condition for a

switched system to be invertible, which says that the subsystems should be invertible and there should be no switch-singular pairs. In case a switched system is invertible, one can build a switched inverse system to reconstruct the switching signal and the input. The setup naturally leads to an algorithm for output generation where a prescribed reference signal is generated using the system dynamics.

In practice, the exact knowledge of the initial condition and the output may be an overly stringent requirement for invertibility of the system. We relax this requirement by allowing disturbances in the output and uncertainties in the knowledge of the initial condition. Using the theory of reachable sets, an alternative formulation for reconstruction of the switching signal is presented. To relieve the computational burden, we utilize the notion of a gap between subspaces for mode detection that involves merely coarse spherical approximation of the reachable set. This approach of using the reachable sets, though applicable to a general class of linear systems, may not reconstruct switching over large time intervals as the uncertainties in the state may grow to an extent that the outputs of the subsystems become indistinguishable. However, if the individual subsystems are assumed to be minimum phase, which is the same as assuming the stability of the minimal order inverse system in the linear case, then the switching signal can be reconstructed for all times under the dwell-time assumption.

Another important property for diagnostic applications and system design is the observability of switched systems. It is seen that the switched systems essentially act as time-varying systems, and in contrast to time-invariant systems, the ability to recover the state either instantaneously or after some time has different meanings as the information available after switching, from another subsystem, may reveal more knowledge about the state. This idea of gathering information from all the active subsystems is formalized to yield a characterization of observability for switched linear systems. A related, but relatively weaker, notion of determinability deals with recovering the value of the state at some time in the future rather than the initial time. This turns out to be particularly useful in the construction of observers, as the estimates generated by the observers are shown to converge asymptotically to the true state when the switched system is determinable. Similar concepts are studied for another class of switched systems where the underlying subsystems are modeled with differential algebraic equations instead of ordinary differential equations, but the observer design remains a topic of further study in such systems.

The problem of observability is also studied in the context of switched nonlinear systems. Because of the rich nature of the dynamics of such systems and the fact that analytical solutions of the nonlinear ordinary differential equations are not always available, the framework of linear systems is not easily extendable. We therefore propose an alternate approach to derive a sufficient condition for observability in nonlinear switched systems. This condition naturally leads to an observer design, and with the help of analysis, it is shown that the corresponding state estimate indeed converges to the actual state of the system. An effort is made to obtain a characterization in the form of a necessary and sufficient condition for observability. Examples are included throughout the text to help understand the underlying concepts.

Having discussed the properties of invertibility and observability from an analytical perspective, we then discuss an application of these theoretical concepts to study the problem of fault detection in electrical energy systems. The tools developed for solving the invertibility and observability problem have been tailored to address the models of voltage converters and their networks. Categorizing soft faults as unknown disturbances and hard faults as unknown mode transitions, we show that such faults can be recovered if the switched system under consideration is invertible. An algorithm for fault detection and results of simulation are included to demonstrate the utility of the proposed framework. Since the invertibility approach requires the knowledge of the initial condition and the derivatives of the output to reconstruct the soft faults, an alternative observer-based approach is presented for detection of soft faults. Because the initial condition is no longer assumed to be known, the observer dynamics first estimate the state of the system, and then we define auxiliary observer outputs that are only sensitive to faults so that the effect of a nonzero fault is reflected in those new outputs.

A significant aspect of structural properties is their utility in solving some of the prominent design problems, and the concepts related to invertibility of switched systems are utilized in designing switching signals and control inputs for generating desired output trajectories. We conclude the document by proposing some synthesis problems using the system inversion tools. A desired property for the control input in output generation and tracking is its boundedness relative to the size of the output. Classically, this is achieved by requiring the inverse system to be stabilizable. We extend this idea to switched systems to propose a preliminary result for computing bounded inputs that generate a desired bounded output trajectory. If the initial condition is not known, then exact output generation may not be possible and in that case, tracking the output asymptotically is the problem of interest. We present our initial approach on how to achieve output tracking in switched systems and outline the methods for our future work related to this problem.

To my parents, and my better half,
for their love and support

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Table of Contents

Chapter 1

Introduction

1.1 Motivation

In systems theory, a system is typically modeled as a set of differential equations that describe the evolution of certain variables, called state variables, in a finite-dimensional space. The structure of these differential equations basically defines the behavior of the state variables over time. The functions appearing in these equations can be linear, nonlinear, time-invariant, time-varying, smooth, non-smooth or even discontinuous in their arguments. Similar formalisms exist in the domain of discrete systems, where instead of differential equations, the system is modeled by difference equations.

Broadly speaking, the thesis deals with a class of dynamical systems described by a set of differential equations that are discontinuous in time. In practice, such systems are encountered when there is a coupling between continuous dynamics and discrete events. With the use of computers becoming extremely popular, digital networks and embedded systems are getting increasingly complex and there is a need to study the interaction between logic-based components and continuous-time physical systems. This leads to a new modeling paradigm to allow analysis, and design of systems that combine continuous dynamics with discrete logic. Systems in which these two types of dynamics coexist and interact are called *hybrid systems*.

The hybrid modeling framework covers a large class of systems, which leads to their application in various fields:

1. *Electrical Energy Systems:* It is natural to think of electrical circuits using switches to direct the flow of electrical current. Systems in power electronics, such as converters, use switching to regulate the voltage levels or change the frequency of the current. The use of a switch causes abrupt changes in the flow of state variables, and the models of such circuits must accommodate the changes caused by the switches [1]. We will study some of these models in Chapter 7.

2. *Robotic Systems:* The output tracking problem in robotic manipulators is of utmost utility in industry. Depending on the task being performed, the inertial dynamics of the robot may change from time to time [2]. Another kind of system that combines discrete events with continuous dynamics is a biped walking robot where the dynamics exhibit switching from flight mode to stance mode while running [3]. The hybrid behavior is also seen in some complex models of bipeds involving feet movement and ankle rotation [4]. Moreover, hybrid strategies have been employed extensively in the control and stabilization of robot dynamics [5, 6].

3. *Multi-agent Networks:* In multi-agent systems, the collective dynamics of the network of agents are obtained by putting together all the agent dynamics, and depend on how the agents are connected to each other [7]. If the network under consideration consists of mobile agents, then it is natural to assume that some of the existing communication links may fail due to an obstacle between two agents or new links may be established over the period of time as the change in position of agents would affect the range of detection between one another. Because of the time-varying nature of the network topology, these networks can be modeled as hybrid systems where the subsystems are the network dynamics with fixed topologies and the discrete logic, in this case, indicates the active topology at every time.

4. *Biological Systems:* Biological cell networks exhibit complex combinations of both discrete and continuous behaviors; the dynamics that govern the spatial and temporal increase or decrease of protein concentration or activity inside a single cell are continuous differential equations, while the activation or deactivation of these continuous dynamics is triggered by switches which encode protein concentrations reaching given thresholds. Hybrid systems theory presents an ideal framework to model and analyze these processes, with the goal of generating predictions that can be experimentally verified. An example dealing with the regulation of intracellular Delta and Notch protein concentrations is considered in [8, 9], where the authors also use the derived hybrid model to compute the reachable sets in Delta-Notch signalling mechanism.

Several other examples of modeling the hybrid systems along with a generalized solution framework, adopted to address asymptotic and robust stability, are discussed in [10]. From the analysis viewpoint, detailed investigation of the discrete behavior has been a lower priority for researchers in the control-theoretic area. For feasibility of analysis, we ignore the details of discrete dynamics and consider a more general class of systems, called *switched* *systems* [11, 12, 13]. Switched systems comprise a family of continuous-time dynamical subsystems (also called modes) and a switching signal that determines the active mode of operation. Switched systems can be broadly classified as time-varying because of the changes induced in the dynamics by the switching signals. However, as the name indicates, it is more appropriate to think of them as systems where transition occurs from one dynamical subsystem to another. This way we can relate the properties of the whole system to the properties of individual subsystems so that the standard tools can be extended for the purpose of analysis.

The most general class of switched systems (with the exception of Chapter 6) considered in this thesis has the following structure:

$$\Gamma_\sigma : \begin{cases} \dfrac{dx}{dt} &= f_\sigma(x) + G_\sigma(x)u(t) = f_\sigma(x) + \sum_{k=1}^{m}(g_k)_\sigma(x)u_k(t), \quad t \neq t_i, \\ x(t_i) &= \psi_{\sigma(t_i^-),\sigma(t_i)}(x(t_i^-)), \\ y(t) &= h_\sigma(x), \end{cases} \tag{1.1}$$

where $\sigma : [t_0, T) \to \mathcal{P}$ is the switching signal that indicates the active subsystem at every time, $\mathcal{P}$ is some finite index set, and f_p, G_p, h_p, where $p \in \mathcal{P}$, define the dynamics of individual subsystems; and the jump map $\psi_{p,q}$ defines the transition in the value of state trajectory when switching from mode p to mode q. The switching signal is a piecewise constant and everywhere right-continuous function that has a finite number of discontinuities on every bounded time interval; the time instants at which these discontinuities occur are denoted by t_i, which we call *switching times*. For any initial state x_0, switching signal $\sigma(\cdot)$, and any admissible input $u(\cdot)$, a solution of (1.1) always exists (in Carathéodory sense) and is unique, provided the flow of every subsystem is well-defined for the time interval during which it is active, i.e., the state trajectories do not blow up in finite time. In fact, this assumption results in state trajectories that are locally absolutely continuous [10, 14]. Since the switching signals are right-continuous, the outputs are also right-continuous (note that, in general, $h_i(x) \neq h_j(x)$, for $i \neq j$) and whenever we take the derivative of an output, we assume it is the right derivative.

1.2 Thesis Overview

Structural properties of a dynamical system are attributed to the interaction between its three components: the state, the inputs, and the outputs. Switched systems represent a class of dynamical systems that arise in practice where the continuous dynamics of the system interact with the discrete variables in the system. Because of their immense utility in the modeling of physical phenomena [15], the design of control strategies [16], and system verification [17], switched systems have attracted the attention of many researchers in the past couple of decades. From analysis standpoint, several structural properties of switched systems, such as stability, controllability, and observability, have been studied where the most intriguing aspect is how the interaction between the state, the inputs and the outputs gets affected due to the presence of a switching signal and whether a property of interest is preserved under certain classes of switching signals.

In this thesis, we treat two of the structural properties of switched systems mentioned above: *invertibility* and *observability*. If the dynamical systems are seen as mappings from the input space to the output space that depend on the internal states of the system, then both these properties relate to the recovery of an unknown entity from the measurements of the outputs. Both invertibility and observability have been used in stabilization of systems using dynamic output feedback [18] and observer-based state feedback [19], respectively. Moreover, their study is particularly useful in diagnostic applications as they have been employed in fault detection algorithms frequently [20, 21, 22, 23]. Thus, these properties are not only theoretically interesting but also possess useful applications. This provides the motivation to study them outside the realm of non-switched systems.

1.2.1 Research on Switched Systems

Below we provide brief notes on the existing literature on structural properties of switched systems:

Stability

Stability of dynamical systems is critical for any application. For this reason, stability of switched systems has received the most attention among all other structural properties. A survey addressing the issues involved in stability and design of switched systems appears in [12] and a tutorial description of some of the existing results appears in [11, Chapters 2 and 3]. One is also referred to [24] for a recent survey on stability and stabilizability of switched linear systems. The two fundamental questions that arise in addressing the stability of switched systems are: *(a)* whether the system is stable under arbitrary switching or *(b)* whether the stability is achieved for a limited class of switching signals. In answering the first question, extending the classical tool set, one seeks a *common Lyapunov function*. Similar to standard Lyapunov stability theorem, the existence of common Lyapunov function proves to be sufficient for stability. Moreover, the converse Lyapunov theorem can also be derived in different settings [25, 26]. Note that the uniform stability with respect to the switching signals requires the individual subsystems to be stable; under an additional condition that requires the vector fields of the subsystems to commute, which in turn leads to commutativity of the flows of corresponding subsystems, the overall system becomes stable. The results based on commutator operation or Lie-algebraic criteria appear in [27, 28, 29].

On the other hand, for stability under constrained switching, one uses the *multiple Lyapunov function* approach [30, 31]. The basic idea that follows in this approach is that if we look at the values of the Lyapunov function associated to each subsystem at every time that subsystem is activated, then those values must form a non-decreasing sequence. One can extend this idea to develop the notion of *dwell-time* and *average dwell-time* stability in switched systems [32, 33] under which the stability among asymptotically stable systems is preserved when the switching is slow enough. This idea of slow switching has been combined with stability notions of nonlinear subsystems to develop conditions for Input-to-state stability [34, 35], Input/Output-to-state stability [36], the concepts which have been applied to feedback stabilization [34] and construction of state-norm estimators for switched systems [37, 38]. It is also possible to stabilize the switched system even when some of the constituent subsystems are not stable. The most common result along these lines is to look for a stable convex combination of the constituent subsystems that can be implemented by switching among the subsystems with the ratio determined by the coefficients of the convex combination [39, 40]. Methods relying on limiting the total activation time of unstable modes appear in [41, 42, 36], and more recently averaging methods have been used to compute switching signals that stabilize the switched system [43] with unstable modes. Some of these stability notions will be applied in Chapter 3.

Controllability

Controllability basically refers to the notion of driving the state from any point in the state space to the origin. Its natural extension to switched systems addresses the question whether there exists a switching signal and an input that can drive an arbitrary state to the origin. If the objective is to drive the state to the origin arbitrarily fast, then the controllability of individual modes is required. However, even if the individual modes are not controllable, under certain conditions, it is possible to drive the state to the origin after some time, with the help of switching. Geometric criteria for controllability over a time interval in terms of controllability subspaces of individual subsystems appear in [44, 45, 46]. Also, look at [13] for a detailed overview. Along the same lines, the recent work of [47], in addition, identifies the minimum number of switches required to drive the state to the origin. A sufficient condition and a necessary condition for controllability independent of switching times was given by [48]. Note that when seeking conditions independent of switching times, there is always a gap between the necessary and sufficient condition. The major motivation for studying controllability is to achieve stabilization, which has been studied by [49] under the assumption that all subsystems are controllable, and by [50] without imposing controllability assumption on individual modes but requiring the switching signal to be periodic such that the switching yields a Hurwitz convex combination of the constituent modes. In case the control inputs are constrained, small-time controllability of the switched system is no longer equivalent to the controllability of individual modes, and characterization of small-time controllability with conic constraints on control inputs has been studied by [51].

Observability

The observability in switched systems has also been investigated by several researchers. In classical linear time-invariant systems, there is a single uniform notion of observability that deals with recovering the state from the knowledge of the output. However, in switched systems the observability problem can be approached in several ways. If we allow for the usage of the differential operator in the observer, then it may be desirable to determine the state of the system instantaneously from the measured output. This in turn requires each subsystem to be observable; however, the problem becomes nontrivial when the switching signal is treated as a discrete state and simultaneous recovery of the discrete and continuous state is required for observability. Some results on this problem are published in [52, 53, 54].

On the other hand, with the knowledge of switching signal, even though the individual modes are not observable, it is possible to recover the initial state $x(t_0)$ when the output is observed over an interval $[t_0, T)$ that involves multiple switching instants. This phenomenon is inherent only to switched systems as the notion of instantaneous observability and observability over an interval coincide for linear time invariant systems. This variant of the observability in switched systems has been studied most notably by [45, 46, 55]. The authors in [56, 57] study the observability problem for the systems that allow jumps in the states, but they do not consider the change in the dynamics that is introduced by switching to different matrices associated with the active mode. Moreover, the observer design has also received some attention in the literature [58, 59, 60, 55]. With the exception of [55], it is assumed that each mode in the system is in fact observable admitting a state observer, and the switching is treated as a source of perturbation effect. This approach immediately incurs the need of a common Lyapunov function for the switched error dynamics, or a fixed amount of dwell-time between switching instants, because it is intrinsically a stability problem of the error dynamics.

Invertibility

System inversion is an interesting problem, not only from a theoretic standpoint but also from practical viewpoint as it finds application in stabilization [61, 62] and output tracking problems [18, 63, 64]. Although the literature on the inversion of non-switched system dates back to the 1960s, the problem of invertibility for switched systema was introduced very recently in [65].

Roughy speaking, the problem of invertibility deals with recovering the input from the knowledge of the output. In non-switched systems, there is no ambiguity about the input space. However, in switched systems, the switching signal can be seen as an exogenous signal acting on the system in case of time-dependent switching or an internal signal when the switching is state-dependent. In the former case, switching signal may be treated as an extended 'input' to the system and in the latter case, it can be regarded as a function of state variables. Since the state is assumed to be known in solving the invertibility, recovering the switching signal or assuming it is known generates two different approaches towards the solution of this problem. The case where the switching signal is assumed to be unknown and the problem of invertibility deals with recovering the input and the switching signal has been treated in [65] for linear systems. A geometric approach towards the solution of this problem appears in [66]. Our results on invertibility of nonlinear switched systems are published in [67]. A geometric heuristic based approach for this problem also appears in [68].

The problem of recovering the input from the output with known switching signal and initial state has been discussed for discrete-time linear switched systems in [69, 70]. The case of quantized measurements of the output under the known switching signal assumption is treated in [71, 72]. Since this is one of the topics explored in this thesis, we give more details in the next section.

1.2.2 Contribution

In the context of switched systems, this thesis aims at studying the structural properties that relate to extracting information about the state and/or the inputs from the knowledge of the output. The first one, commonly called *observability*, is based on investigating the mapping that exists between the state space and the output space. The second property, called *invertibility*, relates to the mapping between the input space and the output space. On an abstract level, the fundamental question in studying either of these properties is whether the underlying mapping is injective (one-to-one); this characteristic determines whether the output can reveal complete information about the state (observability) or the input (invertibility). Both these system properties reveal fundamental characteristics of switched systems, in the spirit of what one can say about the qualitative behavior of the system in the long run or how much one can infer from and influence the system's behavior based on observed data. Also, because of the switching dynamics, the interface between discrete and continuous dynamics not only reveals several novel phenomena, but also provides some new insights into the structure of switched systems. It will turn out that several features of the relevant mappings are attributed to a particular class of switching signals; therefore, it is of interest not only to determine the maximal information that could be obtained from the measured outputs but whether or not this information is preserved for certain classes of switching signals. We now highlight the contribution of our research on the study of these properties.

Invertibility of Switched Systems

For every control system with an output, we have an input-output map and the question of left (resp. right) invertibility is, roughly speaking, that of the injectivity (surjectivity) of this map. When dealing with non-switched systems, the existing literature on invertibility mainly presents conditions and algorithms which make it possible to recover the input from the output using the knowledge of initial state. The problem statement in switched systems framework is analogous to the classical invertibility problem for non-switched systems. As expected, the main difference arises because of the presence of the switching signal, which is viewed as an exogenous signal with time as its only argument. This way the 'input' space for switched systems is augmented, which motivates us to define the invertibility problem as follows: *What is the condition on the subsystems of a switched system so that, given an initial state x_0 and the corresponding output y generated with some switching signal σ and input u, we can recover the switching signal σ and the input u uniquely?*

This problem has only been recently studied for switched linear systems [65]. In our work, we extend their methodology to study the problem of invertibility of continuous-time switched nonlinear systems, which concerns finding the conditions on the subsystems to guarantee unique recovery of the switching signal and the input from the initial state and the output. Necessary and sufficient conditions for invertibility of nonlinear systems, affine in control, are given. Formulae are computed that would lead to the reconstruction of σ and u. Also, an output generation algorithm is presented that computes the switching signal and input that would exactly reproduce a prescribed desired output.

In order to solve the invertibility problem, we use the exact knowledge of the initial condition. In several engineering problems, this may be undesired or the knowledge of initial state may not be precise. Moreover, some error is introduced in the actual values of outputs when they are stored digitally. Thus, it is natural to work out a robust extension of the invertibility problem which deals with perturbations in the values of the output and the initial state. This motivates us to develop the *robust invertibility* framework where we derive sufficient conditions for reconstruction of the original switching signal even in the presence of disturbances. In the general case, this can be done only on a finite time interval as the uncertainties may grow unbounded after some point in time, making it difficult to recover the exact information. However, when the constituent subsystems are assumed to be minimum phase, conditions that would lead to the exact recovery of the switching signal for all times are presented with the underlying assumption that the switching is slow enough.

Note that the inverse systems may produce unbounded inputs even when the outputs are bounded. Since the boundedness of the input is the most desirable property in the practical setup, we show that certain stability assumptions in addition to invertibility lead to bounded inputs when the switching signals are restricted by average dwell-time. In general, it is not possible to generate desired output exactly when the initial condition is not known, and the objective is to track the given signal asymptotically. In non-switched systems, a stabilizing control input (different from the one obtained from the inverse system) can be computed that achieves this objective, but in case of switched systems, the unknown switching signal makes the problem more interesting and nontrivial. Designing control laws for output tracking with switched systems is proposed as a topic of future work.

To study the applications of invertibility of systems, we propose a framework for fault detection and isolation (FDI) in electrical energy systems. Modeling faults as external unknown exogenous signals, the problem of fault detection can then be seen as recovering, reconstructing or identifying these unknown signals. Some of the most commonly used power converters, both small and large scale, can be modeled as switched systems either because of internal switching or occurrence of faults. The invertibility algorithms not only detect the fault but also reveal their magnitude in case the faults are slow and time-varying.

Observability of Switched Systems

Similar to invertibility, the problem of observability deals with extracting information from the output of the system with the roles of input and initial state reversed, that is, the unknown quantity to be recovered is the state of the systems while the input and the switching signal are assumed to be known. The basic difference in the problem formulation is because of the fact that the state evolves in a finite-dimensional space and the value of the state at the initial time determines the value of state trajectory for all times as a solution of differential equations. So, essentially, we are recovering a finite-dimensional unknown variable from the knowledge of an element in an infinite-dimensional space, which is different from the invertibility problem where the unknown input is itself an infinite-dimensional entity and may assume any value at any time. The notions adopted in the observability of switched systems mainly come from nonlinear system theory. Using the terminology adopted in [73], the system is called *large-time observable* if there exists a time after which the state can be determined uniquely, and it is *small-time observable* if the state can be recovered completely on an arbitrarily small time interval. We will also use the term *instantaneous observability* to refer to small-time observability.

As mentioned earlier, the existing literature on observability of switched systems is mainly concentrated on linear systems and deals with small-time as well as large-time observability. Small-time observability mainly employs a differentiable operator on the outputs to compute the state. In the literature on large-time observability, the conditions that guarantee observability basically determine whether there exists a switching signal which makes it possible to recover the state uniquely. Existing observer designs mostly assume that the individual modes are observable so that the classic Luenberger observer for each mode can be constructed; assuming slow switching in this case leads to converging state estimates.

In our work, we start off with linear switched systems that involve state jumps. We assume the switching signal to be known and fixed so that the execution of a switched system can be seen as that of a time-varying system. For this time-varying system, we determine a necessary and sufficient condition for recovering the initial state. Recovering the initial state also determines the value of the entire state trajectory (in theory). The characterization thus obtained depends on the switching times. As a corollary, we derive a necessary condition and a sufficient condition that only depend on the mode sequence and not the switching instants.

In presence of non-invertible jump maps, our knowledge about the state vector essentially collapses to a subspace in the ambient space after switching, so that the information required to compute the state is less than it was before the switching. Recovering the state at some time instant using the knowledge of past outputs is called determinability or reconstructability. This concept may be essential for several practical considerations where the knowledge of state trajectory for all times is not required and it suffices to recover the state from the current time onwards. Based on the characterization of determinability, an observer is constructed which generates state estimates that converge asymptotically to the actual state of the system. This unified approach of constructing a dynamic observer based on the geometric conditions for observability differentiates our work from the existing literature on similar topics.

Continuing with observability, we then move on to nonlinear switched systems. The tool set adopted to solve the linear case is not so easy to generalize for the nonlinear case; however, at a conceptual level we will try to extend the same idea. Our first step is to use the tools from differential geometry to derive a sufficient condition for observability of switched systems that involve state jumps and comprise nonlinear dynamical subsystems affine in control. Without assuming observability of individual modes, the sufficient condition is based on gathering partial information from each mode so that the state is recovered completely after some time. Based on the sufficient condition, an observer is designed which employs a novel 'back-and-forth' technique to generate state estimates. Under the assumption of persistent switching, analysis shows that the estimate converges asymptotically to the actual state of the system. In order to develop results parallel to linear systems, an attempt is made to obtain a characterization for observability of switched systems in the form of a necessary and sufficient condition. The result is presented in the form of a conjecture for a simpler class of subsystems and applied to several examples.

Finally, we deal with another class of switched systems where the dynamical subsystems are modeled as *differential-algebraic equations* (DAEs). DAEs are an important class of mathematical models used to describe several mechanical and electrical systems. Because of their rich solution framework, they need to be treated separately. We study observability of switched linear DAEs, which has not been explored much in the literature. To highlight the differences with the switched ODEs, we first study the simplest case of two modes where the switching signal only involves a single transition. Similar to the ODEs, characterization for the general case with multiple switches is developed using this basic approach. However, observer construction for such systems is still a topic of ongoing work.

1.3 Organization of the Thesis

The remainder of the thesis is structured as follows:

Chapter 2 addresses the invertibility problem for switched nonlinear systems affine in controls. The problem is concerned with reconstructing the input and switching signal uniquely from given output and initial state. We extend the concept of switch-singular pairs, introduced recently, to nonlinear systems and develop a formula for checking if the given state and output form a switch-singular pair. A necessary and sufficient condition for the invertibility of switched nonlinear systems is given, which requires the invertibility of individual subsystems and the nonexistence of switch-singular pairs. When all the subsystems are invertible, we present an algorithm for finding switching signals and inputs that generate a given output in a finite interval when there is a finite number of such switching signals and inputs. Detailed examples are included to illustrate these newly developed concepts.

Chapter 3 deals with the problem of robust invertibility and bounded output generation and tracking. To address practical concerns, we develop the framework of reconstructing the switching signal in the presence of disturbances in the output and uncertainty in the initial condition. Using the notion of distance between subspaces, we give an upper bound on the time interval over which the switching signal can be recovered exactly. In case the underlying subsystems are minimum-phase, that interval can be extended all the way to infinity under certain dwell-time assumption.

Chapter 4 is concerned with the observability of switched linear systems. Necessary and sufficient conditions for observability and determinability are presented which lead to the construction of an observer. Analyses show that the state estimates converge to the actual state.

Chapter 5 is about observability in nonlinear switched systems. We use the differential geometric approach to propose a sufficient condition for observability which leads to the design of an observer for state estimation. The analyses prove the convergence property of the observer and the results are demonstrated with the help of examples.

Chapter 6 discusses the observability of switched linear DAEs. For clarity of presentation, the simplest case of two subsystems with a single mode transition is considered which highlights the differences that arise in obtaining the characterization of observability for this particular class of systems. We then extend the basic results to the general class of switched DAEs with multiple switchings and subsystems.

Chapter 7 presents the applications of system inversion and observability in electrical energy systems. An invertibility-based approach is adopted for detection of soft and hard faults; and to overcome the limitations of invertibility (need of initial condition and output derivatives), an observer-based strategy is proposed for detection of soft faults. We tailor the algorithms to address the particular structure of power-electronic circuits. Several case studies of DC-DC converters are included along with simulation results.

Chapter 8 summarizes the entire thesis and some design problems are presented as a research direction for future work. The problem of generating a prescribed output with bounded inputs is studied. For the related problem of bounded output tracking, problem statements are proposed as an initial stepping stone.

Chapter 2

Invertibility of Switched Nonlinear Systems

2.1 Introduction

System invertibility problems are of great importance from theoretical and practical viewpoints and have been studied extensively for fifty years, after being pioneered by Brockett and Mesarovic [74]. For nonswitched linear systems, the algebraic criterion for invertibility and the construction of inverse systems were given by Silverman [75], and also by Sain and Massey [76]. The systematic study of invertibility for nonswitched nonlinear systems began with Hirschorn, who first studied the single-input single-output (SISO) case [77] and then generalized Silverman's structure algorithm to multiple-input multiple-output (MIMO) nonlinear systems [78]. Singh [79] then modified the algorithm to cover a larger class of systems. Isidori and Moog [80] used this algorithm to calculate zero-output constrained dynamics and reduced inverse system dynamics. The algorithm is also closely related to the dynamic extension algorithm used to solve the dynamic state feedback input-output decoupling problem [63, Sections 8.2 and 11.3]. Geometric methods have been studied in [81]. A higher-level interpretation given by a linear-algebraic framework, which also establishes links between these algorithms and the geometric approach, is presented in [82]. We also recommend a useful survey on various invertibility techniques by Respondek [83].

The problem of invertibility for switched linear systems was introduced very recently by Vu and Liberzon [65], where the authors used Silverman's structure algorithm to formulate conditions for the invertibility of switched systems with continuous dynamics. The problem of invertibility for discrete-time switched linear systems has been discussed in [69, 70], but the authors assume that the switching sequence is known and find the corresponding input. In this chapter, we make no such assumption and adopt an approach similar to [65], to study the invertibility problem for continuous-time switched nonlinear systems, affine in controls,[1]

[1] A related problem is discussed in [68] but it does not follow the same theoretical approach we do, and instead uses a heuristic approach with the purpose of studying a specific application.

using Singh's nonlinear structure algorithm. The concept of singular pairs, conceived in [65], is extended to nonlinear systems; however, in the thesis, such pairs are termed as "switch-singular pairs" to avoid conflict with the singularities of individual nonlinear subsystems. The main contribution of our work lies in the technical details of developing and checking the conditions for invertibility of nonlinear systems. In particular, the use of nonlinear structure algorithm, possibility of finite escape times, and the existence of singularities in state space and output set require more careful analysis and technical rigor as compared to the linear case.

As is the case in the classical setting of nonswitched systems, we start with an output and an initial state, but here there is a set of dynamic models and we wish to recover the switching signal in addition to the input. In the context of hybrid systems, recovering the switching signal is equivalent to the mode identification for hybrid systems or the observability of the discrete state variable (location), which has been studied in [84, 52, 53]. Hence, the inversion of switched systems can also be thought of as doing the mode detection and input recovery simultaneously. Consequently, the basic idea for solving the invertibility problem is to first do the mode identification by utilizing the relationship among the outputs and the states of the subsystems, and then use the nonlinear structure algorithm for the corresponding subsystem to recover the input.

For the case when subsystems are linear, Silverman's structure algorithm seems to be the most convenient tool to formulate invertibility conditions, which leads to a simple and elegant rank test for checking the existence of switch-singular pairs, but in nonlinear systems it is hard to achieve such a level of generality. For this reason, we start with the SISO case to highlight the technical difficulties in moving from linear to nonlinear systems. Discussing the SISO case first also helps in understanding the concepts behind the formula derived for verification of switch-singular pairs.

The remainder of this chapter is organized as follows. Section 2.2 contains the definitions of invertibility and the formal problem statement. The main result on left-invertibility is presented in Section 2.3. We then give a characterization of switch-singular pairs and the construction of inverse systems in Section 2.4. An algorithm for output generation is given in Section 2.5 along with an example.

2.2 Preliminaries

In this section, we develop the required notations and provide some background on invertibility of nonswitched nonlinear systems. Based on that, we develop the definition for the invertibility of switched nonlinear systems followed by the formal problem statement to which we seek solution in the chapter.

2.2.1 Nonswitched Nonlinear Systems

The dynamics of a square nonlinear system, affine in controls, are given by

$$\Gamma := \begin{cases} \dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \\ y = h(x), \end{cases} \tag{2.1}$$

where $x \in \mathbb{M}$, an n-dimensional real connected smooth manifold, for example $\mathbb{R}^n$; f, g_i are smooth vector fields on $\mathbb{M}$; and $h : \mathbb{M} \to \mathbb{R}^m$ is a smooth function. Admissible input signals are locally essentially bounded, Lebesgue measurable functions $u : [t_0, \infty) \to \mathbb{R}^m$. If the two inputs differ on a set of measure zero, i.e. $u_1(t) = u_2(t)$ almost everywhere (a.e.), then they are considered to be equal. We use the notation $u_{[t_0,T)}$ to denote the input u over the time interval $[t_0, T)$; and $\Gamma_{x_0}(u)$ denotes the state trajectory generated by (2.1) after applying the input u with initial condition x_0.

We start off by reviewing classical definitions of invertibility for such systems. For that, consider the input-output map $\mathrm{H}_{x_0} : \mathcal{U} \to \mathcal{Y}$ for some input function space $\mathcal{U}$ and the corresponding output function space $\mathcal{Y}$. H_{x_0} maps an input $u(\cdot)$ to the output $y(\cdot)$ generated by the system driven by $u(\cdot)$ with an initial condition x_0. Since the state trajectories of nonlinear systems may exhibit finite escape times, an input $u_{[t_0,\infty)}$ may not have a well defined image in the output space, over the interval $[t_0, \infty)$, under this map. For this reason, we only consider inputs over a finite interval $[t_0, T)$, which is the maximal interval of existence of solution, such that $\mathrm{H}_{x_0}(u_{[t_0,T)}) = y_{[t_0,T)}$ always exists and is well-defined.

Invertibility[2] of the dynamical system (2.1) basically refers to the injectivity of the map H_{x_0}. Before giving a formal definition, let us have a look at an example first.

[2]Throughout the text, *invertibility* refers to the left-invertibility.

Example 2.1. Consider a nonswitched nonlinear system with two inputs and two outputs,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_1 u_1 \\ x_3 u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbb{M} = \mathbb{R}^3.$$

We then have

$$\dot{y}_1 = x_1 u_1, \tag{2.2a}$$
$$\ddot{y}_2 = \frac{x_3 \ddot{y}_1 - \dot{y}_1 \dot{y}_2 + \dot{y}_1 u_2}{x_1}. \tag{2.2b}$$

It follows that u_1 can be recovered uniquely from $\dot{y}_1$ if $x_1 \neq 0$, and u_2 can be recovered uniquely from $\ddot{y}_2$ if $\dot{y}_1 \neq 0$ and $x_1 \neq 0$. The point $x_1 = 0$ and the function $\dot{y}_1 = 0$ are the singularities in the state space and the output space, respectively. Let $\mathbb{M}^\alpha := \{x \in \mathbb{R}^3 \,|\, x_1 \neq 0\}$, $Y^s = \{z \in \mathbb{R}^2 \,|\, z_1 = 0\}$, and $\mathcal{Y}^s := \{y : [t_0, T) \to \mathbb{R}^2 \,|\, \dot{y}(t) \in Y^s$ for almost all $t \in [t_0, t_0 + \delta) \subseteq [t_0, T)$, where $\delta > 0$ is arbitrary$\}$. In words, $\mathcal{Y}^s$ includes those outputs which remain in singular set for some duration of time. The complement of $\mathcal{Y}^s$ is given by $\mathcal{Y}^\alpha := \{y : [t_0, T) \to \mathbb{R}^2 \,|\, \dot{y}(t) \notin Y_1^s$ for almost all $t \in [t_0, t_0 + \varepsilon)$ and some $\varepsilon > 0\}$. If the system is driven by a class of inputs u such that the resulting motion $\Gamma_{x_0}(u) \in \mathbb{M}^\alpha$ a.e. and $\mathrm{H}_{x_0}(u) \in \mathcal{Y}^\alpha$, then there is a one-to-one relation between the output and input signals provided their domains are restricted to $[t_0, t_0 + \varepsilon)$. In summary, the input can be recovered uniquely using the knowledge of output, its derivatives and possibly some states as long as the output and state trajectories do not hit some singularities. ◁

We now proceed to the formal definition of invertibility for nonswitched systems.

Definition 2.2. *Fix an output set $\mathcal{Y}$ and consider an arbitrary interval $[t_0, T)$. The system (2.1) is* invertible at a point $x_0 := x(t_0) \in \mathbb{M}$ *over $\mathcal{Y}$ if for every $y_{[t_0,T)} \in \mathcal{Y}$, the equality $\mathrm{H}_{x_0}(u_{1[t_0,T)}) = \mathrm{H}_{x_0}(u_{2[t_0,T)}) = y_{[t_0,T)}$ implies that $\exists\ \varepsilon > 0$ such that $u_{1[t_0,t_0+\varepsilon)} = u_{2[t_0,t_0+\varepsilon)}$. The system is* strongly invertible at a point x_0 *if it is invertible for each $x \in N(x_0)$, where N is some open neighborhood of x_0. The system is* strongly invertible *if there exists an open and dense submanifold $\mathbb{M}^\alpha$ such that $\forall x_0 \in \mathbb{M}^\alpha$, the system is strongly invertible at x_0.* ◁

As illustrated in Example 2.1, a system is invertible at x_0 for the class of inputs $u(\cdot)$ such that along the trajectory of the system (2.1), the resulting motion $x(\cdot), y(\cdot)$ does not hit any singularities. It is entirely possible that the state trajectory or the output hits singularity at a time instant $t_0 + \varepsilon$ with $0 < \varepsilon < T - t_0$, thus making it impossible to recover u uniquely beyond $t_0+\varepsilon$; this explains why we require distinct inputs over arbitrarily small time domains in Definition 2.2.

In the most general construction of inverse systems like the one given by [79], there exists a set of *singular outputs* $\mathcal{Y}^s$ such that the system is not invertible for $y \in \mathcal{Y}^s$; and its complement $\mathcal{Y}^\alpha := \mathcal{Y} \backslash \mathcal{Y}^s$ is the set of all outputs on which the system is strongly invertible. Also, in general, the inverses of nonlinear dynamical systems are not defined on the entire state space. If the vector fields $f(x)$, $g(x)$ and the output function $h(x)$ are analytic, then the singular points are reduced to a closed and nowhere dense set comprising zeros of certain analytic functions. Under these assumptions, if the system is invertible then there exists an open and dense subset of $\mathbb{M}$ on which the dynamics of a nonlinear system are invertible; that subset is called the *inverse submanifold* and is denoted by $\mathbb{M}^\alpha$. All these notions will be developed formally in Section 2.4.

Using Definition 2.2, invertibility at x_0 is equivalent to saying that $u_{1[t_0,t_0+\varepsilon)} \neq u_{2[t_0,t_0+\varepsilon)}$ for all $\varepsilon \in (0, T-t_0)$ implies that $\mathrm{H}_{x_0}(u_{1[t_0,T)}) \neq \mathrm{H}_{x_0}(u_{2[t_0,T)})$. This notion was captured by [78]. Our definition is essentially the same as one considered by Hirschorn in the sense that both notions address the injectivity of an input-output map. The difference lies in the fact that Hirschorn considered a class of *analytic* nonlinear systems with analytic inputs and $\mathcal{Y}^s = \emptyset$, an empty set. In that case, if the system is invertible and the state trajectory starts from a nonsingular set, it is possible to recover inputs on a small interval, but because of analyticity, we continue to recover inputs uniquely even after hitting singularity. For if two analytic inputs are different on a subinterval then they are different everywhere; otherwise, their difference (an analytic function) would have an infinite number of zeros on a finite interval. In our work though, we consider non-analytic systems driven by inputs that are not necessarily analytic, so the input recovery can be guaranteed over small time intervals only.

We will now generalize this notion of local invertibility to the switched systems.

2.2.2 Switched Nonlinear Systems

In this chapter, we will consider switched nonlinear systems, affine in controls, that have the following structure:

$$\Gamma_\sigma : \begin{cases} \dot{x} = f_\sigma(x) + G_\sigma(x)u = f_\sigma(x) + \sum_{i=1}^m (g_i)_\sigma(x)u_i, \\ y = h_\sigma(x), \end{cases} \tag{2.3}$$

where $\sigma : [t_0, T) \to \mathcal{P}$ is the switching signal that indicates the active subsystem at every time, $\mathcal{P}$ is some finite index set, and f_p, G_p, h_p, where $p \in \mathcal{P}$, define the dynamics of individual subsystems. The state space $\mathbb{M}$ is a connected real smooth manifold of dimension n, for example $\mathbb{R}^n$; f_p, $(g_i)_p$ are real smooth vector fields on $\mathbb{M}$; and $h_p : \mathbb{M} \to \mathbb{R}^m$ is a smooth function. A switching signal is a piecewise constant and everywhere right-continuous function that has a finite number of discontinuities at t_i, which we call *switching times*, on every bounded time interval. Denote by $\sigma^p_{[t_0,T)}$ the constant switching signal over the interval $[t_0, T)$ such that $\sigma^p(t) := p \in \mathcal{P}$, $\forall\, t \in [t_0, T)$. We assume that all the subsystems are equidimensional, they live in the same state space $\mathbb{M}$, and the state jump at the switching times is disregarded for the time being. For any initial state x_0, switching signal $\sigma(\cdot)$, and any admissible input $u(\cdot)$, a solution of (2.3) always exists (in Carathéodory sense) and is unique, provided the flow of every subsystem is well-defined for the time interval during which it is active; i.e., the state trajectories do not blow up in finite time. In fact, this assumption results in absolutely continuous state trajectories [14]. Denote by $[t_0, T)$ the maximal interval of existence of solution, so that the outputs are well-defined on $[t_0, T)$. Since the switching signals are right-continuous, the outputs are also right-continuous (note that, in general, $h_i(x) \neq h_j(x)$, for $i \neq j$) and whenever we take the derivative of an output, we assume it is the right derivative. For $p \in \mathcal{P}$, denote by $\Gamma_{p,x_0}(u)$ the trajectory of the corresponding subsystem with the initial state x_0 and the input u, and the corresponding output by $\Gamma^O_{p,x_0}(u)$.

We will use $\mathcal{F}^{pc}$ to denote the space of *piecewise right-continuous functions*[3] and $\mathcal{F}^n$ to denote the subset of $\mathcal{F}^{pc}$ whose elements are n times differentiable between two consecutive discontinuities. Likewise, $\mathcal{F}^{AC}$ denotes the subset of $\mathcal{F}^{pc}$ whose elements are absolutely continuous between two consecutive discontinuities. Finally, we use $\oplus$ for the concatenation of two signals.

In case of switched systems (2.3), the map H_{x_0} has an augmented domain; that is, now we have a (switching signal $\times$ input)-output map $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$, where $\mathcal{S}$ is a switching signal set, $\mathcal{U}$ is the input space, and $\mathcal{Y}$ is the output space. Let us first extend the definition of invertibility to switched systems.

Definition 2.3. *Fix an output set $\mathcal{Y}$ and consider an arbitrary interval $[t_0, T)$. A* switched

[3] By piecewise right-continuous functions, we mean that there is a finite number of jump discontinuities in any finite interval; the function is continuous in between any two consecutive discontinuities; and the function is continuous from the right at discontinuities. To avoid excessive rigidness, we will use the term "piecewise continuous" throughout the paper, and it is understood that "piecewise continuous" means "piecewise right-continuous."

system is invertible at a point x_0 *over* $\mathcal{Y}$ *if for every output* $y_{[t_0,T)} \in \mathcal{Y}$*, the equality* $\mathrm{H}_{x_0}(\sigma_{1[t_0,T)}, u_{1[t_0,T)}) = \mathrm{H}_{x_0}(\sigma_{2[t_0,T)}, u_{2[t_0,T)}) = y_{[t_0,T)}$ *implies that* $\exists\, \varepsilon > 0$ *such that* $\sigma_{1[t_0,t_0+\varepsilon)} = \sigma_{2[t_0,t_0+\varepsilon)}$ *and* $u_{1[t_0,t_0+\varepsilon)} = u_{2[t_0,t_0+\varepsilon)}$. *A* switched system is strongly invertible at a point x_0 *if it is invertible at each* $x \in N(x_0)$*, where* N *is some open neighborhood of* x_0. *A* switched system is strongly invertible *if there exists an open and dense submanifold* $\mathbb{M}^\alpha$ *of* $\mathbb{M}$ *such that* $\forall x_0 \in \mathbb{M}^\alpha$*, the system is strongly invertible at* x_0. ◁

For linear switched systems, as discussed by [65], all the notions in Definition 2.3 coincide and a system is termed invertible if the input and switching signal could be recovered uniquely for all x_0.

The invertibility property formulated in Definition 2.3 may fail to hold in two ways: (a) either because there exist two different inputs u_1 and u_2 that yield the same output or (b) because there exist two different switching signals $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ that yield the same output. The first case refers to the notion of classical invertibility as already explained in Definition 2.2 and Section 2.2.1. To address the second possibility, we need the concept of switch-singular pairs which refers to the ability of more than one subsystem to produce a segment of the desired output starting from the same initial condition. The formal definition is given below:

Definition 2.4. *Consider* $x_0 \in \mathbb{M}$ *and* $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$ *on some time interval* $[t_0, T)$*, where* $p, q \in \mathcal{P}$*,* $p \neq q$*. The pair* $(x_0,\, y)$ *is a* switch-singular pair *of the two subsystems* Γ_p*,* Γ_q *if there exist* u_1*,* u_2 *and* $\varepsilon > 0$ *such that* $\Gamma^O_{p,x_0}(u_{1[t_0,t_0+\varepsilon)}) = \Gamma^O_{q,x_0}(u_{2[t_0,t_0+\varepsilon)}) = y_{[t_0,t_0+\varepsilon)}$. ◁

If all subsystems are linear, $x_0 = 0$ and $y \equiv 0$ always form a switch-singular pair regardless of the dynamics of the subsystems. This is because $u \equiv 0$ and any switching signal will produce $y \equiv 0$, that is, $\mathrm{H}_0(\sigma, 0) = 0 \;\forall\, \sigma$, and therefore H_0 is not injective if the zero function belongs to $\mathcal{Y}$. In nonlinear systems, this is not the case in general, and all switch-singular pairs are solely determined by the subsystem dynamics. As stated earlier and will be formally proved below, the switched system is not invertible if $\mathcal{Y}$ contains outputs that form switch-singular pairs with x_0. Thus, if there exist any switch-singular pairs, we have to restrict the output set $\mathcal{Y}$, instead of letting $\mathcal{Y}$ be the set of all possible concatenations of nonsingular output trajectories.

Next, we use the concept of switch-singular pairs to study the invertibility problem of switched systems. Since Definition 2.3 contains different variants of invertibility, we start off with the weakest of them all, i.e., invertibility of a switched system at a point. In particular, we are interested in solving the following fundamental problem: *Find a suitable set* $\mathcal{Y}$ *and* *a condition on the subsystems such that the system is invertible at x_0 over $\mathcal{Y}$.* An abstract characterization of the set $\mathcal{Y}$ and constraints on subsystem dynamics which guarantee invertibility are given in Section 2.3 under Theorem 2.5; Corollary 2.6 and Corollary 2.7 then characterize the set $\mathcal{Y}$ more explicitly (depending on the required variant of invertibility). Later in Section 2.4, we give mathematical formulae (Lemma 2.11 through Lemma 2.22) for checking the abstract conditions given in Section 2.3.

2.3 Characterization of Invertibility

In this section, we describe the output set $\mathcal{Y}$ used in Definition 2.3 and give conditions on the subsystem dynamics so that the switched system is invertible for some sets $\mathcal{S}$, $\mathcal{U}$, and $\mathcal{Y}$. Restricting the outputs to lie in $\mathcal{Y}$ implies a set of restrictions on the set of allowable inputs, but an explicit characterization of such inputs is not always obtainable. That is why we do not explicitly specify what the input sets $\mathcal{U}$ and $\mathcal{S}$ are, but instead specify the set $\mathcal{Y}$ and then $\mathcal{U}$ will be the corresponding set which, together with $\mathcal{S}$, generates $\mathcal{Y}$.

For all $p \in \mathcal{P}$, let $\mathcal{Y}_p$ be the set of smooth outputs[4] that can be generated by Γ_p, and let $\mathcal{Y}^{all}$ be the set of all the possible concatenations of all elements of $\mathcal{Y}_p$, $\forall p \in \mathcal{P}$. Due to the existence of certain singular outputs (for which the system is not invertible), we seek invertibility at a fixed point x_0 over a subset $\mathcal{Y}^\alpha \subseteq \mathcal{Y}^{all}$.

Theorem 2.5. *Consider the switched system (2.3) and an output set $\mathcal{Y}^\alpha \subseteq \mathcal{Y}^{all}$. The switched system is invertible at $x_0 \in \mathbb{M}$ over $\mathcal{Y}^\alpha$ if and only if each subsystem Γ_p is invertible at x_0 over $\mathcal{Y}^\alpha \cap \mathcal{Y}_p$ and for all $y \in \mathcal{Y}^\alpha$, the pairs (x_0, y) are not switch-singular pairs of Γ_p, Γ_q for all $p \neq q$, $p, q \in \mathcal{P}$.*

Proof. Necessity: We show that if any of the subsystems is not invertible at x_0 or if there exist switch-singular pairs (x_0, y), then the switched system is not invertible.

Suppose that a subsystem Γ_p, $p \in \mathcal{P}$, is not invertible at x_0 over $\mathcal{Y}^\alpha \cap \mathcal{Y}_p$; then there exists $y_{[t_0,T)} \in \mathcal{Y}^\alpha \cap \mathcal{Y}_p$ such that $\Gamma^O_{p,x_0}(u_{1[t_0,T)}) = \Gamma^O_{p,x_0}(u_{2[t_0,T)}) = y_{[t_0,T)}$ for some u_1, u_2 and $\forall \varepsilon \in (0, T - t_0)$, $u_1 \neq u_2$ on $[t_0, t_0 + \varepsilon)$. This implies that $\mathrm{H}_{x_0}(\sigma^p_{[t_0,T)}, u_{1[t_0,T)}) = \mathrm{H}_{x_0}(\sigma^p_{[t_0,T)}, u_{2[t_0,T)}) = y_{[t_0,T)}$ and thus, Definition 2.3 implies that the switched system is not invertible at x_0 over $\mathcal{Y}^\alpha$.

[4] This assumption can be relaxed depending upon the system under consideration; see Remarks 2.13 and 2.19 in Section 2.4 for details.

For necessity of the second condition, suppose that $\exists\, y \in \mathcal{Y}^\alpha \cap \mathcal{Y}_p \cap \mathcal{Y}_q$, so that (x_0, y) is a switch-singular pair of Γ_p, Γ_q, $p \neq q$. This means that both subsystems, even though invertible at x_0, can produce this output over the interval $[t_0, t_0+\varepsilon) \subset [t_0, T)$, $\forall \varepsilon > 0$. Consequently, $\exists\; u_{1[t_0,T)}$, $u_{2[t_0,T)}$ (possibly same) such that $\Gamma^O_{p,x_0}(u_{1[t_0,T)}) = \Gamma^O_{q,x_0}(u_{2[t_0,T)}) = y_{[t_0,T)}$. Hence, we have $\mathrm{H}_{x_0}(\sigma^p_{[t_0,T)}, u_{1[t_0,T)}) = \mathrm{H}_{x_0}(\sigma^q_{[t_0,T)}, u_{2[t_0,T)}) = y_{[t_0,T)}$; that is, the preimage of y is not unique as $\sigma^p \neq \sigma^q$ on $[t_0 + t_0 + \varepsilon)$, $\forall\, \varepsilon \in (0, T - t_0)$. This implies that the switched system is not invertible at x_0 for given $\mathcal{Y}^\alpha$.

Sufficiency: Suppose that for the given $x_0 \in \mathbb{M}$, there exist some inputs u_1, u_2 and switching signals σ_1, σ_2 such that $\mathrm{H}_{x_0}(\sigma_1, u_1) = \mathrm{H}_{x_0}(\sigma_2, u_2) = y \in \mathcal{Y}^\alpha$ over $[t_0, T)$. Since (x_0, y) is not a switch-singular pair, there exists ε_1 such that $\sigma_1(t) = \sigma_2(t) = p$, $\forall\, t \in [t_0, t_0 + \varepsilon_1)$[5] and $y_{[t_0,t_0+\varepsilon_1)} \in \mathcal{Y}_p$. Since Γ_p is invertible at x_0, $\exists\, \varepsilon_2 < \varepsilon_1$ such that $u_{1[t_0,t_0+\varepsilon_2)} = u_{2[t_0,t_0+\varepsilon_2)}$ and $\Gamma^O_{p,x_0}(u_{1[t_0,t_0+\varepsilon_2)}) = \Gamma^O_{p,x_0}(u_{2[t_0,t_0+\varepsilon_2)}) = y_{[t_0,t_0+\varepsilon_2)}$. Letting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, it then follows from Definition 2.3 that the switched system is invertible at x_0 over $\mathcal{Y}^\alpha$. $\square$

In the proof of the sufficiency part, the switched system is strongly invertible at x_0 for the signals whose domain is restricted to the interval $[t_0, t_0 + \varepsilon)$, where $t_0 + \varepsilon$ is the time instant at which the state trajectory or the output enters the singular set. If the output y loses continuity over the interval $[t_0, t_0 + \varepsilon)$ because of switching, then $(\sigma_{[t_0,t_0+\varepsilon)}, u_{[t_0,t_0+\varepsilon)}) = (\sigma_{[t_0,t_1)}, u_{[t_0,t_1)}) \oplus \cdots \oplus (\sigma_{[t_k,t_0+\varepsilon)}, u_{[t_k,t_0+\varepsilon)})$, where k is the total number of switches in the interval $[t_0,\, t_0 + \varepsilon)$ and t_i, $i = 1, \cdots, k$, are the switching instants.

Let us now consider a refinement of Theorem 2.5 by characterizing the set $\mathcal{Y}^\alpha$. For all $p \in \mathcal{P}$, let $\mathcal{Y}^s_p$ be the set of singular outputs of Γ_p for which Γ_p is not invertible (see Example 2.1 and Section 2.4.2, or [79]), and let $\mathcal{Y}^\alpha_p = \mathcal{Y}_p \backslash \mathcal{Y}^s_p$ be the set of outputs on which Γ_p is invertible at x_0. Define $\mathcal{Y}^s := \cup_{p\in\mathcal{P}} \mathcal{Y}^s_p$ as the collection of all singular outputs and let $\mathcal{Y}^{all}$ be the set of outputs generated by all the possible concatenations of all elements of $\mathcal{Y}_p$, $\forall p \in \mathcal{P}$. Finally, define $\overline{\mathcal{Y}}^\alpha := \mathcal{Y}^{all} \backslash \mathcal{Y}^s$ as a set of outputs over which we seek invertibility. We now have the following modified version of Theorem 2.5.

Corollary 2.6. *The switched system is invertible at x_0 over the set $\overline{\mathcal{Y}}^\alpha$ if and only if the pairs (x_0, y) are not switch-singular pairs of Γ_p and Γ_q, for all $y \in \overline{\mathcal{Y}}^\alpha$, for all $p \neq q$, $p, q \in \mathcal{P}$.*

Proof. By the application of Theorem 2.5, the desired result is obtained by showing that Γ_p, $\forall\, p \in \mathcal{P}$, is invertible at x_0 over the set $\overline{\mathcal{Y}}^\alpha \cap \mathcal{Y}_p$. By construction, $\mathcal{Y}_p = \mathcal{Y}^\alpha_p \cup \mathcal{Y}^s_p$ and

[5] This argument can also be proved in another way: it will be shown later that the points in state space that form switch-singular pairs are actually a zero set of smooth nonlinear equations. Thus, if x_0 does not form a switch-singular pair with y then there exists a neighborhood $N(x_0)$ such that $\forall\, x \in N(x_0)$, (x, y) is not a switch-singular pair. As there are no switch-singular pairs in $N(x_0)$, $\exists\; \varepsilon_1 > 0$ such that $\sigma_{1[t_0,t_0+\varepsilon_1)} = \sigma_{2[t_0,t_0+\varepsilon_1)}$.

$\overline{\mathcal{Y}}^{\alpha} \cap \mathcal{Y}_p^s = \emptyset$; using these two equalities, it is easy to see that $\overline{\mathcal{Y}}^{\alpha} \cap \mathcal{Y}_p \subseteq \mathcal{Y}_p^{\alpha}$. As each subsystem Γ_p is invertible at x_0 over $\mathcal{Y}_p^{\alpha}$, it follows that each subsystem Γ_p is, in particular, invertible at x_0 over the output set $\overline{\mathcal{Y}}^{\alpha} \cap \mathcal{Y}_p$. □

Corollary 2.7. *Consider the switched system (2.3) and an output set $\mathcal{Y}^{\alpha} \subseteq \mathcal{Y}$. The switched system is strongly invertible at $x_0 \in \mathbb{M}$ over $\mathcal{Y}^{\alpha}$ if and only if each subsystem Γ_p is strongly invertible at x_0 over $\mathcal{Y}^{\alpha} \cap \mathcal{Y}_p$ and there exists a neighborhood $\overline{N}(x_0)$ such that for all $x \in \overline{N}(x_0)$, $y \in \mathcal{Y}^{\alpha}$, the pairs (x, y) are not switch-singular pairs of Γ_p, Γ_q for all $p \neq q$, $p, q \in \mathcal{P}$.*

Proof. Necessity: If the switched system is strongly invertible at x_0, then $\exists\, N(x_0)$ such that the switched system is invertible at every $x \in N(x_0)$ over $\mathcal{Y}^{\alpha}$. Let $\overline{N}(x_0) := N(x_0)$. By Theorem 2.5, each subsystem is invertible at every $x \in N(x_0)$, hence strongly invertible at x_0, and there exist no switch-singular pairs (x, y), for all $x \in \overline{N}(x_0)$, $y \in \mathcal{Y}^{\alpha}$.

Sufficiency: If each subsystem is strongly invertible at x_0, i.e., $\exists\, N_p(x_0)$ such that Γ_p is invertible at every $x \in N_p(x_0)$, then $N^{\alpha} := \bigcap_{p\in\mathcal{P}} N_p$ is an open set on which all subsystems are invertible. If we define $N := N^{\alpha} \cap \overline{N}$, then the switched system is invertible at every $x \in N(x_0)$ over $\mathcal{Y}^{\alpha}$ and hence by Theorem 2.5, strongly invertible at x_0. □

For the strong invertibility of the switched system on an open and dense subset, assume that the vector fields f_p, $(g_i)_p$ and the output function h_p are analytic. Under these assumptions, if a subsystem Γ_p is strongly invertible, then $\mathbb{M}_p^{\alpha}$ denotes the inverse submanifold of Γ_p.

Corollary 2.8. *The switched system (2.3) is strongly invertible, with inverse submanifold $\mathbb{M}^{\alpha} \subseteq \mathbb{M}$, over an output set $\mathcal{Y}^{\alpha} \subseteq \mathcal{Y}$ if and only if each subsystem is strongly invertible over $\mathcal{Y}^{\alpha} \cap \mathcal{Y}_p$ and the subsystem dynamics are such that the pairs (x_0, y) are not switch-singular pairs of Γ_p, Γ_q for all $p \neq q$, $p, q \in \mathcal{P}$, for every $x_0 \in \mathbb{M}^{\alpha}$, and every $y \in \mathcal{Y}^{\alpha}$.*

Proof. Necessity: If the switched system is strongly invertible, then it is strongly invertible at every $x_0 \in \mathbb{M}^{\alpha}$ over $\mathcal{Y}^{\alpha}$. By Corollary 2.7, each subsystem is strongly invertible at every $x_0 \in \mathbb{M}^{\alpha}$, and hence strongly invertible with inverse submanifold $\mathbb{M}^{\alpha}$. Furthermore, there exist no switch-singular pairs (x_0, y), $\forall\, x_0 \in \mathbb{M}^{\alpha}$, $y \in \mathcal{Y}^{\alpha}$.

Sufficiency: Under the given hypothesis, there exists an inverse submanifold $\mathbb{M}_p^{\alpha}$ such that Γ_p is strongly invertible at every $x_0 \in \mathbb{M}_p^{\alpha}$ over $\mathcal{Y}^{\alpha} \cap \mathcal{Y}_p$, for all $p \in \mathcal{P}$. Define $\mathbb{M}^{\alpha} := \bigcap_{p\in\mathcal{P}} \mathbb{M}_p^{\alpha}$; then $\mathbb{M}^{\alpha}$ is an open and dense subset of $\mathbb{M}$ because it is a finite intersection of open and dense subsets. Under relative topology, $\mathbb{M}^{\alpha}$ is a submanifold. Since each subsystem Γ_p is strongly invertible at every $x_0 \in \mathbb{M}^{\alpha}$ over $\mathcal{Y}^{\alpha} \cap \mathcal{Y}_p$ and there exist no switch-singular pairs, application of Corollary 2.7 implies that the switched system is strongly invertible at every $x_0 \in \mathbb{M}^\alpha$ over $\mathcal{Y}^\alpha$. $\square$

In essence, Theorem 2.5, and the related corollaries state that the invertibility of subsystems in a certain sense implies the invertibility of the switched system in a similar sense provided there are no switch-singular pairs between the states and the outputs considered. Before concluding this section, a couple of remarks are in order.

Remark 2.9. For the switched system (2.3), if all the subsystems are globally invertible in addition to the hypothesis of Corollary 2.8, that is, $\mathbb{M}^\alpha = \mathbb{M}$ and $\mathcal{Y}^s = \emptyset$, then it is possible to recover the inputs and switching signals uniquely over the time interval $[t_0, T)$. Also note that T may be arbitrarily large if the state trajectories do not exhibit finite escape time. $\triangleleft$

Remark 2.10. If a subsystem has more inputs than outputs, then it cannot be (left) invertible. On the other hand, if it has more outputs than inputs, then some outputs are redundant (as far as the task of recovering the input is concerned). Thus, the case of input and output dimensions being equal is, perhaps, the most interesting case. $\triangleleft$

2.4 Checking Invertibility

In this section, we address the computational aspect of the concepts introduced in previous sections and develop algebraic criteria for checking the invertibility of switched systems. The first condition in Theorem 2.5 asks for invertibility of subsystems and is verified by the structure algorithm. To put everything into perspective, we provide appropriate background related to the invertibility of nonswitched systems and use it to develop the concept of functional reproducibility. To check if (x_0, y) is a switch-singular pair, we develop a formula using the functional reproducibility criteria of nonswitched systems. After verifying the invertibility of subsystems and nonexistence of switch-singular pairs, we will be able to construct a switched inverse system that recovers the original input and switching signal uniquely.

2.4.1 Single-Input Single-Output (SISO) Systems

We start off with the case when all the subsystems are SISO because it gives more insight into computations and helps understand the concepts which we will later generalize to multivariable systems. To this end, consider a SISO nonlinear system affine in controls

(2.1) with $m = 1$ and assume it has a relative degree r at x_0 [18], i.e., $\exists$ a neighborhood $N(x_0)$ such that $L_g L_f^k h(x) = 0, \forall\, x \in N(x_0), k = 0, \cdots, r-1$ and $L_g L_f^{r-1} h(x_0) \neq 0$, where $L_f^k h(x) = \frac{\partial (L_f^{k-1} h(x))}{\partial x} f(x)$ and $L_f^0 h(x) = h(x)$.

To check if the subsystem is invertible or not, following [77], we first derive an explicit expression for the input u in terms of the output y by computing the derivatives of y as follows:

$$
\begin{aligned}
y(t) &= h(x(t)), && \text{(2.4a)}\\
\dot{y}(t) &= L_f h(x(t)), && \text{(2.4b)}\\
&\vdots \\
y^{(r)}(t) &= L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t)) u(t). && \text{(2.4c)}
\end{aligned}
$$

From the last equation, we can derive an expression for $u(t)$:

$$
u(t) = -\frac{L_f^r h(x(t))}{L_g L_f^{r-1} h(x(t))} + \frac{1}{L_g L_f^{r-1} h(x(t))} y^{(r)}(t). \tag{2.5}
$$

Hence, u can be determined explicitly in terms of the measured output y, and state x. On substituting the expression for u from (2.5) in equation (2.1), one gets the dynamics for the inverse system:

$$
\begin{aligned}
\dot{z} &= f(z) + g(z)\Big(-\frac{L_f^r h(z)}{L_g L_f^{r-1} h(z)} + \frac{1}{L_g L_f^{r-1} h(z)} y^{(r)} \Big),\\
u &= -\frac{L_f^r h(z)}{L_g L_f^{r-1} h(z)} + \frac{1}{L_g L_f^{r-1} h(z)} y^{(r)}.
\end{aligned} \tag{2.6}
$$

The dynamics of this inverse subsystem evolve on the set $\mathbb{M}^\alpha := \{z \in \mathbb{M} \mid L_g L_f^{r-1} h(z) \neq 0\}$. $\mathbb{M}^\alpha$ is open and dense if f, g, h are analytic. Since the inverse system dynamics are driven by $y^{(r)}(\cdot)$ which satisfies equation (2.4c), it is not hard to see that the state trajectories of the inverse system satisfy the differential equation of the original system (2.1) where the input has just been replaced by a function of y. So if the inverse system is initialized with the same initial condition as that of the plant, then both of the systems follow exactly the same trajectory. This discussion motivates the following result:

Lemma 2.11. *A SISO system is strongly invertible at x_0 if the system has a finite relative degree r at x_0.* $\triangleleft$

Remark 2.12. The condition given in Lemma 2.11 for strong invertibility at a point x_0 is only sufficient, and not necessary. As an example, consider $\dot{x} = 1 + xu$, $y = x$, $x \in \mathbb{R}$, $x_0 = 0$; there is no relative degree at x_0, but the system is strongly invertible at x_0 because the trajectory immediately leaves the singularity. In general, this occurs when the first function of the sequence $L_g h(x), L_g L_f h(x), \cdots, L_g L_f^k h(x)$ which is not identically zero (in a neighborhood of x_0) has a zero exactly at the point $x = x_0$. A result somewhat similar to Lemma 2.11 appears in [77, Theorem 2.1], where the author gives a necessary and sufficient condition for strong invertibility of a SISO system but considers only analytic systems with a slightly different notion of relative degree. $\lhd$

Remark 2.13. For SISO systems, the input u appears in the r-th derivative of the output (2.4). Thus the differentiability/smoothness of u will not affect the existence of first $r-1$ derivatives of y. If $u : [t_0, T) \to \mathbb{R}$ is a locally essentially bounded, Lebesgue measurable function, then $y^{(r)}(\cdot)$ exists almost everywhere and $y^{(r-1)}(\cdot)$ is absolutely continuous [14]. So for SISO nonlinear nonswitched systems, $\mathcal{U}$ is defined as the space of functions which are locally essentially bounded and Lebesgue measurable, and $\mathcal{Y}^\alpha$ is the set of corresponding outputs. $\lhd$

We now turn to the concept of functional reproducibility, which in broad terms means the ability to follow a given reference signal. This concept will help us study the existence of switch-singular pairs. We look at the conditions under which a system can produce the desired output y_d over some interval $[t_0, T)$ starting from a particular initial state x_0. To be precise, given the system (2.1) with $m = 1$ and initial state x_0, we want to find out if there exists a control u such that $\Gamma^O_{x_0}(u) = y_d$. The following result was given by [77]:

Lemma 2.14. *If the system (2.1), with $m = 1$ and $x(t_0) = x_0$, has a relative degree $r < \infty$ at x_0, then there exists a control input u such that $\Gamma^O_{x_0}(u) = y_d$ if and only if*

$$y_d^{(k)}(t_0) = L_f^k h(x_0) \qquad \forall\, k = 0, 1, \cdots, r-1. \tag{2.7}$$

This result is easy to comprehend by looking at the expressions for the output derivatives (2.4). As control $u(t)$ does not directly affect $y^{(k)}(t)$, for $k = 1, \cdots, r-1$, their values at t_0 are determined by the initial state. Substituting

$$u(t) = -\frac{L_f^r h(x(t))}{L_g L_f^{r-1} h(x(t))} + \frac{1}{L_g L_f^{r-1} h(x(t))} y_d^{(r)}(t) \tag{2.8}$$

in (2.4c) gives $y^{(r)}(t) = y_d^{(r)}(t)$. Using (2.7), repeated integration yields $y(t) = y_d(t)$.

We can now easily check for the switch-singular pairs among Γ_p, Γ_q with relative degrees r_p, r_q respectively, where p, $q \in \mathcal{P}$.

Lemma 2.15. *For SISO switched systems, (x_0, y) is a switch-singular pair of two subsystems Γ_p and Γ_q if and only if $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$ and*

$$\begin{pmatrix} y \\ \vdots \\ y^{(r_\kappa - 1)} \end{pmatrix} (t_0) = \begin{pmatrix} h_\kappa(x_0) \\ \vdots \\ L_{f_\kappa}^{r_\kappa - 1} h_\kappa(x_0) \end{pmatrix}, \tag{2.9}$$

where $\kappa = p, q$. ◁

The example below illustrates the use of these concepts.

Example 2.16. Consider a SISO switched system with two modes

$$\Gamma_p := \begin{cases} \dot{x} = \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_1 x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ x_2 \end{pmatrix} u, \quad \mathbb{M} = \mathbb{R}^3, \\ y = x_1, \end{cases}$$

$$\Gamma_q := \begin{cases} \dot{x} = \begin{pmatrix} x_2 \\ x_2 x_3 \\ -x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ x_2 \end{pmatrix} u, \quad \mathbb{M} = \mathbb{R}^3. \\ y = 2x_1, \end{cases}$$

If Γ_p is active, then $\dot{y} = x_1 + x_2$; if Γ_q is active, then $\dot{y} = 2x_2$. Both subsystems have relative degree 2 on $\mathbb{R}^3$ which can be verified by taking the second derivative of the output. If there exists $x \in \mathbb{R}^3$ which forms a switch-singular pair with $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$, then the following equality must be satisfied:

$$\begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix},$$

which gives $x_1 = x_2 = 0$. This state constraint yields $y = \dot{y} = 0$. If we let $\overline{\mathcal{Y}}^\alpha := \Big\{ y : [t_0, T) \to \mathbb{R} \,|\, \dot{y}_{[t_0,T)} \in \mathcal{F}^{AC} \text{ and } \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} \neq 0 \text{ for almost all } t \in [t_0, T) \Big\}$, then there exists no switch-singular pair between $x_0 \in \mathbb{R}^3$ and $y \in \overline{\mathcal{Y}}^\alpha$. Theorem 2.5 and Lemma 2.11 infer that the switched system generated by $\{\Gamma_p, \Gamma_q\}$ is strongly invertible with inverse submanifold $\mathbb{R}^3$

over $\overline{\mathcal{Y}}^{\alpha}$. Alternatively, if $x_0 \neq 0$ then (x_0, y) is not a switch-singular pair for any y and the switched system is strongly invertible with inverse submanifold $\mathbb{R}^3 \backslash \{0\}$ over $\mathcal{Y}^{all}$. $\triangleleft$

For general switched nonlinear systems, it is hard to check for the existence of switch-singular pairs. To see this, consider the system (2.3) with $m = 1$. For simplicity, assume that $\mathcal{P} = \{p, q\}$ and the subsystems Γ_p, Γ_q have equal relative degrees, i.e., $r_p = r_q =: r$. Lemma 2.15 states that Γ_p, Γ_q have a switch-singular pair (x_0, y) if and only if

$$\hat{y} = \mathcal{H}_p(x_0) = \mathcal{H}_q(x_0), \tag{2.10}$$

where $\hat{y} = (y, \dot{y}, \cdots, y^{(r-1)})^T$ and $\mathcal{H}_\kappa = (h_p, L_{f_p}h_p, \cdots, L_{f_p}^{r-1}h_p)^T$, $\kappa = \{p, q\}$. To see if there exist any switch-singular pairs between two subsystems, one is interested in solving $\mathcal{H}_p(x_0) = \mathcal{H}_q(x_0)$ for x_0; that is, x_0 that forms switch-singular pair actually lies in the solution space of r-nonlinear equations where each equation itself involves functions of an n-dimensional variable x_0. As it is hard to talk to about the solutions of nonlinear equations in general, investigation into more constructive conditions for checking of switch-singular pairs is a topic of ongoing research. Nonetheless, in case of SISO switched bilinear systems, the nonlinear equations in (2.9) become linear and the task of checking the existence of switch-singular pairs between two subsystems is comparatively easier, as illustrated below.

Example 2.17. Consider a switched system with SISO bilinear subsystems, having the dynamics of the form

$$\begin{aligned} \dot{x} &= A_{\sigma(t)}x + B_{\sigma(t)}xu, \\ y &= C_{\sigma(t)}x, \end{aligned} \tag{2.11}$$

where $\sigma(t) = p \in \mathcal{P}$, $x \in \mathbb{R}^n$, $A_p, B_p \in \mathbb{R}^{n\times n}$, $C_p \in \mathbb{R}^{1\times n}$. Also, $u(t),\ y(t) \in \mathbb{R}$.

If some mode $p \in \mathcal{P}$ is active over a time interval, then at any time t in that interval, the expression for the derivatives of output is

$$\begin{aligned} y(t) &= C_p x(t), \\ \dot{y}(t) &= C_p A_p x(t), \\ &\vdots \\ y^{(r_p-1)}(t) &= C_p A_p^{r_p-1} x(t), \\ y^{(r_p)}(t) &= C_p A_p^{r_p} x(t) + C_p A_p^{r_p-1} B_p x u(t), \end{aligned} \tag{2.12}$$

where r_p denotes the relative degree of subsystem p. If we introduce the notations

$$\hat{y}_p(t) := \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(r_p-1)(t)} \end{pmatrix} \qquad \text{and} \qquad Z_p := \begin{pmatrix} C_p \\ C_pA_p \\ \vdots \\ C_pA_p^{r_p-1} \end{pmatrix},$$

then based on the functional reproducibility criteria, an output $y_{[t_0,t_0+\varepsilon)}$ can be produced by a subsystem p if and only if $\hat{y}_p(t_0) = Z_p x(t_0)$. Consequently, if two subsystems p, q can produce a given segment of output on an interval $[t_0, t_0+\varepsilon)$, then we will have

$$\begin{pmatrix} \hat{y}_p(t_0) \\ \hat{y}_q(t_0) \end{pmatrix} = \begin{pmatrix} Z_p \\ Z_q \end{pmatrix} x(t_0)\ . \tag{2.13}$$

This is equivalent to saying that

$$\left[\frac{I_p}{I_q}\right] \hat{y}(t_0) = \left(\frac{Z_p}{Z_q}\right) x(t_0), \tag{2.14}$$

where $\hat{y} := \left(y, \dot{y}, \cdots, y^{(r-1)}\right)^T$, $r := \max\{r_p, r_q\}$, and for $\kappa = \{p, q\}$, I_κ is an $r_\kappa \times r$ matrix whose ij^{th} element is 1 if $i = j$ and 0 otherwise. Thus, the existence of switch-singular pairs in case of SISO bilinear switched systems implies that the intersection of range spaces of $\begin{pmatrix} I_p \\ I_q \end{pmatrix}$ and $\begin{pmatrix} Z_p \\ Z_q \end{pmatrix}$ is not empty. Since $\begin{pmatrix} I_p \\ I_q \end{pmatrix}$ and $\begin{pmatrix} Z_p \\ Z_q \end{pmatrix}$ are both linear operators acting on linear subspaces, the *zero vector* is always in their range space. Thus, an identically zero output always forms a switch singular pair with the kernel of $\begin{pmatrix} Z_p \\ Z_q \end{pmatrix}$; that is, $(\ker \begin{pmatrix} Z_p \\ Z_q \end{pmatrix}, 0)$ forms a switch-singular pair for such systems. That is the trivial case; for the nontrivial case we check if $\begin{pmatrix} I_p \\ I_q \end{pmatrix}$ and $\begin{pmatrix} Z_p \\ Z_q \end{pmatrix}$ have a nontrivial common range space. So, if there exists a nonzero output that forms a switch-singular pair with some state at time t, then $\hat{y}(t) \in \text{ range} \begin{pmatrix} I_p \\ I_q \end{pmatrix} \cap \text{ range} \begin{pmatrix} Z_p \\ Z_q \end{pmatrix}$, or equivalently $\text{rank} \begin{bmatrix} I_p & Z_p \\ I_q & Z_q \end{bmatrix} < \text{rank} \begin{bmatrix} I_p \\ I_q \end{bmatrix} + \text{rank} \begin{bmatrix} Z_p \\ Z_q \end{bmatrix}$.

In other words, if all the subsystems in (2.11) are invertible and $r := \max_{p\in\mathcal{P}} r_p < \infty$, then for all $x(t_0) := x_0 \in \mathbb{R}^n$ and $y \in \mathcal{Y}^\alpha := \{y \mid y^{(r-1)} \in \mathcal{F}^{AC}$ and $\hat{y}_{[t_0,t_0+\varepsilon)} \not\equiv 0,$ for some $\varepsilon > 0\}$, the pairs (x_0, y) are not switch-singular pairs of Γ_p, Γ_q, if and only if the following rank condition holds:

$$\operatorname{rank}\begin{bmatrix} I_p & Z_p \\ I_q & Z_q \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I_p \\ I_q \end{bmatrix} + \operatorname{rank}\begin{bmatrix} Z_p \\ Z_q \end{bmatrix}, \tag{2.15}$$

for all $p \neq q$, $p, q \in \mathcal{P}$ such that $\mathcal{Y}_p \cap \mathcal{Y}_q \neq \{0\}$.

This condition is similar to the one given in [65, Lemma 3] for checking the existence of switch-singular pairs in switched linear systems. The common framework in both cases is the appearance of linear equations when taking the derivatives of the outputs, which makes it easier to derive the rank conditions. $\triangleleft$

We now have a toolset to check the invertibility conditions given in Theorem 2.5. If these conditions are satisfied and the switched system is strongly invertible, a switched inverse system can be constructed to recover the input and switching signal σ from given output and initial state. For the switched inverse system, define the *index inversion function* $\overline{\Sigma}^{-1} : \mathbb{M}^\alpha \times \mathcal{Y}^\alpha \to \mathcal{P}$ as:

$$\overline{\Sigma}^{-1}(x_0, y) = p \ : \ y \in \mathcal{Y}_p \text{ and } y^{(k)}(t_0) = L^k_{f_p} h_p(x_0), \tag{2.16}$$

where $k = 0, 1, \cdots, r_p - 1$, t_0 is the initial time of y, and $x_0 = x(t_0)$. The function $\overline{\Sigma}^{-1}$ is well-defined since p is unique by the fact that there are no switch-singular pairs. The existence of p is guaranteed because it is assumed that $y \in \mathcal{Y}^\alpha$ is an output. The dynamics of the inverse switched system Γ^{-1}_σ are:

$$\begin{aligned} \sigma(t) &= \overline{\Sigma}^{-1}(z(t), y_{[t,t+\varepsilon)}), \\ \dot{z} &= f_\sigma(z) + g_\sigma(z)\Big(\frac{y^{(r_\sigma)} - L^{r_\sigma}_{f_\sigma} h_\sigma(z)}{L_{g_\sigma} L^{r_\sigma - 1}_{f_\sigma} h_\sigma(z)}\Big), \\ u(t) &= \frac{y^{(r_\sigma)}(t) - L^{r_\sigma}_{f_\sigma} h_\sigma(z(t))}{L_{g_\sigma} L^{r_\sigma-1}_{f_\sigma} h_\sigma(z(t))}. \end{aligned}$$

with the initial condition $z(t_0) = x_0$. The notation $(\cdot)_\sigma$ denotes the object in the parenthesis calculated for the subsystem with index $\sigma(t)$. The initial condition $\sigma(t_0)$ determines the initial active subsystem at the initial time t_0, from which time onwards, the active subsystem indexes and the input as well as the state are determined uniquely and simultaneously.

2.4.2 Multiple-Input Multiple-Output (MIMO) Systems

For multiple-input multiple-output (MIMO) nonlinear systems affine in controls (2.1), one uses the *structure algorithm* to compute the inverse. When a system is invertible, the structure algorithm, or Singh's inversion algorithm, allows us to express the input as a function of the output, its derivatives and possibly some states.

The Structure Algorithm: This version of the algorithm closely follows the construction given by [82], which is a slightly modified version of the algorithm by [79].

Step 1: Calculate

$$\dot{y} = L_f h(x) + L_G h(x) u = \frac{\partial h}{\partial x}[f(x) + G(x)u],$$

and write it as $\dot{y} =: a_1(x) + b_1(x)u$. Define $s_1 := \operatorname{rank} b_1(x)$, which is the rank of $b_1(x)$ in some neighborhood of x_0, denoted as $N_1(x_0)$. Permute, if necessary, the components of the output so that the first s_1 rows of $b_1(x)$ are linearly dependent. Decompose y as

$$\dot{y} = \begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix} = \begin{pmatrix} \tilde{a}_1(x) + \tilde{b}_1(x)u \\ \hat{a}_1(x) + \hat{b}_1(x)u \end{pmatrix},$$

where $\dot{\tilde{y}}_1$ consists of the first s_1 rows of $\dot{y}$. Since the last $m - s_1$ rows of $b_1(x)$ are linearly dependent upon the first s_1 rows, there exists a matrix $F_1(x)$ such that

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ \dot{\hat{y}}_1 &= \hat{h}^1(x, \dot{\tilde{y}}_1) = \hat{a}_1(x) + F_1(x)(\dot{\tilde{y}}_1 - \tilde{a}_1(x)), \end{aligned} \tag{2.17a}$$

where the last equation is affine in $\dot{\tilde{y}}_1$. Finally, set $\widetilde{B}_1(x) := \tilde{b}_1(x)$.

Step k+1: Suppose that in steps 1 through k, $\dot{\tilde{y}}_1, \cdots, \tilde{y}_k^{(k)}, \hat{y}_k^{(k)}$ have been defined so that

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ &\vdots \\ \tilde{y}_k^{(k)} &= \tilde{a}_k(x, \{\tilde{y}_i^{(j)} | \, 1 \le i \le k-1, i \le j \le k\}) \\ &\quad + \tilde{b}_k(x, \{\tilde{y}_i^{(j)} | \, 1 \le i \le k-1, i \le j \le k-1\})u, \\ \hat{y}_k^{(k)} &= \hat{h}^k(x, \{\tilde{y}_i^{(j)} | \, 1 \le i \le k, i \le j \le k\}), \end{aligned}$$

where all the expressions on the right-hand side are rational functions of $\tilde{y}_i^{(j)}$. Suppose also that the matrix $\widetilde{B}_k := [\tilde{b}_1^T, \ldots, \tilde{b}_k^T]^T$ (vertical stacking of the linearly independent rows obtained at each step) has full rank equal to s_k in $N_k(x_0)$. Then calculate

$$\hat{y}_k^{(k+1)} = \frac{\partial \hat{h}^k}{\partial x}[f(x) + G(x)u] + \sum_{i=1}^{k} \sum_{j=i}^{k} \frac{\partial \hat{h}^k}{\partial \tilde{y}_i^{(j)}} \tilde{y}_i^{(j+1)},$$

and write it as

$$\begin{aligned} \hat{y}_k^{(k+1)} &= a_{k+1}(x, \{\tilde{y}_i^{(j)} \,|\, 1 \le i \le k, i \le j \le k+1\}) \\ &\quad + b_{k+1}(x, \{\tilde{y}_i^{(j)} \,|\, 1 \le i \le k, i \le j \le k\})u. \end{aligned} \tag{2.18}$$

Define $B_{k+1} := [\widetilde{B}_k^T, b_{k+1}^T]^T$, and $s_{k+1} := \operatorname{rank} B_{k+1}$. Permute, if necessary, the components of $\hat{y}_k^{(k+1)}$ so that the first s_{k+1} rows of B_{k+1} are linearly independent. Decompose $\hat{y}_k^{(k+1)}$ as

$$\hat{y}_k^{(k+1)} = \begin{pmatrix} \tilde{y}_{k+1}^{(k+1)} \\ \\ \hat{y}_{k+1}^{(k+1)} \end{pmatrix},$$

where $\tilde{y}_{k+1}^{(k+1)}$ consists of the first $(s_{k+1} - s_k)$ rows. Since the last rows of $B_{k+1}(x, \{\tilde{y}_i^{(j)} \,|\, 1 \le i \le k, i \le j \le k\})$ are linearly dependent on the first s_{k+1} rows, we can write

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ &\vdots \\ \tilde{y}_{k+1}^{(k+1)} &= \tilde{a}_{k+1}(x, \{\tilde{y}_i^{(j)} \,|\, 1 \le i \le k, i \le j \le k+1\}) \\ &\quad + \tilde{b}_{k+1}(x, \{\tilde{y}_i^{(j)} \,|\, 1 \le i \le k, i \le j \le k\})u, \\ \hat{y}_{k+1}^{(k+1)} &= \hat{h}^{k+1}(x, \{\tilde{y}_i^{(j)} \,|\, 1 \le i \le k+1, i \le j \le k+1\}), \end{aligned}$$

where once again everything is rational in $\tilde{y}_i^{(j)}$. Finally, set $\widetilde{B}_{k+1} := [\widetilde{B}_k^T, \tilde{b}_{k+1}^T]^T$, which has full rank equal to s_{k+1} locally.

End of Step $k+1$.

By construction, $s_1 \le s_2 \le \cdots \le m$. If for some integer α we have $s_\alpha = m$, then the algorithm terminates and the system is strongly invertible at x_0. We call α the *relative order*[6] of the system. The closed form expression for u is derived from the α-th step of the

[6] The term was coined by [78] and is weaker than the notion of vector relative degree. Parallel to the terminology used in linear system theory, [85] show that α is the *highest order of zeros at infinity*.

structure algorithm, which gives an invertible matrix $\widetilde{B}_\alpha := [\tilde{b}_1^T, \ldots, \tilde{b}_\alpha^T]^T$ having full rank equal to m in a neighborhood $N_\alpha(x_0) =: N(x_0)$, namely,

$$u = \widetilde{B}_\alpha^{-1} \left[\begin{pmatrix} \dot{\tilde{y}}_1 \\ \vdots \\ \tilde{y}_\alpha^{(\alpha)} \end{pmatrix} - \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_\alpha \end{pmatrix} \right] =: \widetilde{B}_\alpha^{-1}[\tilde{Y}_\alpha - \tilde{A}_\alpha]. \tag{2.19}$$

Note that the entries of the matrix $\widetilde{B}_\alpha$ are rational functions of the derivatives of the output and there may exist an output for which the rank of $\widetilde{B}_\alpha$ drops. We denote by Y^s the values of the output and its derivatives, evaluated at a time instant t, for which the rank of $\widetilde{B}_\alpha(x, y(t))$ is less than m, while $x \in N(x_0)$. We can now formally define the sets $\mathcal{Y}^s$ and $\mathcal{Y}^\alpha$ for a subsystem as follows: $\mathcal{Y}^s := \{y : [t_0, T) \to \mathbb{R}^2 \,|\, y(t) \in Y^s$ for almost all $t \in [t_0, t_0 + \delta) \subseteq [t_0, T),$ where $\delta > 0$ is arbitrary$\}$, and $\mathcal{Y}^\alpha := \{y : [t_0, T) \to \mathbb{R}^2 \,|\, y(t) \notin Y^s$ for almost all $t \in [t_0, t_0 + \varepsilon)$ and some $\varepsilon > 0\}$. In other words, $\mathcal{Y}^s$ includes those outputs for which the matrix $\widetilde{B}_\alpha$ is not invertible and $\mathcal{Y}^\alpha$ is its complement. Hence, we work with u such that $\Gamma^O_{x_0}(u) \notin \mathcal{Y}^s$. Comparing to the SISO case, we had $\widetilde{B}_\alpha = L_g L_f^{r-1} h(x)$ which is a function of the state only and thus there exists no singular output for SISO systems. Another useful class of systems for which $\mathcal{Y}^s = \emptyset$ was discussed by [78]. As was the case in SISO systems, substitution of the expression for u from (2.19) in (2.1) gives the dynamics of the inverse system. These dynamics are defined on the set $\mathbb{M}^\alpha := \{x \in \mathbb{M} \,|\, \operatorname{rank} \widetilde{B}_\alpha(x, y(t)) = m, y(t) \notin Y^s\}$, which is open and dense if $f(x), g(x), h(x)$ are analytic functions.

Example 2.18. As an illustration of the structure algorithm, let us revisit the system defined in Example 2.1. Step 1 of the algorithm yields $\dot{y} = \begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_3 & 0 \end{pmatrix} u$. Using $F_1(x) = x_3/x_1$, we get $\dot{\hat{y}}_1 = \dot{y}_2 = (x_3/x_1)\dot{y}_1$. In Step 2, after differentiating $\dot{\hat{y}}_1 = \dot{y}_2$, we get the following set of equations:

$$\begin{aligned} \dot{\tilde{y}}_1 &= \dot{y}_1 = x_1 u_1, \\ \ddot{\tilde{y}}_2 &= \ddot{y}_2 = \frac{x_3 \ddot{y}_1 - \dot{y}_1 \dot{y}_2 + \dot{y}_1 u_2}{x_1} \Rightarrow \widetilde{B}_2 = \begin{pmatrix} x_1 & 0 \\ 0 & \dot{y}_1/x_1 \end{pmatrix}. \end{aligned}$$

So, $\widetilde{B}_2$ has rank 2, the number of inputs. Hence, $\alpha = 2$; $\mathbb{M}^\alpha = \{x \in \mathbb{R}^3 \,|\, x_1 \neq 0\}$; $Y^s = \{z \in \mathbb{R}^2 \,|\, z_1 = 0\}$, and $\mathcal{Y}^s = \{y : [t_0, T) \to \mathbb{R}^2 \,|\, \dot{y}(t) \in Y^s$ for almost all $t \in [t_0, t_0 + \delta) \subseteq [t_0, T),$ where $\delta > 0$ is arbitrary$\}$. $\triangleleft$

Remark 2.19. Unlike in the SISO case, we need some differentiability assumptions on the input signals to characterize the input space for MIMO systems. In the structure algorithm, Step 1 gives $\dot{\tilde{y}}_1$ that has already u on the right-hand side and the α-th step of the algorithm involves $\{\tilde{y}_i^{(j)} | \, 1 \le i \le \alpha - 1, i \le j \le \alpha\}$. Thus $\tilde{y}_i^{(\alpha-1)}$ must be absolutely continuous so that $\tilde{y}_i^{(\alpha)}$ exists almost everywhere. For the input space, it means that $u^{(\alpha-1)}$ must be Lebesgue measurable and locally essentially bounded. These constraints characterize the input space $\mathcal{U}$ for MIMO case and $\mathcal{Y}$ is the corresponding set of outputs. From the structure algorithm, we deduce that the system is invertible on $\mathcal{Y}^\alpha = \mathcal{Y} \backslash \mathcal{Y}^s$. $\lhd$

Based on the structure algorithm, we now study the conditions for functional reproducibility of multivariable nonlinear systems. Using the notation derived in the structure algorithm, denote by Z the vector

$$Z\Big(x, \dot{\tilde{y}}_1, \cdots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\Big) := \begin{pmatrix} h(x) \\ \hat{h}^1(x, \dot{\tilde{y}}_1) \\ \vdots \\ \hat{h}^{\alpha-1}\Big(x, \dot{\tilde{y}}_1, \cdots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\Big) \end{pmatrix}, \tag{2.20}$$

and let

$$\hat{y} := \begin{pmatrix} y \\ \hat{y}_1 \\ \vdots \\ \hat{y}_{\alpha-1}^{(\alpha-1)} \end{pmatrix}, \qquad \hat{y}_d := \begin{pmatrix} y_d \\ \hat{y}_{d_1} \\ \vdots \\ \hat{y}_{d_{\alpha-1}}^{(\alpha-1)} \end{pmatrix}. \tag{2.21}$$

So Z is basically a concatenation of the expressions appearing at each step of Singh's structure algorithm which get differentiated and $\hat{y}$ is the concatenation of the corresponding expressions on the left-hand side so that

$$Z\Big(x, \dot{\tilde{y}}_1, \cdots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\Big) - \hat{y} = 0.$$

The following result is along the same line as Lemma 2.14 and has appeared in [86, Theorem 1]. However, the proof given here is developed differently than [86].

Lemma 2.20. *If the system given by (2.1), with $x(t_0) = x_0$, has a relative order $\alpha < \infty$, then there exists a control input u such that $\Gamma_{x_0}^O(u) = y_d(\cdot)$ if and only if*

$$\hat{y}_d(t_0) = Z\big(x_0, \dot{\tilde{y}}_{d_1}(t_0), \cdots, \tilde{y}_{d_k}^{(k)}(t_0)\big) \qquad \forall k = 0, 1, \cdots, \alpha - 1. \tag{2.22}$$

Proof. Necessity: Supposing $\exists\, \varepsilon > 0$ and input u defined over the interval $[t_0, t_0 + \varepsilon)$, such that $\Gamma^O_{x_0}(u(t)) = y_d(t)$, $\forall t \in [t_0, t_0 + \varepsilon)$, then

$$\begin{aligned} y_d(t_0) &= y(t_0) = h(x_0), \\ \hat{y}_{d_1}(t_0) &= \hat{y}_1(t_0) = \hat{h}^1(x, \dot{\tilde{y}}_1) = \hat{h}^1(x, \dot{\tilde{y}}_{d_1}), \\ &\vdots \\ \hat{y}^{(\alpha-1)}_{d_{\alpha}-1}(t_0) &= \hat{y}^{(\alpha-1)}_{\alpha-1}(t_0) \\ &= \hat{h}^{\alpha-1}\Big(x_0, \dot{\tilde{y}}_1, \cdots, \tilde{y}^{(\alpha-1)}_1, \cdots, \tilde{y}^{(\alpha-1)}_{\alpha-1}\Big) \\ &= \hat{h}^{\alpha-1}\Big(x_0, \dot{\tilde{y}}_{d_1}, \cdots, \tilde{y}^{(\alpha-1)}_{d_1}, \cdots, \tilde{y}^{(\alpha-1)}_{d_{\alpha-1}}\Big), \end{aligned}$$

and hence equation (2.22) is satisfied.

Sufficiency: If we inject $y_d(t)$ into the inverse system, then the control input produced by this inverse system is given by (2.19) with $\tilde{y}$ replaced by $\tilde{y}_d$, and substituting it in the α-th step of the structure algorithm

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ &\vdots \\ \tilde{y}^{(\alpha)}_{\alpha} &= \tilde{a}_{\alpha}(x, \{\tilde{y}^{(j)}_i \,|\, 1 \le i \le \alpha - 1, i \le j \le \alpha\}) \\ &\quad + \tilde{b}_{\alpha}(x, \{\tilde{y}^{(j)}_i \,|\, 1 \le i \le \alpha - 1, i \le j \le \alpha - 1\})u, \end{aligned}$$

we get

$$\dot{\tilde{y}}_1(t) = \dot{\tilde{y}}_{d_1}(t), \qquad \forall\, t \in [t_0, t_0 + \varepsilon)\ \cdot \tag{2.23}$$

Here $t_0 + \varepsilon$ characterizes the time instant at which the trajectory of the inverse system hits the singular point in the state space. As the system is strongly invertible at x_0, it is guaranteed that $\varepsilon > 0$.

Using hypothesis (2.22), we have $h(x_0) = y_d(t_0)$, and integrating (2.23) on both sides over the interval $[t_0, t_0 + \varepsilon)$ to get

$$\tilde{y}_1(t) = \tilde{y}_{d_1}(t), \qquad \forall\, t \in [t_0, t_0 + \varepsilon)\ \cdot \tag{2.24}$$

Using the initial conditions characterized by (2.22), the desired result can now be derived by induction. Suppose Equations (2.23) and (2.24) are true for index k; that is

$$\tilde{y}_k^{(k)}(t) = \tilde{y}_{d_k}^{(k)}(t) \qquad \forall\, t \in [t_0, t_0 + \varepsilon),$$
$$\tilde{y}_k(t) = \tilde{y}_{d_k}(t) \qquad \forall\, t \in [t_0, t_0 + \varepsilon)\,.$$

Since $\tilde{y}_{k+1}^{(k+1)} = \tilde{a}_{k+1}(x, \{\tilde{y}_i^{(j)} |\, 1 \le i \le k, i \le j \le k+1\}) + \tilde{b}_{k+1}(x, \{\tilde{y}_i^{(j)} |\, 1 \le i \le k, i \le j \le k\})u$, substituting u from (2.19) as generated by the inverse system, with $\tilde{y}$ replaced by $\tilde{y}_d$, gives

$$\tilde{y}_{k+1}^{(k+1)}(t) = \tilde{y}_{d_{k+1}}^{(k+1)}(t) \qquad \forall\, t \in [t_0, t_0 + \varepsilon)\,.$$

Again using hypothesis (2.22) and integrating both sides, we get

$$\tilde{y}_{k+1}(t) = \tilde{y}_{d_{k+1}}(t) \qquad \forall\, t \in [t_0, t_0 + \varepsilon)\,.$$

As $y(t) = (\tilde{y}_1(t), \ldots, \tilde{y}_\alpha(t))$, we get $\Gamma_{x_0}^O(u(t)) = y_d(t)$, $\forall\, t \in [t_0, t_0 + \varepsilon)$. □

Another version of this result in terms of jet spaces is given by [83]. Similarly to the SISO case, the idea is that the portion of the output which is not directly affected by u is determined initially by the value of state variables; and the input u, for which $\Gamma_{x_0}^O(u) = y_d(\cdot)$, is given by (2.19) with y replaced by y_d in that formula.

Example 2.21. Consider the system given in Example 2.1. The vector $\hat{y}$ is the portion of the output that gets differentiated, and therefore

$$\hat{y} = \begin{pmatrix} y_1 \\ y_2 \\ \dot{y}_2 \end{pmatrix} \quad \Rightarrow \quad \hat{y}_d = \begin{pmatrix} y_{d_1} \\ y_{d_2} \\ \dot{y}_{d_2} \end{pmatrix}.$$

The vector $Z(x, y_{d_1}, y_2, \dot{y}_{d_1})$ is given by

$$Z(x, y_{d_1}, y_2, \dot{y}_{d_1}) = \begin{pmatrix} x_1 \\ x_2 \\ \dot{y}_{d_1}(x_3/x_1) \end{pmatrix}.$$

Using Lemma 2.20 and calculations in Example 2.18, if we have

$$\hat{y}_d(t_0) = Z\big((x_0, y_{d_1}(t_0), y_2(t_0), \dot{y}_{d_1}(t_0)\big),$$

then the control which produces y_d as an output, on a small interval, is given by

$$u_1 = \frac{\dot{y}_{d_1}}{x_1},$$
$$u_2 = \frac{x_1\ddot{y}_{d_2} - x_3\ddot{y}_{d_1} + \dot{y}_{d_1}\dot{y}_{d_2}}{\dot{y}_{d_1}}.$$

If $y_d(\cdot) \in \mathcal{Y}^\alpha$ for all times and the corresponding state trajectory $x(\cdot) \in \mathbb{M}^\alpha$, then the system can produce $y_d(\cdot)$ as an output over an arbitrary time interval. $\triangleleft$

Lemma 2.20 gives the following condition for the verification of switch-singular pairs.

Lemma 2.22. *For MIMO switched systems, (x_0, y) is a switch-singular pair of two subsystems Γ_p, Γ_q if and only if $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$ and*

$$\begin{pmatrix} y \\ \dot{\hat{y}}_1 \\ \vdots \\ \hat{y}_{(\alpha_\kappa-1)}^{\alpha_\kappa-1} \end{pmatrix} = \begin{pmatrix} h_\kappa(x_0) \\ \hat{h}_\kappa^1(x_0, \dot{\tilde{y}}_1) \\ \vdots \\ \hat{h}_\kappa^{\alpha_\kappa-1}(x_0, \dot{\tilde{y}}_1, \cdots, \tilde{y}_{\alpha_\kappa-1}^{(\alpha_\kappa-1)}) \end{pmatrix}, \tag{2.25}$$

for $\kappa = p, q$, and α_κ denotes the relative order of subsystem Γ_κ. $\triangleleft$

The procedure for constructing the inverse from this point onwards is exactly the same as discussed earlier for the SISO case with u given by (2.19) instead of (2.5).

Remark 2.23. The results in this section can also be extended to include the case when there are state jumps at switching times. Denote by $\psi_{p,q} : \mathbb{M} \to \mathbb{M}$ the *reset map* when switching from subsystem p to subsystem q, p, $q \in \mathcal{P}$. Thus far, we have considered the case of identity reset maps only, where $\psi_{p,q}(x) = x \;\forall\, p, q \in \mathcal{P}$, $\forall x \in \mathbb{M}$. For nonidentity reset maps, Definition 2.4 is modified to "*(x_0, y) is a switch-singular pair of the two subsystems Γ_p, Γ_q if there exist u_1, u_2 and $\varepsilon > 0$ such that $\Gamma^O_{p,x_0}(u_{1[t_0,t_0+\varepsilon)}) = \Gamma^O_{q,\psi_{p,q}(x_0)}(u_{2[t_0,t_0+\varepsilon)}) = y_{[t_0,t_0+\varepsilon)}$ or $\Gamma^O_{p,\psi_{q,p}(x_0)}(u_{1[t_0,t_0+\varepsilon)}) = \Gamma^O_{q,x_0}(u_{2[t_0,t_0+\varepsilon)}) = y_{[t_0,t_0+\varepsilon)}$.*" Essentially, this means that the output is indistinguishable between the two subsystems, taking into account the effect of state jumps. In case of SISO systems, instead of equation (2.9), we check for switch-singular pairs using

$$\begin{pmatrix} y \\ \vdots \\ y^{(r_\kappa-1)} \end{pmatrix}(t_0) = \begin{pmatrix} h_\kappa(\psi_{p,\kappa}(x_0)) \\ \vdots \\ L_{f_\kappa}^{r_\kappa-1} h_\kappa(\psi_{p,\kappa}(x_0)) \end{pmatrix}, \tag{2.26}$$

or

$$\begin{pmatrix} y \\ \vdots \\ y^{(r_\kappa - 1)} \end{pmatrix}(t_0) = \begin{pmatrix} h_\kappa(\psi_{q,\kappa}(x_0)) \\ \vdots \\ L_{f_\kappa}^{r_\kappa - 1} h_\kappa(\psi_{q,\kappa}(x_0)) \end{pmatrix}, \tag{2.27}$$

where $\kappa = p, q \in \mathcal{P}$ and $\psi_{p,p}(x_0) = \psi_{q,q}(x_0) = x_0$, $\forall\ p, q \in \mathcal{P}$. Equation (2.25) would also be modified similarly when dealing with MIMO systems. The statement of Theorem 2.5, in either case, remains unchanged.

Another generalization is to include switching mechanisms, such as *switching surfaces*. Denote by $S_{p,q}$ the switching surface for subsystem p, where the switched system jumps to subsystem q. Then we only need to check for the switch-singularity of $x_0 \in S_{p,q}$ and $x_0 \in S_{q,p}$ instead of $x_0 \in \mathbb{M}$ for the two subsystems Γ_p, Γ_q. $\triangleleft$

2.5 Output Generation

In the previous section, we considered the question of left-invertibility where the objective was to recover $(\sigma,\ u)$ uniquely for all y in some output set $\mathcal{Y}^\alpha$. In this section, we address a different problem which concerns finding $(\sigma,\ u)$ (that may not be unique) such that $\mathrm{H}_{x_0}(\sigma,\ u) = y_d$ for a given function y_d and a state x_0. For the invertibility problem, we found conditions on the subsystems and the output set $\mathcal{Y}$ so that the map H_{x_0} is injective. Here, we are given one particular $(x_0,\ y_d)$ and wish to find its preimage under the map H_{x_0}. For the switched system (2.3), denote by $\mathrm{H}_{x_0}^{-1}(y_d)$ the preimage of a function y_d,

$$\mathrm{H}_{x_0}^{-1}(y_d) := \{(\sigma,\ u) : \mathrm{H}_{x_0}(\sigma,\ u) = y_d\}. \tag{2.28}$$

If y_d is not in the image set of H_{x_0}, then by convention $\mathrm{H}_{x_0}^{-1} = \emptyset$. When $\mathrm{H}_{x_0}^{-1}(y_d)$ is a singleton, the system is invertible at x_0. We want to find conditions and an algorithm to generate $\mathrm{H}_{x_0}^{-1}(y_d)$ when $\mathrm{H}_{x_0}^{-1}(y_d)$ is a finite set.

We require the individual subsystems to be invertible at x_0 because if this is not the case, then the set $\mathrm{H}_{x_0}^{-1}(y_d)$ may be infinite. When a square nonswitched nonlinear system is not invertible, the matrix $\widetilde{B}_\alpha^{-1}$ in (2.19) is not defined and the expression for u is modified to:

$$u(t) = \widetilde{B}_\alpha^{\dagger}[\tilde{Y}_\alpha - \tilde{A}_\alpha] + K(x, \tilde{Y}_{\alpha-1})v, \tag{2.29}$$

where K is a matrix whose columns form a basis for the null space of $\widetilde{B}_\alpha$ and $\widetilde{B}_\alpha^{\dagger} :=$

$\widetilde{B}_\alpha^T(\widetilde{B}_\alpha\widetilde{B}_\alpha^T)^{-1}$ is a right pseudo-inverse of $\widetilde{B}_\alpha$. If an output is generated by some input u obtained from (2.29) with some initial state, then due to arbitrary choice of v, there always exist infinitely many different inputs that generate the same output with the same initial state. Hence to avoid infinite loop reasoning, we will assume that the individual subsystems Γ_p are invertible at x_0 for all $p \in \mathcal{P}$. However, we do not assume that the switched system is invertible as the subsystems may have switch-singular pairs. We will only consider the functions $y_d(\cdot)$ over finite time intervals so that there is only a finite number of switches under consideration.

A switching inversion algorithm for switched systems, similar to the one given by [65], is now presented in Algorithm 1. The algorithm takes the parameters $x_0 \in \mathbb{M}$, $y_d \in \mathcal{F}^{pc}$ (defined over a finite interval) and returns the set $\mathrm{H}_{x_0}^{-1}(y_d)$. It uses the *index-matching map*[7] $\Sigma^{-1} : \mathbb{M} \times \mathcal{F}^{pc} \to 2^{\mathcal{P}}$ defined as $\Sigma^{-1}(x_0, y_d) := \{p \text{ such that } y_d \in \mathcal{Y}_p^\alpha \text{ and } y_d \text{ satisfies (2.22)}\}$, obtained via the structure algorithm of Γ_p. The index-matching map returns the indexes of the subsystems that are capable of generating y_d starting from x_0. If the returned set is empty, no subsystem is able to generate that y_d starting from x_0. Note that the index-matching map Σ^{-1} is defined for every pair (x_0, y_d) and always returns a set, whereas the index inversion function $\overline{\Sigma}^{-1}$ in (2.16) is defined only for (x_0, y_d) which are not switch-singular pairs and returns an element of $\mathcal{P}$.

In the algorithm, $\Gamma_{p,x_0}^{-1,O}(y_d)$ denotes the output of the inverse subsystem Γ_p^{-1}; the symbol "$\leftarrow$" reads "assigned as", and ":=" reads "defined as". The concatenation of an element η and a set S is $\eta \oplus S := \{\eta \oplus \zeta, \zeta \in S\}$. By convention, $\eta \oplus \emptyset = \emptyset$, $\forall \eta$. Finally, the concatenation of two sets S and T is $S \oplus T := \{\eta \oplus \zeta,\ \eta \in S,\ \zeta \in T\}$.

The return set $\mathcal{A}$ is always finite and, if nonempty, it contains the pairs of switching signals and inputs that generate the given y_d starting from x_0. If the return is an empty set, it means that there is no switching signal and input that generate y_d, or there is an infinite number of possible switching times. Also by our concatenation notation, if at any instant of time, the return of the procedure is an empty set, then that branch of the search will be empty because $\eta \oplus \emptyset = \emptyset$.

Based on the semigroup property for the trajectories of dynamical systems, the algorithm determines the switching signal and the input on a subinterval $[t_0, t)$ of $[t_0, T)$ and then concatenates these signals with the corresponding preimage on the rest of the interval $[t, T)$. If t is the first switching time after t_0, then we can find $H_{x_0}^{-1}(y_{d[t_0,t)})$ by singling out which subsystems are capable of generating $y_{d[t_0,t)}$ using the index-matching map. The obvious

[7]The set $2^{\mathcal{P}}$ denotes the set of all subsets of the set $\mathcal{P}$.

Algorithm 1: Output Generation in Nonlinear Switched Systems

begin $\mathrm{H}_{x_0}^{-1}(y_{d[t_0,T)})$
  Let $\overline{\mathcal{P}} := \{p \in \mathcal{P} : y_{d[t_0,t_0+\varepsilon)} \in \mathcal{Y}_p^\alpha$ and $x_0 \in \mathbb{M}_p^\alpha,\ \varepsilon > 0\}$
  Let $t^* := \min\{t \in [t_0, T) : y_{d[t,t+\varepsilon)} \notin \mathcal{Y}_p^\alpha$ for some $p \in \overline{\mathcal{P}},\ \varepsilon > 0\}$ otherwise $t^* = T$.
  Let $\mathcal{P}^* := \Sigma^{-1}(x_0,\ y_{d[t_0,t_0+\varepsilon)})$.
  if $\mathcal{P}^* \neq \emptyset$ **then**
    Let $\mathcal{A} := \emptyset$
    foreach $p \in \mathcal{P}^*$ **do**
      Let $x := \Gamma_{p,x_0}^{-1}(y_{d[t_0,t^*)})$
      if $x \in \mathbb{M}_p^\alpha$ *and* $y_{d[t_0,t^*)} \in \mathcal{Y}_p^\alpha$ **then**
        Let $u := \Gamma_{p,x_0}^{-1,O}(y_{d[t_0,t^*)})$
        $\mathcal{T} := \{t \in (t_0, t^*) : (x(t), y_d(t))$ is a switch- singular pair of Γ_p, Γ_q for some $q \neq p\}$.
        if $\mathcal{T}$ *is a finite set* **then**
          foreach $t_i \in \mathcal{T}$ **do**
            let $\xi := \Gamma_p(u)(t_i)$
            $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,t_i)},\ u_{[t_0,t_i)}) \oplus \mathrm{H}_\xi^{-1}(y_{d[t_i,T)})\}$
        else if $\mathcal{T} = \emptyset$ *and* $t^* < T$ **then**
          let $\xi := \Gamma_p(u)(t^*)$
          $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,t^*)},\ u) \oplus \mathrm{H}_\xi^{-1}(y_{d[t^*,T)})\}$
        else if $\mathcal{T} = \emptyset$ *and* $t^* = T$ **then**
          $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,T)}, u)\}$
        else
          $\mathcal{A} := \emptyset$
      else
        $\mathcal{A} := \emptyset$
  else
    $\mathcal{A} := \emptyset$
  return $\mathrm{H}_{x_0}^{-1}(y_d) := \mathcal{A}$
end

candidate for first switching time, denoted by t^* in the algorithm, is the time at which the output loses smoothness. Note that in the SISO case, t^* is the time at which one of the first $r-1$ derivatives of the output lose continuity (see Section 2.4.1). But, it is entirely possible that we have a switching at some time instant t_i and the output is still smooth (see Example 2.24). In this case, $(x(t_i), y_{[t_i,t_i+\varepsilon)})$ forms a switch-singular which, in the SISO case, can be checked by using (2.9), or for the systems with reset maps, using (2.26) or (2.27).

The algorithm keeps track of all the switch-singular pairs encountered along the trajectory of the motion and uses a switch at a later time to recover a "hidden switch" earlier (e.g. a switch at which the output is smooth). This makes the switching inversion algorithm a recursive procedure calling itself with different parameters within the main algorithm (e.g. the function $\mathrm{H}_{x_0}^{-1}(y_d)$ uses the returns of $\mathrm{H}_{\xi}^{-1}(y_{d[t^*,T)})$).

The following example should help understand this algorithm.

Example 2.24. Consider a SISO switched system with two modes

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad \mathbb{M} = \mathbb{R}^2, \\ y = x_2, \end{cases}$$

$$\Gamma_2 : \begin{cases} \dot{x} = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} + \begin{pmatrix} e^{x_2} \\ e^{x_2} \end{pmatrix} u, \quad \mathbb{M} = \mathbb{R}^2. \\ y = x_1, \end{cases}$$

We wish to reconstruct the switching signal $\sigma(\cdot)$ and the input $u(\cdot)$ which will generate the following output:

$$y_d(t) = \begin{cases} \cos t & \text{if } t \in [0,\, t^*), \\ 2\cos t & \text{if } t \in [t^*,\, T), \end{cases}$$

where $t^* = \pi$ and $T = 4.5$, with the given initial state $x_0 = (0,\, 1)^T$.

In this example, $(x_0, y_{[t_0,t_0+\varepsilon)})$ form a switch-singular pair if, for some $c \in \mathbb{R}$, $x_0 = \begin{pmatrix} c \\ c \end{pmatrix}$ and $y(t_0) = c$.

We now use the above switching inversion algorithm to find $(\sigma,\, u)$ such that $\Gamma_{x_0,\sigma}^{O}(u) = y_d$. We have $\overline{\mathcal{P}} = \{1, 2\}$ and $\mathcal{P}^* := \Sigma^{-1}(x_0,\, y_{d[0,t^*)}) = \{1\}$ by using the index-matching map with given x_0 and $y_d(0) = 1$. The inverse of Γ_1 on $[0,\, t^*)$ is

$$\Gamma_1^{-1} : \begin{cases} \dot{z} = \begin{pmatrix} z_1 z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{y}_d, \quad \mathbb{M}_1^{\alpha} = \mathbb{R}^2, \\ u(t) = -z_2 + \dot{y}_d, \end{cases}$$

with $z(0) = x_0$, which then gives

$$\begin{aligned} z(t) &= \begin{pmatrix} 0 \\ \cos t \end{pmatrix} =: x(t), \\ u(t) &= -\cos t - \sin t, \end{aligned} \tag{2.30}$$

for $t \in [0, t^*)$. We want to find the set $\mathcal{T} := \{t \leq t^* : (x(t), y_{d[t,t^*)})$ is a switch-singular pair of Γ_1, $\Gamma_2\}$, which is equivalent to solving

$$\cos t = x_1(t) = 0, \quad t \in (0,\, t^*).$$

This equation has a solution $t = \pi/2 =: t_1 < t^*$, and hence $\mathcal{T} = \{t_1\}$, a finite set. We have $\xi = x(t_1) = (0, 0)^T$ and we repeat the procedure for the initial state ξ and the output $y_{d[t_1,T)}$ with $\mathcal{P}^* := \Sigma^{-1}(\xi,\, y_{d[t_1,t^*)}) = \{1, 2\}$. We analyze these two cases:

Case 1: $p = 1$. This implies t_1 is not a switching time, i.e., $\sigma(t) = 1$ for $t \in [t_0, t^*)$ and $u(t)$, $x(t)$ are given by (2.30) for $0 \leq t < t^*$, which gives $\xi = x(t^*) = (0, -1)^T$. At t^*, Γ_2 must be active, but then $y(t^*) = x_1(t^*) = 0 \neq -2 = y_d(t^*)$; thus the index-matching map returns an empty set, $\Sigma^{-1}(\xi, y_{d[t^*,T)}) = \emptyset$.

Case 2: $p = 2$, which means that t_1 is a switching instant. So we work with the inverse system of Γ_2,

$$\Gamma_2^{-1} : \begin{cases} \dot{z} = \begin{pmatrix} 0 \\ z_1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \dot{y}_d, \quad \mathbb{M}_1^\alpha = \mathbb{R}^2, \\ u(t) = e^{-z_2}\, \dot{y}_d, \end{cases}$$

with initial state $z(t_1) = \xi$, which gives

$$\begin{aligned} z(t) &= \begin{pmatrix} \cos t \\ \cos t + \sin t - 1 \end{pmatrix} =: x(t), \\ u(t) &= -e^{(1-\cos t - \sin t)}\, \sin t, \end{aligned}$$

for $t \geq t_1$. We find $\mathcal{T} = \{t_1 < t \leq t^* : (x(t), y_{d[t,t^*)})$ is a switch-singular pair of Γ_1, $\Gamma_2\}$, which is equivalent to solving for

$$\cos t = \cos t + \sin t - 1, \qquad \frac{\pi}{2} = t_1 < t \leq t^* = \pi.$$

It is easy to see that this equation has no solution and thus there exist no switch-singular pairs in the interval $(t_1,\, t^*)$. So, we let $\xi = x(t^*) = (-1,\, -2)^T$ and repeat the procedure with ξ and $y_{d[t^*,T)}$, which gives the unique solution $\sigma_{[t^*,T)} = 1$ and $u_{[t^*,T)} = -2(\cos t + \sin t)$.

Thus, the switching inversion algorithm returns $(\sigma,\, u)$, where

$$(\sigma,\, u) = \begin{cases} (1,\ -\cos t - \sin t), & \text{if } 0 \le t < t_1, \\ (2,\ -e^{(1-\cos t - \sin t)}\ \sin t), & \text{if } t_1 \le t < t^*, \\ (1,\ -2(\cos t + \sin t)), & \text{if } t^* \le t \le T\,. \end{cases}$$

In this example, two switches are required to generate the given output. One of the switching instants is t^* as the output loses smoothness at that instant. The other switching instant is t_1 where the output does not lose smoothness. Without the concept of switch-singular pairs, one may try all the four possible combinations with t^* as the only switching instant and arrive at the false conclusion that there is no switching signal and input that generate $y_d(t)$; but instead the use of the switching inversion algorithm allows us to construct the input and switching signal. $\triangleleft$

Chapter 3

Robust Invertibility of Switched Linear Systems

This chapter takes into account the practical considerations of invertibility algorithms. One of the main drawbacks of the problem of invertibility is that it requires precise knowledge of the output and the initial condition in order to recover the switching signal and the input. Due to physical limitations of the sensors and non-uniform/unpredictable operating conditions of the system, it is often the case that these two quantities (output and the initial condition) are not precisely known. Thus, it is natural to ask the question whether it is possible to recover the unknown switching signal and the input with disturbances in the measurement of the output and imprecise knowledge of the initial state.

Motivated by this practical setup, this chapter will address the effects of uncertainties in output measurements and initial conditions on the recovery of the input and the switching signal. We give conditions under which it is possible to recover the exact switching signal over certain time interval, provided the uncertainties are bounded in some sense. If there are no switch-singular pairs under ideal setup, i.e., it is possible to recover the switching signal with exact measurements and initial condition, then a lower bound on the time interval, over which the switching signal can be recovered in the presence of uncertainties, is provided. In addition, we discuss separately the case where each subsystem is minimum-phase and it is possible to recover the exact switching signal globally in time. The input, though, is recoverable only up to a neighborhood of the original input.

For the problem of recovering the switching signal under uncertainties, we will only consider switched linear systems described as:

$$\Gamma_\sigma : \begin{cases} \dot{x} = A_\sigma x + B_\sigma u, \\ y = C_\sigma x + D_\sigma u. \end{cases} \tag{3.1}$$

For each p in the index set $\mathcal{P}$, $A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, $C_p \in \mathbb{R}^{m\times n}$, $D_p \in \mathbb{R}^{m\times m}$ so that $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$; also, state variable $x \in \mathbb{R}^n$. The input and output dimensions are assumed to be the same so that the system is square. Further, $u \in \mathcal{F}^n$, so that the output

$y \in \mathcal{F}^n$, where $\mathcal{F}^n$ denotes the subset of piecewise right-continuous functions whose elements are n-times continuously differentiable between any two consecutive discontinuities.

Notations: We denote by $\Gamma^O_{p,x_0}(u)$ the output of subsystem p with initial condition x_0 and input u. The p-norm of a vector in Euclidean spaces is denoted by $|\cdot|_p$, $1 \leq p \leq \infty$ and the induced p-norm of a matrix is denoted by $\|\cdot\|_p$. If no subscript appears, we mean the 2-norm. For a signal $y : \mathbb{R} \mapsto \mathbb{R}^m$, we denote by $y_\mathcal{I}$ the restriction of the signal over the interval $\mathcal{I} \subseteq \mathbb{R}$. The notation $\|y_\mathcal{I}\|_p$ denotes the p-th norm of the signal defined as:

$$\|y_\mathcal{I}\|_p := \left(\int_\mathcal{I} |y(s)|^p ds\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$

and $\|y_\mathcal{I}\|_\infty := \operatorname{ess\,sup}_{t\in\mathcal{I}} |y(t)|_\infty$. For two matrices, A_1 and A_2, $\mathrm{col}(A_1, A_2)$ denotes the vertical concatenation of A_1 and A_2, that is, $\mathrm{col}(A_1, A_2) := [A_1^\top, A_2^\top]^\top$, where the vectors in $\mathbb{R}^k$ are seen as matrices of dimension $k \times 1$. For a signal $y : \mathbb{R} \mapsto \mathbb{R}^k$, $Y^k := \mathrm{col}(y, \dot{y}, \cdots, y^{(k)})$, so that Y^k is the vertical concatenation of the output and its derivatives. For a vector z in some Euclidean space and a positive scalar r, the ball of radius r centered at z is defined as $\mathcal{B}_r(z) := \{y : |y - z| \leq r\}$. We use the symbol $\langle z, y\rangle$ to denote the inner product of z and y, whenever y and z are the vectors in the same space.

Problem Setup: With $y : \mathbb{R} \to \mathbb{R}^m$ as the exact output of the system, let $Y^k := \mathrm{col}(y, \dot{y}, \ddot{y}, \cdots, y^{(k)})$, $k \in \mathbb{N}$, denote the vector comprising the exact output and its first k-derivatives. For brevity, $Y := Y^n$. Both y and Y^k are considered to be unknown. Let $\widehat{Y}^k := \mathrm{col}(\hat{y}, \hat{\dot{y}}, \hat{\ddot{y}}, \cdots, \hat{y}^{(k)})$ denote the imprecise estimate of the output and its derivatives obtained by inaccurate measurements and numerical differentiation. Several useful techniques for obtaining the estimates of the derivatives, even for noisy signals, have been discussed in the literature; see [87] and references therein. It is assumed that for each t, the uncertainty in the measurement of the output and its derivatives is bounded by some fixed and known number $\varrho > 0$, that is, $|Y(t) - \widehat{Y}(t)| \leq \varrho$. Also, the exact knowledge of the initial state $x_0 := x(t_0)$ is no longer assumed; instead x_0 is assumed to be contained in a known compact and convex set $\mathcal{R}_{t_0}$, so that[1] $\hat{x}_0 \in \mathcal{R}_{t_0}$ is an initial estimate of x_0. *Our objective is to: (a) find conditions on subsystem dynamics and a deterministic function $\widetilde{\Sigma}^{-1}(\hat{x}_0, \hat{y})$ that reconstructs the original value of σ over some time interval, (b) compute the maximum error between the actual and the reconstructed input, (c) find conditions under which $\widetilde{\Sigma}^{-1}(\hat{x}_0, \hat{y})$ yields the actual value of σ at all times for a particular class of systems.*

[1] In order to reduce uncertainty and for reasons that will become clear later on, $\hat{x}_0$ is chosen as: $\hat{x}_0 = \arg\min_{x\in\mathcal{R}_{t_0}} \left\{\min_{y\in\mathcal{R}_{t_0}} |x - y|\right\}$. Thus, for a spherical or an ellipsoidal $\mathcal{R}_{t_0}$, $\hat{x}(t_0)$ is chosen as the center of that sphere or ellipsoid.

After stating the required background and basic terminologies in Section 3.1, the solutions to problems (a) and (b) appear in Section 3.2 as we derive the analytical bounds on time intervals over which exact recovery of the switching signal is possible under certain conditions. If each subsystem is minimum-phase, then we extend these conditions to recover the switching signal globally for all times in Section 3.3 as a solution to (c).

3.1 Background and Preliminaries

As mentioned at the beginning of Chapter 2, invertibility of a switched system (3.1) requires each subsystem to be invertible and the identification of the active mode [65]. To check the former property, i.e., invertibility of a subsystem, one uses Silverman's structure algorithm [75, 88]; this paper, however, uses the notations developed in a terse version of the structure algorithm given in [65]. If a subsystem is invertible, the structure algorithm leads to the construction of an inverse subsystem that reconstructs the original input using the state and the output values. For mode identification in (3.1), we first develop a relationship between the output and the state for each subsystem and then utilize it to determine the active subsystem at each time instant. This relationship is characterized by the range theorem [88] and the characterization uses certain operators L_p and W_p, which are obtained by applying the structure algorithm to each subsystem Γ_p. The formulae for L_p and W_p in terms of system data have been developed in the Appendix A. The exact expressions of these operators are not required in the understanding of this paper, and we refer the reader to Appendix A if such formulae are sought. The following example helps illustrate how these operators show up in computations:

Example 3.1. Consider a non-switched linear SISO system

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u, \\ y = [0 \ 0.5]x. \end{cases}$$

Clearly, y and $\dot{y}$ are independent of the input; that is, $\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} x$, or equivalently $[I_{2\times2} \ 0_{2\times1}]Y = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} x$, where $Y := \mathrm{col}(y, \dot{y}, \ddot{y})$. So, we let $W_1 := [I_{2\times2} \ 0_{2\times1}]$, and $L_1 := \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ and the relation $W_1 Y(t) = L_1 x(t)$ holds $\forall\, t \geq t_0$. Computing the expression for $\ddot{y}$, solving it for u, and plugging the resultant back into the original dynamics yields the corresponding inverse system,

$$\Gamma_1^{-1}: \begin{cases} \dot{\hat{x}} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \hat{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \hat{\hat{y}}, \\ u = [0 \ -4]\hat{x} + 4\hat{\hat{y}}. \end{cases} \qquad \triangleleft$$

For the sake of clear presentation, we introduce the following assumptions to state a simplified version of the range theorem in Proposition 3.2 which characterizes the relationship between the output and the state for a subsystem Γ_p.

Assumption 3.1. Throughout the paper, it is assumed that:

1. each subsystem Γ_p, $p \in \mathcal{P}$, is invertible;
2. and the inputs are such that the output produced by each subsystem is n-times differentiable (i.e., $\mathcal{C}^n$).

Proposition 3.2. *Consider system* (3.1) *with initial state* x_0*. If Assumption 3.1 holds and* $y \in \mathcal{C}^n([t_0, t_1), \mathbb{R}^m)$*, then there exists an input* u *such that* $y_{[t_0,t_1)} = \Gamma^O_{p,x_0}(u_{[t_0,t_1)})$ *if and only if* $W_pY \in \mathcal{C}^0([t_0, t_1), \mathbb{R}^m)$ *and* $W_pY(t_0) = L_px_0$*.* $\triangleleft$

In other words, when the outputs considered are sufficiently smooth on an interval $[t_0, t_1)$, the condition $W_pY(t_0) = L_px_0$ guarantees that the particular y can be generated by subsystem Γ_p on that interval, starting from the particular initial state x_0 at time t_0. It is also noted from the structure algorithm that regardless of what the input is, the output and the state are related by the equation $W_pY(t) = L_px(t)$, for all $t \geq t_0$ when Γ_p is active, and not just at the initial time t_0. When dealing with the switched systems, this is the fundamental idea employed in mode detection and it also leads to the following concept of switch-singular pairs [65].

Definition 3.3. *Let* $x_0 \in \mathbb{R}^n$ *and* $y \in \mathcal{C}^n$ *be an* $\mathbb{R}^m$*-valued function on some time interval. The pair* $(x_0,\ y)$ *is a* switch-singular pair *of the two subsystems* Γ_p, Γ_q *if there exist* u_1, u_2 *such that* $\Gamma^O_{p,x_0}(u_{1_{[t_0,t_0+\epsilon)}}) = \Gamma^O_{q,x_0}(u_{2_{[t_0,t_0+\epsilon)}}) = y_{[t_0,t_0+\epsilon)}$*, for some* $\epsilon > 0$*.* $\triangleleft$

Essentially, if a state and an output (the time domain can be arbitrary) form a switch-singular pair, then there exist inputs for the two subsystems to produce that same output starting from that same initial state. Under Assumption 3.1, it follows from Definition 3.3

and Proposition 3.2 that (x_0, y) is a switch-singular pair for Γ_p, Γ_q if, and only if,

$$\begin{bmatrix} W_p \\ W_q \end{bmatrix} Y(t_0) = \begin{bmatrix} L_p \\ L_q \end{bmatrix} x_0, \tag{3.2}$$

where $x_0 = x(t_0)$, and t_0 is the initial time of y. This condition for verifying the existence of switch-singular pairs can be checked easily for a certain class of outputs using a rank condition. According to [65, Lemma 3], if y is such that $\left[\begin{smallmatrix} W_p \\ W_q \end{smallmatrix}\right] Y(t) \neq 0$, for any $t \geq t_0$, then there exist no switch-singular pairs (x_0, y) between subsystems Γ_p and Γ_q if, and only if,

$$\operatorname{rank} \begin{bmatrix} L_p & W_p \\ L_q & W_q \end{bmatrix} = \operatorname{rank} \begin{bmatrix} L_p \\ L_q \end{bmatrix} + \operatorname{rank} \begin{bmatrix} W_p \\ W_q \end{bmatrix}. \tag{3.3}$$

If $\mathcal{L}_{pq}$ and $\mathcal{W}_{pq}$ denote the range spaces of the matrices $\left[\begin{smallmatrix} L_p \\ L_q \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} W_p \\ W_q \end{smallmatrix}\right]$ respectively, then geometrically, condition (3.3) is equivalent to saying that $\mathcal{L}_{pq} \cap \mathcal{W}_{pq} = \{0\}$.

Next, let $\overline{\mathcal{Y}}$ be the set of piecewise smooth functions such that if $y \in \overline{\mathcal{Y}}$, then $\left[\begin{smallmatrix} W_p \\ W_q \end{smallmatrix}\right] Y(t_0) \neq 0$ for all $p \neq q$, $p, q \in \mathcal{P}$. It has been shown in [65] that, for the output set $\overline{\mathcal{Y}}$, a switched system is invertible if and only if all subsystems are invertible and subsystem dynamics are such that there exist no switch-singular pairs among them. So, if Assumption 3.1 and equation (3.3) hold, then the switched system is invertible for the output set $\overline{\mathcal{Y}}$.

In case a switched system is invertible, a switched inverse system can be constructed to recover the input and the switching signal σ from the given output and the initial state. Towards that end, define the *index-inversion function* $\overline{\Sigma}^{-1} : \mathbb{R}^n \times \overline{\mathcal{Y}} \to \mathcal{P}$ as:

$$\overline{\Sigma}^{-1}(x_0, y) = \{p : W_p Y(t_0) = L_p x_0\}. \tag{3.4}$$

The function $\overline{\Sigma}^{-1}$ is well-defined since p is unique by the fact that there are no switch-singular pairs. The existence of p is guaranteed because it is assumed that $y \in \overline{\mathcal{Y}}$ is an output. Having determined the mode using (3.4), the corresponding inverse system is activated to recover the input. Thus, an inverse switched system Γ_σ^{-1}, with initial condition x_0, is implemented as follows:

$$\sigma(t) = \overline{\Sigma}^{-1}(x(t), y_{[t,t+\epsilon)}), \tag{3.5a}$$

$$\dot{x} = \widehat{A}_{\sigma(t)} x + \widehat{B}_{\sigma(t)} Y, \tag{3.5b}$$

$$u = \widehat{C}_{\sigma(t)} x + \widehat{D}_{\sigma(t)} Y, \tag{3.5c}$$

where $\widehat{A}_\sigma := (A - B\overline{D}_\alpha^{-1}\overline{C}_\alpha)_\sigma$, $\widehat{B}_\sigma := (B\overline{D}_\alpha^{-1}V)_\sigma$, $\widehat{C}_\sigma := (-\overline{D}_\alpha^{-1}\overline{C}_\alpha)_\sigma$, and $\widehat{D}_\sigma := (\overline{D}_\alpha^{-1}V)_\sigma$. The matrices $\overline{C}_\alpha, \overline{D}_\alpha, V$ are defined for each subsystem through the structure algorithm and their formulae are developed in the Appendix A. The notation $(\cdot)_\sigma$ denotes the object in the parenthesis calculated for the subsystem with index $\sigma(t)$. The initial condition $\sigma(t_0)$ determines the initial active subsystem at the initial time t_0, from which time onwards, the active subsystem indexes and the input as well as the state are determined uniquely and simultaneously.

3.2 Inversion under Uncertainties

In the problem setup, it is assumed that x_0 and Y are unknown, and we work with their respective estimates $\hat{x}_0 \in \mathcal{R}_{t_0}$, and $\widehat{Y}$. So, instead of (3.5), the following equations are utilized to get an estimate of the actual state trajectory and the input appearing in (3.1), which are now denoted by $\hat{x}$ and $\hat{u}$ respectively.

$$\dot{\hat{x}}(t) = \widehat{A}_\sigma \hat{x}(t) + \widehat{B}_\sigma \widehat{Y}(t), \tag{3.6a}$$

$$\hat{u}(t) = \widehat{C}_\sigma \hat{x}(t) + \widehat{D}_\sigma \widehat{Y}(t). \tag{3.6b}$$

In (3.6), the switching signal σ remains unknown and the remainder of this section concentrates on recovering the switching signal using $\mathcal{R}_{t_0}$ and $\widehat{Y}$. In Section 3.2.1, we use the concept of reachable sets to compute σ; this method, although conceptually simple and meaningful, is computationally expensive as it requires propagating reachable sets of the switched system and computing distances among certain polygons at each instant in time. A computationally feasible method is then devised in Section 3.2.2, where we introduce the concept of weak switch-singular pair and, using the idea of minimal gap between subspaces, derive the necessary conditions for existence of weak switch-singular pairs. This leads to simpler computations for reconstructing σ for the case $\mathcal{R}_{t_0} = \mathcal{B}_{\delta_0}(\hat{x}_0)$ for some known δ_0. Section 3.2.3 then quantifies the error between the actual input and its estimate.

3.2.1 Switching Signal Recovery using Reachable Sets

In the presence of uncertainties in the output and the state values, the most natural extension of the mode identification method described in the previous section is to compute the set that contains $x(t)$ and look at the intersection of the image of this set under the map L_p

with $W_p(\mathcal{B}_\varrho(\widehat{Y}(t)))$. The subsystem Γ_p is declared active if the corresponding intersection is non-empty.

Let $\tilde{x}(t) := x(t) - \hat{x}(t)$, then the difference of equations (3.5a) and (3.6a) gives,

$$\dot{\tilde{x}}(t) = \widehat{A}_\sigma \tilde{x}(t) + \widehat{B}_\sigma w(t), \tag{3.7}$$

where $w(t) := Y(t) - \widehat{Y}(t)$, and $|w(t)| \leq \varrho$ for each $t \geq t_0$.

Reachable Sets

The following lemma reveals two important properties of the reachable sets of linear time invariant systems which can be generalized to the switched systems.

Lemma 3.4 (Reachable sets). *Consider the linear time invariant system*

$$\dot{x} = Ax + B\eta(t), \tag{3.8}$$

with initial state x_0, $\eta(t)$ contained in some compact, convex set $\mathcal{U}$ for $t \in [t_0, t_1]$; then the reachable set $\mathcal{R}_{t_1} = \{x(t_1) : x(t)$ solves (3.8) with $\eta(t) \in \mathcal{U}, t \in [t_0, t_1], x(t_0) = x_0\}$ is compact, convex and varies continuously with t_1 on $t_1 \geq t_0$. $\triangleleft$

This result is proved in [89]. Since the computation of reachable sets is well-studied topic, several methods for computing the set $\mathcal{R}_t$ are available in literature [90] and we assume that $\mathcal{R}_t$ can be computed.

The statement of Lemma 3.4 also holds if we replace the initial condition $x(t_0) = x_0$ by the condition $x(t_0) \in \mathcal{R}_{t_0}$ where $\mathcal{R}_{t_0}$ is a compact convex set. To show that $\mathcal{R}_{t_1}$ is compact and varies continuously with t_1, one can essentially follow the same arguments as given in the proof of Theorem 1 in [89, Section 2.2]. The convexity part, though obvious from variations of constants formula, is proved formally in [91].

The last part of the theorem states that the correspondence $t \to \mathcal{R}_t$, for $t > t_0$ is a continuous map of the real half line into the metric space of nonempty compact subsets of $\mathbb{R}^n$. If we introduce the following definitions[2]:

$$\epsilon_1 := \min_{x \in \mathcal{R}_{t_1}} \max_{y \in \mathcal{R}_{t_2}} |x - y| \quad ; \quad \epsilon_2 := \min_{y \in \mathcal{R}_{t_2}} \max_{x \in \mathcal{R}_{t_1}} |y - x|,$$

[2]In [89], ϵ_1 and ϵ_2 appear as the definitions of distances between the reachable sets $\mathcal{R}_{t_1}$ and $\mathcal{R}_{t_2}$. To avoid confusion, these definitions are not termed as distances as we will work with the notion of distance given in (3.10).

then for given $\epsilon > 0$, there exists $\delta > 0$ so that the $\min\{\epsilon_1, \epsilon_2\} < \epsilon$ whenever $|t_1 - t_2| < \delta$.

The result of Lemma 3.4 can be extended to the error dynamics of a switched system given in (3.7) when the solution is in the sense of Carathédory. As $w(t)$ is contained in the ball of radius ϱ around the origin, a compact and convex set, and $\tilde{x}(t_0)$ is contained in[3] $\mathcal{R}_{t_0} - \hat{x}(t_0)$, which is also compact and convex, then so is the reachable set $\mathcal{R}_{t_1} - \hat{x}(t_1)$. An inductive argument and repeated application of Lemma 3.4 suggests that $\mathcal{R}_{t_i} - \hat{x}_{t_i}$ is compact and convex at i-th switching instant and $\mathcal{R}_t - \hat{x}(t)$ is also compact and convex for each $t \in [t_i, t_{i+1}]$, $i \geq 0$. Since $x(t) = e(t) + \hat{x}(t)$, $x(t)$ belongs to the set $\mathcal{R}_t$ containing $\hat{x}(t)$, for each $t \geq t_0$. One can also arrive at similar result using Fillipov's theorem for linear time varying systems.

Note that L_p is a linear and continuous operator and since $\mathcal{R}_t$ is a compact and convex set at each time t, $L_p(\mathcal{R}_t)$ is also compact and convex.

Index Matching Function $\widehat{\Sigma}^{-1}$

To compute the value of the switching signal $\sigma(t)$ using index-inversion function (3.4), we find p for which $W_p Y(t) = L_p x(t)$, or alternatively $|W_p Y(t) - L_p x(t)| = 0$. Since $Y(t)$ and $x(t)$ are no longer available, this condition cannot be verified anymore. A new function, that recovers the value of the switching signal, can be computed using the following lemma.

Proposition 3.5. *For system* (3.1)*, if there exists an input u such that $\Gamma^O_{p,x_0}(u) = y$ over an interval* $[t_0, t_1)$*, and $|Y(t) - \widehat{Y}(t)| \leq \varrho$, then $|L_p x(t) - W_p \widehat{Y}(t)| \leq \|W_p\| \varrho$, for each $t \in [t_0, t_1)$.*

Proof. Since $\Gamma^O_{p,x_0}(u) = y$ over the interval $[t_0, t_1)$ for some input u, it follows from Proposition 3.2 that $L_p x(t) = W_p Y(t)$ for each $t \in [t_0, t_1)$. So that,

$$\begin{aligned} |L_p x(t) - W_p \widehat{Y}(t)| &= |\cancel{L_p x(t)} - \cancel{W_p Y(t)} + W_p Y(t) - W_p \widehat{Y}(t)| \\ &= |W_p (Y(t) - \widehat{Y}(t))| \\ &\leq \|W_p\| \, |Y(t) - \widehat{Y}(t)| \leq \|W_p\| \varrho. \end{aligned}$$

□

Note that, rather than the exact value of $x(t)$, it is only known that $x(t) \in \mathcal{R}_t$. If $\mathcal{Z}_p(t) := L_p(\mathcal{R}_t)$, we define the distance between the set $\mathcal{Z}_p(t)$ and the vector $W_p Y(t)$ to be:

$$d_p(t) = \min_{z \in \mathcal{Z}_p(t)} |W_p \widehat{Y}(t) - z| \cdot \tag{3.10}$$

[3]For a given vector z and a set $\mathcal{R}$, we define $\mathcal{R} - z := \{x \,|\, x = r - z, r \in \mathcal{R}\}$.

The set $\mathcal{Z}_p(t)$ is compact and convex and contains $L_p x(t)$, so there exists a unique solution to this optimization problem and according to the *projection theorem* in [92], there exists a unique $z^* \in \mathcal{Z}_p(t)$ that satisfies $\left\langle W_p \widehat{Y}(t) - z^*, z - z^* \right\rangle \leq 0$, for all $z \in \mathcal{Z}_p(t)$ and $d_p(t) = |W_p \widehat{Y}(t) - z^*|$. Use of Proposition 3.5 guarantees that if Γ_p produces the output at time t, then the distance between the set $\mathcal{Z}_p(t)$ and $W_p \widehat{Y}(t)$ is less than $\|W_p\| \varrho$. This motivates us to introduce the following definition of *index-matching function*:

$$\widehat{\Sigma}^{-1}(\mathcal{R}_t, \hat{y}_{[t,t+\epsilon)}) := \{p \,|\, d_p(t) \leq \|W_p\| \varrho\}. \tag{3.11}$$

Next, in Proposition 3.7, it is shown that the distance function (3.10) is continuous locally in time and that the value of the index-matching function (3.11) coincides with the original switching signal. The proof requires the following lemma.

Lemma 3.6. *Let* $z^*_{p,0} = \arg\min_{z \in \mathcal{Z}_p(t_0)} |Y_0 - z|$, *and* $z^*_{p,s} = \arg\min_{z \in \mathcal{Z}_p(s)} |Y_s - z|$, *where* Y_0 *and* Y_s *are some given fixed vectors. Then, for every* $\epsilon > 0$, *there exists* $\rho > 0$ *such that* $|z^*_{p,s} - z^*_{p,0}| < \epsilon$ *whenever* $|s - t_0| < \rho$, *and* $|Y_s - Y_0| \leq \frac{\epsilon}{2}$.

Proof. Step 1: We show that, for a nonempty, compact, convex set $\mathcal{Z}$, the correspondence

$$Y \mapsto \arg\min_{z \in \mathcal{Z}} |Y - z|$$

is continuous. Let $\overline{z}_i = \arg\min_{z \in \mathcal{Z}} |Y_i - z|$, $i = 0, 1$; then according to the projection theorem [92], the following holds for any $z_0, z_1 \in \mathcal{Z}$:

$$\langle Y_0 - \overline{z}_0, z_0 - \overline{z}_0 \rangle \leq 0, \qquad \langle \overline{z}_1 - Y_1, \overline{z}_1 - z_1 \rangle \leq 0.$$

In particular, let $z_0 = \overline{z}_1$ and $z_1 = \overline{z}_0$; then adding the two inequalities and applying the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} \langle \overline{z}_1 - \overline{z}_0, \overline{z}_1 - \overline{z}_0 \rangle &\leq \langle Y_1 - Y_0, \overline{z}_1 - \overline{z}_0 \rangle, \\ |\overline{z}_1 - \overline{z}_0|^2 &\leq |Y_1 - Y_0| \cdot |\overline{z}_1 - \overline{z}_0|, \\ |\overline{z}_1 - \overline{z}_0| &\leq |Y_1 - Y_0|. \end{aligned}$$

Thus, small perturbations in the value of Y result in small changes in the value $\arg\min_{z \in \mathcal{Z}} |Y - z|$, hence proving continuity of the map under consideration.

Step 2: For a fixed vector Y, let $z^*_0 = \arg\min_{z \in \mathcal{Z}_p(t_0)} |Y - z|$, $z^*_s = \arg\min_{z \in \mathcal{Z}_p(s)} |Y - z|$. For a given $\epsilon > 0$, we show that $\exists\, \rho > 0$ such that $|z^*_0 - z^*_s| \leq \epsilon$ whenever $|s - t_0| \leq \rho$. According to the projection theorem, for any $z_0 \in \mathcal{Z}_p(t_0)$, $\langle Y - z_0^*, z_0 - z_0^* \rangle \leq 0$, or equivalently

$$\langle Y - z_0^*, z_s^* - z_0^* \rangle + \langle Y - z_0^*, z_0 - z_s^* \rangle \leq 0. \tag{3.12}$$

Similarly, for any $z_s \in \mathcal{Z}_p(s)$, $\langle z_s^* - Y, z_s^* - z_s \rangle \leq 0$, which leads to

$$\langle z_s^* - Y, z_s^* - z_0^* \rangle + \langle z_s^* - Y, z_s^* - z_1 \rangle \leq 0. \tag{3.13}$$

Adding (3.12) and (3.13), we get

$$\begin{aligned} \langle z_s^* - z_0^*, z_s^* - z_0^* \rangle &\leq \langle z_0^* - Y, z_0 - z_s^* \rangle + \langle Y - z_s^*, z_0^* - z_s \rangle \,, \\ |z_s^* - z_0^*|^2 &\leq |z_0^* - Y| \cdot |z_0 - z_s^*| + |Y - z_s^*| \cdot |z_0^* - z_s|. \end{aligned} \tag{3.14}$$

Let $x_0^* \in \mathcal{R}_0$ (resp. $x_s^* \in \mathcal{R}_s$) be such that $z_0^* = L_p x_0^*$ (resp. $z_s^* = L_p x_s^*$). By Lemma 3.4, there exist $\rho > 0$ and $x_0 \in \mathcal{R}_0$ (resp. $x_s \in \mathcal{R}_s$) so that $|x_0 - x_s^*| < \frac{\epsilon^2}{16\|L_p\||z_0^* - Y|}$ (resp. $|x_s - x_0^*| < \frac{\epsilon^2}{16\|L_p\||Y - z_s^*|}$) whenever $|s - t_0| < \rho$. Choosing $z_0 = L_p x_0$ (resp. $z_s = L_p x_s$), we get $|z_0 - z_s^*| < \frac{\epsilon^2}{16|z_0^* - Y|}$, and $|z_0^* - z_s| < \frac{\epsilon^2}{16|Y - z_s^*|}$. Inequality (3.14) now becomes:

$$|z_s^* - z_0^*| \leq \frac{\varrho}{4} + \frac{\varrho}{4} = \frac{\varrho}{2}.$$

For the desired result, we now combine Step 1 and Step 2. Define $\overline{z}_{p,0} := \arg\min_{z \in \mathcal{Z}_p(t_0)} |Y_s - z|$; then

$$|z_{p,s}^* - z_{p,0}^*| \leq |z_{p,s}^* - \overline{z}_{p,0}| + |\overline{z}_{p,0} - z_{p,0}^*| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

whenever $|s - t_0| \leq \rho$ and $|Y_s - Y_0| \leq \frac{\epsilon}{2}$. □

This lemma leads us to the following result:

Proposition 3.7. *Consider the switched system* (3.1) *with initial condition contained in a compact, convex set $\mathcal{R}_{t_0}$, and measured output $\hat{y}$ over time interval $[t_0, t_1]$. Assume that:*

1. *there exists a unique $p \in \mathcal{P}$ such that $d_p(t_0) \leq \|W_p\|\varrho$,*
2. *for all $q \neq p$, $d_q(t_0) > 3\|W_q\|\,\varrho$;*

then there exists $\rho > 0$ such that $\sigma(t) = \widehat{\Sigma}^{-1}(\mathcal{R}_t, \hat{y}_{[t,t+\epsilon)})$ for all $t \in [t_0, t_0 + \rho)$, where $\widehat{\Sigma}^{-1}$ is defined in (3.11).

Literally speaking, Proposition 3.5 gives the necessary condition for a subsystem to produce an output in presence of bounded disturbances and Proposition 3.7 states that if there is a unique candidate satisfying this necessary condition then the continuity of the reachable sets guarantees uniqueness over some time interval.

Proof of Proposition 3.7. By assumption, $\widehat{\sigma}(t_0) = p$. If $\sigma(t_0) \neq p$, then there exists $q \in \mathcal{P}$ and some u such that $\Gamma^O_{q,x_0}(u_{[t_0,t_0+\epsilon)}) = y_{[t_0,t_0+\epsilon)} \Rightarrow |L_q\hat{x}_0 - W_q\widehat{Y}(t_0)| < \|W_q\|\varrho \Rightarrow d_q(t_0) < \|W_q\|\varrho$, which is a contradiction and hence $\hat{\sigma}(t_0) = p = \sigma(t_0)$.
Let $\bar{t} := \min\{t > t_0 \,|\, u \text{ or } \sigma \text{ is discontinuous at } t\}$. Since Y remains continuous on the interval $[t_0, \bar{t})$, there exists $\rho_1 > 0$ such that if $|t - t_0| < \rho_1$, then $|Y(t) - Y(t_0)| < \epsilon/2$. This gives

$$\begin{aligned} |W_q\widehat{Y}(t) - W_q\widehat{Y}(t_0)| &\leq \|W_q\| \; |\widehat{Y}(t) - \widehat{Y}(t_0)| && (3.15a)\\ &\leq \|W_q\| \; (|\widehat{Y}(t) - Y(t)| + |Y(t) - Y(t_0)| + |Y(t_0) - \widehat{Y}(t_0)|) && (3.15b)\\ &\leq \|W_q\|(\varrho + \varrho) + \frac{\epsilon}{2} = 2\|W_q\| \, \varrho + \frac{\epsilon}{2}, && (3.15c)\end{aligned}$$

for all t such that $|t - t_0| < \rho_1$.
Let $z^*_{q,0}$, $z^*_{q,s}$ be such that $d_p(t_0) = |W_q\widehat{Y}(t_0) - z^*_{q,0}|$ and $d_p(s) = |W_q\widehat{Y}(s) - z^*_{q,s}|$. Lemma 3.6 implies that there exists $\rho_2 > 0$ such that if $|s - t_0| < \rho_2$, then $|z^*_{q,s} - z^*_{q,0}| < \frac{\epsilon}{2}$. Choose $\rho = \min\{\rho_1, \rho_2\}$; then for all s satisfying $|s - t_0| < \rho$, the use of reverse triangle inequality and (3.15) gives:

$$\begin{aligned} |W_q\widehat{Y}(s) - z^*_{q,s}| &= |\big(W_q\widehat{Y}(t_0) - z^*_{q,0}\big) - \big(W_q\widehat{Y}(t_0) - W_q\widehat{Y}(s)\big) - \big(z^*_{q,s} - z^*_{q,0}\big)| && (3.16a)\\ &\geq |W_q\widehat{Y}(t_0) - z^*_{q,0}| - |W_q\widehat{Y}(t_0) - W_q\widehat{Y}(s)| - |z^*_{q,s} - z^*_{q,0}| && (3.16b)\\ &> 3\|W_q\| \, \varrho - 2\|W_q\| \, \varrho - \frac{\epsilon}{2} - \frac{\epsilon}{2} = \|W_q\| \, \varrho - \epsilon. && (3.16c)\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $|W_q\widehat{Y}(s) - z^*_{q,s}| > \|W_q\|\varrho$. This guarantees that if $\hat{\sigma}(t_0) \neq q$, then there exists an interval $[t_0, t_0 + \rho)$ such that $\hat{\sigma}_{[t_0,t_0+\rho)} \neq q$. As the output y is being produced by Γ_p over the interval $[t_0, t_0 + \rho)$, we obtain $\hat{\sigma}_{[t_0,t_0+\rho)} = \sigma_{[t_0,t_0+\rho)} = p$. □

Proposition 3.7 basically suggests that there exists a lower bound on the time interval over which the switching signal can be recovered. If the conditions of this proposition are also satisfied at time $t_0 + \rho$, then there exists $\overline{\rho} > 0$ such that the switching signal can be recovered over the interval $[t_0, t_0 + \rho + \overline{\rho})$. Thus, larger intervals can be obtained by applying Proposition 3.7 inductively. The switching signal $\sigma(\cdot)$ is now recovered by letting:

$$\sigma(t) = \widehat{\Sigma}^{-1}(\mathcal{R}_t, \hat{y}_{[t,t+\epsilon)}).$$

This definition leads to the recovery of switching signal over an interval $[t_0, T)$, where $T := \min\{t \geq t_0 \,|\, \exists p, q \in \mathcal{P}$ satisfying $d_p(t) \leq \varrho\|W_p\|$ and $d_q(t)\varrho\|W_q\|\}$. We now incorporate these results in an example.

Example 3.8. Consider a switched system with following two modes:

$$\Gamma_1 : \begin{cases} \dot{x} &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0.5 \end{bmatrix} x \end{cases} \quad ; \quad \Gamma_2 : \begin{cases} \dot{x} &= \begin{bmatrix} -1 & 1 \\ 1 & -0.5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x. \end{cases}$$

The corresponding inverse systems are:

$$\Gamma_1^{-1} : \begin{cases} \dot{\hat{x}} &= \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \hat{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \hat{\dot{y}} \\ \hat{u} &= \begin{bmatrix} 0 & -4 \end{bmatrix} \hat{x} + 4\hat{\dot{y}} \end{cases} \quad ; \quad \Gamma_2^{-1} : \begin{cases} \dot{\hat{x}} &= \begin{bmatrix} 0.5 & -0.25 \\ 1 & -0.5 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hat{\dot{y}} \\ \hat{u} &= \begin{bmatrix} 0.75 & -0.625 \end{bmatrix} \hat{x} + 0.5\hat{\dot{y}}. \end{cases}$$

For this example, $W_1 = W_2 = I_{2\times 2}$, and

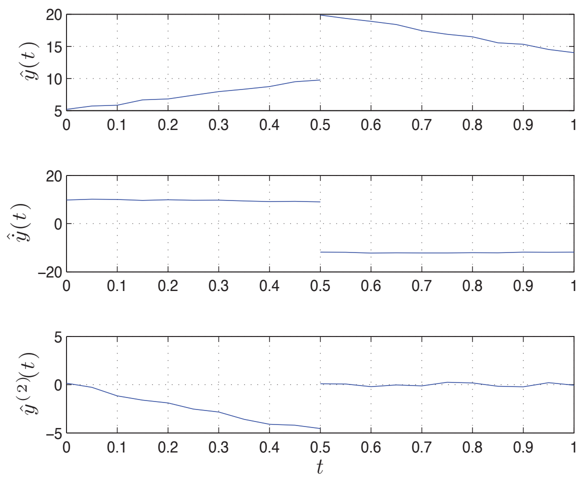
$$L_1 = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \quad ; \quad L_2 = \begin{bmatrix} 0 & 1 \\ 1 & -0.5 \end{bmatrix}.$$

Both the subsystems Γ_1, Γ_2 are invertible and the conditions in Proposition 3.7 hold for $\varrho = 0.25$ and $\mathcal{R}_0 = \{x : (x - 10)^\top (x - 10) = 1)\}$ (a ball of of unit radius centered at col(10, 10)). The results of the simulation are shown in Fig. 3.1.

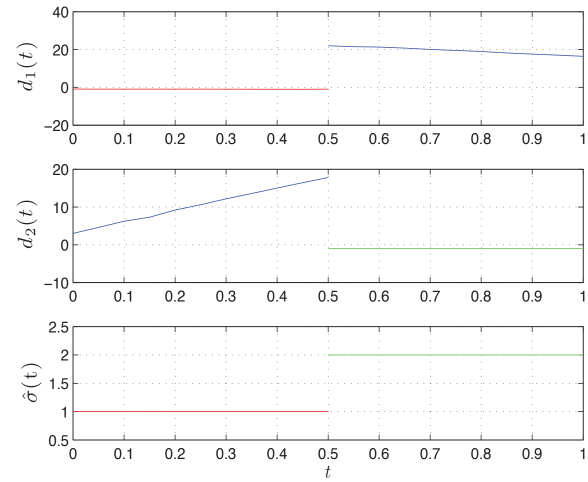
Figure 3.1(a) shows the measured output and its derivatives used in the computation, and Fig. 3.1(b) shows the distance functions and the switching signal obtained by comparing these two distance functions. It can be seen that $\sigma(t) = 1$ when $d_1(t)$ is near zero and $\sigma(t) = 2$ when d_2 is near zero. Figure 3.1(c) plots the set $\mathcal{Z}_1(t) = L_1(\mathcal{R}_t)$ (in red) and $\mathcal{Z}_2(t) = L_2(\mathcal{R}_t)$ (in green) at several distinct points t in the $(y, \dot{y})$ plane. The reachable sets $\mathcal{R}_t$ are obtained by ellipsoidal approximations. $\triangleleft$

3.2.2 Switching Signal Recovery using Approximate Reachable Sets

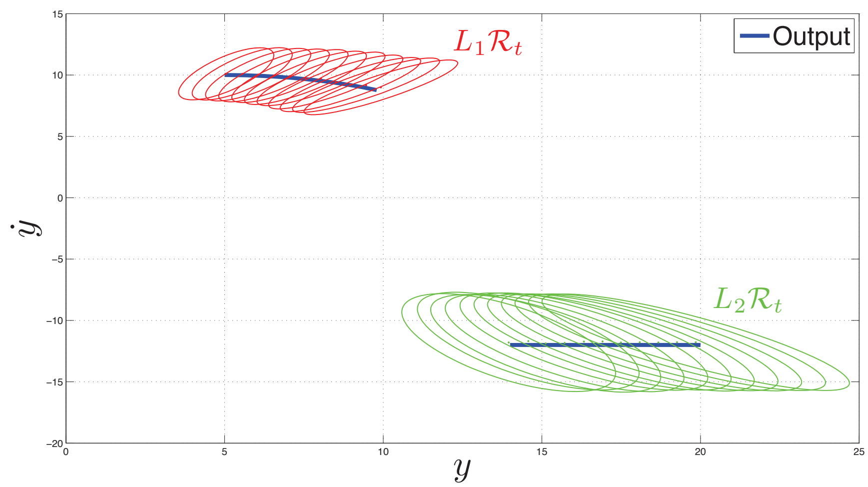
In the previous section, we recovered the switching signal using the index-inversion function $\widehat{\Sigma}^{-1}$, whose arguments were the measured output and the reachable set at each point in



(a) Measured output trajectory and its derivatives.



(b) The distance functions $d_1(t)$ and $d_2(t)$ that lead to recovery of $\sigma(t)$.



(c) Corresponding Reach Sets $\mathcal{R}_t$.

Figure 3.1: Reconstruction of switching signal using reachable sets.

time, and furthermore the function itself involved solving the optimization problem at each instant in time. Clearly, this approach is computationally very expensive. In this section, we derive an alternative formula for the recovery of the switching signal with the help of certain approximations, which relieves the computational burden enormously. The drawback, however, is that the interval over which the switching signal is recovered is smaller. We start off with the definition of $(\mathcal{R}, \varrho)$ *switch-singular pair*:

Definition 3.9 ($(\mathcal{R}, \varrho)$ switch-singular pair)**.** *Let $x_0 = x(t_0)$ be contained in a compact set $\mathcal{R} \subset \mathbb{R}^n$, and y be an $\mathbb{R}^m$-valued function over some time interval with $Y_0 := Y(t_0)$. We say* *that* (x_0, y) *forms an* $(\mathcal{R}, \varrho)$ *switch-singular pair for subsystems* Γ_p, Γ_q *if, for a given* $\varrho > 0$, *there exist* $x_1, x_2 \in \mathcal{R}$ *and* $Y_1, Y_2 \in \mathcal{B}_\varrho(Y_0)$ *such that* $L_p x_1 = W_p Y_1$ *and* $L_q x_2 = W_q Y_2$. $\lhd$

In the sequel, we will also refer to $(\mathcal{R}, \varrho)$ switch-singular pair as the *weak switch-singular pair* when $\mathcal{R}$ and ϱ are clear from the context.

Gap between Subspaces

To study the existence of weak switch-singular pairs, one has to introduce the notion of minimal gap between the subspaces which is defined as follows.

Definition 3.10 (Minimal Gap between subspaces)**.** *Let* $\mathcal{M}, \mathcal{N}$ *be two subspaces of an Euclidean space. The minimal gap* $\alpha(\mathcal{M}, \mathcal{N})$ *is defined as:*

$$\alpha(\mathcal{M}, \mathcal{N}) = \alpha(\mathcal{N}, \mathcal{M}) := \min\{\widehat{\alpha}(\mathcal{M}, \mathcal{N}), \widehat{\alpha}(\mathcal{N}, \mathcal{M})\},$$

where

$$\widehat{\alpha}(\mathcal{M}, \mathcal{N}) := \min_{|x|=1, x \in \mathcal{M}} d(x, \mathcal{N}). \quad \lhd$$

The notion of minimum gap between subspaces has appeared in [93, 94, 95] for spaces other than Euclidean spaces. A dual notion of maximal gap has also been used in robust control, see [96, 97] and references therein.

Proposition 3.11 (Computation of $\widehat{\alpha}(\mathcal{M}, \mathcal{N})$)**.** *Let* $\Pi_\mathcal{N}$ *denote the orthogonal projection on* $\mathcal{N}$ *and matrix* M *be such that its columns are orthonormal vectors that span* $\mathcal{M}$*; then*

$$\widehat{\alpha}(\mathcal{M}, \mathcal{N})^2 = \min_{|x|=1, x \in \mathcal{M}} d^2(x, \mathcal{N}) = 1 - \|\Pi_\mathcal{N} M\|^2.$$

Proof. Using the projection theorem [92], the square of the distance between a point x and a subspace $\mathcal{N}$ is given by $|x|^2 - |\Pi_\mathcal{N} x|^2$. The desired expression can now be derived as follows:

$$\begin{aligned}
\min_{|x|=1, x \in \mathcal{M}} d^2(x, \mathcal{N}) &= \min_{|x|=1, x \in \mathcal{M}} \left\{|x|^2 - |\Pi_\mathcal{N} x|^2\right\} = 1 - \max_{|x|=1, x \in \mathcal{M}} |\Pi_\mathcal{N} x|^2 \\
&= 1 - \max_{|Mz|=1} |\Pi_\mathcal{N} M z|^2 = 1 - \max_{|z|=1} |\Pi_\mathcal{N} M z|^2 \\
&= 1 - \|\Pi_\mathcal{N} M\|^2.
\end{aligned}$$

The first equality uses the fact that $-\min(-A) = \max(A)$ for any set A, and the third equality uses the fact that M is orthonormal, so $\|M\| = 1$, which in turn implies that $|Mz| = 1$ if and only if $|z| = 1$. □

Note that $\alpha(\mathcal{M}, \mathcal{N}) = 0$ if and only if $\mathcal{M} \cap \mathcal{N} \neq \{0\}$, and $\alpha(\mathcal{M}, \mathcal{N}) = 1$ if and only if $\mathcal{M}$ and $\mathcal{N}$ are mutually orthogonal to each other. Roughly speaking, $\alpha(\mathcal{M}, \mathcal{N})$ measures the sine of minimum angle between the subspaces $\mathcal{M}$ and $\mathcal{N}$.

Corollary 3.12. *Suppose $\mathcal{M}$, $\mathcal{N}$ are two subspaces such that $\mathcal{M} \cap \mathcal{N} = \{0\}$. Given $x \in \mathcal{M}$, $z \in \mathcal{N}$, $|x - z| < \epsilon$, only if, $|x| < \dfrac{\epsilon}{\alpha(\mathcal{M}, \mathcal{N})}$.*

Proof. For $x \neq 0$, $x \in \mathcal{M}$ can be written as $x = cy$ where $y \in \mathcal{M}$ has unit norm. Note that $|x| = |c|$. Using reverse triangle inequality, we obtain:

$$\begin{aligned}\epsilon > |x - z| \geq ||x| - |z|| \geq |c| \left| |y| - \left| \frac{z}{c} \right| \right| \geq |c| d(y, \mathcal{N}) \\ \geq |c| \inf_{|y|=1, y \in \mathcal{M}} d(y, \mathcal{N}) = |c| \widehat{\alpha}(\mathcal{M}, \mathcal{N}) \geq |c| \alpha(\mathcal{M}, \mathcal{N}),\end{aligned}$$

whence the desired result follows. □

Necessary Conditions for Weak Switch-Singular Pairs

If for a given $\hat{x}_0$ and $\hat{y}$, there exists $x_0 \in \mathcal{B}_{\delta_0}(\hat{x}_0)$ and $Y_0 \in \mathcal{B}_{\varrho}(\widehat{Y}_0)$, $\widehat{Y}_0 := \widehat{Y}(t_0)$, such that $L_p x_0 = W_p Y_0$, that is, subsystem Γ_p produces the output y with initial condition x_0, then

$$|L_p \hat{x}_0 - W_p \widehat{Y}_0| \leq |L_p \hat{x}_0 - L_p x_0| + \cancel{|L_p x_0 - W_p Y_0|} + |W_p Y_0 - W_p \widehat{Y}_0| \tag{3.17a}$$

$$\leq \|L_p\| \delta_0 + \|W_p\| \varrho. \tag{3.17b}$$

In particular, if $(\hat{x}_0, \hat{y})$ forms an $(\mathcal{B}_{\delta_0}(\hat{x}_0), \varrho)$ switch-singular pair, then

$$\begin{aligned}\left| \begin{bmatrix} L_p \\ L_q \end{bmatrix} \hat{x}_0 - \begin{bmatrix} W_p \\ W_q \end{bmatrix} \widehat{Y}_0 \right| &\leq |L_p \hat{x}_0 - W_p \widehat{Y}_0| + |L_q \hat{x}_0 - W_q \widehat{Y}_0| \\ &\leq (\|L_p\| + \|L_q\|) \delta_0 + (\|W_p\| + \|W_q\|) \varrho =: \kappa^0_{pq}.\end{aligned}$$

With $\mathcal{L}_{pq}$ denoting the range space of $\begin{bmatrix} L_p \\ L_q \end{bmatrix}$ and $\mathcal{W}_{pq}$ denoting the range space of $\begin{bmatrix} W_p \\ W_q \end{bmatrix}$, we have the following result.

Proposition 3.13. *If $\mathcal{L}_{pq} \cap \mathcal{W}_{pq} = \{0\}$, then $(\hat{x}_0, \hat{y})$ forms an $(\mathcal{B}_{\delta_0}(\hat{x}_0), \varrho)$ switch-singular pair for subsystems Γ_p and Γ_q, $p, q \in \mathcal{P}$ only if*

$$\left| \begin{bmatrix} L_p \\ L_q \end{bmatrix} \hat{x}_0 \right| \leq \frac{\kappa^0_{pq}}{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})} \quad \textit{and} \quad \left| \begin{bmatrix} W_p \\ W_q \end{bmatrix} \widehat{Y}_0 \right| \leq \frac{\kappa^0_{pq}}{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})}.$$

Proof. This is a straightforward consequence of Corollary 3.12 applied with $\mathcal{M} := \mathcal{L}_{pq}$ and $\mathcal{N} := \mathcal{W}_{pq}$. □

The above proposition gives two necessary conditions under which subsystems Γ_p, Γ_q, $p, q \in \mathcal{P}$, may form weak switch-singular pairs.

Example 3.14. Consider the second order SISO switched system given in Example 3.8. We showed that $W_1 = W_2 = I_{2\times 2} = L_1 = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$; $L_2 = \begin{bmatrix} 0 & 1 \\ 1 & -0.5 \end{bmatrix}$. The columns of $W_{12} = \text{col}(W_1, W_2)$ and $L_{12} = \text{col}(L_1, L_2)$ span two-dimensional subspaces of $\mathbb{R}^4$ and it can be verified that their intersection is the null vector. In terms of orthonormal basis, we can write

$$\mathcal{W}_{12} = \text{span} \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right\} \quad ; \quad \mathcal{L}_{12} = \text{span} \left\{ \begin{pmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{7} \\ 1/\sqrt{7} \\ 2/\sqrt{7} \\ -1/\sqrt{7} \end{pmatrix} \right\}.$$

Since both W_{12} and L_{12} comprise linearly independent columns, they are left-invertible and the orthogonal projection can be written in terms of left-pseudo inverse (denoted by †), that is,

$$\Pi_{\mathcal{L}_{12}} = L_{12} L_{12}^{\dagger} = L_{12}(L_{12}^{\top} L_1 2)^{-1} L_{12}^{\top} \quad ; \quad \Pi_{\mathcal{W}_{12}} = W_{12} W_{12}^{\dagger} = W_{12}(W_{12}^{\top} W_1 2)^{-1} W_{12}^{\top}.$$

From these matrices we can now compute the gap between $\mathcal{L}_{12}$ and $\mathcal{W}_{12}$:

$$\alpha(\mathcal{W}_{12}, \mathcal{L}_{12}) = \widehat{\alpha}(\mathcal{W}_{12}, \mathcal{L}_{12}) = \widehat{\alpha}(\mathcal{L}_{12}, \mathcal{W}_{12}) = 0.1368.$$

Moreover, with $\varrho = \delta_0 = 0.25$, we get $\kappa^0_{12} = 1.0224$. Considering the data of Example 3.8 at initial time t_0 with $\hat{x}_0 = \text{col}(10, 10)$, we get $\left| \begin{bmatrix} W_p \\ W_q \end{bmatrix} \widehat{Y} \right| = 11.45 > \frac{\kappa^0_{pq}}{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})} = 7.02$, and

$\left|\begin{bmatrix} L_p \\ L_q \end{bmatrix} \hat{x}_0\right| = 11.18$. Both necessary conditions are violated, so there are no switch-singular pairs at time t_0. Using certain approximations, we will show that there is a subinterval over which the nonexistence of switch-singular pairs can be guaranteed. $\triangleleft$

Spherical Approximation and Elimination of Switch-Singular Pairs over an Interval

Proposition 3.13 provides necessary conditions for the existence of weak switch-singular pairs at time instant t_0 when the uncertainty in the initial state is given by a ball of radius δ_0. Note that, even though the output is changing with time, there is a constant upper bound on the uncertainties in the output ϱ, whereas the uncertainty in the state variable, denoted by $\delta(t)$ ($\delta_0 := \delta(t_0)$), is a function of time. This is because, in absence of uncertainties, the inverse system gives the state trajectory of the original system. But now since the inverse system is being driven by the corrupted output, the accuracy of the state estimate changes with time. To guarantee that state $\hat{x}(t)$ does not form $(\mathcal{B}_{\delta(t)}(\hat{x}(t)), \varrho)$ switch-singular pairs with the output $\hat{y}(t)$, one must verify that, at time t, the following inequality holds:

$$\left|\begin{bmatrix} L_p \\ L_q \end{bmatrix} \hat{x}(t) - \begin{bmatrix} W_p \\ W_q \end{bmatrix} \widehat{Y}(t)\right| \geq (\|L_p\| + \|L_q\|)\delta(t) + (\|W_p\| + \|W_q\|)\varrho =: \kappa_{pq}^t. \tag{3.18}$$

Our goal now is to determine whether, under certain conditions, it is possible to rule out the existence of weak switch-singular pairs over some time interval. The following result specifies the length of such time intervals during which the output and state form no switch-singular pairs.

Theorem 3.15. *Consider the switched system in* (3.1) *and assume that the following hold:*

1. $\operatorname{rank}\begin{bmatrix} L_p & W_p \\ L_q & W_q \end{bmatrix} = \operatorname{rank}\begin{bmatrix} L_p \\ L_q \end{bmatrix} + \operatorname{rank}\begin{bmatrix} W_p \\ W_q \end{bmatrix}$ (i.e., $\mathcal{L}_{pq} \cap \mathcal{W}_{pq} = \{0\}$),
2. $\min_{t \geq t_0} |W_p \widehat{Y}(t)| \geq \beta > \varrho$ *for each* $p \in \mathcal{P}$.

Moreover, let ϱ *be the maximum amount of error in the output measurements, i.e.,* $|\widehat{Y}_\alpha(t) - Y_\alpha(t)| \leq \varrho$ *for each* $t \geq t_0$*; then there exist*[4] $\delta_0 > 0$, $T > t_0$, *such that if* $x(t_0) \in \mathcal{B}_{\delta_0}(\hat{x}(t_0))$ *then* $(x(t), y(t))$ *do not form a weak switch-singular pair for any* $t \in [t_0, T)$.

[4]The explicit expressions for δ_0 and T appear in the proof.

Proof. Let $\tilde{x} = x - \hat{x}$, where x indicates the original state trajectory of the plant (3.5a) and $\hat{x}$ is the state trajectory obtained by simulating the inverse system (3.6a) with corrupted outputs and uncertain initial condition $\hat{x}_0 = \hat{x}(t_0)$; then the following expression is obtained by solving (3.7):

$$
\begin{aligned}
\tilde{x}(t) = {} & e^{\widehat{A}_{p_{k+1}}(t-t_k)} e^{\widehat{A}_{p_k}(t_k-t_{k-1})} \cdots e^{\widehat{A}_{p_1}(t_1-t_0)} \tilde{x}(t_0) \\
& + \int_{t_0}^{t_1} e^{\widehat{A}_{p_{k+1}}(t-t_k)} e^{\widehat{A}_{p_k}(t_k-t_{k-1})} \cdots e^{\widehat{A}_{p_1}(t_1-s)} \widehat{B}_{p_1} w(s) ds \\
& + \int_{t_1}^{t_2} e^{\widehat{A}_{p_{k+1}}(t-t_k)} e^{\widehat{A}_{p_k}(t_k-t_{k-1})} \cdots e^{\widehat{A}_{p_2}(t_2-s)} \widehat{B}_{p_2} w(s) ds \\
& + \cdots + \int_{t_{k-1}}^{t_k} e^{\widehat{A}_{p_{k+1}}(t-t_k)} e^{\widehat{A}_{p_k}(t_k-s)} \widehat{B}_{p_k} w(s) ds + \int_{t_k}^{t} e^{\widehat{A}_{p_{k+1}}(t-s)} \widehat{B}_{p_{k+1}} w(s) ds.
\end{aligned}
\tag{3.19}
$$

For each $p \in \mathcal{P}$, there exists $\lambda_p, a_p \in \mathbb{R}$ such that $\| \exp(\widehat{A}_p t) \| \leq e^{(a_p + \lambda_p t)}$. Define $\lambda := \max_{p \in \mathcal{P}} \lambda_p$, $a := \max_{p \in \mathcal{P}} a_p$ and $b := \max_{p \in \mathcal{P}} \|\widehat{B}_p\|$. We consider two cases where $\lambda \neq 0$ and $\lambda = 0$.

Case 1: $\lambda \neq 0$**;** In this case (3.19) leads to:

$$
\begin{aligned}
|\tilde{x}(t)| \leq {} & e^{(k+1)a} e^{\lambda(t-t_0)} |\tilde{x}(t_0)| + e^{(k+1)a} e^{\lambda t} \left(\frac{e^{-\lambda t_0} - e^{-\lambda t_1}}{\lambda} \right) b\varrho + e^{ka} e^{\lambda t} \left(\frac{e^{-\lambda t_1} - e^{-\lambda t_2}}{\lambda} \right) b\varrho \\
& + \cdots + e^{2a} e^{\lambda t} \left(\frac{e^{-\lambda t_{k-1}} - e^{-\lambda t_k}}{\lambda} \right) b\varrho + e^{a} e^{\lambda t} \left(\frac{e^{-\lambda t_k} - e^{-\lambda t}}{\lambda} \right) b\varrho \\
\leq {} & e^{(k+1)a} e^{\lambda(t-t_0)} |\tilde{x}(t_0)| + e^{(k+1)a} e^{\lambda t} \left(\frac{e^{-\lambda t_0} - e^{-\lambda t}}{\lambda} \right) b\varrho \\
\leq {} & e^{a + ka + \lambda(t-t_0)} \left(\delta_0 + \frac{b\varrho}{\lambda} \right) - \frac{e^{(a+ka)} b\varrho}{\lambda} =: \delta(t).
\end{aligned}
\tag{3.20}
$$

Condition 2 of the theorem statement and use of Proposition 3.13 imply that if a subsystem Γ_p producing the output forms a weak switch-singular pair with Γ_q at any time instant $t \in [t_k, t_{k+1})$, then

$$
\beta \leq |W_p \widehat{Y}(t)| \leq \left| \begin{bmatrix} W_p \\ W_q \end{bmatrix} \widehat{Y}(t) \right| \leq \frac{(\|L_p\| + \|L_q\|)\delta(t) + (\|W_p\| + \|W_q\|)\varrho}{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})}.
$$

So the minimal uncertainty in the state variable, denoted by $\delta(t)$, that allows the existence of weak switch-singular pairs between subsystems Γ_p and Γ_q is:

$$\delta(t) \geq \frac{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})\beta - (\|W_p\| + \|W_q\|)\varrho}{(\|L_p\| + \|L_q\|)} =: \Omega_{pq}. \tag{3.21}$$

Let

$$\Omega := \max_{p,q \in \mathcal{P}} \Omega_{pq};$$

then $(\hat{x}(t), \hat{y}(t))$ do not form a $(\mathcal{B}_{\delta(t)}(\hat{x}(t)), \varrho)$ switch-singular pair if

$$\delta(t) < \Omega.$$

As $\mathcal{R}_t \subseteq \mathcal{B}_{\delta(t)}(\hat{x}(t))$, the above condition also guarantees that $(\hat{x}(t), \hat{y}(t))$ do not form a $(\mathcal{R}_t, \varrho)$ switch-singular pair. Substituting the expression of $\delta(t)$ from (3.20) on the left-hand side, one gets the following inequality after simple algebraic manipulations:

$$\lambda(t - t_0) < \log\left(\frac{\lambda\Omega + e^{(a+ka)}b\varrho}{\lambda\delta_0 + b\varrho}\right) - (k+1)a. \tag{3.22}$$

It is easy to verify that right hand side is strictly positive if

$$\delta_0 < \frac{\Omega}{e^{(a+ka)}}. \tag{3.23}$$

Under the constraint imposed by (3.23), the inequality (3.22) can now be solved for the minimal time interval $[t_0, T)$, which does not allow the existence of weak switch-singular pairs, where

$$T < t_0 + \frac{1}{\lambda}\log\left(\frac{\lambda\Omega + e^{(a+ka)}b\varrho}{\lambda\delta_0 + b\varrho}\right) - \frac{(k+1)a}{\lambda}. \tag{3.24}$$

Case 2: $\lambda = 0$; In this case, the expression for $\delta(t)$ is derived as follows:

$$\begin{aligned} |\tilde{x}(t)| &\leq e^{(k+1)a}|\tilde{x}(t_0)| + e^{(k+1)a}b\varrho(t_1 - t_0) + e^{ka}b\varrho(t_2 - t_1) \\ &\qquad + \cdots + e^{2a}b\varrho(t_k - t_{k-1}) + e^{a}b\varrho(t - t_k) \\ &\leq e^{(k+1)a}|\tilde{x}(t_0)| + e^{(k+1)a}b\varrho(t - t_0) \\ &\leq e^{(k+1)a}(\delta_0 + b\varrho(t - t_0)) =: \delta(t). \end{aligned} \tag{3.25}$$

In order to have $\delta(t) < \Omega$, at time t, we must have

$$T < t_0 + \frac{1}{b\varrho}\left(\frac{\Omega}{e^{((k+1)a)}} - \delta_0\right). \tag{3.26}$$

As before, the right-hand side is strictly positive if (3.23) holds. Under this constraint, the above inequality gives the minimal length of the interval over which there exist no switch-singular pairs. □

Based on the result of the Theorem 3.15, one can formulate an alternative index-matching function $\widetilde{\Sigma}^{-1}$ as follows:

$$\widetilde{\Sigma}^{-1}(\hat{x}(t), y_{[t,t+\epsilon)}) = \{p \mid L_p\hat{x}(t) - W_p\widehat{Y}(t)| \leq \|L_p\|\delta(t) + \|W_p\|\varrho\}, \tag{3.27}$$

where $\hat{x}(t)$ is obtained by solving (3.6) and δ is obtained from (3.20). It is guaranteed by Theorem 3.15 that $\sigma(t) = \widetilde{\Sigma}^{-1}(\hat{x}(t), y_{[t,t+\epsilon)})$ is well defined over the interval $[t_0, T)$ where the expression for T is either given by (3.24) or (3.26). The result of Theorem 3.15 could be applied to reconstruct σ from (3.27) using the following algorithm:

Algorithm 2: Reconstruction of switching signal

```
Input: x̂_0, ŷ
Initialization: σ(t_0) ← Σ̃^{-1}(x̂_0, ŷ_[t_0, t_0+ε)), i = 0.
while σ(t_i) is not multi-valued do
    k = 0, t_0 = t_i, δ_0 = δ(t_i)
    if δ_0 satisfies (3.23) then
        compute T using (3.24) or (3.26)
        while σ(t) = σ(t^-) and t < T do
            compute δ(t) using (3.20) or (3.25)
            compute x̂(t) from (3.6a).
            σ(t) ← Σ̃^{-1}(x̂(t), y_[t,t+ε))
        end while
        t_{i+1} = t; i = i + 1.
    end if
end while
```

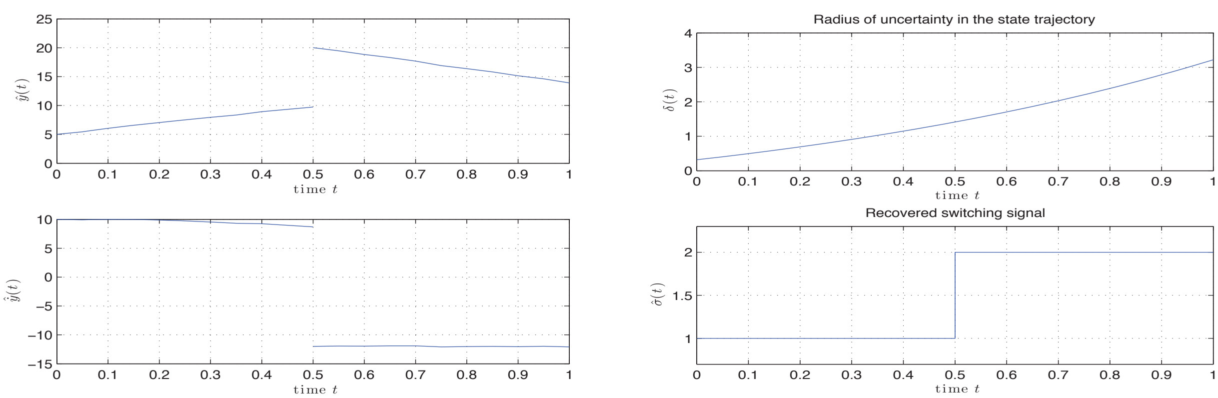
If we compare the two functions $\widehat{\Sigma}^{-1}$ in (3.11) and $\widetilde{\Sigma}^{-1}$ in (3.27), then it is observed that the mode detection through $\widehat{\Sigma}^{-1}$ requires the computation of minimum distance between the reachable sets and the measured output at each instant in time, whereas the function $\widetilde{\Sigma}^{-1}$

only requires coarse spherical approximation of the reachable set, which can be obtained analytically as shown in the proof of Theorem 3.15. The interval over which the switching signal can be constructed is, in general, larger with $\widehat{\Sigma}^{-1}$ than with $\widetilde{\Sigma}^{-1}$. To obtain a larger time interval for reconstruction of switching signal with light computation, one may combine the index-matching function $\widetilde{\Sigma}^{-1}$ in (3.27) with the computation of tightly approximated reachable sets. This can be done by resetting the value of $\delta(t_i)$ to a number that tightly approximates the radius of reachable sets at t_i in Step 2 of Algorithm 2.

Example 3.16. (Simulation of Theorem 3.15) Once again, consider the system given in Example 3.8. It was shown in Example 3.14 that $\alpha(\mathcal{L}_{12}, \mathcal{W}_{12}) = 0.1368$. Further computations give:

$$b = 2 \quad ; \quad \lambda_1 = 1 \;,\; \lambda_2 = 0.625 \quad \Rightarrow \quad \lambda = 1 \quad ; \quad a_1 = 0.9624 \;,\; a_2 = 0.4949 \quad \Rightarrow \quad a = 0.9624.$$

We consider the same output trajectory considered in Example 3.8, Fig. 3.1(a), for which $\beta = 11.18$. With $\varrho = 0.1$, one gets $\Omega_{12} = 0.636$. Now starting with $\delta_0 = 0.05$, and $\varrho = 0.1$, we get $T = 0.17$. The recovered switching signal over this interval is given in Fig. 3.2(b). The simulation results show that no switch-singular pairs were encountered up till $t = 0.17$.



(a) Measured output trajectory and its derivative.

(b) Uncertainty in state trajectory and recovered switching signal.

Figure 3.2: Simulation results for illustration of Theorem 3.15.

The conservativeness of this method can be seen from the fact that even after $t = 0.17$, one continues to recover the switching signal over the entire interval $[0, 1]$ with the same method because the necessary condition for producing the output is satisfied by only one subsystem, and not the other. $\triangleleft$

3.2.3 Input Recovery

The input reconstructed using the measured output is given by (3.6b). Using the exact expression for u in (3.5c), the input estimation error, $\tilde{u} := u - \hat{u}$, is given by

$$\tilde{u}(t) = (-\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)}\tilde{x}(t) + (\overline{D}_\alpha^{-1}V)_{\sigma(t)}(\widehat{Y}(t) - Y(t)).$$

Using the notation $d_c := \max_{p\in\mathcal{P}} \|(\overline{D}_\alpha^{-1}\overline{C}_\alpha)_p\|$ and $d_v := \max_{p\in\mathcal{P}} \|(\overline{D}_\alpha^{-1}V)_p\|$, the maximal error in the reconstruction of u at any time t is given by

$$|\tilde{u}(t)| \leq d_c\delta(t) + d_v\varrho. \tag{3.28}$$

3.3 Minimum-Phase Systems

In the previous section, the results were stated for general linear systems. In classical linear systems theory, the stability of the inverse system is closely related to minimum-phase property of the system. Using this idea, we derive conditions under which it is possible to recover the switching signal over the interval $[t_0, \infty)$.

For each subsystem Γ_p, the matrix L_p has r_p rows and rank r_p. So, there exists an $(n-r_p)\times n$ matrix $\overline{T}_p$ such that $T_p := \begin{bmatrix} L_p \\ \overline{T}_p \end{bmatrix}$ and $L_pT_p^{-1} = [I_{r_p\times r_p}\ 0_{r_p\times(n-r_p)}]$. The matrix T_p defines a coordinate transformation for the subsystem Γ_p and the transformed matrices are : $A_p^* = T_pA_pT_p^{-1}$, $B_p^* = T_pB_p$, $C_p^* = C_pT_p^{-1}$, $D_p^* = D_p$, and $L_p = [I_{r_p\times r_p}\ 0_{r_p\times(n-r_p)}]$. Apply the structure algorithm in the new coordinates, and let Q_p be the matrix formed from the last $n-r_p$ rows and columns of $(A_p^* - B_p^*D_{\alpha_p}^{*-1}C_{\alpha_p}^*)$, let G_p^1 be the matrix formed from the first r_p columns and the last $(n-r_p)$ rows of $(A_p^* - B_p^*D_{\alpha_p}^{*-1}C_{\alpha_p}^*)$, and let G_p^2 be the matrix formed by the last $(n-r_p)$ rows of $B_p^*D_{\alpha_p}^{*-1}$. If $z_p = T_px$ denotes the new state variable, then

$$z_p^1 := [(z_p)_1, \cdots, (z_p)_{r_p}]^\top = [\tilde{y}_0^\top, \cdots, \tilde{y}_{\alpha_p-1}^\top],$$

and for the remaining $(n-r_p)$ state variables denoted by z_p^2, the dynamic equation is:

$$\dot{z}_p^2 = F_pz_p^2 + G_p^1z_p^1 + G_p^2\overline{y}_{\alpha_p}.$$

Let $\hat{z}_p$ be an estimate of z_p, and let $\tilde{z}_p(t) = \hat{z}_p(t) - z_p(t)$ denote the error between actual state trajectory and the simulated one; then $|\tilde{z}_p^1| \leq \varrho$ and

$$\dot{\tilde{z}}_p^2 = F_p \tilde{z}_p^2 + G_p w,$$

where $G_p = [G_p^1 \; G_p^2]$ and $|w| \leq \varrho$.

Definition 3.17 (Minimum-phase system)**.** *The subsystem Γ_p is called minimum-phase if F_p is Hurwitz.* ◁

Our next goal is to obtain a uniform bound on the reachable sets of switched systems in Proposition 3.20. This is done using the following lemma, which basically states how the level sets of a Lyapunov function (associated to one subsystem) could be tightly packed inside the level set of another Lyapunov function (possibly associated to another subsystem).

Lemma 3.18. *Given two positive definite functions $V_1 = x^\top P_1 x$ and $V_2 = x^\top P_2 x$ with P_1 and P_2 symmetric positive definite matrices, the minimal level set of V_2 that contains the set $\{x \,|\, V_1(x) \leq c\}$ is given by $\{x \,|\, V_2(x) \leq \lambda_{\max}(H^{-1^\top} P_2 H^{-1})c\}$, where the matrix H is an upper triangular matrix obtained from Cholesky decomposition of P_1.*

Proof. The matrix P_1 admits Cholesky decomposition given by $P_1 = H^\top H$, where H is an upper triangular matrix. It follows that $H^{-1^\top} P_1 H^{-1} = I$. Let $z = Hx$; in the new coordinates defined by z, the level sets of V_1 are spheres of dimension $n-1$ embedded in $\mathbb{R}^n$. Consider the region $\mathcal{R} = \{z \,|\, z^\top H^{-1^\top} P_2 H^{-1} z \leq \lambda_{\max}(H^{-1^\top} P_2 H^{-1})c\}$. If $|z|_2^2 \leq c$, then $z \in \mathcal{R}$. Moreover, if z is in the span of eigenvector corresponding to $\lambda_{\max}(H^{-1^\top} P_2 H^{-1})$ with $|z|_2^2 = c$, then z is also on the boundary of $\mathcal{R}$, implying that the bounding region $\mathcal{R}$ wraps the level sets of V_1 tightly. Applying the coordinate transformation, $x = H^{-1} z$ gives the desired result. □

Remark 3.19. In the literature on dynamical systems that involves Lyapunov based analysis, we often encounter the inequality

$$V_1(x) \leq \frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_2)} V_2(x), \qquad P_1, P_2 > 0,$$

to bound the values of the positive definite function $V_1(x) := x^\top P_1 x$ in terms of the values of another positive definite function $V_2(x) := x^\top P_2 x$. However, Lemma 3.18 serves the same purpose by providing the following tighter bound:

$$V_1(x) \leq \lambda_{\max}(H^{-1^\top} P_1 H^{-1}) V_2(x), \qquad P_1 > 0, \; H^\top H = P_2 > 0.$$

This way, Lemma 3.18 has the potential to be useful in several other results for switched systems; see for example [11, Chapter 5] for the utility of this result. $\triangleleft$

We now use Lemma 3.18 to derive a bound on the reachable sets under the dwell-time assumption in the following proposition. The result conceptually relates to the incremental input-to-state stability (ISS) property of the system (3.7). Incremental stability properties for switched systems have been studied in [98] for homogenous systems, but here, the formulation takes into account the disturbances due to measurement uncertainties and the bounds on input-to-state gains are also computed.

Proposition 3.20. *Consider the system* (3.1) *and assume that Γ_p is minimum-phase for each $p \in \mathcal{P}$ and that the maximum disturbance in the measurement of output and its derivatives is bounded by $\varrho > 0$. Then there exist constants*[5] *$\delta > 0$ and $\tau_d > 0$ such that $x(t) \in \mathcal{B}_\Delta(\hat{x}(t))$ for some $\Delta > 0$ and all $t \geq t_0$, provided the initial state $x(t_0)$ is contained in $\mathcal{B}_\delta(\hat{x}_0)$ and $t_{i+1} - t_i \geq \tau_d$, for every switching instant t_i.*

Proof. For each subsystem Γ_p, there exists an $(n - r_p) \times (n - r_p)$ matrix P_p such that $V_p : \mathbb{R}^{r_p} \to \mathbb{R}$ defined as $V_p(\tilde{z}_p^2) = \tilde{z}_p^{2\top} P_p \tilde{z}_p^2$ is a Lyapunov function for $\tilde{z}_p^2$ and there exists a positive definite matrix Q_p such that $F_p^\top P_p + P_p F_p = -Q_p$. Also,

$$\begin{aligned}\dot{V}_p &= -\tilde{z}_p^{2\top} Q_p \tilde{z}_p^2 + 2\tilde{z}_p^{2\top} P_p R_p w \\ &\leq -\lambda_{\min}(Q_p)|\tilde{z}_p^2|^2 + 2|\tilde{z}_p^2|\|P_p R_p\|\varrho \\ &= -|\tilde{z}_p^2|\lambda_{\min}(Q_p)(|\tilde{z}_p^2| - \Theta_p \varrho),\end{aligned}$$

where

$$\Theta_p := \frac{2\|P_p R_p\|}{\lambda_{\min}(Q_p)}.$$

For a small enough $\varepsilon > 0$, we have

$$|\tilde{z}_p^2| > \Theta_p \varrho(1+\varepsilon) \quad \Rightarrow \quad \dot{V}_p < -|\tilde{z}_p^2|\lambda_{\min}(Q_p)\Theta_p \varrho \varepsilon.$$

Next, for each $p \in \mathcal{P}$, define $\overline{P}_p := \left[\begin{array}{c|c} I_{r_p} & 0 \\ \hline 0 & P_p \end{array}\right]$. Let $\Theta := \min_{p\in\mathcal{P}} \Theta_p$; let $\hat{\delta} := \min\{\varrho, \Theta\varrho(1+\varepsilon)\}$.

Define $\delta := \frac{\hat{\delta}}{\|T\|}$, where $\|T\| := \max_{p\in\mathcal{P}} \|T_p\|$. Let us consider the evolution of state trajectory for the switched system with $|\tilde{z}(t_0)| \leq \delta$. Let t_i denote the i-th switching instant,

[5] There is an upper bound on δ and a lower bound on τ_d, Δ in the proof.

and denote the active subsystem on $[t_0, t_1)$ by Γ_{p_1} and the one on $[t_1, t_2)$ by Γ_{p_2}. Using Fig. 3.3 as a guideline, consider the following region:

$$\mathcal{R}_{p_1} := \{x : x^\top \overline{P}_{p_1} x \leq \varrho^2 + \lambda_{\max}(P_{p_1})\Theta_{p_1}^2 \varrho^2(1+\varepsilon)^2\}.$$

Note that $|\tilde{z}_{p_1}(t_0)| \leq \|T_{p_1}\||\tilde{z}(t_0)| < \hat{\delta}$, so for each $t \in [t_0, t_1)$, $|\tilde{z}^1_{p_1}(t)| \leq \varrho$ and $\tilde{z}^2_{p_1}(t) \in \{z \in \mathbb{R}^{(n-r_{p_1})} \,|\, z^\top P_{p_1} z \leq \lambda_{\max}(P_{p_1})\Theta_{p_1}^2 \varrho^2(1+\varepsilon)^2\}$, the invariant set in $\mathbb{R}^{(n-r_{p_1})}$ containing the ball of radius $\Theta_{p_1}\varrho(1+\varepsilon)$. This gives $\tilde{z}_{p_1}^\top \overline{P}_{p_1} \tilde{z}_{p_1} = |\tilde{z}^1_{p_1}|^2 + \tilde{z}^2_{p_1} P_{p_1} \tilde{z}^2_{p_1} \leq \varrho^2 + \lambda_{\max}(P_{p_1})\Theta_{p_1}^2 \varrho^2(1+\varepsilon)^2$, so that $\tilde{z}_{p_1}(t) \in \mathcal{R}_{p_1}$ for each $t \in [t_0, t_1)$.

Let $T_{ij} := T_{p_j} T_{p_i}^{-1}$, $i, j = \{1, 2\}$, $i \neq j$. At switching instant t_1, we have $\tilde{z}_{p_2}(t_1) = T_{12}\tilde{z}_{p_1}(t_1^-)$. Substituting $\tilde{z}_{p_1}(t_1^-) = T_{21}\tilde{z}_{p_2}(t_1)$ in the expression for $\mathcal{R}_1$, it follows that $\tilde{z}_{p_2}(t_1) \in \mathcal{R}_{12}$, where

$$\mathcal{R}_{12} := \{x : x^\top T_{21}^\top \overline{P}_{p_1} T_{21} x \leq \varrho^2 + \lambda_{\max}(P_{p_1})\Theta_{p_1}^2 \varrho^2(1+\varepsilon)^2\}.$$

Let H_{21} be an upper triangular matrix obtained from the Cholesky decomposition such that $T_{21}^\top \overline{P}_{p_1} T_{21} = H_{21}^\top H_{21}$. So, $\tilde{z}_{p_2}(t_1) \in \{x \in \mathbb{R}^n \,|\, x^\top \overline{P}_{p_2} x \leq \lambda_{\max}(H^{-1^\top} \overline{P}_{p_2} H^{-1})(\varrho^2 + \lambda_{\max}(P_{p_1})\Theta_{p_1}^2 \varrho^2(1+\varepsilon)^2)\} =: \mathcal{R}^o_{p_2}$.

The structure of the problem reveals that for all $t \geq t_1$, $|\tilde{z}^1_{p_2}(t)| \leq \varrho$, and for $t \geq t_1$ large enough, $\tilde{z}^2_{p_2}(t) \in \{z \in \mathbb{R}^{(n-r_{p_2})} \,|\, z^\top P_{p_2} z \leq \lambda_{\max}(P_{p_2})\Theta_{p_2}^2 \varrho^2(1+\varepsilon)^2\}$, the invariant set in $\mathbb{R}^{(n-r_{p_2})}$ containing the ball of radius $\Theta_{p_2}\varrho(1+\varepsilon)$, and consequently $\tilde{z}_{p_2}(t)$ is contained in the invariant region $\mathcal{R}_{p_2}$, where

$$\mathcal{R}_{p_2} := \{x : x^\top \overline{P}_{p_2} x \leq \varrho^2 + \lambda_{\max}(P_{p_2})\Theta_{p_2}^2 \varrho^2(1+\varepsilon)^2\}.$$

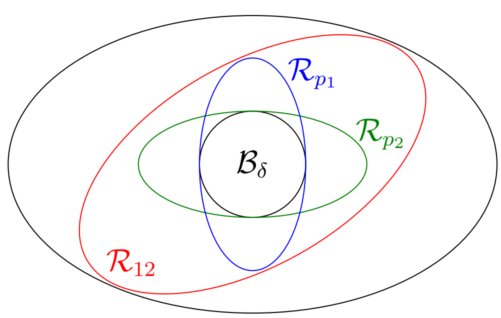


Figure 3.3: The regions used in the proof of Proposition 3.20.

The maximum time $\tilde{z}_{p_2}(t_1)$ takes to reach $\mathcal{R}_{p_2}$, starting from $\mathcal{R}^o_{p_2}$, is:

$$\tau_{12} := \frac{\lambda_{\max}(H_{21}^{-1^\top}\overline{P}_{p_2}H_{21}^{-1})\varrho^2 + \varrho^2(1+\varepsilon)^2(\lambda_{\max}(H_{21}^{-1^\top}\overline{P}_{p_2}H_{21}^{-1})\lambda_{\max}(P_{p_1})\Theta_{p_1}^2 - \lambda_{\max}(P_{p_2})\Theta_{p_1}^2)}{\Theta_{p_2}^2\varrho^2(1+\varepsilon)\lambda_{\min}(Q_{p_2})\varepsilon}.$$

This shows that $|\tilde{z}(t)|^2$ is bounded by the following constant term:

$$\frac{\lambda_{\max}(H_{21}^{-1^\top}\overline{P}_{p_2}H_{21}^{-1})}{\lambda_{\min}(\overline{P}_{p_2})}(\varrho^2 + \lambda_{\max}(P_{p_1})\Theta_{p_1}^2\varrho^2(1+\varepsilon)^2),$$

for each t in the time interval $[t_0, t_1 + \tau_{12}]$. The argument can now be repeated for future switching instants. Let H_{qp} denote an upper triangular matrix that satisfies $T_{qp}^\top\overline{P}_pT_{qp} = H_{qp}^\top H_{qp}$, where $T_{qp} := T_pT_q^{-1}$. If the dwell time between the switching times is defined as

$$\tau_d := \max_{p,q\in\mathcal{P}} \frac{\lambda_{\max}(H_{qp}^{-1^\top}\overline{P}_qH_{qp}^{-1})\varrho^2 + \varrho^2(1+\varepsilon)^2(\lambda_{\max}(H_{qp}^{-1^\top}\overline{P}_qH_{qp}^{-1})\lambda_{\max}(P_p)\Theta_{p_1}^2 - \lambda_{\max}(P_q)\Theta_p^2)}{\Theta_q^2\varrho^2(1+\varepsilon)\lambda_{\min}(Q_q)\varepsilon}, \tag{3.29}$$

then, for all times, the error vector is contained in a ball of radius Δ which is defined as:

$$\Delta := \max_{p,q\in\mathcal{P}} \frac{\lambda_{\max}(H_{qp}^{-1^\top}\overline{P}_qH_{qp}^{-1})}{\lambda_{\min}(\overline{P}_q)}(\varrho^2 + \lambda_{\max}(P_p)\Theta_p^2\varrho^2(1+\varepsilon)^2). \tag{3.30}$$

This completes the proof. □

The result in Proposition 3.20 can now be used to develop a statement parallel to Theorem 3.15 for switched systems with minimum-phase subsystems.

Theorem 3.21. *For system* (3.1)*, if each subsystem* Γ_p *is minimum-phase and moreover, the following holds:*

1. *For each* $p, q \in \mathcal{P}$, $\operatorname{rank}\begin{bmatrix} L_p & W_p \\ L_q & W_q \end{bmatrix} = \operatorname{rank}\begin{bmatrix} L_p \\ L_q \end{bmatrix} + \operatorname{rank}\begin{bmatrix} W_p \\ W_q \end{bmatrix}$ *(i.e.,* $\mathcal{L}_{pq} \cap \mathcal{W}_{pq} = \{0\}$*),*
2. *Measured output* $\hat{y}$ *is such that* $\min_{t\geq t_0}|W_p\widehat{Y}(t)| \geq \beta > \varrho$ *for each* $p \in \mathcal{P}$,
3. *For each* $p, q \in \mathcal{P}$, $\Delta < \dfrac{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})\beta - (\|W_p\| + \|W_q\|)\varrho}{(\|L_p\| + \|L_q\|)}$*, where* Δ *is given by* (3.30)*,*
4. *The switching signal* σ *has the dwell-time* τ_d *given in* (3.29)*,*

then $\sigma(t) = \{p : |L_p\hat{x}(t) - W_p\widehat{Y}(t)| \leq \|L_p\|\Delta + \|W_p\|\varrho\}$ *for all* $t \in [t_0, \infty)$*. Moreover,* $\|\tilde{u}\|_\infty = d_c\Delta + d_v\varrho$.

Proof. The result of proposition 3.20 guarantees that $x(t) \in \mathcal{B}_\Delta(\hat{x}(t))$. Condition 3 of the theorem statement implies that the inequality (3.21) in the proof of Theroem 3.15 is violated for all times and this in turn implies that $\hat{x}(t)$ does not form $(\mathcal{B}_\Delta(\hat{x}(t)), \varrho)$ switch-singular pair with $\hat{y}(t)$ for any $t \geq t_0$. Thus, the index-matching function of (3.27) is well-defined and reconstructs the original switching signal. The uniform upper bound on $\tilde{u}$ is obtained from (3.28). □

Thus, we have arrived at a result parallel to Theroem 3.15 with the additional benefit of being able to reconstruct the switching signal at all times. The development inherently relied on the stability of minimal order inverses (the minimum-phase assumption) in order to generate stable error dynamics. These dynamics were then shown to converge to zero under dwell-time assumption.

Chapter 4

Observability of Switched Linear Systems

The problem of observability deals with extracting information about the state of the system from the measured output while the input and the switching signal are assumed to be known. This property plays a fundamental role in state estimation as the construction of state estimators, or observers, requires the actual system to be observable. Even though the basic problem of observability has been well studied for non-switched systems, the presence of the switching signal and the hybrid nature of switched systems make the problem more significant. In general, the individual subsystems of a switched system may not be observable, so the information about the state cannot be estimated in arbitrarily fast time using the classical methods, such as Luenberger observers. Thus, an interesting question is how the partial information about the state, that is available from each mode, can be combined to recover the complete information about the state even though the individual subsystems are unobservable. Moreover, if the system is observable, how can one go about designing the state estimators that are feasible for implementation on digital computers? Fundamental questions of this sort, i.e., seeking observability of the entire system especially with unobservable modes and constructing asymptotic observers for estimating state are not only theoretically interesting but also possess significant applicability.

For our work, we seek a unified approach in which the geometric conditions lead to the design of asymptotic observers without any restrictive assumptions. Also, by tackling the problem for a larger class of systems such as nonlinear ODEs and switched DAEs, a lot more breadth could be seen in our work.

This chapter presents a characterization of observability and an observer design method for switched linear systems with state jumps. A necessary and sufficient condition is presented for observability, globally in time, when the system evolves under predetermined mode transitions. Because this characterization depends upon the switching signal under consideration, the existence of singular switching signals is studied alongside the development of a sufficient condition that guarantees uniform observability with respect to switching times. Furthermore, while taking state jumps into account, a relatively weaker characterization is given for determinability, the property that is concerned with recovery of the original state at some time rather than at all times. Assuming determinability of the system, a hybrid observer is designed for the most general case to estimate the state of the system, and it is shown that the estimation error decays exponentially. Since the individual modes of the switched system may not be observable, the proposed strategy for designing the observer is based upon a novel idea of accumulating the information from individual subsystems. Contrary to the usual approach, dwell-time between switchings is not necessary, but the proposed design does require persistent switching. For practical purposes, the calculations also take into account the time consumed in performing computations.

4.1 Introduction

This chapter studies observability conditions and observer construction for switched linear systems described as

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \qquad t \neq \{t_q\}, \tag{4.1a}$$

$$x(t_q) = E_{\sigma(t_q^-)}x(t_q^-) + F_{\sigma(t_q^-)}v_q, \qquad q \geq 1, \tag{4.1b}$$

$$y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t), \qquad t \geq t_0, \tag{4.1c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^{d_y}$ is the output, $v_q \in \mathbb{R}^{d_v}$ and $u(t) \in \mathbb{R}^{d_u}$ are the inputs, and $u(\cdot)$ is a locally bounded measurable function. For some index set $\mathcal{I}$, the switching signal $\sigma : \mathbb{R} \mapsto \mathcal{I}$ is a piecewise constant and right-continuous function that changes its value at switching times $\{t_q\}$, $q \in \mathbb{N}$. In our notation, if a function exhibits discontinuity at time instant t_q, we evaluate that function at t_q^- to represent its value prior to discontinuity, and at t_q to indicate its value right after the jump. It is assumed that there are a finite number of switching times in any finite time interval; thus we rule out the Zeno phenomenon in our problem formulation. The switching mode $\sigma(t)$ and the switching times $\{t_q\}$ may be governed by a supervisory logic controller, or determined internally depending on the system state, or considered as an external input. In any case, it is assumed in this chapter that the signal $\sigma(\cdot)$ (and thus, the active mode and the switching times $\{t_q\}$ as well) is known over the interval of interest. For estimation of the switching signal $\sigma(t)$, one may refer to, e.g., [60, 52, 99, 65].

In the past decade, the structural properties of switched systems have been investigated by many researchers and observability along with observer construction has been one of them.

In switched systems, the observability can be studied from various perspectives. If we allow for the use of the differential operator in the observer, then it may be desirable to determine the continuous state of the system instantaneously from the measured output and inputs. This in turn requires each subsystem to be observable, and the problem becomes nontrivial when the switching signal is treated as an unknown discrete state and simultaneous recovery of the discrete and continuous state is desired. Some results on this problem are published in [52, 53, 54].

On the other hand, if the mode transitions are represented by a known switching signal, then, even though the individual subsystems are not observable, it is still possible to recover the initial state $x(t_0)$ by appropriately processing the measured signals over an interval $[t_0, T)$ that involves multiple switching instants. This phenomenon is of particular interest for switched systems or systems with state jumps as the notion of instantaneous observability and observability over an interval coincide for non-switched linear time invariant systems. This variant of the observability problem in switched systems has been studied most notably by [46, 55, 13]. The authors in [56, 57] have studied the similar problem for the systems that allow jumps in the states, but they do not consider the change in the dynamics that is introduced by switching to different matrices associated with the active mode. The observer design has also received some attention in the literature [58, 59, 60], where the authors have assumed that each mode in the system is in fact observable admitting a state observer, and have treated the switching as a source of perturbation effect. This approach not only has limited applicability but it also incurs the need of a common Lyapunov function for the switched error dynamics, or a fixed amount of dwell-time between switching instants, because it is intrinsically a stability problem of the error dynamics.

The main contribution of this chapter is to present a characterization of observability and an observer design for the systems represented by (4.1), where the subsystems are no longer required to be observable. So the notion of observability adopted in this chapter is related to [55, 46] in the sense that we also consider observability over an interval. However, the authors in [55] only present a coordinate-dependent sufficient condition that leads to the construction of an observer; and the work of [46] only focuses on a necessary and sufficient condition under which there exists a switching signal that makes it possible to recover $x(t_0)$, without any discussion on design of observers. This chapter fills the void by constructing an asymptotic observer based on a necessary and sufficient condition. To the best of the author's knowledge, the considered class of linear systems is the most general for this purpose in the literature.

Similar to our recent work in [100], the switching signal is considered to be known and fixed, so that the trajectory of the system satisfies a time-varying linear differential equation with state jumps. Then for that particular trajectory, we answer the question whether it is possible to recover $x(t_0)$ from the knowledge of measured inputs and outputs. We present a necessary and sufficient condition for observability over an interval, which is independent of coordinate transformations. Since this condition depends upon the switching times, we study the denseness property of a set of switching signals with a fixed mode sequence such that system (4.1) satisfies the observability condition for each switching signal in that set. Furthermore, sufficient conditions guaranteeing uniform observability with respect to switching times are developed. For the sake of completeness, a necessary condition, which can be verified independently of switching times, is derived as a corollary to the main result. Also, with a similar tool set, the notion of determinability, which is more in the spirit of recovering the current state based on the knowledge of inputs and outputs in the past, is developed. Moreover, a hybrid observer for system (4.1) is designed based on the proposed necessary and sufficient condition. Since the observers are useful for various engineering applications, their utility mainly lies in their online operation method. This thought is essentially rooted in the idea for observer construction adopted in this chapter: the idea of combining the partial information available from each mode and processing this collected information at one instance of time to get the estimate of the state. For real-time implementation, the time required for processing this information is also taken into account in our design. We show that under mild assumptions, such an estimate converges to the actual state of the plant and the state estimation error satisfies an exponentially decaying bound.

More emphasis will be given to the case when the individual modes of system (4.1) are not observable (in the classical sense of linear time-invariant systems theory) since it is obvious that the system becomes immediately observable when the system switches to an observable mode. In such cases, the switching signal plays a pivotal role as the observability of the switched system depends upon not only the mode sequence but also the switching times. In order to facilitate our understanding of this matter, let us begin with an example.

Example 4.1. Consider a switched system characterized by:

$$A_a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_b = \begin{bmatrix} \epsilon & 1 \\ -1 & \epsilon \end{bmatrix},$$
$$C_a = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_b = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

with $E_i = I$, $F_i = 0$, $B_i = 0$, $D_i = 0$ for $i \in \mathcal{I} := \{a, b\}$, and ϵ is a constant. It is noted that neither of the two pairs, (A_a, C_a) or (A_b, C_b), is observable. However, if the switching signal $\sigma(t)$ changes its value in the order $a \to b \to a$ at times t_1 and t_2, then we can recover the state. In fact, it turns out that at least two switchings are necessary and the switching sequence should contain the subsequence of modes (a, b, a). For instance, if the switching happens as $a \to b \to a$, the outputs at time t_1^- and t_2 are: $y(t_1^-) = C_a x(t_1^-) = x_1(t_0)$, and $y(t_2) = C_a e^{A_b \tau} x(t_0) = e^{\epsilon\tau} \cos\tau \cdot x_1(t_0) + e^{\epsilon\tau} \sin\tau \cdot x_2(t_0)$, where $x(t_0) = [x_1(t_0), x_2(t_0)]^\top$ is the initial condition and $\tau = t_2 - t_1$. Then, it is obvious that $x(t_0)$ can be recovered from two measurements $y(t_1^-)$ and $y(t_2)$ if $\tau \neq k\pi$ with $k \in \mathbb{N}$. On the other hand, any switching signal whose duration for the mode b is an integer multiple of π is a 'singular' switching signal (whose precise meaning will be given in Section 4.2.2). $\lhd$

Notations: For a square matrix A and a subspace $\mathcal{V}$, we denote by $\langle A|\mathcal{V}\rangle$ the smallest A-invariant subspace containing $\mathcal{V}$, and by $\langle \mathcal{V}|A\rangle$ the largest A-invariant subspace contained in $\mathcal{V}$. (See Property 7 in Appendix B for their computation.) With a matrix A, $\mathcal{R}(A)$ denotes the column space (range space) of A. The sum of two subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ is defined as $\mathcal{V}_1 + \mathcal{V}_2 := \{v_1 + v_2 : v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2\}$. For a possibly non-invertible matrix A, the pre-image of a subspace $\mathcal{V}$ under A is given by $A^{-1}\mathcal{V} = \{x : Ax \in \mathcal{V}\}$. Let $\ker A := A^{-1}\{0\}$; then it is seen that $A^{-1} \ker C = \ker(CA)$ for a matrix C. For convenience of notation, let $A^{-\top}\mathcal{V} := (A^\top)^{-1}\mathcal{V}$ where $A^\top$ is the transpose of A, and it is understood that $A_2^{-1}A_1^{-1}\mathcal{V} = A_2^{-1}(A_1^{-1}\mathcal{V})$. Also, we denote the products of matrices A_i as $\prod_{i=j}^{k} A_i := A_j A_{j+1} \cdots A_k$ when $j < k$, and $\prod_{i=j}^{k} A_i := A_j A_{j-1} \cdots A_k$ when $j > k$. The notation $\mathrm{col}(A_1, \ldots, A_k)$ means the vertical stack of matrices $A_1, \cdots, A_k$, that is, $[A_1^\top, \ldots, A_k^\top]^\top$.

4.2 Geometric Conditions

To make precise the notions of observability and determinability considered in this chapter, let us introduce the formal definitions.

Definition 4.2. *Let $(\sigma^i, u^i, v^i, y^i, x^i)$, for $i = 1, 2$, be the signals that satisfy* (4.1) *over an interval*[1] $[t_0, T^+)$. *We say that the system* (4.1) *is* $[t_0, T^+)$-observable *if the equality*

[1]The notation $[t_0, T^+)$ is used to denote the interval $[t_0, T + \varepsilon)$, where $\varepsilon > 0$ is arbitrarily small. In fact, because of the right-continuity of the switching signal, the output $y(T)$ belongs to the next mode when T is the switching instant. Then, the point-wise measurement $y(T)$ is insufficient to contain the information for the new mode, and thus, it is imperative to consider the output signal over the interval $[t_0, T + \varepsilon)$ with $\varepsilon > 0$. This definition implicitly implies that the observability property does not change for sufficiently small ε (which is true, and becomes clear shortly).

$(\sigma^1, u^1, v^1, y^1) = (\sigma^2, u^2, v^2, y^2)$ *implies that* $x^1(t_0) = x^2(t_0)$. *Similarly, the system* (4.1) *is said to be* $[t_0, T^+)$-determinable *if the equality* $(\sigma^1, u^1, v^1, y^1) = (\sigma^2, u^2, v^2, y^2)$ *implies that* $x^1(T) = x^2(T)$.

Since the initial state $x(t_0)$, the switching signal σ, and the inputs (u, v) uniquely determine $x(t)$ on $[t_0, T^+)$ by (4.1), observability is achieved if and only if the state trajectory $x(t)$, for each $t \in [t_0, T^+)$, is uniquely determined by the inputs, the output, and the switching signal. Obviously, observability implies determinability by forward integration of (4.1), but the converse is not true due to the possibility of non-invertible matrices E_σ. In case there is no jump map (4.1b), or each E_σ is invertible, observability and determinability are equivalent. The notion of determinability has also been called reconstructability in [13].

The use of Definition 4.2 leads to the following proposition which will be useful in deriving conditions for observability.

Proposition 4.3. *For a switching signal* σ*, the system* (4.1) *is* $[t_0, T^+)$*-observable (or, determinable) if, and only if, zero inputs and zero output on the interval* $[t_0, T^+)$ *imply that* $x(t_0) = 0$ *(or,* $x(T) = 0$*).*

Proof. Since the zero solution with the zero inputs yields the zero output, the necessity follows from the fact that $x(t_0)$ (or, $x(T)$) is uniquely determined from the inputs and the outputs. For the sufficiency, suppose that the system (4.1) is not $[t_0, T^+)$-observable (or determinable); that is, there exist two different states $x^1(t_0)$ and $x^2(t_0)$ (or, $x^1(T)$ and $x^2(T)$) that yield the same output y over the interval $[t_0, T^+)$, under the same inputs (u, v). Let $\tilde{x}(t) := x^1(t) - x^2(t)$, where $x^i(t)$, $i = 1, 2$, is the solution of (4.1) which takes the value $x^i(t_0)$ at initial time t_0 (or, $x^i(T)$ at terminal time T). Then, by linearity, it follows that $\dot{\tilde{x}} = A_\sigma \tilde{x}$, $\tilde{x}(t_q) = E_\sigma \tilde{x}(t_q^-)$, and $C_\sigma \tilde{x} = C_\sigma x^1 - C_\sigma x^2 = y - y = 0$, which is the same as system (4.1) with zero inputs and zero outputs; but $\tilde{x}(t_0) = x^1(t_0) - x^2(t_0) \neq 0$ (or, $\tilde{x}(T) = x^1(T) - x^2(T) \neq 0$). Hence, zero inputs and zero output do not imply $x(t_0) = 0$ (or, $x(T) = 0$), and the sufficiency holds. □

Because of Proposition 4.3, we are motivated to introduce the following homogeneous switched system, which has been obtained by setting the inputs (u, v) equal to zero in (4.1):

$$\dot{x}(t) = A_{\sigma(t)} x(t), \quad y(t) = C_{\sigma(t)} x(t), \quad t \in [t_{q-1}, t_q) \tag{4.2a}$$

$$x(t_q) = E_{\sigma(t)} x(t_q^-). \tag{4.2b}$$

If this homogeneous system is observable (or, determinable) with a given σ, then $y \equiv 0$ implies that $x(t_0) = 0$ (or, $x(T) = 0$), and in terms of description of system (4.1), this means that zero inputs and zero output yield $x(t_0) = 0$ (or, $x(T) = 0$); hence, (4.1) is observable (or, determinable) because of Proposition 4.3. On the other hand, if the system (4.1) is observable (or, determinable), then it is still observable (or, determinable) with zero inputs, which is described as system (4.2). Thus, the observability (or, determinability) of systems (4.1) and (4.2) are equivalent.

Before going further, let us rename the switching sequence for convenience. For system (4.1), when the switching signal $\sigma(t)$ takes the mode sequence $\{q_1, q_2, q_3, \cdots\}$, we rename them as increasing integers $\{1, 2, 3, \cdots\}$, which is ever increasing even though the same mode is revisited; for convenience, this sequence is indexed by q and not by $\sigma(t)$. Moreover, it is often the case that the mode of the system changes without the state jump (4.1b), or the state jumps without switching to another mode. In the former case, we can simply take $E_q = I$ and $F_q = 0$, and in the latter case, we increase the mode index by one and take $A_q = A_{q+1}$, $B_q = B_{q+1}$ and so on. In this way various situations fit into the description of (4.1) with increasing mode sequence. The switching time t_q is the instant when transition from mode q to mode $q + 1$ takes place.

4.2.1 Necessary and Sufficient Condition for Observability

In this section, we present a characterization of the unobservable subspace for system (4.2) with a given switching signal. Towards this end, let $\mathcal{N}_q^m$ ($m \geq q$) denote the set of states at $t = t_{q-1}$ for system (4.2) that generate identically zero output over $[t_{q-1}, t_{m-1}^+)$. Then, for fixed switching times, it is easily seen that $\mathcal{N}_q^m$ is actually a subspace due to linearity of (4.2), and we call $\mathcal{N}_q^m$ the *unobservable subspace for* $[t_{q-1}, t_{m-1}^+)$. It can be seen that the system (4.2) is an LTI system between two consecutive switching times, so that its unobservable subspace on the interval $[t_{q-1}, t_q)$ is simply given by the largest A_q-invariant subspace contained in $\ker C_q$, i.e., $\langle \ker C_q | A_q \rangle = \ker G_q$ where

$$G_q := \mathrm{col}(C_q, C_q A_q, \cdots, C_q A_q^{n-1}).$$

So it is clear that $\mathcal{N}_q^q = \ker G_q$. Now, when the measured output is available over the interval $[t_{q-1}, t_{m-1}^+)$ that includes switchings at $t_q, t_{q+1}, \ldots, t_{m-1}$, more information about the state is obtained in general so that $\mathcal{N}_q^m$ gets smaller as the difference $m - q$ gets larger, and we claim that the subspace $\mathcal{N}_q^m$ can be computed recursively as follows:

$$\begin{aligned}\mathcal{N}_m^m &= \ker G_m,\\ \mathcal{N}_q^m &= \ker G_q \cap e^{-A_q\tau_q}E_q^{-1}\mathcal{N}_{q+1}^m, \quad 1 \le q \le m-1,\end{aligned} \tag{4.3}$$

where $\tau_q := t_q - t_{q-1}$. The following theorem presents a necessary and sufficient condition for observability of the system (4.1) while proving the claim in the process.

Theorem 4.4. *For system* (4.2) *with a switching signal* $\sigma_{[t_0,t_{m-1}^+)}$, *the unobservable subspace for* $[t_0, t_{m-1}^+)$ *at* t_0 *is given by* $\mathcal{N}_1^m$ *from* (4.3). *Therefore, system* (4.1) *is* $[t_0, t_{m-1}^+)$*-observable if, and only if,*

$$\mathcal{N}_1^m = \{0\}. \tag{4.4}$$

From (4.3), it is not difficult to arrive at the following formula for $\mathcal{N}_q^m$:

$$\begin{aligned}\mathcal{N}_q^m &= \ker G_q \cap \left(\bigcap_{i=q+1}^{m} \prod_{j=q}^{i-1} e^{-A_j\tau_j}E_j^{-1} \ker G_i \right) & \text{(4.5a)}\\ &= \ker G_q \cap \left(\bigcap_{i=q+1}^{m} \ker \left(G_i \prod_{l=i-1}^{q} E_l e^{A_l\tau_l} \right)\right). & \text{(4.5b)}\end{aligned}$$

In order to inspect the observability of the system (4.2), one can compute $\mathcal{N}_1^m$ using (4.3), (4.5a), or (4.5b). It is easily seen that $\mathcal{N}_1^{m_1} \supseteq \mathcal{N}_1^{m_2}$ if $m_1 \le m_2$ (with arbitrary $\tau_i > 0$, $i \in \mathbb{N}$).

Proof of Theorem 4.4. *Sufficiency.* Using the result of Proposition 4.3, it suffices to show that the identically zero output of (4.2) implies $x(t_0) = 0$. Assume that $y \equiv 0$ on $[t_0, t_{m-1}^+)$. Then, it is immediate that $x(t_{m-1}) \in \mathcal{N}_m^m = \ker G_m$. We next apply the inductive argument to show that $x(t_{q-1}) \in \mathcal{N}_q^m$ for $1 \le q \le m-1$. Suppose that $x(t_q) \in \mathcal{N}_{q+1}^m$, then $x(t_{q-1}) \in e^{-A_q\tau_q}E_q^{-1}\mathcal{N}_{q+1}^m$ since $x(t)$ is the solution of (4.2). Zero output on the interval $[t_{q-1}, t_q)$ also implies that $x(t_{q-1}) \in \ker G_q$. Therefore,

$$x(t_{q-1}) \in \ker G_q \cap e^{-A_q\tau_q}E_q^{-1}\mathcal{N}_{q+1}^m.$$

From (4.3), it follows that $x(t_{q-1}) \in \mathcal{N}_q^m$. This induction proves the claim that $\mathcal{N}_q^m$ is given by (4.3). With $q = 1$, it is seen that $x(t_0) \in \mathcal{N}_1^m = \{0\}$, which proves the sufficiency.

Necessity. Assuming that $\mathcal{N}_1^m \ne \{0\}$, we show that a non-zero initial state $x(t_0) \in \mathcal{N}_1^m$ yields the solution of (4.2) such that $y \equiv 0$ on $[t_0, t_{m-1}^+)$, which implies unobservability. First, we show the following implication:

$$x(t_{q-1}) \in \mathcal{N}_q^m \quad \Rightarrow \quad x(t_q) \in \mathcal{N}_{q+1}^m, \qquad q < m. \tag{4.6}$$

Indeed, assuming that $x(t_{q-1}) \in \mathcal{N}_q^m$ with $q < m$, it follows that $x(t_q) = E_q e^{A_q \tau_q} x(t_{q-1})$, which further gives

$$\begin{aligned} x(t_q) &\in E_q e^{A_q\tau_q}\mathcal{N}_q^m = E_q e^{A_q\tau_q}\left(\ker G_q \cap e^{-A_q\tau_q} E_q^{-1}\mathcal{N}_{q+1}^m\right) \\ &\subseteq E_q \ker G_q \cap E_q E_q^{-1}\mathcal{N}_{q+1}^m = E_q \ker G_q \cap \mathcal{N}_{q+1}^m \cap \mathcal{R}(E_q) \subseteq \mathcal{N}_{q+1}^m \end{aligned}$$

by using (4.3) and Properties 2, 3, and 11 in Appendix B. Therefore, for $0 \le q \le m-1$, $x(t_q) \in \mathcal{N}_{q+1}^m \subseteq \ker G_{q+1}$, and the solution $x(t) = e^{A_{q+1}(t-t_q)}x(t_q)$ for $t \in [t_q, t_{q+1})$ satisfies that $y(t) = C_{q+1}x(t) = 0$ for $t \in [t_q, t_{q+1})$ due to A_{q+1}-invariance of $\ker G_{q+1}$. □

4.2.2 Denseness of Regular Switching Signals

The observability condition (4.4) given in Theorem 4.4 is dependent upon a particular switching signal under consideration, and it is entirely possible that the system is observable for certain switching signals and unobservable for others (*cf.* Example 4.1). However, it would be more useful to know whether the observability property holds for a particular class of switching signals. While addressing this issue in the current section, we show that if there is a switching signal that satisfies (4.4), then the set of switching signals, with the same mode sequence, for which (4.4) does not hold, is nowhere dense.

To formalize this argument, consider the set $\mathcal{S}$ consisting of all switching signals σ (over a possibly different time domain) with a fixed mode sequence $(1, 2, \cdots, m)$ and switching times t_i such that $t_0 < t_1 < \cdots < t_{m-1}$. Then, for each $\sigma \in \mathcal{S}$, there is a corresponding vector $\tau = \mathrm{col}(\tau_1, \cdots, \tau_{m-1}) \in \mathcal{T} := \{\tau \in \mathbb{R}^{m-1} : \tau_i > 0\}$ with $\tau_i = t_i - t_{i-1}$ being the activation period for mode i under σ. We now introduce the metric $d(\cdot, \cdot)$ on the set $\mathcal{S}$ as follows: for any $\sigma^1, \sigma^2 \in \mathcal{S}$,

$$d(\sigma^1, \sigma^2) := \|\tau^1 - \tau^2\|_1 = \sum_{i=1}^{m-1} |\tau_i^1 - \tau_i^2|,$$

where $\tau^1, \tau^2 \in \mathcal{T}$ and $\|\cdot\|_1$ denotes the usual ℓ_1 norm in $\mathbb{R}^{m-1}$. It can be shown that $d(\cdot, \cdot)$ indeed satisfies all the hypotheses of a metric, that is, positive definiteness, symmetry, and triangle inequality. Note that an ε-neighborhood of σ is obtained by perturbing the activation period of each mode, τ_i, by $\varepsilon_i \geq 0$ such that $\sum_{i=1}^{m-1} \varepsilon_i < \varepsilon$; that is, the set of σ' whose corresponding $\tau' \in \mathcal{B}_\varepsilon(\tau) := \{\tau' \in \mathcal{T} : \|\tau' - \tau\|_1 < \varepsilon\}$. From now on, we denote $\mathcal{N}_1^m$ by $\mathcal{N}_1^m(\tau)$ for $\tau \in \mathcal{T}$ to emphasize the dependence of $\mathcal{N}_1^m$ on the switching times.

Theorem 4.5. *Let $\mathcal{S}^* := \{\sigma \in \mathcal{S} :$ system (4.1) is $[t_0, t_{m-1}^+)$-observable with $\sigma\}$. If the set $\mathcal{S}^*$ is non-empty, then it is an open and dense subset of $\mathcal{S}$ under the topology induced by the metric $d(\cdot,\cdot)$.*

A consequence of Theorem 4.5 is that if there exists $\sigma' \in \mathcal{S} \backslash \mathcal{S}^*$, then we can find an element $\sigma'' \in \mathcal{S}^*$ by introducing arbitrarily small perturbations in the vector τ' that corresponds to σ'. We call such σ' a *singular* switching signal and the ones contained in the set $\mathcal{S}^*$ are called *regular* switching signals.

Proof of Theorem 4.5. It is first shown that, for any $\sigma^* \in \mathcal{S}^*$, a neighborhood of σ^* is also contained in $\mathcal{S}^*$. Recalling the expression for $\mathcal{N}_1^m$ from (4.5b), introduce the following matrix:

$$W(\tau) := \operatorname{col}(G_1, G_2 E_1 e^{A_1 \tau_1}, \cdots, G_m \prod_{l=m-1}^{1} E_l e^{A_l \tau_l}),$$

and let $\overline{W}(\tau) := W^\top(\tau) W(\tau)$. Note that $\mathcal{N}_1^m(\tau) = \ker W(\tau)$, so that $\mathcal{N}_1^m(\tau) = \{0\}$ if, and only if, $W(\tau)$ has full column rank, or equivalently $\psi(\tau) \neq 0$, where $\psi : \mathcal{T} \to \mathbb{R}$ denotes the determinant of the matrix $\overline{W}(\tau)$. Since $\overline{W}(\tau)$ comprises analytic functions of τ, the determinant $\psi(\tau)$ is also an analytic function. It is well-known that an analytic function is either identically zero, or the set comprising zeros of an analytic function has an empty interior [101, Chapter 4]. Therefore, the set $\mathcal{Z} := \{\tau \in \mathcal{T} : \psi(\tau) = 0\}$ has an empty interior (with respect to the topology induced by ℓ_1 norm), and is closed. Hence, the set $\mathcal{T} \setminus \mathcal{Z}$ is open and there exists an $\varepsilon > 0$ such that $\psi(\tau) \neq 0$ for each $\tau \in \mathcal{B}_\varepsilon(\tau^*)$ where τ^* is associated with σ^* and satisfies $\psi(\tau^*) \neq 0$ since $\sigma^* \in \mathcal{S}^*$. Now pick any $\sigma \in \mathcal{S}$ such that $d(\sigma, \sigma^*) < \varepsilon$. Then, the corresponding τ belongs to $\mathcal{B}_\varepsilon(\tau^*)$, which implies $\sigma \in \mathcal{S}^*$ showing that $\mathcal{S}^*$ is open.

Next, to show the denseness of $\mathcal{S}^*$, we pick $\sigma' \in \mathcal{S} \setminus \mathcal{S}^*$, and show that σ' is the limit point of $\mathcal{S}^*$. In this case, $\psi(\tau') = 0$, and $\tau' \in \mathcal{Z}$. Since $\mathcal{Z}$ has an empty interior, for every $\varepsilon > 0$, there exists $\tau'' \in \mathcal{B}_\varepsilon(\tau')$ such that $\psi(\tau'') \neq 0$. Let σ'' be the switching signal corresponding to τ''; then $\sigma'' \in \mathcal{S}^*$, proving that every neighborhood of σ', with respect to metric $d(\cdot,\cdot)$, has a non-empty intersection with $\mathcal{S}^*$. □

4.2.3 Conditions Independent of Switching Times

Existence of singular switching signals naturally raises the question whether, under certain conditions, observability holds uniformly with respect to switching times. In other words, it is desirable to know whether the observability could be verified for a given mode sequence independently of the switching times. For this, we again consider the sets $\mathcal{S}$ and $\mathcal{T}$ for a given mode sequence $(1, 2, \cdots, m)$. Then, the following corollary is immediate.

Corollary 4.6. *The switched system* (4.1) *is uniformly observable for all switching signals $\sigma \in \mathcal{S}$ (i.e., $\mathcal{S}^* = \mathcal{S}$) if, and only if,*

$$\mathcal{V}_1^m := \cup_{\tau_1>0} \cup_{\tau_2>0} \cdots \cup_{\tau_{m-1}>0} \mathcal{N}_1^m(\tau) = \{0\}. \tag{4.7}$$

By using the distributive property of intersection over union of sets, one can compute $\mathcal{V}_1^m$ by proceeding in the sequential manner as before:

$$\begin{aligned} \mathcal{V}_m^m &:= \ker G_m \\ \mathcal{V}_q^m &:= \ker G_q \cap (\cup_{\tau_q>0} e^{-A_q\tau_q} E_q^{-1} \mathcal{V}_{q+1}^m), \qquad 1 \le q \le m-1. \end{aligned} \tag{4.8}$$

However, in order to check condition (4.7) in practice, a difficulty arises due to the fact that $\mathcal{V}_1^m$ is not a subspace in general. (This is because the set $\cup_{\tau>0} e^{A\tau}\mathcal{V}$ is not necessarily a subspace although $\mathcal{V}$ is.) To avoid this difficulty, an over-approximation of each $\mathcal{V}_q^m$ is considered, so that a sufficient condition is obtained for uniform observability with respect to switching times.

Corollary 4.7. *Let $\overline{\mathcal{N}}_1^m$ be defined as follows:*

$$\begin{aligned} \overline{\mathcal{N}}_m^m &:= \ker G_m, \\ \overline{\mathcal{N}}_q^m &:= \left\langle A_q \middle| \ker G_q \cap E_q^{-1} \overline{\mathcal{N}}_{q+1}^m \right\rangle, \qquad 1 \le q \le m-1. \end{aligned}$$

Then, $\overline{\mathcal{N}}_1^m$ is an over-approximation of $\mathcal{V}_1^m$, and thus, the system (4.1) *is uniformly observable for all $\sigma \in \mathcal{S}$ if $\overline{\mathcal{N}}_1^m = \{0\}$.*

By construction, the subspace $\overline{\mathcal{N}}_1^m$ is also an over-approximation of $\mathcal{N}_1^m$ so that it serves a sufficient condition for (4.4) as well as for (4.7).

Proof. The proof is completed by showing that $\mathcal{V}_q^m \subseteq \overline{\mathcal{N}}_q^m$ for $1 \le q \le m$. First, note that $\mathcal{V}_m^m = \overline{\mathcal{N}}_m^m$. Assuming that $\mathcal{V}_{q+1}^m \subseteq \overline{\mathcal{N}}_{q+1}^m$ for $1 \le q \le m-1$, we now claim that $\mathcal{V}_q^m \subseteq \overline{\mathcal{N}}_q^m$.

Indeed, by Properties 3, 9, and 11 in Appendix B, and the recursion equation (4.8), we obtain

$$\begin{aligned}\mathcal{V}_q^m &= \ker G_q \cap \left(\cup_{\tau_q>0} e^{-A_q\tau_q} E_q^{-1}\mathcal{V}_{q+1}^m\right) \\ &= \cup_{\tau_q>0} e^{-A_q\tau_q}\left(\ker G_q \cap E_q^{-1}\mathcal{V}_{q+1}^m\right) \\ &\subseteq \left\langle A_q \middle| \ker G_q \cap E_q^{-1}\mathcal{V}_{q+1}^m\right\rangle \\ &\subseteq \left\langle A_q \middle| \ker G_q \cap E_q^{-1}\overline{\mathcal{N}}_{q+1}^m\right\rangle = \overline{\mathcal{N}}_q^m, \quad 1 \le q \le m-1.\end{aligned} \tag{4.9}$$

Therefore, the condition $\overline{\mathcal{N}}_1^m = \{0\}$ implies (4.7). □

Since $\overline{\mathcal{N}}_1^m$ is an over-approximation of $\mathcal{V}_1^m$, it would be of interest to investigate how far the statement of Corollary 4.7 is from necessity.

Lemma 4.8. *For each* $1 \le q \le m-1$*, if* $\mathcal{V}_{q+1}^m = \overline{\mathcal{N}}_{q+1}^m$*, then* $\overline{\mathcal{N}}_q^m$ *is the smallest subspace containing the set* $\mathcal{V}_q^m$*.*

Proof. From the inclusion relation of (4.9), it suffices to show that $\langle A_q|\mathcal{V}\rangle$ is the smallest subspace containing $\cup_{\tau_q>0}e^{-A_q\tau_q}\mathcal{V}$ where $\mathcal{V} := \ker G_q \cap E_q^{-1}\mathcal{V}_{q+1}^m = \ker G_q \cap E_q^{-1}\overline{\mathcal{N}}_{q+1}^m$. Let $\mathcal{W}$ denote the smallest subspace containing $\cup_{\tau_q>0}e^{-A_q\tau_q}\mathcal{V}$. Then, since $\langle A_q|\mathcal{V}\rangle$ is a subspace containing $\cup_{\tau_q>0}e^{-A_q\tau_q}\mathcal{V}$ by Property 9 in Appendix B, it follows that $\mathcal{W} \subseteq \langle A_q|\mathcal{V}\rangle$. Next, pick any $x \in \mathcal{W}^\perp$ and let V be a matrix such that $\mathcal{V} = \mathcal{R}(V)$. From the definition of $\mathcal{W}$, it follows that $x^\top e^{-A_q\tau_q}V = 0$ for all $\tau_q > 0$, but by continuity, it also holds for all $\tau_q \ge 0$. Repeated differentiation of both sides at $\tau_q = 0$ leads to $x^\top A_q^i V = 0$ for $i = 0, 1, \ldots, n-1$, or equivalently $x \in \langle A_q|\mathcal{V}\rangle^\perp$ by Property 7 in Appendix B. This shows that $\mathcal{W}^\perp \subseteq \langle A_q|\mathcal{V}\rangle^\perp$, and hence, $\mathcal{W} = \langle A_q|\mathcal{V}\rangle$. □

The above discussion can be summarized as follows, which suggests when the condition in Corollary 4.7 becomes necessary.

Corollary 4.9. *If each* $\mathcal{V}_q^m$*, for* $1 \le q \le m-1$*, is a subspace, then system* (4.1) *is uniformly observable with respect to the switching times if and only if* $\overline{\mathcal{N}}_1^m = \{0\}$*.*

Example 4.10. The switched system considered in Example 4.1, with the mode sequence (a, b, a) and an arbitrary constant ϵ, serves an example for Corollary 4.9. It is seen that each $\mathcal{V}_q^m$ is a subspace as $\overline{\mathcal{N}}_3^3 = \mathcal{V}_3^3 = \text{span}\{\text{col}(0,1)\}$, $\overline{\mathcal{N}}_2^3 = \mathcal{V}_2^3 = \mathbb{R}^2$, and $\mathcal{N}_1^3 = \mathcal{V}_1^3 = \text{span}\{\text{col}(0,1)\} \ne \{0\}$. Indeed, the switched system in Example 4.1 is not uniformly observable for the mode sequence (a, b, a), as seen by the existence of the singular switching signals.

Now, let us consider an additional subsystem indexed by c, where $A_c := \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $E_c = I_{2\times 2}$, and B_c, C_c, D_c, F_c are zero matrices with appropriate dimensions. For the switched system with the mode sequence (a, c, a), and $m = 3$, we get $\overline{\mathcal{N}}_3^3 = \operatorname{span}\{\operatorname{col}(0,1)\}$, $\overline{\mathcal{N}}_2^3 = \mathbb{R}^2$, and $\overline{\mathcal{N}}_1^3 = \operatorname{span}\{\operatorname{col}(0,1)\} \neq \{0\}$. The sufficient condition in Corollary 4.7 is violated but the resulting switched system is observable for all $\tau_1, \tau_2 > 0$ (which can be seen from Theorem 4.4). The source of this gap is the fact that the set $\mathcal{V}_2^3 = \cup_{\tau_2>0} e^{-A_c\tau_2}(\ker G_c \cap \ker G_a) = \cup_{\tau_2>0} e^{-A_c\tau_2} \operatorname{span}\{\operatorname{col}(0,1)\}$ is not a subspace. It is seen that the smallest subspace containing $\mathcal{V}_2^3$, in this case, is $\overline{\mathcal{N}}_2^3 = \mathbb{R}^2$. ◁

Having studied the uniform observability for the switching times, we also discuss existence of switching times for observability under a given mode sequence. This is to see whether the set $\mathcal{S}^*$ is empty or not when the mode sequence is given (or, the set $\mathcal{S}$ is given). Regarding this question, the idea of under-approximating $\mathcal{N}_1^m$ yields the following necessary condition.

Corollary 4.11. *Let $\underline{\mathcal{N}}_1^m$ be defined as follows:*

$$\begin{aligned} &\underline{\mathcal{N}}_m^m := \ker G_m, \\ &\underline{\mathcal{N}}_q^m := \left\langle \ker G_q \cap E_q^{-1} \underline{\mathcal{N}}_{q+1}^m | A_q \right\rangle, \qquad 1 \le q \le m-1. \end{aligned}$$

Then, $\underline{\mathcal{N}}_1^m$ is an under-approximation of $\mathcal{N}_1^m(\tau)$ for all $\tau \in \mathcal{T}$, and thus, if there exists a vector $\tau \in \mathcal{T}$ such that $\mathcal{N}_1^m(\tau) = \{0\}$, that is, system (4.1) is $[t_0, t_{m-1}^+)$-observable (or, equivalently $\mathcal{S}^*$ is non-empty), then $\underline{\mathcal{N}}_1^m = \{0\}$.

Proof. For each $\tau \in \mathcal{T}$, the proof proceeds similar to Corollary 4.7. With $\mathcal{N}_m^m = \underline{\mathcal{N}}_m^m$, we assume that $\mathcal{N}_{q+1}^m \supseteq \underline{\mathcal{N}}_{q+1}^m$ for $1 \le q \le m-1$, and claim that $\mathcal{N}_q^m \supseteq \underline{\mathcal{N}}_q^m$. Again by Properties 3, 9, and 11 in Appendix B, and employing equation (4.3), we obtain

$$\begin{aligned} \mathcal{N}_q^m &= e^{-A_q\tau_q}\left(\ker G_q \cap E_q^{-1}\mathcal{N}_{q+1}^m\right) \\ &\supseteq \left\langle \ker G_q \cap E_q^{-1}\mathcal{N}_{q+1}^m | A_q \right\rangle \\ &\supseteq \left\langle \ker G_q \cap E_q^{-1}\underline{\mathcal{N}}_{q+1}^m | A_q \right\rangle = \underline{\mathcal{N}}_q^m, \quad 1 \le q \le m-1. \end{aligned} \tag{4.10}$$

The condition $\underline{\mathcal{N}}_1^m = \{0\}$ is then implied by $\mathcal{N}_1^m(\tau) = \{0\}$. □

As a matter of fact, it can be shown that[2]

$$\underline{\mathcal{N}}_1^m = \cap_{\tau_1>0} \cdots \cap_{\tau_{m-1}>0} \mathcal{N}_1^m(\tau). \tag{4.11}$$

Since the right-hand side of (4.11) can become $\{0\}$ even though $\mathcal{N}_1^m(\tau) \neq \{0\}$ for any $\tau \in \mathcal{T}$, it is clear that the condition $\underline{\mathcal{N}}_1^m = \{0\}$ is much weaker than requiring the existence of τ having the property that $\mathcal{N}_1^m(\tau) = \{0\}$. This can also be seen in the following example, but Corollary 4.11 is still useful when verifying unobservability of a given switched system with a mode sequence for arbitrary switching times.

Example 4.12. Suppose that, for the switched system of Example 4.1, the mode sequence (b, a) is given. Then, with $m = 2$, we obtain that $\underline{\mathcal{N}}_2^2 = \ker G_a = \mathrm{span}\{\mathrm{col}(0, 1)\}$ and $\underline{\mathcal{N}}_1^2 = \langle \ker G_b \cap \ker G_a | A_b \rangle = \{0\}$. However, it is verified that $\mathcal{N}_1^2(\tau_1) = \ker\, [e^{\epsilon\tau_1} \cos\tau_1, e^{\epsilon\tau_1} \sin\tau_1] = \mathrm{col}(\cos\tau, -\sin\tau) \neq \{0\}$, so that (4.4) does not hold for any $\tau_1 > 0$, showing that the system is not observable even though the necessary condition of Corollary 4.11 is satisfied. On the other hand, if we consider a different mode sequence (a, b), then $\underline{\mathcal{N}}_2^2 = \ker G_b = \mathbb{R}^2$ and $\underline{\mathcal{N}}_1^2 = \mathrm{span}\{\mathrm{col}(0, 1)\}$. This indicates that there is no possibility of having a set of switching times that yields observability with mode sequence (a, b). Indeed, this is the case since $\mathcal{N}_1^2(\tau_1) = \mathrm{span}\{\mathrm{col}(0, 1)\} \neq \{0\}$ for all $\tau_1 > 0$. $\lhd$

Remark 4.13. By taking orthogonal complements of $\mathcal{N}_q^m$, $\overline{\mathcal{N}}_q^m$ and $\underline{\mathcal{N}}_q^m$, respectively, we get dual conditions for observability, using Properties 5, 6, 8, and 10 in Appendix B, as follows. System (4.1) is $[t_0, t_{m-1}^+)$-observable if and only if $\mathcal{P}_1^m = \mathbb{R}^n$ where

$$\mathcal{P}_1^m := (\mathcal{N}_1^m)^\perp = \mathcal{R}(G_1^\top) + \sum_{i=2}^{m} \prod_{j=1}^{i-1} e^{A_j^\top \tau_j} E_j^\top \mathcal{R}(G_j^\top).$$

Similarly, one can state Corollaries 4.7 and 4.11 in alternate forms. System (4.1) is uniformly observable if $\underline{\mathcal{P}}_1^m = \mathbb{R}^n$, where $\underline{\mathcal{P}}_1^m$ is computed as:

$$\underline{\mathcal{P}}_m^m = (\overline{\mathcal{N}}_m^m)^\perp = \mathcal{R}(G_m^\top)$$
$$\underline{\mathcal{P}}_q^m = (\overline{\mathcal{N}}_q^m)^\perp = \left\langle \mathcal{R}(G_q^\top) + E_q^\top \underline{\mathcal{P}}_{q+1}^m | A_q^\top \right\rangle,\ 1 \le q \le m-1.$$

[2] Indeed, this follows from (4.10) and from the claim that $\cap_{\tau>0} e^{-A\tau}\mathcal{V} = \langle \mathcal{V}|A\rangle$ for any subspace $\mathcal{V}$ and a matrix A, which is proved as follows. Since $e^{-A\tau}\mathcal{V} \supseteq \langle \mathcal{V}|A\rangle$ for all τ, we have that $\cap_{\tau>0} e^{-A\tau}\mathcal{V} \supseteq \langle \mathcal{V}|A\rangle$. On the other hand, since $(\cap_{\tau>0} e^{-A\tau}\mathcal{V})^\perp$ is a subspace containing $\cup_{\tau>0} e^{A^\top\tau}\mathcal{V}^\perp$, it holds that $(\cap_{\tau>0} e^{-A\tau}\mathcal{V})^\perp \supseteq \langle A^\top|\mathcal{V}^\perp\rangle$ because $\langle A^\top|\mathcal{V}^\perp\rangle$ is the smallest subspace containing $\cup_{\tau>0} e^{A^\top\tau}\mathcal{V}^\perp$ (see Lemma 4.8). Taking orthogonal complement, we get $\cap_{\tau>0} e^{-A\tau}\mathcal{V} \subseteq \langle \mathcal{V}|A\rangle$ (by Property 10 in Appendix B), which proves the claim.

Also, if system (4.1) is $[t_0, t_{m-1}^+)$-observable with a switching signal, then $\overline{\mathcal{P}}_1^m = \mathbb{R}^n$, where $\overline{\mathcal{P}}_1^m$ is defined sequentially as:

$$\begin{aligned}\overline{\mathcal{P}}_m^m &= (\underline{\mathcal{N}}_m^m)^\perp = \mathcal{R}(G_m^\top) \\ \overline{\mathcal{P}}_q^m &= (\underline{\mathcal{N}}_q^m)^\perp = \left\langle A_q^\top | \mathcal{R}(G_q^\top) + E_q^\top \overline{\mathcal{P}}_{q+1}^m \right\rangle, \; 1 \le q \le m-1. \end{aligned} \qquad \triangleleft$$

4.2.4 Necessary and Sufficient Conditions for Determinability

In order to study determinability of system (4.1) and arrive at a result parallel to Theorem 4.4, our first goal is to develop an object similar to $\mathcal{N}_q^m$. So, for system (4.2) with a given switching signal, let $\mathcal{Q}_q^m$ be the set of states at time t_{m-1} (or t_{m-1}^+) such that its corresponding solution $x(t)$ produces zero output on the interval $[t_{q-1}, t_{m-1}^+)$. We call $\mathcal{Q}_q^m$ the *undeterminable subspace for* $[t_{q-1}, t_{m-1}^+)$. Then, it can be shown that $\mathcal{Q}_q^m$ is computed recursively as follows:

$$\begin{aligned}\mathcal{Q}_q^q &:= \ker G_q \\ \mathcal{Q}_q^k &:= \ker G_k \cap E_{k-1} e^{A_{k-1}\tau_{k-1}} \mathcal{Q}_q^{k-1}, \quad q+1 \le k \le m. \end{aligned} \tag{4.12}$$

These sequential definitions lead to another equivalent expression for $\mathcal{Q}_q^m$:

$$\mathcal{Q}_q^m = \ker G_m \cap E_{m-1} \ker(G_{m-1}) \;\cap \left(\bigcap_{i=q}^{m-2} \prod_{l=m-1}^{i+1} E_l e^{A_l \tau_l} E_i \ker G_i \right). \tag{4.13}$$

In fact, the subspace $\Pi_{l=m-1}^{i+1} E_l e^{A_l \tau_l} E_i \ker G_i$ indicates the set of states obtained by propagating the unobservable states of the mode i (where $q \le i \le m-2$) to the time t_{m-1} under the dynamics of system (4.2). Intersection of these subspaces with $E_{m-1} \ker G_{m-1}$ and $\ker G_m$ shows that $\mathcal{Q}_q^m$ is the set of states that cannot be determined at time t_{m-1} from the zero output for the interval $[t_{q-1}, t_{m-1}^+)$. Therefore, the determinability of system (4.2) can now be characterized as in the following theorem.

Theorem 4.14. *For system* (4.2) *and a given switching signal* $\sigma_{[t_0, t_{m-1}^+)}$*, the undeterminable subspace for* $[t_0, t_{m-1}^+)$ *at* t_{m-1} *is given by* $\mathcal{Q}_1^m$ *of* (4.13)*. Therefore, system* (4.1) *is* $[t_0, t_{m-1}^+)$*-determinable if and only if*

$$\mathcal{Q}_1^m = \{0\}. \tag{4.14}$$

The condition (4.14) is equivalent to (4.4) when all E_q matrices, $q = 1, \dots, m-1$, are invertible because of the relation

$$\mathcal{Q}_1^m = \prod_{l=m-1}^{1} E_l e^{A_l \tau_l} \mathcal{N}_1^m.$$

Example 4.15. If any of the jump maps E_q of (4.2), $q = 1, \ldots, m-1$, is a zero matrix, then (4.14) trivially holds regardless of whether (4.4) holds or not. This is intuitively clear because we can uniquely determine $x(t_{m-1}) = 0$ even if $x(t_0)$ cannot be determined. $\lhd$

Recalling that $\mathcal{S}$ is the set of switching signals σ with mode sequence $(1, 2, \ldots, m)$ and $\tau \in \mathcal{T}$, the following two corollaries parallel Corollaries 4.7 and 4.11, and are given for completeness. Proofs are omitted but can be done based on the property that $\underline{\mathcal{Q}}_1^q \subseteq \mathcal{Q}_1^q(\tau) \subseteq \cup_{\tau \in \mathcal{T}} \mathcal{Q}_1^q(\tau) \subseteq \overline{\mathcal{Q}}_1^q$.

Corollary 4.16. *System* (4.1) *is uniformly determinable for all* $\sigma \in \mathcal{S}$, *i.e.,* $\mathcal{Q}_1^m(\tau) = \{0\}$ *for all* $\tau \in \mathcal{T}$, *if* $\overline{\mathcal{Q}}_1^m = \{0\}$, *where* $\overline{\mathcal{Q}}_1^m$ *is computed by*

$$\begin{aligned} \overline{\mathcal{Q}}_1^1 &:= \ker G_1 \\ \overline{\mathcal{Q}}_1^q &:= E_{q-1} \left\langle A_{q-1} | \overline{\mathcal{Q}}_1^{q-1} \right\rangle \cap \ker G_q, \quad 2 \le q \le m. \end{aligned}$$

Corollary 4.17. *If there exists a vector* $\tau \in \mathcal{T}$ *such that* $\mathcal{Q}_1^m(\tau) = \{0\}$, *i.e., system* (4.1) *is* $[t_0, t_{m-1}^+)$*-determinable for some* $\sigma \in \mathcal{S}$, *then* $\underline{\mathcal{Q}}_1^m = \{0\}$, *where* $\underline{\mathcal{Q}}_1^m$ *is computed by*

$$\begin{aligned} \underline{\mathcal{Q}}_1^1 &:= \ker G_1 \\ \underline{\mathcal{Q}}_1^q &:= E_{q-1} \left\langle \underline{\mathcal{Q}}_1^{q-1} | A_{q-1} \right\rangle \cap \ker G_q, \quad 2 \le q \le m. \end{aligned}$$

Remark 4.18. An alternative dual characterization of determinability is possible by inspecting whether the complete state information is available while going forward in time. This is achieved in terms of the subspace $\mathcal{M}_q^m$, obtained by taking the orthogonal complement of $\mathcal{Q}_q^m$. Using Properties 5, 6, 8, and 10 in Appendix B, it follows from (4.13) that

$$\mathcal{M}_q^m := (\mathcal{Q}_q^m)^\perp = \sum_{i=q}^{m-2} \prod_{l=m-1}^{i+1} E_l^{-\top} e^{-A_l^\top \tau_l} E_i^{-\top} \mathcal{R}(G_i^\top) + E_{m-1}^{-\top} \mathcal{R}(G_{m-1}^\top) + \mathcal{R}(G_m^\top).$$

Note that, with $m > q$, the set-valued map $\prod_{j=q}^{m-1} e^{-A_j \tau_j} E_j^{-1} x$ pulls the state x at t_{m-1} back in time at t_{q-1}. This map was used in (4.5a) to pull $\mathcal{N}_m^m$ back at t_{q-1}. Since the dual of $\mathbb{R}^n$ is also $\mathbb{R}^n$, the adjoint of this map pushes the row vectors forward in time from t_{q-1}

to t_{m-1}. In other words, $\mathcal{M}_q^m$ is the set of states at time instant $t = t_{m-1}$ that can be identified, modulo the unobservable subspace at t_{m-1}, from the information of y over the interval $[t_{q-1}, t_{m-1}^+)$. Therefore, the dual statement for determinability is that system (4.1) is $[t_0, t_{m-1}^+)$-determinable if and only if

$$\mathcal{M}_1^m = \mathbb{R}^n. \tag{4.15}$$

It is noted that a recursive expression for $\mathcal{M}_1^m$ is given by

$$\begin{aligned} &\mathcal{M}_1^1 = \mathcal{R}(G_1^\top) \\ &\mathcal{M}_1^q = E_{q-1}^{-\top} e^{-A_{q-1}^\top \tau_{q-1}} \mathcal{M}_1^{q-1} + \mathcal{R}(G_q^\top), \quad 2 \leq q \leq m, \end{aligned}$$

and the dual statements of Corollaries 4.16 and 4.17, that are independent of switching times, are given as follows: system (4.1) is uniformly determinable for all $\sigma \in \mathcal{S}$ if $\underline{\mathcal{M}}_1^m = \mathbb{R}^n$, where

$$\begin{aligned} &\underline{\mathcal{M}}_1^1 := (\overline{\mathcal{Q}}_1^m)^\perp = \mathcal{R}(G_1^\top), \\ &\underline{\mathcal{M}}_1^q := (\overline{\mathcal{Q}}_1^q)^\perp = E_{q-1}^{-\top} \left\langle \underline{\mathcal{M}}_1^{q-1} | A_{q-1}^\top \right\rangle + \mathcal{R}(G_q^\top), \quad 2 \leq q \leq m. \end{aligned}$$

Similarly, if there exists a $\sigma \in \mathcal{S}$ such that system (4.1) is $[t_0, t_{m-1}^+)$-determinable then $\overline{\mathcal{M}}_1^m = \mathbb{R}^n$, where $\overline{\mathcal{M}}_1^m$ is computed as follows:

$$\begin{aligned} &\overline{\mathcal{M}}_1^1 := (\underline{\mathcal{Q}}_1^m)^\perp = \mathcal{R}(G_1^\top), \\ &\overline{\mathcal{M}}_1^q := (\underline{\mathcal{Q}}_1^q)^\perp = E_{q-1}^{-\top} \left\langle A_{q-1}^\top | \overline{\mathcal{M}}_1^{q-1} \right\rangle + \mathcal{R}(G_q^\top), \quad 2 \leq q \leq m. \end{aligned} \qquad \triangleleft$$

4.3 Observer Design

In engineering practice, an observer is designed to provide an estimate of the actual state value at current time. In this regard, determinability (weaker than observability according to Definition 4.2) is a suitable notion for switched systems. Based on the conditions obtained for determinability in the previous section, an asymptotic observer is designed for system (4.1) in this section. By asymptotic observer, we mean that the estimate $\hat{x}(t)$ converges to the plant state $x(t)$ as $t \to \infty$.

4.3.1 Observer Overview

In order to construct an observer for system (4.1), we introduce the following assumptions.

Assumption 4.1. *1. The switching is persistent in the sense that there exists a $T_D > 0$ such that a switch occurs at least once in every time interval of length T_D; that is,*

$$t_q - t_{q-1} \le T_D, \qquad \forall\, q \in \mathbb{N}. \tag{4.16}$$

In addition, there are a bounded number of switchings in any finite time interval; i.e., there is a function $J_{\max}(\cdot)$ such that the number of switchings in any time interval of duration T is less than or equal to $J_{\max}(T)$.

2. The system is persistently determinable in the sense that there exists an $N \in \mathbb{N}$ such that

$$\dim \mathcal{M}_{q-N}^{q} = n, \qquad \forall\, q \ge N+1. \tag{4.17}$$

(The integer N is interpreted as the minimal number of switches required to gain determinability.)

3. There are constants b_A and b_E such that $\|A_q\| \le b_A$ and $\max\{1, \|E_q\|\} \le b_E$ for all $q \in \mathbb{N}$ (which is always the case when A_q and E_q belong to a finite set).

The observer we propose is a hybrid dynamical system of the form

$$\dot{\hat{x}}(t) = A_q\hat{x}(t) + B_q u(t), \qquad t \in [t_{q-1}, t_q),\ t \ne \hat{t}_k, \tag{4.18a}$$

$$\hat{x}(t_q) = E_q\hat{x}(t_q^-) + F_q v_q, \quad q \ge 1, \tag{4.18b}$$

$$\hat{x}(\hat{t}_k) = \hat{x}(\hat{t}_k^-) - \xi_k, \quad k \ge 1, \tag{4.18c}$$

with an arbitrary initial state $\hat{x}(t_0) \in \mathbb{R}^n$. Here, $\hat{t}_k$ is the time for the k-th estimation update (see Fig. 4.2), and we assume that $\hat{t}_k \ne t_q$ for any k and q because these updates (4.18b) and (4.18c) are executed sequentially in a digital computer. Figure 4.1, together with Fig. 4.2, provides an illustration of how the proposed observer (4.18) is executed in practice. It is seen that the observer consists of a system copy and an estimate update law by a correction vector ξ_k. The vector ξ_k can be thought of as an approximation of the state estimation error and our goal is to design ξ_k such that $\hat{x}(t) \to x(t)$. In order to construct ξ_k, external signals available from the actual plant are gathered and stored over a time interval encompassing N switches. This is in contrast with the conventional observers where the observer usually runs in parallel with the actual plant and the information is not stored. In our case, however, not only is there a dynamic part of the observer running synchronously with the plant, but in addition the stored knowledge is processed to update the state estimates; see Fig. 4.1. The processing of the information starts with running another dynamic observer for partial states of the most recent $N+1$ active modes. The partial state information, thus accumulated, is used in getting an approximation of the estimation error at switching times using some inversion formula. This approximation is then transported to the current time through a *catch-up* process. Execution of these procedures needs some time for computation, and unlike our conference paper [102] we no longer assume that these computations are performed instantaneously; rather the calculations required to update the state estimate are carried out over an interval and the length of that interval is assumed to be no longer than a computation time $T_C > 0$. For example, in Fig. 4.2, having gathered the information of external signals over the interval $[t_0, t_3)$, the aforementioned computation starts at t_3 and the update in state estimate is introduced at $\hat{t}_1 \leq t_3 + T_C$. This process continues every time some new information appears from the new mode after a switch. In the case that some switches occur while performing computation, we just look at the past $N+1$ active modes when performing the next update. In particular, if the number of switches exceeds N while computing the state update based on the past information, then we ignore some switches; see the information processed after $\hat{t}_3$ in Fig. 4.2.

We remark that the idea of post-processing the stored information is really significant for switched systems with unobservable modes. It will turn out that the gains for partial

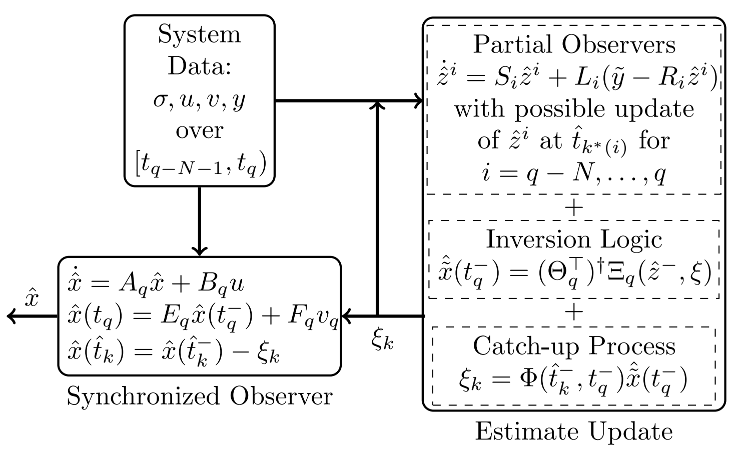


Figure 4.1: The flow diagram of the proposed observer. The stored information is processed in the Estimate Update block to generate the updating signal ξ_k, which is passed on to the Synchronized Observer running parallel to the plant.

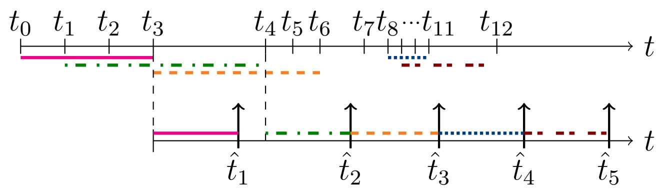


Figure 4.2: Assuming that $N = 2$ in this figure, the computation of ξ_1 begins at time t_3. By processing the data between t_0 and t_3, the computation completes at $\hat{t}_1$ and the estimate $\hat{x}(\hat{t}_1)$ is updated by (4.18c). By assuming that T_C is the maximum computation time, $\hat{t}_1 \leq t_3 + T_C$. The computation for ξ_3 begins at $\hat{t}_2$ which is delayed because the previous computation for ξ_2, having started at t_4, does not finish when the new switchings occur at t_5 and t_6. While computing ξ_4, only the data between t_8 and t_{11} is processed.

observers of individual modes cannot be computed a priori since they require the knowledge of some switching times from the past.

4.3.2 Observer Implementation

In the sequel, the above thought process is formalized by setting up a machinery to compute the correction vector ξ_k as indicated in Fig. 4.1. Based on these computations, a procedure for implementing the hybrid observer, according to the scheme shown in Fig. 4.2, is outlined in Algorithm 3. It is then shown in Theorem 4.19 that the state estimate computed according to the parameter bounds given in Algorithm 3 indeed converges to the actual state of the system.

With $\tilde{x} := \hat{x} - x$, the error dynamics are described by

$$\dot{\tilde{x}}(t) = A_q \tilde{x}(t), \tag{4.19a}$$

$$\tilde{x}(t_q) = E_q \tilde{x}(t_q^-), \tag{4.19b}$$

$$\tilde{x}(\hat{t}_k) = \tilde{x}(\hat{t}_k^-) - \xi_k. \tag{4.19c}$$

The output error is defined as $\tilde{y}(t) := C_q \hat{x}(t) + D_q u(t) - y(t) = C_q \tilde{x}(t)$. Since estimating $\tilde{x}$ is equivalent to the estimation of x (obtained by subtracting $\tilde{x}$ from $\hat{x}$ of (4.18)), our design begins with the estimation of $\tilde{x}$ at time t_q^-, when there is enough information available at t_q^-. In order to estimate the observable part of $\tilde{x}$ from each mode q, let us design partial observers using the Kalman observability decomposition [19] and the dynamics in (4.19). Choose a matrix Z^q such that its columns are an orthonormal basis of $\mathcal{R}(G_q^\top)$, that is, $\mathcal{R}(Z^q) = \mathcal{R}(G_q^\top)$. Further, choose a matrix W^q such that its columns are an orthonormal basis of $\ker G_q$. From the construction, there are matrices $S_q \in \mathbb{R}^{r_q \times r_q}$ and $R_q \in \mathbb{R}^{d_y \times r_q}$, where $r_q = \operatorname{rank} G_q$, such that $Z^{q\top} A_q = S_q Z^{q\top}$ and $C_q = R_q Z^{q\top}$, and the pair (S_q, R_q) is observable. Let $z^q := Z^{q\top}\tilde{x}$ and $w^q := W^{q\top}\tilde{x}$, so that z^q (resp. w^q) denotes the observable (resp. unobservable) states of the mode q. Thus, for $z^q \in \mathbb{R}^{r_q}$, the error dynamics in (4.19) satisfy

$$\begin{aligned} \dot{z}^q(t) &= S_q z^q(t), \quad \tilde{y}(t) = R_q z^q(t), \quad t \in [t_{q-1}, t_q) \\ z^q(\hat{t}_k) &= z^q(\hat{t}_k^-) - Z^{q\top}\xi_k, \qquad \text{if } \hat{t}_k \in (t_{q-1}, t_q) \end{aligned} \tag{4.20}$$

with the initial condition $z^q(t_{q-1}) = Z^{q\top}\tilde{x}(t_{q-1})$. Since z^q is observable over the interval $[t_{q-1}, t_q)$, a standard Luenberger observer, whose role is to estimate $z^q(t_q^-)$ at the end of the interval, is designed as:

$$\begin{aligned} \dot{\hat{z}}^q(t) &= S_q \hat{z}^q(t) + L_q(\tilde{y}(t) - R_q \hat{z}^q(t)), \quad t \in [t_{q-1}, t_q) \\ \hat{z}^q(\hat{t}_k) &= \hat{z}^q(\hat{t}_k^-) - Z^{q\top}\xi_k, \qquad \text{if } \hat{t}_k \in (t_{q-1}, t_q) \end{aligned} \tag{4.21}$$

with the initialization $\hat{z}^q(t_{q-1}) = 0$, where L_q is a matrix such that $(S_q - L_q R_q)$ is Hurwitz. Note that we have fixed the initial condition of the estimator to be zero for each interval.

Now let us define the state transition matrix $\Phi(s, r)$, $s > r$, that results in $\tilde{x}(s) = \Phi(s, r)\tilde{x}(r)$ along the dynamics (4.19a) and (4.19b) (but not (4.19c)). For example, when $s = t_j^-$ and $r = t_i^-$ $(j > i)$ are switching instants, we have that

$$\Phi(t_j^-, t_i^-) = e^{A_j \tau_j} E_{j-1} e^{A_{j-1}\tau_{j-1}} E_{j-2} \cdots e^{A_{i+1}\tau_{i+1}} E_i =: \Psi_i^j \tag{4.22}$$

in which Ψ_i^j is defined for convenience in the future. Note that Ψ_i^j is computed using the knowledge of the switching periods $\{\tau_{i+1}, \cdots, \tau_j\}$ which will be denoted simply by $\tau_{\{i+1,j\}}$, and note also that $\Psi_i^i := I$.

We now define a matrix Θ_i^q with $i \le q$ whose columns form a basis of the subspace $\mathcal{R}(\Psi_i^q W^i)^\perp$; that is,

$$\mathcal{R}(\Theta_i^q) = \mathcal{R}(\Psi_i^q W^i)^\perp, \quad i = q - N, \cdots, q.$$

By construction, each column of Θ_i^q is orthogonal to the subspace $\ker G_i$ that has been transported from t_i^- to t_q^- along the error dynamics (4.19a) and (4.19b). This matrix Θ_i^q will be used for filtering out the unobservable component in the state estimate obtained from the mode i after being transported to the time t_q^-. As a convention, we take Θ_i^q to be a null matrix whenever $\mathcal{R}(\Psi_i^q W^i)^\perp = \{0\}$. Using the determinability of the system (Assumption

4.1.2), it will be shown later in the proof of Theorem 4.19 that the matrix

$$\Theta_q := [\Theta_q^q \vdots \cdots \vdots \Theta_{q-N}^q] \tag{4.23}$$

has rank n. Equivalently, $\Theta_q^\top$ has n independent columns and is left-invertible, so that $(\Theta_q^\top)^\dagger = (\Theta_q \Theta_q^\top)^{-1}\Theta_q$, where $\dagger$ denotes the left-pseudo-inverse. Introduce the notation,

$$\mathcal{K}_i^j := \{k \in \mathbb{N} : \hat{t}_k \in (t_i, t_j)\}, \quad \xi_{\{i,j\}} := \{\xi_k : k \in \mathcal{K}_i^j\}, \quad \text{and} \quad \hat{z}^-_{\{i,j\}} := \{\hat{z}^i(t_i^-), \ldots, \hat{z}^j(t_j^-)\}.$$

Let us also define the vector Ξ_q as

$$\Xi_q(\hat{z}^-_{\{q-N,q\}}, \xi_{\{q-N,q\}}) := \begin{bmatrix} \Theta_q^{q^\top} \Psi_q^q Z^q \hat{z}^q(t_q^-) \\ \Theta_{q-1}^{q^\top} \left(\Psi_{q-1}^q Z^{q-1} \hat{z}^{q-1}(t_{q-1}^-) - \sum_{k \in \mathcal{K}_{q-1}^q} \Phi(t_q^-, \hat{t}_k^-)\xi_k \right) \\ \vdots \\ \Theta_{q-N}^{q^\top} \left(\Psi_{q-N}^q Z^{q-N} \hat{z}^{q-N}(t_{q-N}^-) - \sum_{k \in \mathcal{K}_{q-N}^q} \Phi(t_q^-, \hat{t}_k^-)\xi_k \right) \end{bmatrix}.$$

The matrices M_i^q with $i = q-N, \cdots, q$ are defined such that M_i^q is a null matrix when Θ_i^q is null, and the following holds:

$$[M_q^q, M_{q-1}^q, \cdots, M_{q-N}^q] := (\Theta_q^\top)^\dagger \times \operatorname{blockdiag}\left(\Theta_q^{q^\top}\Psi_q^q, \Theta_{q-1}^{q^\top}\Psi_{q-1}^q, \cdots, \Theta_{q-N}^{q^\top}\Psi_{q-N}^q\right). \tag{4.24}$$

Each non-empty M_i^q is an n by n matrix whose argument is $\tau_{\{q-N+1,q\}}$ in general (due to the inversion of $\Theta_q^\top$), while the argument of both Θ_i^q and Ψ_i^q is $\tau_{\{i+1,q\}}$.

Finally, let $T_B := \max\{T_D, T_C\}$, where T_C is the upper bound on computation time, and define

$$\bar{b} := e^{b_A \cdot (2T_C + T_B)} \cdot b_E^{J_{\max}(2T_C) + J_{\max}(T_B)}. \tag{4.25}$$

Pick any number $\alpha \in (0,1)$ and compute the injection gain L_i such that

$$\bar{b}\|M_i^q(\tau_{\{q-N+1,q\}}) Z^i e^{(S_i - L_i R_i)\tau_i} Z^{i\top}\| \leq \frac{\alpha}{N+2}. \tag{4.26}$$

(One constructive way to compute such an L_i is from the *squashing lemma* [103, Lemma 1].) Using the information over the interval $[t_{q-N-1}, t_q)$, the error correction vector ξ_k in (4.18c) is now computed as:

$$\xi_k = \Phi(\hat{t}_k^-, t_q^-)\hat{\tilde{x}}(t_q^-), \tag{4.27}$$

where

$$\hat{\hat{x}}(t_q^-) = (\Theta_q^\top)^\dagger \Xi_q(\hat{z}^-_{\{q-N,q\}}, \xi_{\{q-N,q\}}). \tag{4.28}$$

The following algorithm summarizes these calculations for ξ_k and also illustrates how the schematics of Fig. 4.1 and Fig. 4.2 could be implemented. We remark that if the computation time is to be ignored, then the implementation becomes much simpler and for that case we refer to the conference version [102].

Algorithm 3: Implementation of the hybrid observer

Input: σ, u, v, y

Initialization: $\hat{x}(t_0) \in \mathbb{R}^n$, $q = N$, $k = 0$, `Update` = idle, `swCount` = 0

Synchronized Observer Loop

- Run the observer (4.18) synchronously to plant (4.1).
- **if** *switching occurs* **then** increment `swCount`.
- **if** `Update` == idle *and* $q <$ `swCount` **then**
  - $q \leftarrow$ `swCount` and call **Estimate Update**.

Loop end

Estimate Update

- `Update` $\leftarrow$ active
- **for** $i = q - N$ **to** q **do**
  - Compute the gains L_i satisfying (4.26).
  - Obtain $\hat{z}^i(t_i^-)$ by running the individual observer (4.21) for the i-th mode.
- Compute $\hat{\hat{x}}$ from (4.28).
- Increment k and set $\hat{t}_k \leftarrow$ `CurrentTime`.
- Set $\xi_k \leftarrow \Phi(\hat{t}_k^-, t_q^-)\hat{\hat{x}}(t_q^-)$ and update $\hat{x}$ by (4.18c).
- `Update` $\leftarrow$ idle

end

Algorithm 3 comprises two processes running in parallel, *Synchronized Observer* and *Estimate Update*. Whenever the switching happens, the Synchronized Observer calls the Estimate Update if the latter is not already occupied with computation from previous switch. If the switch occurs while the Estimate Update is active, we wait for it to finish the previous computation and then look at the information from last $N + 1$ active modes for the next update.

4.3.3 Analysis of Error Convergence

The following theorem shows that the above implementation indeed guarantees the convergence of the state estimation error to zero.

Theorem 4.19. *Under Assumption 4.1, consider the hybrid observer* (4.18) *in which the estimate update ξ_k is computed through* (4.27) *and introduced at $\hat{t}_k$, according to Algorithm 3. If the gains L_i, for each $i = q - N, \cdots, q$, are chosen so that* (4.26) *holds for any choice of $\alpha \in (0,1)$, then $\lim_{t\to\infty} |\hat{x}(t) - x(t)| = 0$. Furthermore, the estimation error satisfies the following exponential convergence rate:*

$$|\tilde{x}(t)| \leq h(\alpha) e^{-\gamma \ln(\alpha^{-1})(t-t_0)} |\tilde{x}(t_0)|, \qquad \forall\, t \geq t_0, \tag{4.29}$$

where γ is a positive constant and the function $h : (0,1) \to \mathbb{R}$ has the property that $h(\alpha) \to \infty$ when $\alpha \to 0$.

Proof. From Assumption 4.1 and Algorithm 3 it can be seen that

$$\hat{t}_{k+1} - \hat{t}_k \leq \max\{T_D, T_C\} = T_B, \quad \forall k \geq 1.$$

Hence, it suffices to show that $\lim_{k\to\infty} |\tilde{x}(\hat{t}_k)| = 0$ because Assumptions 4.1.1 and 4.1.3 imply that

$$|\tilde{x}(t)| \leq e^{b_A T_B} \cdot b_E^{J_{\max}(T_B)} |\tilde{x}(\hat{t}_k)|, \quad \forall t \in [\hat{t}_k, \hat{t}_{k+1}). \tag{4.30}$$

In the remainder of this proof, an expression for $\tilde{x}(\hat{t}_k)$ is derived whose norm is shown to converge to zero. For this purpose, fix $k > N + 3$ and suppose that the k-th estimate update process for ξ_k completes at time $\hat{t}_k$ after having processed the data on the interval $[t_{q-N-1}, t_q)$ (where it is possible that there is another switch between t_q and $\hat{t}_k^-$).

The error at t_q^-, $\tilde{x}(t_q^-)$, can be written as

$$\tilde{x}(t_q^-) = \begin{bmatrix} Z^{q\top} \\ W^{q\top} \end{bmatrix}^{-1} \begin{bmatrix} z^q(t_q^-) \\ w^q(t_q^-) \end{bmatrix} = Z^q z^q(t_q^-) + W^q w^q(t_q^-). \tag{4.31}$$

The matrix Ψ_i^j with $j > i$, defined in (4.22), transports $\tilde{x}(t_i^-)$ to $\tilde{x}(t_j^-)$ along (4.19) by

$$\tilde{x}(t_j^-) = \Psi_i^j \tilde{x}(t_i^-) - \sum_{k \in \mathcal{K}_i^j} \Phi(t_j^-, \hat{t}_k^-) \xi_k. \tag{4.32}$$

We now have the following series of equivalent expressions for $\tilde{x}(t_q^-)$:

$$\begin{aligned}\tilde{x}(t_q^-) &= Z^q z^q(t_q^-) + W^q w^q(t_q^-)\\ &= \Psi_{q-1}^q Z^{q-1} z^{q-1}(t_{q-1}^-) + \Psi_{q-1}^q W^{q-1} w^{q-1}(t_{q-1}^-) - \sum_{k\in\mathcal{K}_{q-1}^q} \Phi(t_q^-, \hat{t}_k^-)\xi_k\\ &= \Psi_{q-2}^q Z^{q-2} z^{q-2}(t_{q-2}^-) + \Psi_{q-2}^q W^{q-2} w^{q-2}(t_{q-2}^-) - \sum_{k\in\mathcal{K}_{q-2}^q} \Phi(t_q^-, \hat{t}_k^-)\xi_k\\ &\ \ \vdots\\ &= \Psi_{q-N}^q Z^{q-N} z^{q-N}(t_{q-N}^-) + \Psi_{q-N}^q W^{q-N} w^{q-N}(t_{q-N}^-) - \sum_{k\in\mathcal{K}_{q-N}^q} \Phi(t_q^-, \hat{t}_k^-)\xi_k.\end{aligned} \tag{4.33}$$

To appreciate the implication of this equivalence, we first note that for each $q - N \leq i \leq q$, the term $\Psi_i^q Z^i z^i(t_i^-)$ transports the observable information of the i-th mode from the interval $[t_{i-1}, t_i)$ to the time instant t_q^-. This observable information is corrupted by the unknown term $w^i(t_i^-)$, but since the information is being accumulated at t_q^- from modes $i = q - N, \cdots, q$, the idea is to combine the partial information from each mode to recover $\tilde{x}(t_q^-)$. This is where we use the notion of determinability. By Properties 1, 5, and 6 in Appendix B, and the fact that $\mathcal{R}(W^i)^\perp = (\ker G_i)^\perp = \mathcal{R}(G_i^\top)$ and $e^{-A_q^\top \tau_q}\mathcal{R}(G_q^\top) = \mathcal{R}(G_q^\top)$, it follows under Assumption 4.1.2 that

$$\begin{aligned}&\mathcal{R}(W^q)^\perp + \mathcal{R}(\Psi_{q-1}^q W^{q-1})^\perp + \cdots + \mathcal{R}(\Psi_{q-N}^q W^{q-N})^\perp\\ &= e^{-A_q^\top \tau_q}\Big(\mathcal{R}(G_q^\top) + E_{q-1}^{-\top}\mathcal{R}(G_{q-1}^\top) + \sum_{i=q-N}^{q-2}\prod_{l=q-1}^{i+1} E_l^{-\top} e^{-A_l^\top \tau_l} E_i^{-\top}\mathcal{R}(G_i^\top)\Big)\\ &= e^{-A_q^\top \tau_q}\mathcal{M}_{q-N}^q = \mathbb{R}^n.\end{aligned}$$

This equation shows that the matrix Θ_q defined in (4.23) has rank n, and is left-invertible. Keeping in mind that the range space of each Θ_i^q is orthogonal to $\mathcal{R}(\Psi_i^q W^i)$, each equality in (4.33) leads to the following relation:

$$\Theta_i^{q\top}\tilde{x}(t_q^-) = \Theta_i^{q\top}\left(\Psi_i^q Z^i z^i(t_i^-) - \sum_{k\in\mathcal{K}_i^q}\Phi(t_q^-, \hat{t}_k^-)\xi_k\right) \tag{4.34}$$

for $i = q - N, \cdots, q$. Stacking (4.34) from $i = q$ to $i = q - N$, and employing the left-inverse of $\Theta_q^\top$, we obtain that

$$\tilde{x}(t_q^-) = (\Theta_q^\top)^\dagger \Xi_q(z^-_{\{q-N,q\}}, \xi_{\{q-N,q\}}) \tag{4.35}$$

where $z^-_{\{q-N,q\}}$ denotes $\{z^{q-N}(t^-_{q-N}), \dots, z^q(t_q^-)\}$. It is seen from (4.35) that, if we were able to estimate $z^-_{\{q-N,q\}}$ without error, then the plant state $x(t_q^-)$ would be exactly recovered by (4.35) because $x(t_q^-) = \hat{x}(t_q^-) - \tilde{x}(t_q^-)$ and both entities on the right side of the equation are known. However, since this is not the case, $z^-_{\{q-N,q\}}$ is replaced with its estimate $\hat{z}^-_{\{q-N,q\}}$ in (4.28), and $\hat{\tilde{x}}(t_q^-)$ is set as an estimate of $\tilde{x}(t_q^-)$, as done in (4.28).

Using the linearity of Ξ_q in its arguments, and substituting ξ_k from (4.27) in (4.19c), we get

$$\begin{aligned} \tilde{x}(\hat{t}_k) &= \tilde{x}(\hat{t}_k^-) - \Phi(\hat{t}_k^-, t_q^-)\hat{\tilde{x}}(t_q^-) \\ &= \Phi(\hat{t}_k^-, t_q^-)(\Theta_q^\top)^\dagger \Big(\Xi_q(z^-_{\{q-N,q\}}, \xi_{\{q-N,q\}}) - \Xi_q(\hat{z}^-_{\{q-N,q\}}, \xi_{\{q-N,q\}}) \Big) \\ &= -\Phi(\hat{t}_k^-, t_q^-)(\Theta_q^\top)^\dagger \Xi_q(\tilde{z}^-_{\{q-N,q\}}, 0) \end{aligned} \tag{4.36}$$

where $\tilde{z}^-_{\{q-N,q\}}$ denotes $\{\tilde{z}^{q-N}(t^-_{q-N}), \dots, \tilde{z}^q(t_q^-)\}$ with $\tilde{z}^i(t_i^-) = \hat{z}^i(t_i^-) - z^i(t_i^-)$. It follows from (4.20) and (4.21) that $\dot{\tilde{z}}^i = (S_i - L_i R_i)\tilde{z}^i$ for $t \in [t_{i-1}, t_i)$ with

$$\tilde{z}^i(t_{i-1}) = \hat{z}^i(t_{i-1}) - z^i(t_{i-1}) = 0 - Z^{i\top}\tilde{x}(t_{i-1}),$$

which implies that

$$\tilde{z}^i(t_i^-) = e^{(S_i - L_i R_i)\tau_i}\tilde{z}^i(t_{i-1}) = -e^{(S_i - L_i R_i)\tau_i} Z^{i\top}\tilde{x}(t_{i-1}).$$

Plugging this expression in (4.36), and using the definition of M_i^q $(i = q-N, \dots, q)$ from (4.24), we get

$$\tilde{x}(\hat{t}_k) = \Phi(\hat{t}_k^-, t_q^-) \sum_{i=q-N}^{q} M_i^q(\tau_{\{q-N+1,q\}}) Z^i e^{(S_i - L_i R_i)\tau_i} Z^{i\top}\tilde{x}(t_{i-1}). \tag{4.37}$$

For each $i = q-N-1, \dots, q-1$, let $k^*(i) := \max\{k : \hat{t}_k < t_i\}$. Then it follows that

$$\tilde{x}(t_i) = \Phi(t_i, \hat{t}_{k^*(i)})\tilde{x}(\hat{t}_{k^*(i)}).$$

From Fig. 4.2 and Algorithm 3, it is seen that

$$\hat{t}_k^- - t_q^- \leq 2T_C \quad \text{and} \quad t_i - \hat{t}_{k^*(i)} \leq T_B.$$

Thus, $\|\Phi(\hat{t}_k^-, t_q^-)\| \cdot \|\Phi(t_i, \hat{t}_{k^*(i)})\| \leq \bar{b}$, $\forall\, i = q-N-1, \ldots, q-1$, where $\overline{b}$ is defined in (4.25).

Moreover, with k and q considered above, it can be seen that, for each $i = q - N - 1, \ldots, q-1$, it holds that $k - N - 3 \leq k^*(i) \leq k-2$ (since $k^*(q-1)$ either equals $k-2$ or $k-3$). Then, from the selection of gains L_i's satisfying (4.26), it is seen that

$$|\tilde{x}(\hat{t}_k)| \leq \frac{\alpha}{N+2} \sum_{i=q-N-1}^{q-1} |\tilde{x}(\hat{t}_{k^*(i)})| \leq \alpha \max_{k-N-3 \leq i \leq k-2} |\tilde{x}(\hat{t}_i)|, \tag{4.38}$$

where $0 < \alpha < 1$. Finally, applying the statement of Lemma 4.20 to (4.38) aids us in the completion of the proof as it shows that $|\tilde{x}(\hat{t}_k)| \to 0$ as $k \to \infty$.

In order to compute the exponential decay bound, note that equation (4.44) in the statement of Lemma 4.20, with $a_i = \tilde{x}(\hat{t}_i)$, leads to the following inequality:

$$|\tilde{x}(\hat{t}_i)| \leq \frac{1}{\alpha} \exp\left(-\frac{\ln(\alpha^{-1})}{(N+3)T_B}(\hat{t}_i - \hat{t}_1)\right) \max_{1 \leq i \leq N+3} |\tilde{x}(\hat{t}_i)|, \quad i \geq 1, \tag{4.39}$$

because $\hat{t}_{k+1} - \hat{t}_k \leq T_B$ and $0 < \alpha < 1$. And, since $\exp\left(-\frac{\ln(\alpha^{-1})}{(N+3)T_B}(t - \hat{t}_i)\right) \geq \alpha$ for $t \geq \hat{t}_i$, it follows from (4.30) that

$$|\tilde{x}(t)| \leq e^{b_A T_B} \cdot b_E^{J_{\max}(T_B)} \cdot \frac{1}{\alpha} \exp\left(-\frac{\ln(\alpha^{-1})}{(N+3)T_B}(t - \hat{t}_i)\right) |\tilde{x}(\hat{t}_i)|, \qquad t \in [\hat{t}_i, \hat{t}_{i+1}). \tag{4.40}$$

Combining (4.39) and (4.40), it holds that

$$|\tilde{x}(t)| \leq e^{b_A T_B} \cdot b_E^{J_{\max}(T_B)} \cdot \frac{1}{\alpha^2} \exp\left(-\frac{\ln(\alpha^{-1})}{(N+3)T_B}(t - \hat{t}_1)\right) \max_{1 \leq i \leq N+3} |\tilde{x}(\hat{t}_i)|, \qquad t \geq \hat{t}_1. \tag{4.41}$$

On the other hand, since $t_{N+1} < \hat{t}_1 \leq t_{N+1} + T_C$, an over-approximation of the error on the interval $[t_0, \hat{t}_{N+3}]$ is obtained, by ignoring error updates, as

$$\max_{t \in [t_0, \hat{t}_{N+3}]} |\tilde{x}(t)| \leq e^{b_A((N+1)T_D + T_C + (N+2)T_B)} \cdot b_E^{N+1+J_{\max}(T_C+(N+2)T_B)} \cdot |\tilde{x}(t_0)| =: \tilde{c} \cdot |\tilde{x}(t_0)|. \tag{4.42}$$

From (4.41) and (4.42), we arrive at

$$|\tilde{x}(t)| \le e^{b_A T_B} \cdot b_E^{J_{\max}(T_B)} \cdot \frac{1}{\alpha^2} \exp\left(-\frac{\ln(\alpha^{-1})}{(N+3)T_B}(t-\hat{t}_1)\right) \cdot \tilde{c} \cdot |\tilde{x}(t_0)|, \qquad t \ge t_0, \qquad (4.43)$$

in which it should be noted that the inequality holds for all $t \ge t_0$ because the right-hand side is greater than $\tilde{c}|\tilde{x}(t_0)|$ for $t \in [t_0, \hat{t}_1]$. Taking $\gamma = 1/((N+3)T_B)$ and $h(\alpha) = (e^{b_A T_B} b_E^{J_{\max}(T_B)} \tilde{c})/\alpha^2 \cdot e^{-\gamma \ln(\alpha^{-1})(t_0 - \hat{t}_1)}$, the proof is completed. □

Lemma 4.20. *Suppose that the sequence* $\{a_k\}$ *satisfies*

$$|a_k| \le \alpha \max_{k-N-3 \le i \le k-2} |a_i|, \qquad k > N+3,$$

where $0 < \alpha < 1$. *Then the following holds:*

$$\max_{k \le i \le k+N+2} |a_i| \le \alpha \max_{k-N-3 \le i \le k-1} |a_i|, \qquad k > N+3, \qquad (4.44)$$

which implies that the maximum value of the sequence $\{a_k\}$ *over a window of length* $N+3$ *is strictly decreasing and converging to zero, and thus,* $\lim_{k \to \infty} a_k = 0$.

Proof of Lemma 4.20. By putting $|a_{k-1}|$ into the right-hand side it is clear that

$$|a_k| \le \alpha \max_{k-N-3 \le i \le k-1} |a_i|. \qquad (4.45)$$

Similarly, it follows that

$$\begin{aligned} |a_{k+1}| &\le \alpha \max_{k-N-2 \le i \le k} |a_i| \le \alpha \max\left\{ |a_{k-N-3}|, \max_{k-N-2 \le i \le k-1} |a_i|, |a_k| \right\} \\ &\le \alpha \max_{k-N-3 \le i \le k-1} |a_i|, \end{aligned}$$

where the last inequality follows from (4.45). By induction, this leads to

$$\max_{k \le i \le k+N+2} |a_i| \le \alpha \max_{k-N-3 \le i \le k-1} |a_i|.$$

□

Example 4.21. We demonstrate the operation of the proposed observer for the switched system considered in Example 4.1 with $\epsilon > 0$. We assume that each mode is activated for τ

seconds and $\tau \neq \kappa\pi$ for any $\kappa \in \mathbb{N}$, so that the persistent switching signal is:

$$\sigma(t) = \begin{cases} a & \text{if } t \in [2m\tau, (2m+1)\tau), \\ b & \text{if } t \in [(2m+1)\tau, (2m+2)\tau), \end{cases} \qquad m : \text{nonnegative integer.} \tag{4.46}$$

As mentioned earlier, the system is observable (and thus, determinable) with this switching signal if the mode sequence $a \to b \to a$ is contained in a time interval. Hence, we pick $N = 3$ in order to include both sequences (a, b, a, b) and (b, a, b, a), so that Assumption 4.1.2 holds. For simplicity, it is assumed that $0 < T_C < \tau$ and that the computations always end at $\hat{t}_k = t_q + T_C$ with $q = k + N$ (because $\hat{t}_1 = t_4 + T_C$). Let us call $[2m\tau, (2m+1)\tau)$, the *odd* interval, and $[(2m+1)\tau, (2m+2)\tau)$, the *even* interval. With an arbitrary initial condition $\hat{x}(0)$, the observer to be implemented is:

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= A_a \hat{x}(t) \\ \hat{y}(t) &= C_a \hat{x}(t) \end{aligned}\right\}, \quad t \in [2m\tau, (2m+1)\tau), \tag{4.47a}$$

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= A_b \hat{x}(t) \\ \hat{y}(t) &= C_b \hat{x}(t) \end{aligned}\right\}, \quad t \in [(2m+1)\tau, (2m+2)\tau), \tag{4.47b}$$

$$\hat{x}(\hat{t}_k) = \hat{x}(\hat{t}_k^-) - \xi_k, \quad \hat{t}_k = t_{k+3} + T_C, \ k \in \mathbb{N}. \tag{4.47c}$$

In order to determine the value of ξ_k, we start off with the estimators for the observable part of each subsystem, denoted by z^q in (4.20). Note that mode a has a one-dimensional unobservable subspace whereas for mode b, the unobservable subspace is $\mathbb{R}^2$. Since mode a is active on every odd interval and mode b on every even interval, z^q for every odd q represents the partial information obtained from mode a, and z^q for every even q is a null vector as no information is gathered from mode b. So the one-dimensional partial observer in (4.21) is implemented only for odd intervals. For odd q, we compute from the mode a

$$G_q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{R}(G_q^\top) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, W^q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Z^q = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that $S_q = 0$ and $R_q = 1$, which yields the observer in (4.21) as

$$\begin{aligned} \dot{\hat{z}}^q &= -l_q \hat{z}^q + l_q \tilde{y}, && t \in [(q-1)\tau, q\tau), \quad q\text{: odd}, \\ \hat{z}^q(\hat{t}_k) &= \hat{z}^q(\hat{t}_k^-) - \xi_k^1, && \hat{t}_k \in [(q-1)\tau, q\tau), \end{aligned}$$

with the initial condition $\hat{z}^q((q-1)\tau) = 0$, and $\tilde{y}$ being the difference between the measured output and the estimated output of (4.47). The notation ξ_k^1 denotes the first component of the vector ξ_k. The gain l_q will be chosen later by (4.48). For q even, we take $W^q = I_{2\times 2}$, and $G_q = 0_{2\times 2}$, so that Z^q, S_q, and R_q are null-matrices.

The next step is to use the value of $\hat{z}^q(t_q^-)$ to compute ξ_k. The matrices appearing in the computation of ξ_k are given as follows. For every *even* $q > 3$:

$$\begin{aligned}
\Psi_{q-3}^q &= e^{2\epsilon\tau}\begin{bmatrix}\cos 2\tau & \sin 2\tau\\ -\sin 2\tau & \cos 2\tau\end{bmatrix} \Rightarrow \mathcal{R}\left(\Psi_{q-3}^q W^{q-3}\right)^{\perp} = \mathcal{R}\left(\begin{bmatrix}\cos 2\tau\\ -\sin 2\tau\end{bmatrix}\right),\\
\Psi_{q-2}^q &= e^{\epsilon\tau}\begin{bmatrix}\cos \tau & \sin \tau\\ -\sin \tau & \cos \tau\end{bmatrix} \Rightarrow \mathcal{R}\left(\Psi_{q-2}^q W^{q-2}\right)^{\perp} = \{0\},\\
\Psi_{q-1}^q &= e^{\epsilon\tau}\begin{bmatrix}\cos \tau & \sin \tau\\ -\sin \tau & \cos \tau\end{bmatrix} \Rightarrow \mathcal{R}\left(\Psi_{q-1}^q W^{q-1}\right)^{\perp} = \mathcal{R}\left(\begin{bmatrix}\cos \tau\\ -\sin \tau\end{bmatrix}\right),\\
\Psi_q^q &= I_{2\times 2} \Rightarrow \mathcal{R}\left(\Psi_q^q W^q\right)^{\perp} = \{0\}.
\end{aligned}$$

These subspaces directly lead to the expressions for Θ_j^q, $j = q-3, \ldots, q$, so that

$$\Theta_q = \begin{bmatrix}\Theta_{q-1}^q & \Theta_{q-3}^q\end{bmatrix} = \begin{bmatrix}\cos\tau & \cos 2\tau\\ -\sin\tau & -\sin 2\tau\end{bmatrix}, \text{ and } \Theta_q^{-\top} = \begin{bmatrix}\frac{\sin 2\tau}{\sin\tau} & -1\\ \frac{\cos 2\tau}{\sin\tau} & -\frac{\cos\tau}{\sin\tau}\end{bmatrix}$$

for $q = 4, 6, 8, \ldots$, where, as a convention, we have taken Θ_i^q as a null matrix whenever $\mathcal{R}(\Psi_i^q W^i)^{\perp} = \{0\}$. Hence, the error correction term can be computed recursively for every even $q > 3$ by the formula:

$$\hat{\tilde{x}}(t_q^-) = \Theta_q^{-\top}\begin{bmatrix} e^{\epsilon\tau}\hat{z}^{q-1}(t_{q-1}^-) - [\cos\tau \;\; -\sin\tau]e^{A_b(\tau-T_C)}\xi_{q-4}\\ e^{2\epsilon\tau}\hat{z}^{q-3}(t_{q-3}^-) - [\cos 2\tau \;\; -\sin 2\tau](e^{A_b(\tau-T_C)}\xi_{q-4} + e^{A_b\tau}\xi_{q-5} + e^{A_b(2\tau-T_C)}\xi_{q-6})\end{bmatrix}$$

with $\xi_k = 0$ for $k \le 0$. Since for every even q, $\Phi(\hat{t}_k^-, t_q^-) = I_{2\times 2}$ with $k = q-3$, we get $\xi_k = \hat{\tilde{x}}(t_{k+3}^-)$. Also, for even q, we obtain that M_q^q, M_{q-2}^q are null matrices, and

$$M_{q-1}^q = e^{\epsilon\tau}\begin{bmatrix}\frac{\sin 2\tau}{\sin\tau} & 0\\ \frac{\cos 2\tau}{\sin\tau} & 0\end{bmatrix} \quad \text{and} \quad M_{q-3}^q = e^{2\epsilon\tau}\begin{bmatrix}-1 & 0\\ -\frac{\cos\tau}{\sin\tau} & 0\end{bmatrix}.$$

Next, for every *odd* $q > 3$, we repeat the same calculations to get:

$$\Psi^q_{q-3} = e^{\epsilon\tau}\begin{bmatrix}\cos\tau & \sin\tau\\ -\sin\tau & \cos\tau\end{bmatrix} \Rightarrow \mathcal{R}\left(\Psi^q_{q-3}W^{q-3}\right)^\perp = \{0\},$$
$$\Psi^q_{q-2} = e^{\epsilon\tau}\begin{bmatrix}\cos\tau & \sin\tau\\ -\sin\tau & \cos\tau\end{bmatrix} \Rightarrow \mathcal{R}\left(\Psi^q_{q-2}W^{q-2}\right)^\perp = \mathcal{R}\left(\begin{bmatrix}\cos\tau\\ -\sin\tau\end{bmatrix}\right),$$
$$\Psi^q_{q-1} = I_{2\times 2} \Rightarrow \mathcal{R}\left(\Psi^q_{q-1}W^{q-1}\right)^\perp = \{0\},$$
$$\Psi^q_{q} = I_{2\times 2} \Rightarrow \mathcal{R}\left(\Psi^q_{q}W^{q}\right)^\perp = \mathcal{R}\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right).$$

Once again, using the expressions for Θ^q_j, $j = q-3,\ldots,q$, based on these subspaces, one gets

$$\Theta_q = \begin{bmatrix}\Theta^q_q & \Theta^q_{q-2}\end{bmatrix} = \begin{bmatrix}1 & \cos\tau\\ 0 & -\sin\tau\end{bmatrix}, \quad \Theta_q^{-\top} = \begin{bmatrix}1 & 0\\ \frac{\cos\tau}{\sin\tau} & -\frac{1}{\sin\tau}\end{bmatrix}$$

so that, for $q = 5, 7, 9, \ldots$,

$$\hat{\hat{x}}(t_q^-) = \Theta_q^{-\top}\begin{bmatrix}\hat{z}^q(t_q^-)\\ e^{\epsilon\tau}\hat{z}^{q-2}(t_{q-2}^-) - [\cos\tau \;\; -\sin\tau](\xi_{q-1} + e^{A_b(\tau-T_C)}\xi_{q-2})\end{bmatrix},$$

and $\xi_k = e^{A_bT_C}\hat{\hat{x}}(t_{k+3}^-)$. Again, we obtain for odd q that M^q_{q-1} and M^q_{q-3} are null matrices, and

$$M^q_q = \begin{bmatrix}1 & 0\\ \frac{\cos\tau}{\sin\tau} & 0\end{bmatrix}, \quad \text{and} \quad M^q_{q-2} = \begin{bmatrix}0 & 0\\ -\frac{e^{\epsilon\tau}}{\sin\tau} & 0\end{bmatrix}.$$

By taking l_j equal to l for every odd j, and computing the induced 2-norm of the matrix, it is seen that, $\max_{q-3\le j\le q, j:\text{odd}, q>3} \|M^q_j Z^j e^{(S_j - lR_j)\tau} Z^{j\top}\| = e^{(2\epsilon-l)\tau}/|\sin\tau|$. Also, $\overline{b} = e^{\sqrt{1+\epsilon^2}(2T_C+\tau)}$. So, the lower bound for the gain l, is obtained as follows:

$$\overline{b}\,\frac{e^{(2\epsilon-l)\tau}}{|\sin\tau|} < \frac{1}{N+2} = \frac{1}{5} \quad \Rightarrow \quad l > 2\epsilon + \frac{1}{\tau}\ln\frac{5\overline{b}}{|\sin\tau|}\,. \tag{4.48}$$

Once again it can be seen that the singularity occurs when τ is an integer multiple of π. Moreover, if τ approaches this singularity, then the gain required for convergence gets arbitrarily large. This shows that, although the condition $\sin\tau \neq 0$ guarantees observability, it may cause some difficulty in practice if $\sin\tau \approx 0$. This also explains why the knowledge of the switching signal is required in general to compute the observer gains.

The results of simulations with $\tau = 1$, $T_C = 0.5\tau$, $\epsilon = 0.1$ and $l = 20$, are illustrated

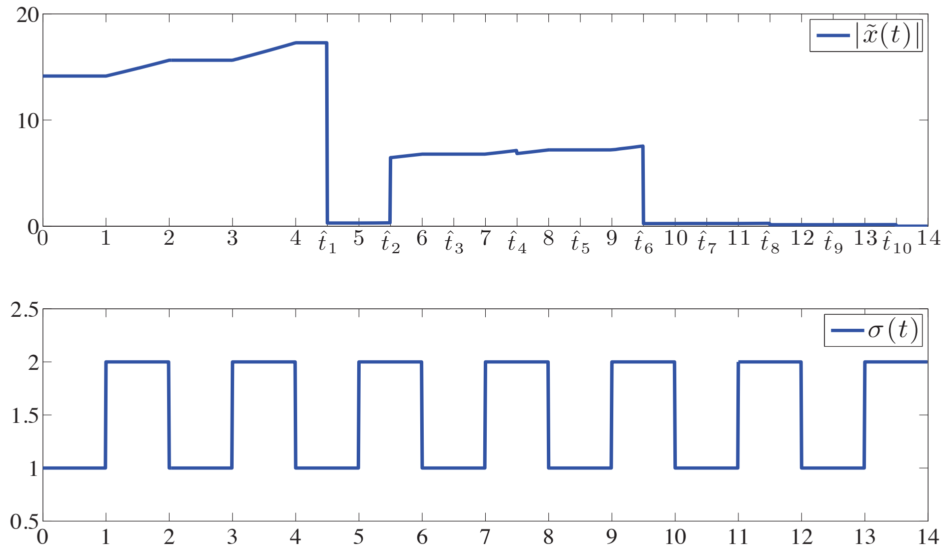


Figure 4.3: Size of state estimation error and the switching signal.

in Fig. 4.3. The error initially evolves according to the unstable system dynamics as no correction is applied till $t_4 + T_C$. The figure clearly shows the hybrid nature of the proposed observer, which is caused by the jump discontinuity in the error signal. The error grows between the error updates because the subsystem at mode b has unstable dynamics, but $\max_{k \le i \le k+N+2} |\tilde{x}(\hat{t}_i)|$ indeed gets smaller as k increases. $\triangleleft$

4.4 Conclusion

This chapter addressed the characterization of observability and determinability in switched linear systems with state jumps. It was shown that, for a fixed mode sequence, the set of switching signals over which these properties hold is either empty or dense under a certain metric topology. To study when the properties hold uniformly with respect to switching times, we derived separate sufficient and necessary conditions as corollaries to the main result. Later, using the property of determinability, an asymptotic observer was constructed that combines the partial information obtained from each mode to get an estimate of the state vector. For practical considerations, the proposed observer takes into account the time consumed in processing the information. Under the assumption of persistent switching, the error analysis shows that the estimate indeed converges to the actual state exponentially.

As an extension to the current work, it may be interesting to investigate how far these ideas carry over to nonlinear systems. The proposed method for observer design relies on the linearity of the system (4.1). In fact, it is seen in (4.33) that the transportation of the partially observable state information (represented by z), obtained at each mode, can be computed even with some unobservable information (by w). Since linearity guarantees that the observable information is not altered by this transportation process, the unobservable components are simply filtered out after the transportation. We emphasize that this idea may not be transparently applied to nonlinear systems, and may need a different approach as in [104].

Chapter 5

Observability of Switched Nonlinear Systems

Continuing with the observability problem, we now address another class of switched systems where the continuous dynamics are modeled by first order nonlinear differential equations. The tool set adopted in Chapter 4 to solve the linear case is not so easy to generalize for the nonlinear case; however, at a conceptual level we will try to extend the same idea. This chapter first presents a sufficient condition for observability of switched systems that involve state jumps and comprise nonlinear dynamical subsystems affine in control. Without assuming observability of individual modes, the sufficient condition is based on gathering partial information from each mode so that the state is recovered completely after some time. Based on the sufficient condition, an observer is designed which employs a novel 'back-and-forth' technique to generate state estimates. Under the assumption of persistent switching, analysis shows that the estimate converges asymptotically to the actual state of the system.

Towards the end of the chapter, we try to work out a condition which is both necessary and sufficient for observability in nonlinear switched systems. At the moment our findings are presented in the form of a conjecture, with a sketch of proof and we highlight some of the issues involved in obtaining such characterization.

5.1 Introduction and Background

We study observability conditions and an observer design for a class of switched nonlinear systems Σ, described as

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) + g_{\sigma(t)}(x(t))u(t), \quad t \neq \{t_q\}, \tag{5.1a}$$

$$x(t_q) = p_{\sigma(t_q^-)}(x(t_q^-)), \tag{5.1b}$$

$$y(t) = h_{\sigma(t)}(x(t)), \tag{5.1c}$$

where $x : \mathbb{R} \mapsto \mathbb{R}^n$ is the state trajectory, $y : \mathbb{R} \mapsto \mathbb{R}^{d_y}$ is the output, the measurable function $u : \mathbb{R} \mapsto \mathbb{R}^{d_u}$ is the input belonging to some input class $\mathcal{U}$ of interest, and $\sigma : \mathbb{R} \mapsto \mathbb{N}$ is the switching signal that is right-continuous and changes its value at switching times $\{t_q\}$, $q \in \mathbb{N}$. Let t_0 be the initial time and the jump map (5.1b) applies at $t = t_q$, $q \geq 1$. It is assumed that there are a finite number of switching times in any finite time interval. The switching mode σ and the switching times $\{t_q\}$ may come from a supervisory logic controller, or may be determined internally depending on the system state. In any case, we treat them as a known, external input in this chapter, leaving aside the study for estimating the switching signal to [105]. It is assumed that the solution $x(t)$ remains in a compact set $\mathcal{X} \subset \mathbb{R}^n$ on the time interval of interest. This is because the observer design that we are going to present is not a global one, and capturing the solution within the set $\mathcal{X}$ is a control problem, which is not of our concern. All the vector fields and functions are assumed to be smooth, and therefore, the existence and uniqueness of the solution, for all times, are guaranteed by the fact that the solution remains in a compact set.

When dealing with observability of nonlinear systems, there are different notions that are involved. The works [18, 106] talk about observability in local neighborhoods of the state space. The authors in [73] describe the notion of 'large-time' versus 'small-time' observability where the difference lies in whether it is possible to recover the state instantaneously in time or the system becomes observable after certain time interval. If the system description has exogenous inputs acting on it, then the question arises whether observability holds for all inputs [107, 108]; if it does, the system is called uniformly observable.

The concept of observability studied in this chapter is a refinement of the 'large-time observability' already considered in the literature (e.g., [73]) and the 'uniform observability' studied in [107, 108]. Switched systems can be thought of as a family of dynamical subsystems, where a switching signal determines the active subsystem at each time instant. It is entirely possible that none of these subsystems is observable in the sense that information about the full state is not immediate in the output signal [106, 18]. But the information available from each mode can be combined in a certain manner so that under some conditions, it is possible to recover the state vector completely after some time. This explains how the concept of 'large-time' comes into the picture when dealing with switched systems, and our goal is to derive conditions that make the system large-time observable on a given set $\mathcal{X}$. Moreover, since we are interested in an observer construction at the end, the observability for all inputs (i.e., uniform observability) is of concern in order for the observer to be independent of particular inputs.

For switched systems, among other structural properties, observability and observer design for linear case have been actively studied during the past decade. Some initial observer results on switched systems, such as [58, 59] for linear case and [109] for nonlinear case, have assumed that each mode in the system is in fact observable admitting a state observer, and have treated the switching as a source of perturbation effect. This approach immediately incurs the need of a common Lyapunov function for the switched error dynamics, or a fixed amount of dwell-time between switching instants, because it is intrinsically a stability problem of the error dynamics. More relaxed approaches do not assume observability of the individual modes, and the notion of gaining observability for linear systems by switching has appeared in, e.g., [52, 54, 46]. The sufficient conditions proposed in [52] imply that the full state information is recovered after one switching, which is extended in [54] where the conditions for recovery of state after multiple switches are proposed. Both papers use outputs and their derivatives to recover the state. The work of [46] gives geometric conditions under which there exists at least one switching signal that makes the system observable. Even though limited to the linear case, it is not clear how the conditions in [46, 52, 54] can lead to feasible observer design. On the other hand, there is not much literature on the observability of switched nonlinear systems. We mention [110] treating this topic but the notions of observability considered in that chapter is entirely different than ours.

The main contribution of this chapter lies in the unified treatment of observability conditions and observer design which has not been discussed in literature for nonlinear systems, to the author's knowledge. For the observer design, our approach shares the same spirit with [55], and the results in this chapter can be regarded as an extension of [55], in the sense that a coordinate-independent condition is derived for observability and nonlinear systems are treated with a new observer design strategy.

In Chapter 4, while deriving the conditions for observability and designing observers for switched linear systems, we used the knowledge of flow of linear systems and the results were dependent on the switching signal. In nonlinear systems, however, it is difficult to compute the analytical expression for flow of the system. This observation motivates us to seek sufficient conditions which guarantee large-time observability without having to solve the nonlinear differential equations so that the result holds independently of switching times. Consequently, the proposed observer design based on this condition allows arbitrary inputs u and all switching signals with particular mode sequence regardless of switching times, in order to generate converging estimates. In this chapter, we present such a sufficient condition, as well as an observer design technique based on the proposed condition. This means that observability with respect to our sufficient condition is uniform with respect to the switching times. In this way, we can deal with the design of observers allowing arbitrary inputs u and switching signals with particular mode sequence, which is more suitable (than asking for observability to be uniform both in the input u and the switching signal σ) for engineering needs in practice.

For development of the results related to observability, we recall some basic definitions from [111, 18]:

Definition 5.1 (Distribution)**.** *A k-dimensional distribution Δ on an n-dimensional manifold $\mathbb{M}$ is an assignment of a k-dimensional subspace $\Delta(x)$ of $T_x\mathbb{M}$ for each $x \in \mathbb{M}$, where $T_x\mathbb{M}$ denotes the tangent space to $\mathbb{M}$ at x. A vector field f on $\mathbb{M}$ is said to belong to (or lie in) the distribution Δ ($f \in \Delta$) if $f(x) \in \Delta(x)$ for each $x \in \mathbb{M}$.* $\triangleleft$

Definition 5.2 (Involutive Distribution)**.** *A distribution Δ is called* involutive *if the Lie bracket $[\nu_1, \nu_2] \in \Delta$ whenever ν_1 and ν_2 belong to Δ.* $\triangleleft$

Definition 5.3 (Integral Manifold of a Distribution)**.** *A submanifold $\mathcal{M}$ of $\mathbb{M}$ is an* integral submaifold *of the distribution Δ if, for every $x \in \mathcal{M}$, the tangent space $T_x\mathcal{M}$ to $\mathcal{M}$ at x coincides with the subspace $\Delta(x)$ of $T_x\mathbb{M}$.* $\triangleleft$

Definition 5.4 (Integrable Distribution)**.** *A nonsingular d-dimensional distribution Δ, defined on an open set $U \subset \mathbb{M}$, is said to be* completely integrable *if, for each point x^o of U, there exist a neighborhood U^o of x^o, and $n-d$ real-valued smooth functions $\lambda_1, \lambda_2, \cdots, \lambda_{n-d}$, all defined on U^o, such that*

$$\operatorname{span}\{d\lambda_1, \ldots, d\lambda_{n-d}\} = \Delta^\perp$$

on U^o. $\triangleleft$

Definition 5.5 (Invariant Distribution)**.** *A distribution Δ is said to be invariant under a vector field f if the Lie bracket $[f, \nu]$ of f with every vector field ν of Δ is again a vector field of Δ, i.e., if*

$$\nu \in \Delta \Rightarrow [f, \nu] \in \Delta. \qquad \triangleleft$$

We will be using the notation $\langle \nu_1, \ldots, \nu_q | \Delta \rangle$ to denote the smallest distribution which contains Δ and is invariant under the vector fields $\nu_1, \ldots, \nu_q$.

Definition 5.6 (Differential)**.** *Let* $\Phi : \mathbb{M} \mapsto \mathbb{M}'$ *be a smooth mapping between smooth manifolds* $\mathbb{M}$ *and* $\mathbb{M}'$*. The* differential *of* Φ *at a point* $x \in \mathbb{M}$ *is a linear mapping*

$$\Phi_{*_x} : T_x\mathbb{M} \mapsto T_{\Phi(x)}\mathbb{M}'$$

defined as:

$$\Phi_{*_x}\left(\left.\frac{d\gamma}{dt}\right|_{t=0}\right) = \left.\frac{d}{dt}\right|_{t=0} \Phi(\gamma(t)),$$

where

$$\gamma : (-\varepsilon, \varepsilon) \mapsto \mathbb{M}, \qquad \gamma(0) = x,$$

is a smooth curve in $\mathbb{M}$ *starting at the point* x*.* ◁

Moving on from the basic definitions of differential geometry, we define the notion of observability that is considered in the nonlinear case.

Definition 5.7. *A system* Σ *with a switching signal* $\sigma(\cdot)$ *is* large-time uniformly observable on a set $\mathcal{X} \subset \mathbb{R}^n$ *if there exist a finite time* $T > t_0$ *and an operator* $\mathcal{E}$ *yielding the state* $x(T) = \mathcal{E}(y_{[t_0,T]}, u_{[t_0,T]}, \sigma_{[t_0,T]})$ *for any measurable input* $u_{[t_0,T]}$*, when the state* $x(t)$ *remains in* $\mathcal{X}$ *for* $[t_0, T]$*. If the time* T *can be chosen arbitrarily with* $T > t_0$*, then the system* Σ *is called* small-time uniformly observable on a set $\mathcal{X}$*.* ◁

In case of no jump map (5.1b), knowledge of $x(T)$, $\sigma_{[t_0,T]}$, and $u_{[t_0,T]}$ determines $x_{[t_0,T]}$ uniquely. This is not the case in general because the jump map (5.1b) may not be reversible. The notion of observability studied in this chapter is also referred to as 'determinability' in [46, 102] and 'reconstructability' in [13], where the systems considered are linear. From the definition, if a certain mode of system Σ is small-time observable and the switching signal activates that mode at certain time, then the system is automatically large-time observable. Note that the operator $\mathcal{E}$ in the definition may include differentiation (although differentiation should be not used in the observer construction).

It is noted that, although the observability in Definition 5.7 is uniform with respect to the input u, uniformity with respect to the switching signal σ is not required. To appreciate its implication, let us first observe the effect of the switching signal on the observability.

Notation: The notation used in this chapter is summarized as follows. $\mathcal{R}(A)$ implies the range space of the columns of matrix A, and $A^\top$ is the transpose of A. We denote $[x_1^\top, x_2^\top]^\top$ simply by $\text{col}(x_1, x_2)$, and $\lambda_{1\sim k} := \text{col}(\lambda_1, \ldots, \lambda_k)$. The time interval $\{t : t_1 < t < t_2\}$ and $\{t : t_1 \le t \le t_2\}$ are expressed by (t_1, t_2) and $[t_1, t_2]$, respectively. And, for signal

$x(t)$, $x_{[t_1,t_2]}$ means $\{(x(t),t) : t_1 \le t \le t_2\}$. The differential of a map p acting on the vector field v is denoted by $p_* v$. For a distribution $\mathcal{W}$, $p_*\mathcal{W} = \{p_* v \,|\, v \in \mathcal{W}\}$. We call a distribution $\mathcal{W}$ at x^o 'nonsingular' when $\dim \mathcal{W}$ is constant in a neighborhood of x^o. The notation l.com $\{\lambda_1(x), \dots, \lambda_k(x)\}$ means a set of linear combinations of the functions λ_i with constant coefficients, i.e., $\{\sum_{i=1}^k c_i \lambda_i(x) : c_i \in \mathbb{R}\}$. Now let $\mathcal{X}$ be a set in $\mathbb{R}^n$, and whenever we say a property holds 'on $\mathcal{X}$,' we mean that it holds for every $x \in \mathcal{X}$. Smooth functions $\lambda_1(x), \dots, \lambda_k(x)$, defined on $\mathcal{X}$, are said to be *independent on* $\mathcal{X}$ if their differential one-forms, $d\lambda_1(x), \dots, d\lambda_k(x)$ are linearly independent on $\mathcal{X}$. In addition, if there exist $n-k$ smooth functions $\lambda_{k+1}, \dots, \lambda_n$ such that $\mathrm{col}(\lambda_1(x), \dots, \lambda_n(x))$ becomes a diffeomorphism from $\mathcal{X}$ to $\mathbb{R}^n$, then we say that $\lambda_1, \dots, \lambda_k$ are *potentially diffeomorphic on* $\mathcal{X}$.

5.2 Motivating Example

Example 5.8. Let $\mathcal{X} := \{x \in \mathbb{R}^3 : x_1, x_3 \ge 0\} \subset \mathbb{R}^3$ and consider that the state x, with initial condition $x_0 := \mathrm{col}(x_{10}, x_{20}, x_{30})$ in $\mathcal{X}$, evolves according to the following dynamics:

$$\Gamma_1 : \begin{cases} \dot{x} = f_1(x) := \begin{pmatrix} 0.1x_3 \\ x_1^2 - x_3^2 + 2x_1 \\ 0.1(x_1+1) \end{pmatrix} \\ y = h_1(x) := x_2 \end{cases} ; \qquad \Gamma_2 : \begin{cases} \dot{x} = f_2(x) := \begin{pmatrix} x_3 \\ -(x_1^2 - x_3^2 + 2x_1)x_2 \\ x_1 + 1 \end{pmatrix} \\ y = h_2(x) := x_1^2 - x_3^2 + 2x_1 \end{cases} ;$$

$$\Gamma_3 : \begin{cases} \dot{x} = f_3(x) := \begin{pmatrix} x_2^2 \\ -\frac{1}{2}x_2 \\ 0 \end{pmatrix} \\ y = h_3(x) := x_1 + x_2^2. \end{cases}$$

Also, the state jumps at t_1 according to the following equation:

$$x(t_1) = p_1(x(t_1^-)) := \mathrm{col}(x(t_1^-), 2x_2(t_1^-), x_3(t_1^-)).$$

Each of the three modes is large-time unobservable. For mode 1, $L_{f_1}h_1(x) = x_1^2 - x_3^2 + 2x_1$, and $L_{f_1}^k h_1(x) = 0$, $\forall k \ge 2$. This gives, $d\mathcal{O}_1 = \mathrm{span}\{\mathrm{col}(dh_1(x), dL_{f_1}h_1(x))\} = \mathrm{span}\left\{\begin{bmatrix} 0 & 1 & 0 \\ x_1+1 & 0 & -x_3 \end{bmatrix}\right\}$. So that we recover information about the state up to a one-dimensional manifold from this mode. Similarly, for mode 2 and mode 3, we can compute the following:

$$
\begin{aligned}
d\mathcal{O}_2 &= \text{span}\left\{\begin{bmatrix} x_1+1 & 0 & -x_3 \end{bmatrix}\right\}, \\
d\mathcal{O}_3 &= \text{span}\left\{\begin{bmatrix} 1 & 2x_2 & 0 \end{bmatrix}\right\}.
\end{aligned}
$$

Once again, it can be seen that the complete information about the state is not obtained from either mode 2 or mode 3 as both of them confine the state to a two-dimensional manifold.

Next, it is claimed that the switching between these three modes, $1 \to 2 \to 3$, makes it possible to recover complete information about the state. Assume that a particular execution of this switched system with mode sequence $\{1, 2, 3\}$ has been observed on some time interval $[0, T)$, $0 < t_1 < t_2 < T$. At time t_1^-, $y(t_1^-) = h_1(x(t_1^-))$, and $\dot{y}(t_1^-) = L_{f_1} h_1(x(t_1^-))$, so that the state at time t_1^- is constrained by the following equations:

$$y(t_1^-) = x_2(t_1^-), \tag{5.2a}$$

$$\dot{y}(t_1^-) = x_1^2(t_1^-) - x_3^2(t_1^-) + 2x_1(t_1^-). \tag{5.2b}$$

Similarly, at t_2^-, the only constraint on the state is that $y(t_2^-) = x_1^2(t_2^-) - x_3^2(t_2^-) + 2x_1(t_2^-)$. It is noted that $\ddot{y}(t) = 0$, for $0 \le t < t_1$ and $\dot{y}(t) = 0$, for $t_1 \le t < t_2$, and also that $y(t_1) = \dot{y}(t_1^-)$. This way $y(t_2^-) = \dot{y}(t_1^-) = \dot{y}(t_0) = x_{10}^2 - x_{30}^2 + 2x_{10}$. Since there is no state jump at t_2 we can rewrite the constraint imposed in (5.2b) as

$$\dot{y}(t_0) = x_1^2(t_2) - x_3^2(t_2) + 2x_1(t_2). \tag{5.3}$$

When mode 3 is activated, the information obtained from the output is:

$$y(t_2^+) = x_1(t_2) + x_2^2(t_2), \tag{5.4}$$

where $x(t_2^+)$ has been replaced by $x(t_2)$ as the state is continuous at t_2. Combining (5.2a), (5.3), and (5.4), alongside the fact that $x_2(t_2) = 2e^{-\dot{y}(t_0)\tau_2} x_2(t_1^-)$, the only possible solutions for $x_1(t_2)$, $x_2(t_2)$, and $x_3(t_2)$ are:

$$x_2(t_2) = 2e^{-\dot{y}(t_0)\tau_2} y(t_1^-), \tag{5.5a}$$

$$x_1(t_2) = y(t_2) - x_2^2(t_2), \tag{5.5b}$$

$$x_3(t_2) = \pm\sqrt{x_1^2(t_2) + 2x_1(t_2) - \dot{y}(t_0)}\ . \tag{5.5c}$$

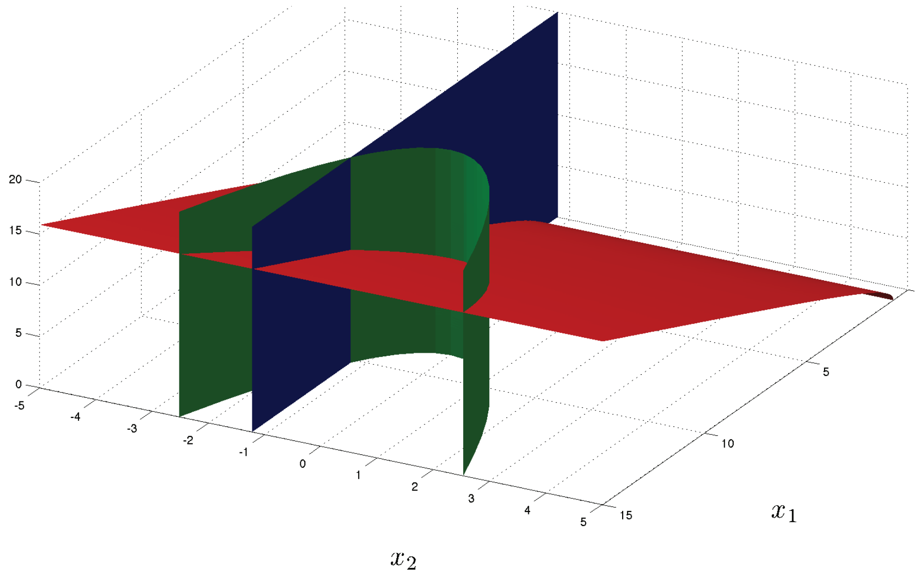


Figure 5.1: Intersection of three unobservable manifolds that combines the information available from each mode at t_2. These manifolds are represented by the solution sets of equations (5.3), (5.4), and (5.5a). Since their intersection is a point, the vector $x(t_2)$ is recovered completely.

To see that the solution exists and is well-defined, note that $x_1^2(t_2)+2x_1(t_2)-\dot{y}(t_0)$ is strictly positive if $x_1(t_2) > x_{10}$. This indeed follows from the system dynamics as $x_{10}, x_{30} \geq 0$, and solving the differential equations for Γ_1, Γ_2 gives

$$x_1(t_2) = (x_{10}+1)\cosh(0.1\tau_1+\tau_2) + x_{30}\sinh(0.1\tau_1+\tau_2) - 1,$$

so that $x_1(t_2) > x_{10}, \forall\, t_2 > 0$. Similarly,

$$x_3(t_2) = (x_{10}+1)\sinh(0.1\tau_1+\tau_2) + x_{30}\sinh(0.1\tau_1+\tau_2),$$

so that $x_3(t_2) > 0, \forall\, t_2 > 0$. Graphically, the solution of (5.5) has been verified through simulation in Fig. 5.1. This shows that there is a unique solution for each component of the state at time t_2. Furthermore, since there is only one invertible jump map, the values of the state obtained at time t_2 can be propagated forward and backward in time to obtain the values of state trajectory at all times. $\triangleleft$

Let us highlight some interesting aspects of this example in order to motivate the technical details that follow:

1. The main idea of the example was to illustrate that even though the individual modes

of the system are not observable, it is possible to extract partial information about the state from these modes. Under certain constraints on the dynamics of the system, it is then possible to accumulate all the information at some time instant in future so that it becomes possible to determine complete knowledge of the state of the system. In this example, we saw that information from mode 1 and mode 2 is combined with that of mode 3 at time t_2 to recover the state at that time instant.

2. The information about x_2 (one of the observable components of mode 1) obtained at t_1^- is preserved at time t_2^- only because $L_{f_2} dx_2 = \text{span}\{[2x_2(x_1{+}1)\; x_1^2{-}x_3^2{+}2x_1\; -2x_2x_3]\} \in d\mathcal{O}_2 + dx_2$; that is, the evolution of $x_2(t)$ along the dynamics of mode 2 depends on the past values of x_2 and the observable components of mode 2. Also, the initial condition for x_2 at t_1 is obtained directly from the knowledge of $x_2(t_1^-)$ which is one of the observable components of mode 1. In other words, even though x_2 is unobservable for mode 2, it does not interact with other unobservable components of mode 2, and hence the information recovered about x_2 from mode 1 is preserved under the new dynamics after switching.

3. The jump map at first switching instant does not combine the known (x_2) and unknown information (x_1, x_3) about the state. If, after the jump, $x_2(t_1)$ were a function of $x_1(t_1^-)$ and $x_3(t_1^-)$ (say, $p_1(x) := \text{col}(x_1, x_1 + x_3, x_3)$), then it would have been not possible to preserve the information about x_2 till time t_2 and consequently the solution would not exist.

The above mentioned arguments underline the basic ideas behind our solution to the observability problem. This approach of combining the available information from various modes and preserving parts of it which do not interact with unknown components of the state, leads to a sufficient condition for observability, which will be formalized in Section 5.4.3, Theorem 5.12. Also, in the above example, we arrived at the solution by solving differential equations and a set of algebraic equations. As this approach is highly impractical and unfruitful, more often than not, our main goal is to design appropriate dynamical observers. Apart from guaranteeing the existence of solution to observability problem, the most useful aspect of Theorem 5.12, however, is that the proposed sufficient condition renders a particular canonical structure to the system dynamics at every mode. This structure is presented in the next section, and we show how it leads to designing the observers.

5.3 Observer Synopsis

In this section, we discuss our primary approach towards designing observers that would be detailed in Section 5.5. The key aspect of our approach is the transformation of each subsystem's dynamics to particular canonical structure and then use that structure to construct appropriate observers. Existence of such a structure in each subsystem is presented as an assumption here in order to highlight the underlying idea of observer design. Later, in Section 5.4.3, a sufficient condition that leads to this particular structure is presented.

5.3.1 Property of Individual Modes

Before dealing with the switched case, let us consider the system (5.1a) and (5.1c) for a fixed mode q, without the jump map (5.1b) for now. In particular, we note that the individual subsystems may not be observable, which calls for the classical Kalman decomposition [18]: changing the coordinates so that the system is explicitly split into the observable part and the unobservable part. Then, we assume the following for each mode q.

Assumption 5.1. There exist potential coordinate functions $\lambda_{q,1\sim n}(x)$ that yield a diffeomorphism $\lambda_q := \mathrm{col}(\lambda_{q,1}(x), \ldots, \lambda_{q,n}(x))$ on $\mathcal{X}$ and that the system (5.1a) and (5.1c) for mode q is represented in the new coordinates $\mathrm{col}(\xi_q', \xi_q)$ with $\xi_q' = \lambda_{q,(k_q+1)\sim n}(x) \in \mathbb{R}^{n-k_q}$ and $\xi_q = \lambda_{q,1\sim k_q}(x) \in \mathbb{R}^{k_q}$ as follows:

$$\dot{\xi}_q' = F_q'(\xi_q', \xi_q) + G_q'(\xi_q', \xi_q)u, \tag{5.6a}$$

$$\dot{\xi}_q = F_q(\xi_q) + G_q(\xi_q)u, \tag{5.6b}$$

$$y = H_q(\xi_q). \tag{5.6c}$$

Furthermore, this representation is valid on the set $\lambda_q(\mathcal{X})$. ◁

Assumption 5.2. For each mode q, the reduced-order subsystem (5.6b) and (5.6c) is small-time uniformly observable on the set $\Xi_q := \lambda_{q,1\sim k_q}(\mathcal{X})$, which is the projection of $\lambda_q(\mathcal{X})$ onto the ξ_q-coordinates. ◁

5.3.2 Canonical Structure for the Switched System

Assumption 5.3. There exists a diffeomorphism $(\lambda_{1\sim k_q}, w_{1\sim \bar{l}_q}, \lambda_{k_q+\bar{l}_q+1\sim n})$, such that with the new coordinates $\xi_q := \lambda_{1\sim k_q}(x) \in \mathbb{R}^{k_q}$, $z_q := w_{1\sim \bar{l}_q}(x) \in \mathbb{R}^{\bar{l}_q}$, and $\xi_q' := \lambda_{k_q+\bar{l}_q+1\sim n}(x) \in$

$\mathbb{R}^{n-k_q-\bar{l}_q}$, the dynamics of system (5.1) take the following form:

$$\dot{\xi}_q = F_q(\xi_q) + G_q(\xi_q)u, \qquad \xi_q \in \mathbb{R}^{k_q}, \tag{5.7a}$$

$$y = H_q(\xi_q), \tag{5.7b}$$

$$\dot{z}_q = F_q^*(z_q, \xi_q) + G_q^*(z_q, \xi_q)u, \qquad z_q \in \mathbb{R}^{\bar{l}_q}, \tag{5.7c}$$

$$z_q(t_{q-1}) = S_q^*\Big(\xi_q(t_{q-1}), P_{q-1}\big(\xi_{q-1}(t_{q-1}^-), z_{q-1}(t_{q-1}^-), \xi_q(t_{q-1})\big)\Big), \tag{5.7d}$$

$$\dot{\xi}_q' = F_q'(\xi_q', \xi_q, z_q) + G_q'(\xi_q', \xi_q, z_q)u, \qquad \xi_q' \in \mathbb{R}^{n-k_q-\bar{l}_q}, \tag{5.7e}$$

where z_1 is taken as a null vector. $\lhd$

In equation (5.7), ξ_q denotes the observable component of subsystem q as introduced in Assumption 5.1 and 5.2. The vector z_q represents the additional information accumulated from other modes over the interval $[t_0, t_{q-1})$. The corresponding dynamics indicate that the vector z_q propagates under the dynamics of subsystem q without interacting with its unobservable components. Also, since the initial condition, $z_q(t_{q-1})$ is a function of known components, it becomes possible to know the value of $z_q(t_q)$ and pass it onto the next mode. This way, we keep accumulating uncorrupted information from the past modes.

If for some $m \geq 1$, $k_m + \bar{l}_m = n$, then it means that sufficient information about the state is available from system dynamics. The state can now be recovered using the observers for (5.7a) and (5.7c) and then using the corresponding inverse transformation.

If system (5.1) admits the structure (5.7), then we construct separate observers for the component ξ_q and z_q. Since ξ_q is observable for each mode, it is possible to get a good estimate of ξ_q. The variable z_q, on the other hand, is not an observable quantity for the mode q. Intuitively speaking, the role of z_q-observer is not to reduce the error $\tilde{z}_q(t) := \hat{z}_q(t) - z_q(t)$, but to deliver the estimates $\hat{\xi}_{q-1}(t_{q-1}^-)$ and $\hat{z}_{q-1}(t_{q-1}^-)$, that are obtained from the previously active mode and are encoded in the initial condition (5.7d), to the next mode through $\hat{z}_q(t)$. Suppose that, seen at time $t = t_m^-$, an ideal observer provides the exact information of $\xi_q(t)$ on each interval $[t_{q-1}, t_q)$, $q = 1, \ldots, m$, using the stored input u and the output y with the model (5.7a) and (5.7b). For example, with exact values of $\hat{\xi}_1(t_1^-) = \xi_1(t_1^-)$ and $\hat{\xi}_2(t_1) = \xi_2(t_1)$, we obtain the exact value of $\hat{z}_2(t_1) = z_2(t_1)$ by (5.7d). Then, integration of (5.7c) for $q = 2$ results in exact values of $\hat{z}_2(t) = z_2(t)$ on $[t_1, t_2)$. This process repeats until we get $\hat{\xi}_m(t_m^-) = \xi_m(t_m^-)$ and $\hat{z}_m(t_m^-) = z_m(t_m^-)$. Assuming that $k_m + \bar{l}_m = n$, $x(t_m^-)$ is now determined uniquely from $\xi_m(t_m^-)$ and $z_m(t_m^-)$, as the map

$$x(t_m^-) \mapsto \begin{bmatrix} \chi(\xi_m(t_m^-)) \\ z_m(t_m^-) \end{bmatrix} \tag{5.8}$$

is invertible, for some known potential coordinate function χ. The details of the implementation of this idea will be given in Section 5.5.

5.4 Geometric Conditions

5.4.1 Useful Lemmas

The following lemmas will be frequently used in the chapter.

Lemma 5.9. *Consider a codistribution $\mathcal{W}$ generated by exact one-forms, that is, $\mathcal{W} = \operatorname{span}\{d\lambda_1, \ldots, d\lambda_k\}$ with $1 \leq k \leq n$, where $\lambda_1, \ldots, \lambda_k$ are potentially diffeomorphic smooth functions defined on a set $\mathcal{X} \subset \mathbb{R}^n$.*

1. *If the codistribution $\mathcal{W}$ is invariant with respect to a smooth vector field $f(x)$, i.e.,*

$$L_f\mathcal{W} \subset \mathcal{W}$$

on the set $\mathcal{X}$, then there exists a smooth vector field F on the set $\lambda_{1\sim k}(\mathcal{X})$ such that $\left.\frac{\partial \lambda_{1\sim k}}{\partial x}\right|_x \cdot f(x) = F(\lambda_{1\sim k}(x))$ on $\mathcal{X}$.

2. *If a smooth function $h : \mathcal{X} \to \mathbb{R}$ satisfies*

$$dh \in \mathcal{W}$$

on $\mathcal{X}$, then there exists a smooth function $H : \lambda_{1\sim k}(\mathcal{X}) \to \mathbb{R}$ such that $h(x) = H(\lambda_{1\sim k}(x))$ on $\mathcal{X}$.

3. *Let $\mathcal{W}'$ be another codistribution such that $\dim(\mathcal{W}+\mathcal{W}')$ is constant on $\mathcal{X}$, and suppose that $\mathcal{W} + \mathcal{W}' = \operatorname{span}\{d\lambda_{1\sim k}, d\mu'_j : j = 1, \ldots, \bar{r}'\}$ where $\bar{r}' = \dim(\mathcal{W} + \mathcal{W}') - \dim \mathcal{W}$ and the elements of $\{\lambda_{1\sim k}, \mu'_{1\sim \bar{r}'}\}$ are smooth and potentially diffeomorphic on $\mathcal{X}$. If a smooth map $p : \mathcal{X} \to p(\mathcal{X})$ satisfies*

$$p_*(\ker \mathcal{W} \cap \ker \mathcal{W}') \subset \ker \mathcal{W}$$

on $\mathcal{X}$, then there exists a smooth map $P : \lambda_{1\sim k}(\mathcal{X}) \times \mu'_{1\sim\bar{r}'}(\mathcal{X}) \to \mathbb{R}^k$ such that the relation $\lambda_{1\sim k}(p(x)) = P(\lambda_{1\sim k}(x), \mu'_{1\sim\bar{r}'}(x))$ holds, while x and $p(x)$ are contained in $\mathcal{X}$.

Proof. Since λ_i, $i = 1, \ldots, k$, are potentially diffeomorphic on $\mathcal{X}$, we can find $\lambda_{k+1}, \ldots, \lambda_n$ such that $z_i = \lambda_i(x)$, $i = 1, \ldots, n$, becomes a diffeomorphism on $\mathcal{X}$, and let $\lambda(x) = \lambda_{1\sim n}(x)$ for simplicity. In the z-coordinates, it is seen that $\mathcal{W} = \text{span}\{dz_1, \ldots, dz_k\}$, and thus,

$$\ker \mathcal{W} = \text{span}\left\{\frac{\partial}{\partial z_{k+1}}, \cdots, \frac{\partial}{\partial z_n}\right\}. \tag{5.9}$$

Also, the vector field $f(x)$ is represented in z-coordinates as:

$$\frac{\partial \lambda}{\partial x}\Big|_{x=\lambda^{-1}(z)} \cdot f(\lambda^{-1}(z)) = \bar{f}(z) = \begin{bmatrix} \bar{f}_a(z_a, z_b) \\ \bar{f}_b(z_a, z_b) \end{bmatrix},$$

where $z_a = [z_1, \ldots, z_k]^\top$, $z_b = [z_{k+1}, \ldots, z_n]^\top$, $\bar{f}_a(z) \in \mathbb{R}^k$, and $\bar{f}_b(z) \in \mathbb{R}^{n-k}$. Then, since $\mathcal{W}$ is invariant w.r.t. f, the distribution $\ker \mathcal{W}$ is also invariant w.r.t. f on $\mathcal{X}$.[1] Since $\ker \mathcal{W}$ is invariant w.r.t. f on $\mathcal{X}$, or equivalently w.r.t. $\bar{f}$ on $\lambda(\mathcal{X})$, it follows that

$$\left[\bar{f}, \frac{\partial}{\partial z_i}\right] = -\sum_{j=1}^{n} \frac{\partial \bar{f}_j}{\partial z_i} \frac{\partial}{\partial z_j} \in \ker \mathcal{W}, \qquad i = k+1, \ldots, n.$$

Hence,

$$\frac{\partial \bar{f}_j}{\partial z_i} = 0, \quad {}^\forall j = 1, \ldots, k, \;\; i = k+1, \ldots, n.$$

This implies $\bar{f}_a(z_a, z_b) = \bar{f}_a(z_a)$. Taking $F = \bar{f}_a$ proves the item 1.

For proving the item 2, the function h is represented in the z-coordinates as $\bar{h}(z) = h \circ \lambda^{-1}(z)$. Since

$$d\bar{h} = \frac{\partial \bar{h}}{\partial z_1} dz_1 + \cdots + \frac{\partial \bar{h}}{\partial z_n} dz_n \in \mathcal{W}, \qquad \text{on } \lambda(\mathcal{X}),$$

it is seen that $\frac{\partial \bar{h}}{\partial z_i} = 0$, $i = k+1, \ldots, n$. Taking $H = \bar{h}$ proves the item 2.

For the item 3, we find $\lambda_{k+\bar{r}'+1}, \ldots, \lambda_n$ such that

$$z = \lambda(x) := \text{col}(\lambda_{1\sim k}(x), \mu'_{1\sim\bar{r}'}(x), \lambda_{k+\bar{r}'+1}(x), \ldots, \lambda_n(x))$$

becomes a diffeomorphism on $\mathcal{X}$. Then, in z-coordinate, we obtain $\mathcal{W} = \text{span}\{dz_1, \ldots, dz_k\}$

[1] By the equality $L_f(\sigma \cdot v) = (L_f \sigma) \cdot v + \sigma \cdot [f, v]$, it is seen with $\sigma \in \mathcal{W}$ and $v \in \ker \mathcal{W}$ that $\sigma \cdot v = 0$ and $(L_f \sigma) \cdot v = 0$. Hence, $\sigma \cdot [f, v] = 0$ [63].

and $\mathcal{W}+\mathcal{W}' = \text{span}\{dz_1, \ldots, dz_{k+\bar{r}'}\}$ so that $\ker \mathcal{W}$ and $\ker \mathcal{W} \cap \ker \mathcal{W}' = (\mathcal{W}+\mathcal{W}')^\perp$ can be equivalently written as $\text{span}\{e_{k+1}, \ldots, e_n\}$ and $\text{span}\{e_{k+\bar{r}'+1}, \ldots, e_n\}$, respectively, on $\lambda(\mathcal{X})$, where e_j is the elementary basis vector (i.e., all elements are zero except the j-th element which is one). With $\bar{p}(z) = \lambda \circ p \circ \lambda^{-1}(z)$, the condition $p_*(\ker \mathcal{W} \cap \ker \mathcal{W}') \subset \ker \mathcal{W}$ implies that for $j = k + \bar{r}' + 1, \ldots, n$,

$$
\begin{aligned}
\frac{\partial \bar{p}}{\partial z} e_j &= \frac{\partial \lambda}{\partial x}\Big|_{x=p(\lambda^{-1}(z))} \circ \frac{\partial p}{\partial x}\Big|_{x=\lambda^{-1}(z)} \circ \frac{\partial \lambda^{-1}}{\partial z} e_j \\
&\in \text{span}\{e_{k+1}, \ldots, e_n\},
\end{aligned}
$$

on $\lambda(\mathcal{X})$. This implies that the upper k functions of $\bar{p}$ do not depend on $z_{k+\bar{r}'+1}, \ldots, z_n$, so that we obtain $P(z_1, \ldots, z_{k+\bar{r}'}) = \bar{p}_{1\sim k}(z)$; that completes the proof. □

5.4.2 Assumption on Individual Modes

Let the observation space $\mathcal{O}_q$ be the linear space of functions over $\mathbb{R}$ containing $h_{q,i}$ (where $h_{q,i}$ is the i-th element of h_q) and all repeated Lie derivatives $L_{v_1} L_{v_2} \cdots L_{v_k} h_{q,i}$ with $v_l \in \{f_q, g_{q,1}, \ldots, g_{q,d_u}\}$ (where $g_{q,j}$ is the j-th column of g_q).

Proposition 5.10. *Assume that the codistribution $d\mathcal{O}_q = \text{span}\{d\lambda : \lambda \in \mathcal{O}_q\}$ has constant dimension k_q; $\dim d\mathcal{O}_q = k_q$ on the set $\mathcal{X}$. In addition, there are potentially diffeomorphic k_q smooth functions $\lambda_{q,j}$, $j = 1, \ldots, k_q$, such that*

$$
d\mathcal{O}_q = \text{span}\{d\lambda_{q,1}, \ldots, d\lambda_{q,k_q}\} \qquad \text{on } \mathcal{X}.
$$

Then, for mode q, there exist functions $\lambda_{q,1\sim n}(x)$ such that under the transformation given by $\text{col}(\xi_q, \xi'_q) := \text{col}(\lambda_{q,1\sim k_q}(x), \lambda_{q,k_q+1\sim n}(x))$, for some $k_q \geq 0$, the system dynamics (5.1a) *and* (5.1c) *admit the structure in* (5.6).

Remark 5.11. *As a matter of fact, the simple condition that $\dim d\mathcal{O}_q(x) = k_q$ in a neighborhood of some $x^o \in \mathcal{X}$ guarantees, by Frobenius theorem, that there exists a local neighborhood $\mathcal{X}' \subset \mathcal{X}$ of x^o such that Assumption 5.1 holds. Compared to this local observability (studied in, e.g., [106, 63, 18]), Assumptions 5.1 and 5.2 may be thought of as global versions ("global" in the sense of the whole region $\mathcal{X}$).* ◁

As long as we restrict our attention to (5.6b) and (5.6c) for each mode, Assumption 5.2 becomes the standard uniform observability assumption that has often been studied in the literature (see [112] and references therein). Assumption 5.2 can be checked in various ways; for instance, if the class of inputs $\mathcal{U}$ consists of smooth functions only, then one may try to find a function $\mathcal{E}$ such that

$$\xi_q = \mathcal{E}(y, \dot{y}, \dots, y^{(n_y-1)}, u, \dot{u}, \dots, u^{(n_u-1)}),$$

where $n_y \in \mathbb{N}$ and $n_u \in \mathbb{N}$, and that the function $\mathcal{E}(\,\cdot\,, u, \dot{u}, \dots, u^{(n_u-1)})$ is surjective onto Ξ_q for all $u(\cdot) \in \mathcal{U}$. The existence of such a function $\mathcal{E}$ is used as the definition of uniform observability in [113, 18].

5.4.3 Sufficient Condition for Observability

In deriving the sufficient condition for observability, we do not assume the individual modes of the system to be observable. So, in order to recover the system state $x(t)$, partial information obtained from each mode is accumulated. This partial information is quantified in terms of the maximal integral submanifold of the distribution $d\mathcal{O}_q^\perp$ which has the property that the states on the slices of this submanifold are not distinguishable by the output of mode q. As soon as a switch occurs, the indistinguishable states must be contained in the intersection of the integral submanifolds of the corresponding modes, which in turn reduces the uncertainty in the state. Continuing in this manner, with subsequent switching, we expect to reduce the size of submanifold that contains the state. Eventually, if the corresponding intersections reduce to a point, we can recover the state completely. However, while the intersections are taken *at the same time*, the information contained in the integral submanifolds is scattered in time because each one of them becomes available sequentially as time goes on. This suggests that the partial information, obtained at each mode, should evolve along the system dynamics uncorrupted until all the information is gathered to compute the state. Inspired by this intuition, we present structural conditions which guarantee that the evolution of the partial information is feasible without being affected by the unknown quantities of subsequent modes.

Before presenting the condition, let us rename the switching sequence for convenience. That is, when the switching signal $\sigma(t)$ takes the mode sequence $\{q_1, q_2, \cdots\}$, we rename them as increasing integers $\{1, 2, 3, \cdots\}$ which is ever increasing even though the same mode is revisited. This description also incorporates cases where there is a state jump without change in dynamics or the mode change does not involve state jumps.

Theorem 5.12. *Suppose that Assumptions 5.1 and 5.2 hold, and define* $d\mathcal{O}'_q := \mathrm{span}\{d(\lambda_{q,i} \circ p_{q-1}) : i = 1, \dots, k_q\}$ *for each* $q \geq 2$. *On* $\mathcal{X}$, *define a sequence of codistributions* $\mathcal{W}_q$, *with* $\mathcal{W}_0 := \{0\}$, *as:*

> $\mathcal{W}_q$ *is the largest nonsingular and involutive*[2] *codistribution, invariant with respect to* f_q *and* g_q, *contained in* $(d\mathcal{O}_q + \mathcal{W}_{q-1})$ *such that* $(p_q)_*(\ker \mathcal{W}_q \cap \ker d\mathcal{O}'_{q+1}) \subset \ker \mathcal{W}_q$.

If

1. *there exists* $m \geq 1$ *such that, on* $\mathcal{X}$,

$$\dim(d\mathcal{O}_m + \mathcal{W}_{m-1}) = n,$$

   with $\mathcal{W}_m := d\mathcal{O}_m + \mathcal{W}_{m-1}$.

2. *The codistributions* $\mathcal{W}_q$ $(1 \leq q \leq m)$, $d\mathcal{O}_q + \mathcal{W}_{q-1}$ $(2 \leq q \leq m)$, *and* $\mathcal{W}_q + d\mathcal{O}'_{q+1}$ $(1 \leq q \leq m-1)$ *are nonsingular on* $\mathcal{X}$. *Moreover,*

   (a) *there exist potentially diffeomorphic smooth functions* $\{\phi_{q,i}, \omega_{q,j} : i = 1, \dots, \bar{k}_q, j = 1, \dots, \bar{l}_q, \bar{k}_q + \bar{l}_q = \dim \mathcal{W}_q\}$ *on* $\mathcal{X}$ *such that*

$$\begin{aligned} \mathcal{W}_q &= \mathrm{span}\{d\phi_{q,1}, \cdots, d\phi_{q,\bar{k}_q}, d\omega_{q,1}, \cdots, d\omega_{q,\bar{l}_q}\}, \\ d\phi_{q,i} &\in d\mathcal{O}_q, \qquad d\omega_{q,j} \notin d\mathcal{O}_q, \end{aligned} \tag{5.10}$$

   (b) *there exist potentially diffeomorphic smooth functions* $\{\mu_{q,i} : i = 1, \dots, \bar{r}_q, \bar{r}_q = \dim(d\mathcal{O}_q + \mathcal{W}_{q-1})\}$ *on* $\mathcal{X}$ *such that*

$$d\mathcal{O}_q + \mathcal{W}_{q-1} = \mathrm{span}\{d\mu_{q,1}, \dots, d\mu_{q,\bar{r}_q}\}, \tag{5.11}$$

$$\begin{aligned} \mu_{q,i} \in \mathrm{l.com}\, \{&\lambda_{q,1}, \dots, \lambda_{q,k_q}, \phi_{q-1,1}, \cdots, \\ &\phi_{q-1,\bar{k}_{q-1}}, \omega_{q-1,1}, \cdots, \omega_{q-1,\bar{l}_{q-1}}\}, \end{aligned} \tag{5.12}$$

   (c) *there exist potentially diffeomorphic smooth functions* $\{\mu'_{q,j} : j = 1, \dots, \bar{r}'_q, \bar{r}'_q =$

[2] Nonsingularity and involutivity are implied by item 2(a). In fact, involutivity of a codistribution is determined by the involutivity of its kernel which is a distribution. A codistribution generated by the exact one-forms is always involutive.

$\dim(\mathcal{W}_q + d\mathcal{O}'_{q+1}) - \dim \mathcal{W}_q\}$ *on* $\mathcal{X}$ *such that*

$$\begin{aligned} \mathcal{W}_q + d\mathcal{O}'_{q+1} = \operatorname{span}\{&d\phi_{q,1}, \cdots, d\phi_{q,\bar{k}_q}, \\ &d\omega_{q,1}, \cdots, d\omega_{q,\bar{l}_q}, d\mu'_{q,1}, \ldots, d\mu'_{q,\bar{r}'_q}\}, \end{aligned} \tag{5.13}$$

$$\mu'_{q,j} \in \text{l.com}\,\{\lambda_{q+1,1} \circ p_q, \ldots, \lambda_{q+1,k_{q+1}} \circ p_q\}, \tag{5.14}$$

then the system (5.1) *is large-time uniformly observable on* $\mathcal{X}$ *for all the switching signals containing the consecutive subsequence* $\{1, 2, \ldots, m\}$. ◁

The following observations are immediate: (a) $d\mathcal{O}_q$ itself is invariant with respect to f_q and g_q by construction, (b) if $p_q(x) = x$, so that there is no state jump, then the condition $(p_q)_*(\ker \mathcal{W}_q \cap \ker d\mathcal{O}'_{q+1}) \subset \ker \mathcal{W}_q$ automatically holds, (c) involutivity and invariance of a codistribution generated by exact one-forms is closed under the addition, and if two smooth nonsingular codistributions $\mathcal{W}_a$ and $\mathcal{W}_b$ satisfy $p_*(\ker \mathcal{W}_i \cap \mathcal{D}) \subset \ker \mathcal{W}_i$ where $i \in \{a, b\}$ for any differentiable map p and any distribution $\mathcal{D}$, then $p_*(\ker(\mathcal{W}_a + \mathcal{W}_b) \cap \mathcal{D}) \subset \ker(\mathcal{W}_a + \mathcal{W}_b)$.[3] Therefore, the "largest" codistribution in the assumption of Theorem 5.12 is well-defined.

The compactness of the set $\mathcal{X}$ guarantees the solution without finite escape time, and will be used for observer construction in the next section. If all the assumptions hold with $\mathcal{X} = \mathbb{R}^n$, then the observability property becomes global in case the solution has no finite escape time. On the other hand, if local observability is of interest, then the assumptions get simpler by removing the items 2(a), 2(b), and 2(c).

Corollary 5.13. *Suppose that Assumptions 5.1 and 5.2 hold in a neighborhood of a point* $x^o \in \mathcal{X}$. *If each of the codistributions* $\mathcal{W}_q$, $d\mathcal{O}_q + \mathcal{W}_{q-1}$, *and* $\mathcal{W}_q + d\mathcal{O}'_{q+1}$ *are nonsingular at* x^o, $\mathcal{W}_q$ *is smooth and involutive at* x^o, *and* $\dim(d\mathcal{O}_m + \mathcal{W}_{m-1})(x^o) = n$, *then the system is large-time uniformly observable in some neighborhood of* x^o *for all the switching signals containing the consecutive subsequence* $\{1, 2, \ldots, m\}$.

Proof. Since the smooth codistribution $\mathcal{W}_q$ is nonsingular and involutive, Frobenius theorem provides, in a local neighborhood of x^o, independent smooth functions $\phi_{q,1\sim\bar{k}_q}$ and $\omega_{q,1\sim\bar{l}_q}$ of the item 2(a) of Theorem 5.12. With those functions for $\mathcal{W}_q$, we can pick more independent functions in $d\mathcal{O}_{q+1}$ and $d\mathcal{O}'_{q+1}$, by which the items 2(b) and 2(c) are satisfied, respectively, in a neighborhood of x^o. □

[3] For each $i \in \{a, b\}$, $\ker(\mathcal{W}_a + \mathcal{W}_b) \subseteq \ker \mathcal{W}_i$, so that $(\ker(\mathcal{W}_a + \mathcal{W}_b) \cap \mathcal{D}) \subseteq \ker \mathcal{W}_i \cap \mathcal{D}$, which in turn implies that $p_*(\ker(\mathcal{W}_a + \mathcal{W}_b) \cap \mathcal{D}) \subseteq p_*(\ker \mathcal{W}_i \cap \mathcal{D}) \subseteq \ker \mathcal{W}_i$ by the assumption. Therefore, we have that $p_*(\ker(\mathcal{W}_a + \mathcal{W}_b) \cap \mathcal{D}) \subseteq \ker \mathcal{W}_a \cap \ker \mathcal{W}_b = \ker(\mathcal{W}_a + \mathcal{W}_b)$.

Now we present the proof of Theorem 5.12, which is constructive in the sense that a technique to recover $x(t)$ at some time $t = T > t_{m-1}$ is revealed (rather than discussing the indistinguishability of two different states). This way paves the road to the observer design in the next section.

Proof of Theorem 5.12. Consider the interval prior to the first switching $[t_0, t_1)$. Since $\mathcal{W}_1 \subset d\mathcal{O}_1$, we have that

$$\mathcal{W}_1 = \operatorname{span}\{d\phi_{1,1}, d\phi_{1,2}, \ldots, d\phi_{1,\bar{k}_1}\}, \qquad \bar{k}_1 \le k_1.$$

Because $d\phi_{1,i}$, for each $i = 1, \ldots, \bar{k}_1$, is an element of $d\mathcal{O}_1$ that is generated by the differentials of $\lambda_{1,l}$, $l = 1, \ldots, k_1$, and $\lambda_{1,l}$'s are potentially diffeomorphic on $\mathcal{X}$ (by Assumption 5.1), the function $\phi_{1,i}$ is a function of $\lambda_{1,1\sim k_1}$ only (by Lemma 5.9.2). Since $\xi_1 := \lambda_{1,1\sim k_1}(x)$ is small-time uniformly observable on $\lambda_{1,1\sim k_1}(\mathcal{X})$ (by Assumption 5.2), the value of the vector $\xi_1(t) = \lambda_{1,1\sim k_1}(x(t))$, and thus, $\phi_{1,1\sim \bar{k}_1}(x(t))$ are recovered for $t \in [t_0, t_1)$.

Now Lemma 5.9.3, with the item 2(c) and $(p_1)_*(\ker \mathcal{W}_1 \cap \ker d\mathcal{O}_2') \subset \ker \mathcal{W}_1$, implies the existence of a function $\breve{P}_1$ (and then P_1 below, since $\phi_{1,i}$ is a function of $\lambda_{1,1\sim k_1}$) such that

$$\begin{aligned}
\phi_{1,1\sim\bar{k}_1}(x(t_1)) &= \phi_{1,1\sim\bar{k}_1}(p_1(x(t_1^-))) \\
&= \breve{P}_1(\phi_{1,1\sim\bar{k}_1}(x(t_1^-)), \mu'_{1,1\sim\bar{r}'_1}(x(t_1^-))) \\
&= P_1(\lambda_{1,1\sim k_1}(x(t_1^-)), \lambda_{2,1\sim k_2} \circ p_1(x(t_1^-))) \\
&= P_1(\lambda_{1,1\sim k_1}(x(t_1^-)), \lambda_{2,1\sim k_2}(x(t_1))),
\end{aligned} \tag{5.15}$$

where the third equality follows from (5.10) and (5.14). Next, consider the interval $[t_1, t_2)$. For $i = 1, \ldots, \bar{k}_2$, using Lemma 5.9.2, the condition $d\phi_{2,i} \in d\mathcal{O}_2 = \operatorname{span}\{d\lambda_{2,1}, \ldots, d\lambda_{2,k_2}\}$ guarantees that $\phi_{2,i}$ is a function of $\lambda_{2,1\sim k_2}$ only. Again by Assumption 5.2, the vector $\xi_2(t) := \lambda_{2,1\sim k_2}(x(t))$, and thus, $\phi_{2,1\sim\bar{k}_2}(x(t))$ are known for the interval $[t_1, t_2)$.

Now observing that $\mathcal{W}_2 = \operatorname{span}\{d\phi_{2,1}, \ldots, d\phi_{2,\bar{k}_2}, d\omega_{2,1}, \ldots, d\omega_{2,\bar{l}_2} : \bar{k}_2 + \bar{l}_2 = \dim \mathcal{W}_2\}$ is invariant w.r.t. f_2 and g_2, and $\{\phi_{2,1\sim\bar{k}_2}, \omega_{2,1\sim\bar{l}_2}\}$ are potentially diffeomorphic on $\mathcal{X}$, we apply Lemma 5.9.1 and obtain smooth vector fields F_2^* and G_2^* such that, with $z_2 := \omega_{2,1\sim\bar{l}_2}(x)$,

$$\begin{aligned}
\dot{z}_2 &= \frac{\partial \omega_{2,1\sim\bar{l}_2}}{\partial x}(x) \cdot (f(x) + g(x)u) \\
&= \breve{F}_2^*(z_2, \phi_{2,1\sim\bar{k}_2}(x)) + \breve{G}_2^*(z_2, \phi_{2,1\sim\bar{k}_2}(x))u \\
&= F_2^*(z_2, \lambda_{2,1\sim k_2}(x)) + G_2^*(z_2, \lambda_{2,1\sim k_2}(x))u \\
&= F_2^*(z_2, \xi_2) + G_2^*(z_2, \xi_2)u,
\end{aligned} \tag{5.16}$$

over $[t_1, t_2)$. In this interval, the vector $\xi_2(t) = \lambda_{2,1\sim k_2}(x(t))$ is known. Hence, if the initial condition of $z_2(t_1) = \omega_{2,1\sim\bar{l}_2}(x(t_1))$ is known, then the vector $z_2(t)$ on the interval $[t_1, t_2)$ is also available by solving the differential equation (5.16).

Note that $d\omega_{2,j} \in \mathcal{W}_2 \subset (d\mathcal{O}_2 + \mathcal{W}_1) = \operatorname{span}\{d\mu_{2,1}, \ldots, d\mu_{2,\bar{r}_2}\}$, $j = 1, \ldots, \bar{l}_2$, by the definition of $\mathcal{W}_2$ and the item 2(b). Therefore, by Lemma 5.9.2, $\omega_{2,j}$ can be written as a function of $\mu_{2,i}$'s, which leads to

$$
\begin{aligned}
z_2(t_1) = \omega_{2,1\sim\bar{l}_2}(x(t_1)) &= \breve{S}_2^*(\mu_{2,1\sim\bar{r}_2}(x(t_1))) \\
&= S_2^*(\xi_2(t_1), \phi_{1,1\sim\bar{k}_1}(x(t_1))) \\
&= S_2^*(\xi_2(t_1), P_1(\lambda_{1,1\sim k_1}(x(t_1^-)), \lambda_{2,1\sim k_2}(x(t_1)))) \\
&= S_2^*(\xi_2(t_1), P_1(\xi_1(t_1^-), \xi_2(t_1))),
\end{aligned}
\tag{5.17}
$$

in which the third equality uses (5.12) with μ_1 being null, and the fourth equality follows from (5.15).

This process is repeated to find F_q^*, G_q^*, S_q^*, and P_q. For instance, we can find P_2 such that

$$
\begin{aligned}
\begin{bmatrix} \phi_{2,1\sim\bar{k}_2}(x(t_2)) \\ \omega_{2,1\sim\bar{l}_2}(x(t_2)) \end{bmatrix} &= \begin{bmatrix} \phi_{2,1\sim\bar{k}_2}(p_2(x(t_2^-))) \\ \omega_{2,1\sim\bar{l}_2}(p_2(x(t_2^-))) \end{bmatrix} \\
&= \breve{P}_2(\phi_{2,1\sim\bar{k}_2}(x(t_2^-)), \omega_{2,1\sim\bar{l}_2}(x(t_2^-)), \mu'_{2,1\sim\bar{r}'_2}(x(t_2^-))) \\
&= P_2(\lambda_{2,1\sim k_2}(x(t_2^-)), \omega_{2,1\sim\bar{l}_2}(x(t_2^-)), \lambda_{3,1\sim k_3} \circ p_2(x(t_2^-))) \\
&= P_2(\xi_2(t_2^-), z_2(t_2^-), \xi_3(t_2)),
\end{aligned}
$$

and find S_3^* such that

$$
\begin{aligned}
z_3(t_2) = \omega_{3,1\sim\bar{l}_3}(x(t_2)) &= \breve{S}_3^*(\mu_{3,1\sim\bar{r}_3}(x(t_2))) \\
&= S_3^*(\xi_3(t_2), \phi_{2,1\sim\bar{k}_2}(x(t_2)), \omega_{2,1\sim\bar{l}_2}(x(t_2))) \\
&= S_3^*(\xi_3(t_2), P_2(\xi_2(t_2^-), z_2(t_2^-), \xi_3(t_2))).
\end{aligned}
$$

In summary, for each time interval $[t_{q-1}, t_q)$, $q = 1, \ldots, m$, we have the following differential equations (with z_1 being null):

$$\dot{\xi}_q = F_q(\xi_q) + G_q(\xi_q)u, \qquad \xi_q \in \mathbb{R}^{k_q}, \tag{5.18a}$$

$$y = H_q(\xi_q), \tag{5.18b}$$

$$\dot{z}_q = F_q^*(z_q, \xi_q) + G_q^*(z_q, \xi_q)u, \qquad z_q \in \mathbb{R}^{\bar{l}_q}, \tag{5.18c}$$

$$z_q(t_{q-1}) = S_q^*\Big(\xi_q(t_{q-1}), P_{q-1}\big(\xi_{q-1}(t_{q-1}^-), z_{q-1}(t_{q-1}^-), \xi_q(t_{q-1})\big)\Big), \tag{5.18d}$$

in which $\xi_q(t)$ and $z_q(t)$ are completely known. The above equations match the structure proposed initially in (5.7).

At any time $t = T > t_{m-1}$, it follows under the assumption in item 1, i.e. $\dim \mathcal{W}_m = n$, that the vectors $\xi_m(T)$ and $z_m(T)$ are completely known or equivalently the vector $\mathrm{col}(\phi_{m,1\sim\bar{k}_m}(x(T)), \omega_{m,1\sim\bar{l}_m}(x(T)))$ is known. This way $x(T)$ is recovered uniquely from the inverse mapping because of the potential diffeomorphic property in item 2(a). $\square$

Example 5.14. For the switched system considered in Example 5.8, it is seen that

$$k_1 = 2; \quad \lambda_{1,1}(x) = x_2; \quad \lambda_{1,2} = x_1^2 - x_3^2 + 2x_1,$$

$$k_2 = 1; \lambda_{2,1} = x_1^2 - x_3^2 + 2x_1; \quad d\mathcal{O}_2' = d\mathcal{O}_2,$$

$$k_3 = 1; \lambda_{3,1} = x_1 + x_2^2; d\mathcal{O}_3' = d\mathcal{O}_3.$$

Starting with $\mathcal{W}_0 = \{0\}$, and $(p_1)_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we can pick $\mathcal{W}_1 = d\mathcal{O}_1$. So that $\mathcal{W}_1 + d\mathcal{O}_2' = \mathcal{W}_1$ as $d\mathcal{O}_2' \subseteq d\mathcal{O}_1$. Also, let $\mu_{1,i} = \phi_{1,i} = \lambda_{1,i}$, $i = 1, 2$, and μ_1' is null.

In the next step, $\mathcal{W}_2 = \mathcal{W}_1$ satisfies the required assumptions. This time, we may pick $\phi_{2,1} = \lambda_{2,1}$, $w_{2,1} = \lambda_{1,1} = x_2$, $\mu_{2,1} = \lambda_{2,1}$, $\mu_{2,2} = \phi_{1,1}$, and $\mu_{2,1}' = \lambda_{3,1}$.

Finally, $\mathcal{W}_3 = d\mathcal{O}_3 + \mathcal{W}_2$ has constant rank 3 on $\mathcal{X}$ and the switched system is observable. For the sake completeness, we may pick $\mu_{3,1} = \phi_{3,1} := \lambda_{3,1}$; $\mu_{3,2} = w_{3,1} := w_{2,1}$, $\mu_{3,3} = w_{3,2} := \phi_{2,1}$. $\triangleleft$

5.5 Observer Design

Based on the study of large-time observability, let us now discuss the design of an asymptotic observer for the system (5.1). By asymptotic observer, we mean that the estimate $\hat{x}(t)$ that it generates converges to the plant state $x(t)$ as time tends to infinity. In order to achieve this, we introduce the following assumptions.

Assumption 5.4. *In the construction of observer, we assume that the following hold:*

1. *The switching is persistent and happens within the duration D; that is,*

$$t_q - t_{q-1} \leq D, \qquad \forall q \in \mathbb{N}, \tag{5.19}$$

where t_q is the switching time.

2. *The solution $x(t)$ of the plant* (5.1) *remains in a compact set $\mathcal{X} \in \mathbb{R}^n$, and the input $u(t)$ is uniformly bounded; $|u(t)| \leq M_u$.*

3. *The plant* (5.1) *is large-time observable on $\bar{\mathcal{X}}$, properly containing $\mathcal{X}$, in the sense that there is an $m \in \mathbb{N}$ such that the assumption of Theorem 5.12 holds, and the mode sequence repeats the particular modes $\{1, 2, \ldots, m\}$; that is, $\sigma(t) = ((q-1) \bmod m) + 1$ for $[t_{q-1}, t_q)$, $q \in \mathbb{N}$.* ◁

We do not consider the time consumed for computation by assuming that the data processor is fairly fast compared to the plant process. The computation time, however, needs to be considered in real-time application if the plant itself is fast.

The observer we propose is of hybrid-type, and has the form

$$\begin{aligned} \dot{\hat{x}}(t) &= \bar{f}_q(\hat{x}(t)) + \bar{g}_q(\hat{x}(t))u(t), \qquad t \in [t_{q-1}, t_q), \\ \hat{x}(t_q) &= \begin{cases} \bar{p}_q(\hat{x}(t_q^-)), & (q \bmod m) \neq 0, \\ \bar{p}_q(\mathcal{L}_q(y_{[t_{q-m}, t_q)}, u_{[t_{q-m}, t_q)})), & (q \bmod m) = 0, \end{cases} \end{aligned} \tag{5.20}$$

with an initial condition $\hat{x}(t_0) \in \mathcal{X} \subset \bar{\mathcal{X}}$, where $\bar{f}_q$, $\bar{g}_q$, and $\bar{p}_q$ are globally Lipschitz and they have the same values as f_q, g_q, and p_q, respectively, inside the compact set $\mathcal{X}$. Their global Lipschitz property can always be obtained by modifying them outside the set $\mathcal{X}$, using the so-called 'Lipschitz extension'.[4] It is seen that the observer consists of a plant copy with an estimate update law by some operator $\mathcal{L}_q$, which we design in this section. In fact, we present a design of $\mathcal{L}_q$, using some dynamic observers for partial states at each mode and an inversion algorithm logic in order to achieve

$$|\tilde{x}(t_m)| \leq \gamma |\tilde{x}(t_0)|, \tag{5.21}$$

[4] Since the plant state $x(t)$ remains in $\mathcal{X}$, this modification can also be applied to the plant model (5.1). This modification can be found in [107] to obtain globally Lipschitz vector fields for observer construction. Detailed procedures to the modification have been discussed in [114, 115].

where $0 < \gamma < 1$ and $\tilde{x} := \hat{x} - x$. The Lipschitz property of (5.20), the fact that $x(t) \in \mathcal{X}$, and Assumption 5.4.1 guarantee that $\sup_{t \in [t_{(j-1)m}, t_{jm})} |\tilde{x}(t)| \leq \Gamma |\tilde{x}(t_{(j-1)m})|$ with a constant Γ and $j \in \mathbb{N}$. In this way, if (5.21) holds then its repeated application leads to $\lim_{t \to \infty} |\tilde{x}(t)| = 0$.

The proposed observer construction is based on the representation (5.7) of the plant (5.1). The idea is that, for each interval $[t_{q-1}, t_q)$, $q = 1, \dots, m$, a conventional nonlinear observer, which we call ξ_q-observer, is employed to obtain the estimate $\hat{\xi}_q(t)$ for that interval. At the same time, a z_q-observer, replicating (5.7c) and (5.7d), is constructed as follows:

$$\dot{\hat{z}}_q = \bar{F}_q^*(\hat{z}_q, \hat{\xi}_q) + \bar{G}_q^*(\hat{z}_q, \hat{\xi}_q) u, \qquad 2 \leq q \leq m, \tag{5.22}$$

with the initial condition given by:

$$\hat{z}_q(t_{q-1}) = \bar{S}_q^*\big(\hat{\xi}_q(t_{q-1}), \bar{P}_{q-1}\big(\hat{\xi}_{q-1}(t_{q-1}^-), \qquad \hat{z}_{q-1}(t_{q-1}^-), \hat{\xi}_q(t_{q-1})\big)\big),$$

and $\hat{z}_1 := 0$ for convenience. Here $\bar{F}_q^*$ is the Lipschitz extension of F_q^* with respect to the set $Z_q \times \Xi_q$ and so on (Z_q is the image of $\mathcal{X}$ through $\omega_{q,1\sim\bar{l}_q}$, and Ξ_q through $\lambda_{q,1\sim k_q}$). In fact, the variable z_q is not an observable quantity for the mode q. Intuitively speaking, the role of z_q-observer is not to reduce the error $\tilde{z}_q(t) := \hat{z}_q(t) - z_q(t)$, but to deliver the estimates $\hat{\xi}_{q-1}(t_{q-1}^-)$ and $\hat{z}_{q-1}(t_{q-1}^-)$, that are obtained from the previously active mode and are encoded in the initial condition (5.5), to the next mode through $\hat{z}_q(t)$. Suppose that, seen at time $t = t_m^-$, an ideal observer provides the exact information of $\xi_q(t)$ on each interval $[t_{q-1}, t_q)$, $q = 1, \dots, m$, using the stored input u and the output y with the model (5.7a) and (5.7b). For example, with exact values of $\hat{\xi}_1(t_1^-) = \xi_1(t_1^-)$ and $\hat{\xi}_2(t_1) = \xi_2(t_1)$, we obtain the exact value of $\hat{z}_2(t_1) = z_2(t_1)$ by (5.5). Then, integration of (5.22) for $q = 2$ results in exact values of $\hat{z}_2(t) = z_2(t)$ on $[t_1, t_2)$. This process repeats until we get $\hat{\xi}_m(t_m^-) = \xi_m(t_m^-)$ and $\hat{z}_m(t_m^-) = z_m(t_m^-)$. With Assumption 5.4.3, i.e. $\dim \mathcal{W}_m = n$, $x(t_m^-)$ is now determined uniquely from $\xi_m(t_m^-)$ and $z_m(t_m^-)$, as the map

$$x(t_m^-) \mapsto \begin{bmatrix} \phi_{m,1\sim\bar{k}_m}(x(t_m^-)) \\ \omega_{m,1\sim\bar{l}_m}(x(t_m^-)) \end{bmatrix} = \begin{bmatrix} \chi(\xi_m(t_m^-)) \\ z_m(t_m^-) \end{bmatrix} \tag{5.23}$$

is invertible; here χ is a function such that $\chi(\lambda_{m,1\sim k_m}(x)) = \phi_{m,1\sim\bar{k}_m}(x)$ whose existence is guaranteed by Lemma 5.9.2. For convenience let us denote the inverse map by Ψ, so that $x(t_m^-) = \Psi(\xi_m, z_m)$. As a result, we choose the estimate update law in (5.20) to be:

$$\hat{x}(t_m^-) = \bar{\Psi}(\hat{\xi}_m(t_m^-), \hat{z}_m(t_m^-)) =: \mathcal{L}_q(y_{[t_0,t_m)}, u_{[t_0,t_m)}), \tag{5.24}$$

where $\bar{\Psi}$ is Lipschitz extension of Ψ. Through this relation, the plant state is recovered as $\hat{x}(t_m^-) = x(t_m^-)$ with exact information $\hat{\xi}_m(t_m^-) = \xi_m(t_m^-)$ and $\hat{z}_m(t_m^-) = z_m(t_m^-)$.

However, asymptotic observers in practice inevitably introduce some error in $\hat{\xi}_q(t)$ while estimating $\xi_q(t)$. Moreover, the estimation of $\xi_q(t)$ on the *entire interval* $[t_{q-1}, t_q)$ needs more attention because the conventional observers, initiated at the time $t = t_{q-1}$, often experience the transient overshoot before they converge to the proper estimates. Reducing the transient period by increasing observer gain may worsen the situation because of the peaking phenomenon [116]; that is, the peaking in $\hat{\xi}_q(t)$ may damage the role of (5.22) because large error in $|\hat{z}_q(t_q^-) - z_q(t_q^-)|$ may occur in spite of small error in $|\hat{z}_q(t_{q-1}) - z_q(t_{q-1})|$.

We overcome this obstacle by employing the so-called back-and-forth observer technique [117]. Suppose that the ξ_q-observer over the interval $[t_{q-1}, t_q)$ is written as

$$\begin{aligned} \dot{\hat{\xi}}_q^f &= F_q(\hat{\xi}_q^f) + G_q(\hat{\xi}_q^f)u + K_q^f(\hat{\xi}_q^f, u, y)(y - H_q(\hat{\xi}_q^f)), \\ \hat{\xi}_q^f(t_{q-1}) &= \bar{\lambda}_{q,1\sim k_q}(\hat{x}(t_{q-1})), \qquad 1 \le q \le m, \end{aligned} \tag{5.25}$$

where $\bar{\lambda}_{q,1\sim k_q}$ is the Lipschitz extension of $\lambda_{q,1\sim k_q}$ and superscript 'f' indicates 'forward', the meaning of which will soon become clear from the context.

Assumption 5.5. Under the assumption that $\xi_q(t) \in \Xi_q$ for $[t_{q-1}, t_q)$, $q = 1, \ldots, m$, the subsystem (5.7a) and (5.7b) admits an observer of the form (5.25), which can be made to converge to the state $\xi_q(t)$ arbitrarily fast; that is, for arbitrarily small constants $b > 0$ and $c > 0$, there exist an injection gain $K_q^f(\cdot)$ and a class-$\mathcal{KL}$ function $\beta^{f,q}$ satisfying

$$\beta^{f,q}(a, t) < ca \quad \text{for all } a > 0 \text{ and } b \le t \le \tau_q, \tag{5.26}$$

$$|\hat{\xi}_q^f(t) - \xi_q(t)| \le \beta^{f,q}(|\hat{\xi}_q^f(t_{q-1}) - \xi_q(t_{q-1})|, t - t_{q-1}), \tag{5.27}$$

for $t \in [t_{q-1}, t_q)$. ◁

Remark 5.15. *Many results in the literature, such as [118, 107], yield an observer satisfying Assumption 5.5 with $\beta^{f,q}$ being an exponential function.* ◁

Now consider another (backward) observer described as

$$\begin{aligned} \dot{\hat{\xi}}_q^b &= -F_q(\hat{\xi}_q^b) - G_q(\hat{\xi}_q^b)u(t_q - t) - K_q^b(\hat{\xi}_q^b, u(t_q - t), y(t_q - t))(y(t_q - t) - H_q(\hat{\xi}_q^b)), \\ \hat{\xi}_q^b(0) &= \hat{\xi}_q^f(t_q^-), \qquad t \in (0, \tau_q], \qquad 1 \le q \le m. \end{aligned} \tag{5.28}$$

Actually, the trajectory $\xi_q^b(t) := \xi_q(t_q - t)$ satisfies the differential equation

$$\dot{\xi}_q^b = -F_q(\xi_q^b) - G_q(\xi_q^b)u(t_q - t), \quad y(t_q - t) = H_q(\xi_q^b),$$

with $\xi_q^b(0) = \xi_q(t_q^-)$, for $t \in (0, \tau_q]$, and therefore, (5.28) can be thought of as one possible observer for it. We further assume that Assumption 5.5 holds for this case as well, with $\hat{\xi}_q^f$, ξ_q, $\beta^{f,q}$ and K_q^f replaced by $\hat{\xi}_q^b$, ξ_q^b, $\beta^{b,q}$ and K_q^b, respectively. Once Assumption 5.5 holds for (5.25), this additional requirement is mild. For example, the designs of [118] and [107] readily satisfy this requirement. Then, using the input u and the output y stored over the interval $[t_{q-1}, t_q)$, we run the observer (5.25) first from the initial condition $\hat{\xi}_q^f(t_{q-1}) = \bar{\lambda}_{q,1\sim k_q}(\hat{x}(t_{q-1}))$, followed by integrating (5.28) from 0 to τ_q. After that, we take our final estimate $\hat{\xi}_q(t)$ as

$$\hat{\xi}_q(t) = \begin{cases} \hat{\xi}_q^b(t_q - t), & t \in [t_{q-1}, t_{q-1} + \tau_q/2), \\ \hat{\xi}_q^f(t), & t \in [t_{q-1} + \tau_q/2, t_q). \end{cases} \tag{5.29}$$

From Assumption 5.5 let us assume that, with $b = \tau/2$ and a given $c \in (0, 1)$, both K_q^f and K_q^b are designed. With $\tilde{\xi}_q := \hat{\xi}_q - \xi_q$, $\tilde{\xi}_q^f := \hat{\xi}_q^f - \xi_q$, and $\tilde{\xi}_q^b := \hat{\xi}_q^b - \xi_q^b$, it is seen that

$$\sup_{t \in [t_{q-1} + \frac{\tau_q}{2}, t_q)} |\tilde{\xi}_q(t)| = \sup_{t \in [t_{q-1} + \frac{\tau_q}{2}, t_q)} |\tilde{\xi}_q^f(t)| \leq \sup_{t \in [t_{q-1} + \frac{\tau_q}{2}, t_q)} \beta^{f,q}(|\tilde{\xi}_q^f(t_{q-1})|, t - t_{q-1}) \leq c|\tilde{\xi}_q^f(t_{q-1})|,$$

and

$$\begin{aligned} \sup_{t \in [t_{q-1}, t_{q-1} + \frac{\tau_q}{2})} |\tilde{\xi}_q(t)| &= \sup_{t \in [t_{q-1}, t_{q-1} + \frac{\tau_q}{2})} |\tilde{\xi}_q^b(t_q - t)| \\ &\leq \sup_{t \in [t_{q-1}, t_{q-1} + \frac{\tau_q}{2})} \beta^{b,q}(\beta^{f,q}(|\tilde{\xi}_q^f(t_{q-1})|, \tau_q), t_q - t) \\ &\leq c^2|\tilde{\xi}_q^f(t_{q-1})| \leq c|\tilde{\xi}_q^f(t_{q-1})|. \end{aligned}$$

Therefore, implementation of the observers in (5.25) and (5.28) leads to

$$\sup_{t \in [t_{q-1}, t_q)} |\tilde{\xi}_q(t)| \leq c|\tilde{\xi}_q^f(t_{q-1})| = c|\bar{\lambda}_{q,1\sim k_q}(\hat{x}(t_{q-1})) - \lambda_{q,1\sim k_q}(x(t_{q-1}))|. \tag{5.30}$$

We now claim that, with sufficiently small $c > 0$, the inequality (5.21) holds. Let L_f be the maximum Lipschitz constant of $\bar{f}_q$, $q = 1, \ldots, m$, and L_g and L_p be defined similarly for $\bar{g}_q$ and $\bar{p}_q$. Then, it can be shown by Gronwall-Bellman's inequality and the Lipschitz property of (5.20) that

$$|\tilde{x}(t_q)| \le (L_p)^q \exp\left((L_f + M_u L_g)\sum_{j=1}^{q}\tau_j\right)|\tilde{x}(t_0)| =: M_q|\tilde{x}(t_0)|, \quad q = 1, \dots, m-1. \quad (5.31)$$

This in turn implies from (5.30) that

$$\sup_{t\in[t_{q-1},t_q)} |\tilde{\xi}_q(t)| \le cL_\lambda M_{q-1}|\tilde{x}(t_0)|, \quad q = 1, \dots, m, \quad (5.32)$$

where $M_0 := 1$ for convenience, L_λ is the maximum Lipschitz constant of $\bar{\lambda}_{q,1\sim k_q}$, and we used the fact that $\lambda_{q,1\sim k_q}(x) = \bar{\lambda}_{q,1\sim k_q}(x)$ if $x \in \mathcal{X}$. Let L_{F^*} and L_{G^*} be the maximum Lipschitz constants of $\bar{F}_q^*$ and $\bar{G}_q^*$, respectively. Then, it follows from (5.22) that $|\dot{\tilde{z}}_q| \le L_z(|\tilde{z}_q| + |\tilde{\xi}_q|)$, where $L_z := L_{F^*} + M_u L_{G^*}$. For $q = 2, \dots, m$, this leads to

$$\begin{aligned}|\tilde{z}_q(t_q^-)| &\le e^{L_z\tau_q}|\tilde{z}_q(t_{q-1})| + (e^{L_z\tau_q} - 1)\sup_{t\in[t_{q-1},t_q)}|\tilde{\xi}_q(t)| \\ &\le e^{L_z\tau_q}|\tilde{z}_q(t_{q-1})| + (e^{L_z\tau_q} - 1)cL_\lambda M_{q-1}|\tilde{x}(t_0)|. \quad (5.33)\end{aligned}$$

From (5.5) and (5.32), for $q = 2, \dots, m$, with $\tilde{z}_1 \equiv 0$,

$$\begin{aligned}|\tilde{z}_q(t_{q-1})| &\le (L_{S^*} + L_{S^*}L_P)|\tilde{\xi}_q(t_{q-1})| + L_{S^*}L_P|\tilde{\xi}_{q-1}(t_{q-1}^-)| + L_{S^*}L_P|\tilde{z}_{q-1}(t_{q-1}^-)| \\ &\le c[(L_{S^*} + L_{S^*}L_P)L_\lambda M_{q-1} + L_{S^*}L_P L_\lambda M_{q-2}]|\tilde{x}(t_0)| + L_{S^*}L_P|\tilde{z}_{q-1}(t_{q-1}^-)|. \quad (5.34)\end{aligned}$$

Putting (5.34) into (5.33), we obtain, for $q = 2, \dots, m$,

$$|\tilde{z}_q(t_q^-)| \le L_{S^*}L_P e^{L_z\tau_q}|\tilde{z}_{q-1}(t_{q-1}^-)| + c\bar{M}_{q-1}|\tilde{x}(t_0)|,$$

where $\bar{M}_{q-1} = (e^{L_z\tau_q} - 1)L_\lambda M_{q-1} + e^{L_z\tau_q}[(L_{S^*} + L_{S^*}L_P)L_\lambda M_{q-1} + L_{S^*}L_P L_\lambda M_{q-2}]$. From this, and $\tilde{z}_1 \equiv 0$, it is not difficult to derive that

$$|\tilde{z}_q(t_q^-)| \le c\mathcal{M}_q|\tilde{x}(t_0)|, \quad (5.35)$$

where

$$\mathcal{M}_q = \bar{M}_{q-1} + \sum_{i=1}^{q-2} L_{S^*}^i L_P^i \exp\left(L_z \sum_{j=0}^{i-1}\tau_{q-j}\right)\bar{M}_{q-i-1}.$$

So far, we have obtained $|\tilde{\xi}_m(t_m^-)| \le cL_\lambda M_m|\tilde{x}(t_0)|$ and $|\tilde{z}_m(t_m^-)| \le c\mathcal{M}_m|\tilde{x}(t_0)|$ from (5.32) and (5.35), which finally leads to

$$\begin{aligned}|\tilde{x}(t_m)| &\leq L_p|\bar{\Psi}(\hat{\xi}_m(t_m^-), \hat{z}_m(t_m^-)) - \Psi(\xi_m(t_m^-), z_m(t_m^-))| \\ &\leq L_p L_\Psi(|\tilde{\xi}_m(t_m^-)| + |\tilde{z}_m(t_m^-)|) \\ &\leq c L_p L_\Psi(L_\lambda M_m + \mathcal{M}_m)|\tilde{x}(t_0)|.\end{aligned}$$

Taking c less than $1/(L_p L_\Psi(L_\lambda M_m + \mathcal{M}_m))$, while $\hat{x}(t_m) \in \bar{\mathcal{X}}$, we arrive at (5.21).

Example 5.16. Let us continue with the system given in Example 5.8 and apply the observer design scheme. We first arrive at the canonical form proposed in Assumption 5.3. For mode 1, let $\xi_{1,1} := h_1(x)$, $\xi_{1,2} := L_{f_1}h_1(x)$, and $\xi_1' := x_1 - x_3$; then we get:

$$\begin{aligned}\dot{\xi}_1' &= -0.1(\xi_1' + 1), \\ \dot{\xi}_{1,1} &= \xi_{1,2}, \\ \dot{\xi}_{1,2} &= 0, \\ y &= \xi_{1,1}.\end{aligned}$$

Similarly, for mode 2, we can introduce the following coordinates: $\xi_2 := \lambda_{2,1} = h_2(x)$, $z_2 := \xi_{1,1} = x_2$, and $\xi_2' = x_1 - x_3$. The dynamics of mode 2 then take the following form:

$$\begin{aligned}\dot{\xi}_2' &= -\xi_2' - 1, \\ \dot{\xi}_2 &= 0, \\ \dot{z}_2 &= -z_2\xi_2 \qquad z_2(t_1) = 2\xi_{1,1}(t_1^-), \\ y &= \xi_2.\end{aligned}$$

Finally, for the third mode, the new coordinates are $\xi_3 := h_3(x)$, $z_{3,1} := \xi_{1,1} = x_2$, $z_{3,2} := \xi_{2,1} = x_1^2 - x_3^2 + 2x_1$; and the resulting system dynamics become:

$$\begin{aligned}\dot{\xi}_3 &= 0, \\ \dot{z}_{3,1} &= -\frac{1}{2}z_{3,1}, & z_{3,1}(t_2) &= z_2(t_2^-), \\ \dot{z}_{3,2} &= 2(\xi_3 - z_{3,1}^2 + 1)z_{3,1}^2, & z_{3,2}(t_2) &= \xi_2(t_2^-), \\ y &= \xi_3.\end{aligned}$$

It is seen that the system dynamics in the new coordinates indeed follow the structure prescribed in (5.7).

Using this canonical form, the forward and backward observers for each of the mode are designed as follows:
Mode 1:

$$\begin{pmatrix} \dot{\hat{\xi}}^f_{1,1} \\ \dot{\hat{\xi}}^f_{1,2} \end{pmatrix} = \begin{pmatrix} \hat{\xi}^f_{1,2} \\ 0 \end{pmatrix} + K^f_1(y - \hat{\xi}^f_{1,1}), \qquad \begin{pmatrix} \dot{\hat{\xi}}^b_{1,1} \\ \dot{\hat{\xi}}^b_{1,2} \end{pmatrix} = -\begin{pmatrix} \hat{\xi}^b_{1,2} \\ 0 \end{pmatrix} - K^f_1 b(y - \hat{\xi}^b_{1,1}).$$

Mode 2:

$$\dot{\hat{\xi}}^f_2 = K^f_2(y - \hat{\xi}^f_2), \qquad \dot{\hat{\xi}}^b_2 = -K^b_2(y - \hat{\xi}^b_2).$$

Mode 3:

$$\dot{\hat{\xi}}^f_3 = K^f_3(y - \hat{\xi}^f_3), \qquad \dot{\hat{\xi}}^b_3 = -K^b_3(y - \hat{\xi}^b_3).$$

The estimates of the corresponding z-components are obtained by simulating the following dynamics:
Mode 2:

$$\dot{\hat{z}}_2 = -\hat{z}_2\hat{\xi}_2, \qquad \hat{z}_2(t_1) = 2\hat{\xi}_{1,1}(t_1^-).$$

Mode 3:

$$\begin{aligned} \dot{\hat{z}}_{3,1} &= -\frac{1}{2}\hat{z}_{3,1}, & \hat{z}_{3,1}(t_2) &= \hat{z}_2(t_2^-), \\ \dot{\hat{z}}_{3,2} &= 2(\hat{\xi}_3 - \hat{z}_{3,1}^2 + 1)\hat{z}_{3,1}^2, & \hat{z}_{3,2}(t_2) &= \hat{\xi}_2(t_2^-). \end{aligned}$$

With these observer dynamics, we generate estimates of the state in (ξ, z) coordinates. As a final step, the inverse of the map $(\xi, z) \mapsto x$, denoted by $\chi(\xi, z)$ and defined as follows, is used at time instant t_k to update the state estimate,

$$\begin{aligned} \hat{x}_2(t_k) &= \hat{z}_{3,1}(t_k), \\ \hat{x}_1(t_k) &= \hat{\xi}_{3,1}(t_k) - \hat{x}_2^2(t_k), \\ \hat{x}_3(t_k) &= \sqrt{\hat{x}_1^2(t_k) + 2\hat{x}_1(t_k) - \hat{z}_{3,2}(t_k)}\,, \end{aligned}$$

where k is such that $k \mod 3 = 0$.

The results of the simulation are given in Fig. 5.2, where we have taken $K^f_1 = \mathrm{col}(2, 1)$, $K^f_2 = 2$, $K^f_3 = 2$, and $K^b_1 = \mathrm{col}(-2, 1)$, $K^f_2 = -2$, $K^f_3 = -2$. ◁

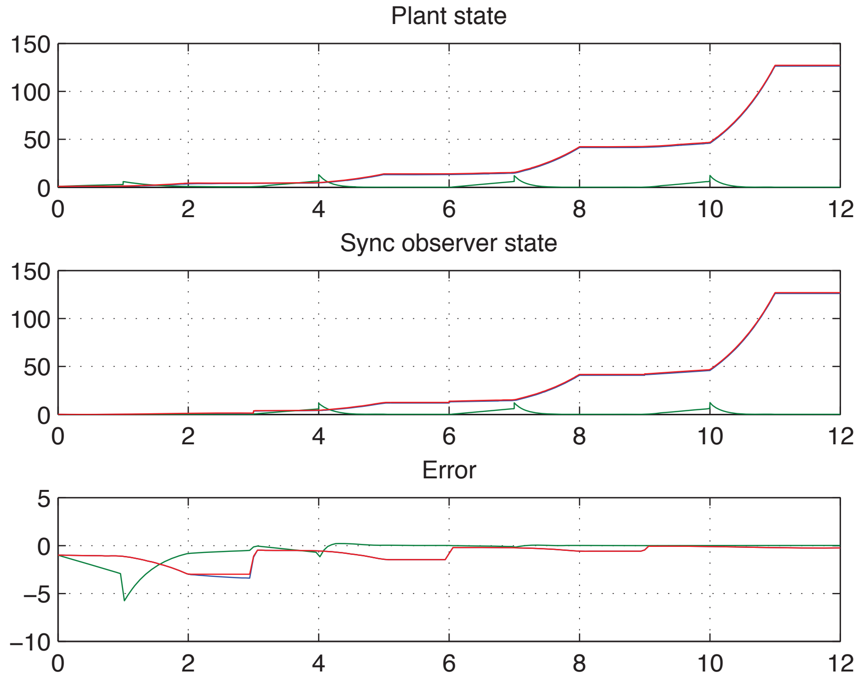


Figure 5.2: Simulation results for observer design in switched nonlinear systems. The top figure shows the actual states of the system. The middle figure shows the state simulated by the observer. And the bottom plot quantifies the state estimation error over time.

5.6 Conjecture for Characterization

So far in this chapter, we derived a sufficient condition for observability which led to the design of an observer. And, just like the linear systems, we are now interested in deriving conditions that are both necessary and sufficient for observability of switched nonlinear systems. But the tool set adopted to solve the linear case is not so easy to generalize for nonlinear case; however, at a conceptual level we will try to extend the same idea. Our first step is to use the tools from differential geometry to arrive at conjecture for characterization of observability. One drawback of the results derived in this section is that they require analytical solution of the flow of the nonlinear ODEs which comprise subsystem dynamics. Because of this requirement, it is not clear how one can proceed with designing the observers. These limitations restrict the applicability of the results proposed in this section, but nonetheless, they are interesting from theoretical standpoint.

In the current section, we limit ourselves to the systems without inputs and jump maps:

$$\dot{x} = f_\sigma(x), \tag{5.36a}$$

$$y = h_\sigma(x). \tag{5.36b}$$

The state space is a smooth differentiable manifold $\mathbb{M}$, the function $x : \mathbb{R} \mapsto \mathbb{M}$ is the state trajectory, the measurable function $y : \mathbb{R} \mapsto \mathbb{R}^{d_y}$ is the output, and $\sigma : \mathbb{R} \mapsto \mathbb{N}$ is the switching signal that is right-continuous; the vector fields f_p and the functions h_p are assumed to be smooth, and they determine the active subsystem over the interval $[t_{i-1}, t_i)$, where t_i is the i-th switching instant. Also, the activation time of each subsystem is denoted by $\tau_i = t_i - t_{i-1}$.

Let Φ_t^f denote the *flow* of the vector field f, i.e. the smooth function of t and x with the property that $x(t) = \Phi_t^f(x_0)$ solves the differential equation

$$\dot{x} = f(x),$$

with initial condition $x(0) = x_0$. In other words, $\Phi_t^f(x)$ is a smooth function of t and x satisfying

$$\frac{\partial}{\partial t}\Phi_t^f(x) = f(\Phi_t^f(x)), \qquad \Phi_0^f(x) = x.$$

Under the smoothness assumption, for any fixed x_0, there is a sufficiently small $t > 0$ such that the mapping

$$\Phi_t^f : x \mapsto \Phi_t^f(x),$$

is defined for all x in the neighborhood of x_0, and is a local diffeomorphism (onto its image).

5.6.1 Necessary and Sufficient Condition for Observability

In this section, we first give a characterization of the unobservable submanifold, comprising indistinguishable states, for one particular execution of a switched system (5.36) when the mode sequence and the transition times are fixed. While arriving at a characterization for observability, we do not assume the individual modes of the system to be observable. So, in order to recover the system state $x(t)$, partial information obtained from each mode is accumulated. This partial information is quantified in terms of the maximal integral submanifold of the distribution $Q_p := \langle f_p | dh_p \rangle^\perp$ which has the property that the states on the slices of this submanifold are not distinguishable by the output of mode p, as $L_{f_p}^k h_p(x)$, $k =$

$0, 1, 2, \cdots$, remains constant for all x contained in that submanifold [18, Chapters 1 and 2]. We will call the integral submanifold of Q_p an unobservable submanifold for mode p. As soon as a switch occurs, the indistinguishable states must be contained in the intersection of the integral submanifolds of the corresponding modes, which in turn reduces the uncertainty in the state. Continuing in this manner, with subsequent switching, we expect to reduce the size of the submanifold that contains the state. Eventually, if the corresponding intersections reduce to a point, we can recover the state completely. Inspired by this intuition, we present a structural condition that characterizes the unobservable manifold of a switched system for a fixed switching signal. The statement and the proof make use of the following notation: for each $m \in \mathbb{N}$, define

$$\mathcal{N}_m^m = Q_m \tag{5.37a}$$

$$\mathcal{N}_{k-1}^m = Q_{k-1} \cap (\Phi_{-\tau_k}^{f_k})_*(\mathcal{N}_k^m), \quad k = m, \cdots, 2. \tag{5.37b}$$

Note that $\mathcal{N}_{m-1}^m = Q_{m-1} \cap Q_m$ because Q_m is invariant under f_m. The motivation for defining $\mathcal{N}_k^m$ in this manner is that the output y and its derivatives remain constant if, and only if, $x(t_k)$ is contained in the integral manifold of $\mathcal{N}_k^m$. In order to avoid technicalities, we will work under the assumption that each element of the sequence $\mathcal{N}_m^k$ for a fixed m has constant rank over the domain of interest.

Assumption 5.6. For each k, $\mathcal{N}_k^m$ in (5.37) has constant rank on some set $\mathcal{X} \subset \mathbb{M}$.

Remark 5.17. It may be the case that the intersection of two smooth distributions, denoted by Δ, is non-smooth. We will replace such non-smooth distributions by the largest smooth distribution contained in Δ without any change of notation. Note that this distribution is well defined as the family of all smooth distributions contained in Δ has a unique maximal element with respect to addition of distributions. For example, the intersection of $\Delta_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ and $\Delta_2 = \left\{ \begin{pmatrix} 1 + x_1 \\ 1 \end{pmatrix} \right\}$ is given by $\Delta(x) = \begin{cases} \{0\} & \text{if } x_1 \neq 0 \\ \Delta_1(x) = \Delta_2(x) & \text{if } x_1 = 0 \end{cases}$, which is non-smooth. The smooth distribution spanning the intersection of Δ_1 and Δ_2 is the identically zero vector field. ◁

Conjecture 5.18 (Characterization of Observability)**.** *Consider the switched system* (5.36) *with a fixed switching signal σ, and the sequence of distributions defined in* (5.37) *on a set $\mathcal{X}$. The system is large-time observable on a set $U \subset \mathcal{X}$ if, and only if, there exists an integer* *m such that*

$$\dim \mathcal{N}_1^m(x) = 0 \qquad \forall\ x \in X. \tag{5.38}$$

Sketch of Proof. Sufficiency: Assume that $y(t)$ for $t \in [t_0, T')$ is given and there exists an $m \in \mathbb{N}$, such that (5.38) holds. We show that the set of initial conditions that can produce this output are contained in a zero-dimensional manifold (a set of isolated points). By assumption, $x(t_{m-1})$ belongs to the integral submanifold $\mathcal{Q}_m$ of Q_m that is determined by the value of the output and its derivatives at time instant t_{m-1}, that is, $\mathcal{Q}_m = \{x \mid L_{f_m}^k h_m(x) = y^{(k)}(t_{m-1}^+), k = 1, \cdots, n-1\}$. Similarly, the active subsystem on the interval $[t_{m-2}, t_{m-1})$ determines that $x(t_{m-1})$ is contained in the integral submanifold of Q_{m-1}, defined as $\mathcal{Q}_{m-1} = \{x \mid L_{f_{m-1}}^k h_{m-1}(x) = y^{(k)}(t_{m-1}^-), k = 1, \cdots, n-1\}$. This implies that $x(t_{m-1})$ is contained in one of the connected components of $\mathcal{Q}_{m-1} \cap \mathcal{Q}_m$, which we denote by $\overline{\mathcal{Q}}_{m-1}$. Note that $\overline{\mathcal{Q}}_{m-1}$ coincides with the integral submanifold of $\mathcal{N}_{m-1}^m$ that passes through $x(t_{m-1})$. This way, $x(t_{m-2}) \in \Phi_{-\tau_{m-1}}^{f_{m-1}}(\overline{\mathcal{Q}}_{m-1})$, which is again a connected component of $\Phi_{-\tau_{m-1}}^{f_{m-1}}(\mathcal{Q}_{m-1}) \cap \Phi_{-\tau_{m-1}}^{f_{m-1}}(\mathcal{Q}_m)$. We claim that the integral submanifold of $Q_{m-1} \cap (\Phi_{-\tau_{m-1}}^{f_{m-1}})_* Q_m$ passing through $x(t_{m-2})$ coincides with $\Phi_{-\tau_{m-1}}^{f_{m-1}}(\overline{\mathcal{Q}}_{m-1})$. Since Q_{m-1} is f_{m-1} invariant, $\Phi_{-\tau_{m-1}}^{f_{m-1}}(\mathcal{Q}_{m-1})$ is also an integral submanifold of Q_{m-1}, but it is determined by the value of the outputs at time instant t_{m-2}. That is,

$$\Phi_{-\tau_{m-1}}^{f_{m-1}}(\mathcal{Q}_{m-1}) = \{x \mid L_{f_{m-1}}^k h_{m-1}(x) = y^{(k)}(t_{m-2}^+), k = 1, \cdots, n-1\},$$

and also, the space tangent to $\Phi_{-\tau_{m-1}}^{f_{m-1}}(\mathcal{Q}_m)$ at $x(t_{m-2}) = \Phi_{-\tau_{m-1}}^{f_{m-1}}(x(t_{m-1}))$ is given by $(\Phi_{-\tau_{m-1}}^{f_{m-1}})_* Q_m$. Continuing this argument inductively, we deduce that $x_0 = x(t_0)$ is contained in one of the connected components of $\Phi_{-\tau_1}^{f_1}(\mathcal{Q}_1 \cap \Phi_{-\tau_2}^{f_2}(\mathcal{Q}_2 \cap \cdots \cap \Phi_{-\tau_{m-1}}^{f_{m-1}}(\mathcal{Q}_{m-1}) \cap \mathcal{Q}_m) \cdots))$, the dimension of which is zero if (5.38) holds. Zero-dimensional manifolds are isolated points, so there exists an open set around every such point which does not contain the other point and if the trajectories of the system remain in this set, then after time $T = t_{m-1}$, the system becomes observable.

Necessity: Suppose that (5.38) does not hold for any $m \in \mathbb{N}$. We compute a set such that each pair of states in that set is indistinguishable. Note that $\mathcal{N}_k^{m+1} \subseteq \mathcal{N}_k^m$ for all $m \in \mathbb{N}$ and $1 \le k \le m-1$. Let $\mathcal{N}_k := \bigcap_{m>k} \mathcal{N}_k^m = \mathcal{N}_k^{k+1} \cap \mathcal{N}_k^{k+2} \cap \cdots$; then by finite dimensionality of $\mathbb{M}$, it follows that $\mathcal{N}_1 \neq \{0\}$. If $\mathcal{M}_k$ denotes the integral manifold of $\mathcal{N}_k$ passing through $x(t_k)$, then we show that the nonzero-dimensional submanifold $\Phi_{-\tau_1}^{f_1}(\mathcal{M}_1)$ is the required set of indistinguishable states.

First, note that each $\mathcal{M}_k$ is contained in the integral submanifold of Q_{k+1}. So for each $x \in$

$\mathcal{M}_k$, the output $y(t) = h_{k+1}(x(t))$ is identical for all $t \in [t_k, t_{k+1})$. Moreover, the implication $x(t_k) \in \mathcal{M}_k \Rightarrow x(t_{k+1}) \in \mathcal{M}_{k+1}$ holds because if $x(t_k) \in \mathcal{M}_k$, then $x(t_{k+1}) = \Phi^{f_{k+1}}_{\tau_{k+1}} x(t_k)$, and using the fact that $\Phi^{f_{k+1}}_{\tau_{k+1}}(\mathcal{M}^m_k) \subset \mathcal{M}^m_{k+1}$, it follows that $x(t_{k+1}) \in \mathcal{M}^m_{k+1}$.

Since $\mathcal{M}_1$ is contained in the integral submanifold of Q_1, each $x_0 \in \Phi^{f_1}_{\tau_1}(\mathcal{M}_1)$ produces the identical output and $x(t_1) \in \mathcal{M}_1$. From the above implication, $x(t_k) \in \mathcal{M}_k$ for each $k > 0$ which in turn leads to same output on each interval $[t_k, t_{k+1})$. □

Let us apply the results of this conjecture to a couple of examples.

Example 5.19. Consider the switched system with following two modes:

$$\Gamma_1 : \begin{cases} \dot{x}_1 = \exp(x_2), \\ \dot{x}_2 = 1, \\ y = x_1 - \exp(x_2) + 1, \end{cases} \quad ; \quad \Gamma_2 : \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1, \\ y = x_1^2 + x_2^2. \end{cases}$$

For these subsystems, $\langle f_1 | dh_1 \rangle = \operatorname{span}\{(1 \ \ -\exp(x_2))\}$, and $\langle f_2 | dh_2 \rangle = \operatorname{span}\{(2x_1 \ 2x_2)\}$. So that, $Q_1 = \operatorname{span}\left\{ \begin{pmatrix} \exp(x_2) \\ 1 \end{pmatrix} \right\}$, and $Q_2 = \operatorname{span}\left\{ \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \right\}$. This implies that

$$Q_2 \cap Q_1 = \begin{cases} \{0\} & \text{if } x_2 + x_1 \exp(x_2) \neq 0, \\ Q_1(x) = Q_2(x) & \text{if } x_2 + x_1 \exp(x_2) = 0. \end{cases}$$

Choose $\mathcal{X} = \{x \in \mathbb{R}^2 \mid x_2 + x_1 \exp(x_2) > 0\}$; then the switched system is large-time observable on every open subset of $\mathcal{X}$, provided the switching signal involves at least one mode transition. Geometrically, one can verify that the output of Γ_2 reveals the radius of the circle that contains the state. The unobservable submanifold of Γ_1 intersects these circles transversally and hence the resultant intersection is a set of two points on either side of the curve $x_2 + x_1 \exp(x_2) = 0$. If the solution is known to lie on one side of that curve, then the intersection would be a unique point as shown in Fig. 5.3. This unique point can now be computed from the flow equations of individual subsystems. ◁

Example 5.20. Consider the switched system considered in Example 5.8. It was claimed that the switching between the three modes, $1 \to 2 \to 3$, makes it possible to recover complete information about the state. Assume that a particular execution of this switched system with mode sequence $\{1, 2, 3\}$ has been observed on some time interval $[0, T)$, $0 < t_1 < t_2 < T$. In order to verify observability using Conjecture 5.18, let us take $\left\{ \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix} \right) \right\}$, $\left\{ \left(\begin{smallmatrix} x_3 \\ 0 \\ x_1+1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix} \right) \right\}$, and $\left\{ \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 2x_2 \\ 1 \\ 0 \end{smallmatrix} \right) \right\}$ as the basis for $Q_1 = \langle f_1 | dh_1 \rangle^\perp$, $Q_2 = \langle f_2 | dh_2 \rangle^\perp$ and

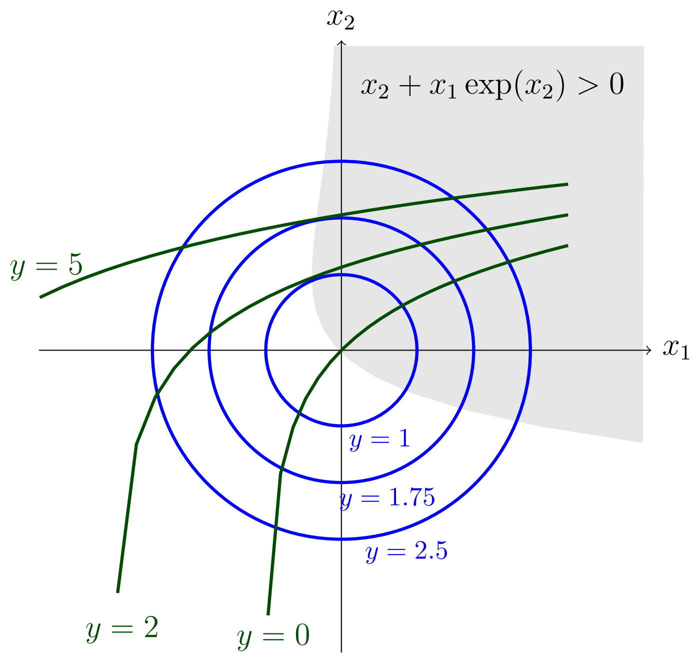


Figure 5.3: Intersection of unobservable submanifolds of two subsystems in Example 5.19. The green curves indicate the unobservable manifold for Γ_1 and the blue circles indicate the unobservable manifold for Γ_2. The output of the corresponding subsystem remains constant on these submanifolds. In the shaded region, a green curve intersects a blue circle at most once, which makes it possible to determine the value of the state completely.

$Q_3 = \langle f_3 | dh_3 \rangle^\perp$ respectively. Also, we can compute the differentials of the flow maps for mode 1 and mode 2 as follows:

$$(\Phi^{f_2}_{-\tau_1})_* = \begin{bmatrix} \cosh 0.1\tau_1 & 0 & -\sinh 0.1\tau_1 \\ 0 & e^{-\tau_2} & 0 \\ -\sinh 0.1\tau_1 & 0 & \cosh 0.1\tau_1 \end{bmatrix}, \qquad (\Phi^{f_2}_{-\tau_2})_* = \begin{bmatrix} \cosh \tau_2 & 0 & -\sinh \tau_2 \\ 0 & e^{-c\tau_2} & 0 \\ -\sinh \tau_2 & 0 & \cosh \tau_2 \end{bmatrix},$$

where c is a constant depending on the initial conditions. It can then be verified that condition (5.37) holds true as long as $x_1(t) > 0, x_3(t) > 0$, for each $t \geq 0$, which actually follows from the system dynamics. Thus, according to Conjecture 5.18 the system is observable, which was verified directly by computing the solution analytically in Example 5.8. $\triangleleft$

5.7 Discussion and Conclusion

This chapter presented a sufficient condition for large-time observability of nonlinear switched systems. Compared to the existing literature on linear systems, this condition is independent of switching times and depends primarily on the mode sequence determined by the switching signal, and the proof reveals how the partial information available from each mode can be combined to recover the state. The observer construction, based on the proof of Theorem 5.12, generates an estimate that converges to the actual state of the system.

Although some ideas are inspired by our recent paper [102], the observability conditions and the observer design in this chapter are quite different from the linear case in [102]. This is because, for the linear system, some partial knowledge of the plant state, say x_o, obtained from a particular mode, can be transported over time simply by multiplying with the easily computable state-transition matrix. Although x_o is transported in combination with some unknown quantity x_u, as $x_0 + x_u$, the unknown part x_u can easily be filtered out by using a matrix whose range-space is orthogonal to x_u. This is possible because the linear dynamics preserves the structural relation between x_o and x_u, which is not the case for nonlinear systems.

In order to verify the condition proposed in Conjecture 5.18, and compute the distributions given in (5.37), one must be able to compute the flow of the vector field f_k analytically, which may not always be obtainable. Also, if we continue in the manner similar to the linear systems for the design of observer, then once again, the analytical solution of nonlinear ODEs is required to propagate the partial information obtained from one mode under the dynamics of another mode to accumulate all the information at one time instant. Thus, Conjecture 5.18 has some limitations with regards to applicability, but nonetheless it reveals some ideas on solving the state estimation problem in switched nonlinear systems.

Chapter 6

Observability of Switched DAEs

Continuing with observability, we deal with another class of switched systems where the dynamical subsystems are modeled as *differential-algebraic equations* (DAEs). So far we have only used ordinary differential equations to model the dynamical behavior of a system. However, the evolution of the states in a physical system may be constrained, e.g., current and voltage in electrical circuits due to Kirchoff's laws, or position variables in coupled mechanical systems. In the modeling of physical systems, it is important to take into account the algebraic constraints imposed on the state variables alongside some differential equations that govern the evolution of these state variables. One such framework is the complementarity modeling of the hybrid systems, discussed in [119], which deals with certain types of algebraic inequalities and equalities. The algebraic equalities are more generally studied under the framework of DAEs. Such constraints are defined by certain algebraic equations and thus the complete model of such systems consists of differential and algebraic equations. And if the system involves interaction between several sets of differential-algebraic equations (DAEs), then it is natural to consider switched DAEs. The most general form of a switched DAE is

$$F_\sigma(t, x, \dot{x}) = 0, \qquad\qquad y = h_\sigma(x).$$

Switched DAEs are an important class of mathematical models and because of their rich solution framework, they need to be treated separately. Structural properties of switched DAEs have not been investigated in much detail; and continuing on our recent work related to structural properties of switched systems [105, 102, 65], we propose to study the observability and observer design in switched DAEs. In this chapter, we consider the switched DAEs of the following form:

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, \\ y &= C_\sigma x, \end{aligned} \tag{6.1}$$

where $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal, and $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times d_u}$, $C_p \in \mathbb{R}^{d_y\times n}$, for $p \in \mathbb{N}$. In general, a switched DAE (6.1) exhibits jumps (or even impulses) in the solution, so it cannot be expected that a classical solution exists; therefore we adopt the *piecewise-smooth distributional solution framework* introduced in [120]; i.e., the state x and the external signals u and y are assumed to be piecewise-smooth distributions. We study *observability* of the switched DAE (6.1) where we call (6.1) observable when the knowledge of the external signals, σ, u and y, allows for a unique reconstruction of the state x.

DAEs arise naturally in modeling physical systems where the state variables satisfy certain algebraic constraints alongside some differential equations that govern the evolution of these state variables. It is a common practice to eliminate the algebraic constraints to arrive at a system description given by ordinary differential equations (ODEs). However, these eliminations are in general different for each subsystem of a switched system, so a description as a switched ODE with *common* state variables is in general not possible. This problem can be overcome by studying the switched DAE (6.1) directly. The motivation for studying DAEs in a distributional framework is the fact that the distributional solution of state variables provides more knowledge about the system, one application of which can be seen in [121].

Switched DAEs are a fairly new topic and not many papers investigate the properties of such systems. Results on stability of switched DAEs have been published very recently in [122], and the only ones (to the best knowledge of the author) related to controllability and observability are reported in [123] and [124], respectively.

In the non-switched case, observability of DAEs has been studied by [125, 126]. As pointed out in [120, Thm. 5.2.5], the observability definitions from [125, 126] can be characterized by a certain *pointwise* observability definition if the problem is embedded into the piecewise-smooth distributional framework. Hence, the non-switched framework discussed so far only focuses on pointwise observability. This is very different from the approach adopted in the switched framework because the switch itself might provide more information about the state trajectory. So, even if the individual subsystems are not observable pointwise in time, it may be possible to achieve global observability due to switching.

Our approach for solving the problem of observability of switched DAEs is in principle different to the existing approach of [124]. In [124], a switched DAE is considered observable if there exists at least one switching signal that makes it possible to recover the state trajectory. In our approach, we consider the switching signal to be known and fixed, which makes the system time-varying. For this time-varying system, we answer the question whether it is possible to recover the state trajectory.

The first result discussed in this section provides a complete characterization for global observability of a switched DAE with two subsystems where the switching signal is restricted to comprise a single switching instant to highlight the difference between switched ODEs and switched DAEs. The distributional framework allows us to incorporate the knowledge provided by the jump and the impulsive part of the output for obtaining information about the state trajectory. This result is then generalized to obtain necessary and sufficient conditions for the general class of switched DAEs with multiple switching instants. If it is not possible to recover the value of the state trajectory at all times, a weaker characterization is provided for determinability, where we only aim to recover the state trajectory after the switching instant in the single switch case. Moreover, the observability conditions are given in terms of *differential* and *impulse* projectors, which present a novel concept of characterizing impulses and derivatives of state trajectories. The definition of these projectors not only makes the development of results parallel to the ODE case but also leads to conditions that are easily verifiable in terms of original system matrices.

6.1 Preliminaries

6.1.1 Properties and Definitions for Regular Matrix Pairs

In the following, we collect important properties and definitions for matrix pairs (E, A). We only consider *regular* matrix pairs, i.e. for which the polynomial $\det(sE - A)$ is not the zero polynomial. A very useful characterization of regularity is the following well-known result.

Proposition 6.1 (Regularity and quasi-Weierstrass form)**.** *A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{6.2}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $0 \leq n_1 \leq n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix. ◁

In view of [127], we call the decomposition (6.2) *quasi-Weierstrass form.* An easy way to calculate the transformation matrices S and T for (6.2) is to use the following so-called

Wong sequences [128, 127][1]:

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \qquad i = 0, 1, \cdots$$
$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), \qquad i = 0, 1, \cdots$$

The Wong sequences are nested and get stationary after finitely many steps. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \quad \mathcal{W}^* := \bigcup_i \mathcal{W}_i.$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (6.2) holds.

Based on the Wong-sequences we define the following "projectors".

Definition 6.2 (Consistency, differential and impulse projectors)**.** *Consider the regular matrix pair* (E, A) *with corresponding quasi-Weierstrass form* (6.2)*. The* consistency projector *of* (E, A) *is given by*

$$\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the differential projector *is given by*

$$\Pi^{\text{diff}}_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

and the impulse projector *is given by*

$$\Pi^{\text{imp}}_{(E,A)} = T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S. \qquad \triangleleft$$

Note that only the consistency projector is a projector in the usual sense (i.e. $\Pi_{(E,A)}$ is an idempotent matrix); the differential and impulse projectors are not projectors in the usual

[1] The problem of finding a canonical form for a pair of matrices simultaneously is also considered in [129]. We choose to work with quasi-Weierstrass form not only because it is easier to compute and work with, but also because the Wong sequences have some useful properties with regard to the solution and structure of the system that are discussed in [120].

sense, because, in general, $\Pi^{\text{diff}}_{(E,A)}\Pi^{\text{diff}}_{(E,A)} \neq \Pi^{\text{diff}}_{(E,A)}$ and the same holds for $\Pi^{\text{imp}}_{(E,A)}$. Let

$$\mathfrak{C}_{(E,A)} := \left\{ \, x_0 \in \mathbb{R}^n \;\middle|\; \exists x \in \mathcal{C}^1 : E\dot{x} = Ax \wedge x(0) = x_0 \, \right\}$$

be the *consistency space* of the DAE $E\dot{x} = Ax$, where $\mathcal{C}^1$ is the space of differentiable functions $x : \mathbb{R} \to \mathbb{R}^n$. Then the following observations hold [127]:

1. All solutions $x \in \mathcal{C}^1$ of $E\dot{x} = Ax$ evolve within $\mathfrak{C}_{(E,A)}$,
2. $\mathfrak{C}_{(E,A)} = \mathcal{V}^*$, i.e. the first Wong-sequence converges to the consistency space,
3. $\operatorname{im}\Pi_{(E,A)} = \mathcal{V}^* = \mathfrak{C}_{(E,A)}$, hence the consistency projector maps onto the consistency space.

The following lemma motivates the name of the differential projector.

Lemma 6.3. *Consider the DAE $E\dot{x} = Ax$ with regular matrix pair (E, A). Then any solution $x \in \mathcal{C}^1$ of $E\dot{x} = Ax$ fulfills*

$$\dot{x} = \Pi^{\text{diff}}_{(E,A)}Ax =: A^{\text{diff}}x.$$

Proof. Let the variables in the quasi-Weierstrass form (6.2) be denoted by v and w, i.e. $x = T\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$. Using the fact that all solutions evolve within the consistency space, we obtain $w = \dot{w} = 0$, and hence

$$\begin{aligned}
\dot{x} &= T\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = T\begin{pmatrix} Jv \\ 0 \end{pmatrix} = T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\begin{pmatrix} v \\ w \end{pmatrix} \\
&= T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}T^{-1}T\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}T^{-1}T\begin{pmatrix} v \\ w \end{pmatrix} \\
&= \Pi_{(E,A)}TSAx = \Pi^{\text{diff}}_{(E,A)}Ax.
\end{aligned}$$

□

For studying impulsive solutions, we consider the space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ from [130] as the solution space; that is, we seek a solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ to the following initial-trajectory problem (ITP):

$$\begin{aligned}
x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\
(E\dot{x})_{[0,\infty)} &= (Ax)_{[0,\infty)},
\end{aligned} \tag{6.3}$$

where $x^0 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ is some initial trajectory, and $f_{\mathcal{I}}$ denotes the restriction of a piecewise-smooth distribution f to an interval $\mathcal{I}$. In [130], it is shown that the ITP (6.3) has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. It is also shown there, that the ITP for the *pure DAE* $N\dot{w} = w$, where $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix, has the unique solution

$$w = \sum_{i=0}^{n_2-1} (N_{[0,\infty)} \tfrac{\mathrm{d}}{\mathrm{d}t})^i (w^0_{(-\infty,0)}).$$

Using the calculus of piecewise-smooth distributions, the expression for the impulsive part of w at $t = 0$, denoted by $w[0]$, is obtained as follows:

$$w[0] = -\sum_{i=0}^{n_2-2} N^{i+1} w^0(0-)\delta^{(i)} = \sum_{i=0}^{n_2-2} N^{i+1} \Delta_0(w)\delta_0^{(i)},$$

where $\delta_0^{(i)}$ denotes the i-th derivative of the Dirac-impulse at zero and $\Delta_0(w) := w(0+) - w(0-)$. To express the impulsive part of the distributional solution x of the ITP (6.3) we need the impulse projector:

Lemma 6.4 (Impulses)**.** *Consider the ITP* (6.3) *with regular matrix pair* (E, A) *and corresponding impulse projector* $\Pi^{\mathrm{imp}}_{(E,A)}$ *with rank* $n_2 \in \mathbb{N}$*. Let* $E^{\mathrm{imp}} := \Pi^{\mathrm{imp}}_{(E,A)} E$*; then any solution* $x \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ *of* (6.3) *fulfills*

$$x[0] = \sum_{i=0}^{n_2-2} (E^{\mathrm{imp}})^{i+1} \Delta_0(x)\delta_0^{(i)}.$$

Proof. First note that all solutions $v \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n_1}$ of the ITP for the ODE $\dot{v} = Jv$ fulfill $v[0] = 0$ and $\Delta_0(v) = 0$; hence

$$\begin{aligned} x[0] = T \begin{pmatrix} v[0] \\ w[0] \end{pmatrix} &= T \sum_{i=0}^{n_2-2} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N^{i+1} \end{bmatrix} \begin{pmatrix} \Delta_0(v) \\ \Delta_0(w) \end{pmatrix} \delta_0^{(i)} \\ &= \sum_{i=0}^{n_2-2} T \begin{bmatrix} 0 & 0 \\ 0 & N^{i+1} \end{bmatrix} T^{-1} \Delta_0(x) \delta_0^{(i)} \\ &= \sum_{i=0}^{n_2-2} (\Pi^{\mathrm{imp}}_{(E,A)} E)^{i+1} \Delta_0(x)\delta_0^{(i)}, \end{aligned}$$

where the last equality follows from the fact that $E = S^{-1}\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right]T^{-1}$ and

$$T\begin{bmatrix} 0 & 0 \\ 0 & N^i \end{bmatrix}T^{-1} = \left(T\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}SS^{-1}\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}T^{-1}\right)^i.$$

□

Since the consistency projector specifies the jump from $x(0-)$ to $x(0+)$ for any solution $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ of the ITP (6.3), we have the following corollary.

Corollary 6.5. *From the notation in Lemma 6.4 and the corresponding consistency projector* $\Pi_{(E,A)}$, *it follows that*

$$x[0] = \sum_{i=0}^{n-1}(E^{\mathrm{imp}})^{i+1}(\Pi_{(E,A)} - I)x(0-)\delta_0^{(i)}. \qquad \triangleleft$$

6.2 Observability Conditions

The concepts introduced in the previous section are now utilized to obtain necessary and sufficient conditions for observability and determinability of switched DAEs. In order to use the piecewise-smooth distributional solution framework and to avoid technical difficulties in general, we only consider switching signals that are right continuous with a locally finite number of jumps; i.e., we exclude an accumulation of switching times.

As was the case in switched ODEs, instead of pointwise observability, we adopt the notion of global obervability in order to extract information from the switching.

Definition 6.6 (Observability)**.** *The switched DAE* (6.1) *with some fixed switching signal* σ, *is called* (globally) observable *if for every pair of inputs and outputs* $(y, u) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{d_y+d_u}$ *there exists at most one* $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ *which solves* (6.1). ◁

The following proposition will be helpful in developing the main result.

Proposition 6.7 (Observability of zero)**.** *The switched DAE* (6.1) *is observable if, and only if,* $y \equiv 0$ *and* $u \equiv 0$ *implies* $x \equiv 0$.

Proof. Necessity is obvious. Assume now that (6.1) is not observable; hence, there exists an external signal (y, u) for which there exist different solutions $x_1, x_2 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ of (6.1). By linearity, it follows that $x = x_1 - x_2 \neq 0$ solves $E_\sigma\dot{x} = A_\sigma x$ and $C_\sigma x = C_\sigma x_1 - C_\sigma x_2 = y - y = 0$; hence, $y \equiv 0$ and $u \equiv 0$ does not imply $x \equiv 0$. □

The above result justifies that we can ignore the input when studying observability of (6.1); hence in what follows, the following homogeneous switched DAE is considered:

$$E_\sigma \dot{x} = A_\sigma x, \quad y = C_\sigma x. \tag{6.4}$$

Furthermore, in order to highlight the major difference between switched ODEs and switched DAEs, we restrict our attention in the remainder of the section to the special switching signal given by:

$$\sigma(t) = 1 \text{ for } t < 0 \quad \text{ and } \quad \sigma(t) = 2 \text{ for } t \geq 0. \tag{6.5}$$

That is, we only consider one switch from some initial subsystem given by $(C_-, E_-, A_-) := (C_1, E_1, A_1)$ – active before the switch – to some other subsystem given by $(C_+, E_+, A_+) := (C_2, E_2, A_2)$ that is active after the switch. Denote the corresponding consistency projectors by Π_-, Π_+ and analogs for the differential and impulse projectors. Let $\mathfrak{C}_\pm := \mathfrak{C}_{(E_\pm, A_\pm)}$ be the consistency spaces of the corresponding subsystems; then $y \equiv 0$, in particular $y^{(i)}(0\pm) = 0$ for all $i \in \mathbb{N}$, together with Lemma 6.3 implies

$$x(0-) \in \mathfrak{C}_- \cap \bigcap_{i \in \mathbb{N}} \ker C_-(\Pi_-^{\text{diff}} A_-)^i$$

and

$$x(0+) \in \mathfrak{C}_+ \cap \bigcap_{i \in \mathbb{N}} \ker C_+(\Pi_+^{\text{diff}} A_+)^i.$$

Define the observability matrices

$$O_\pm := \operatorname{col}(C_\pm, C_\pm A_\pm^{\text{diff}}, C_\pm (A_\pm^{\text{diff}})^2, \cdots, C_\pm (A_\pm^{\text{diff}})^{n-1}), \tag{6.6}$$

where $A_\pm^{\text{diff}} := \Pi_\pm^{\text{diff}} A_\pm$ for any two matrices M_1, M_2 of suitable sizes. Invoking the Cayley-Hamilton-Theorem, see e.g. [131, Thm. X.2.3], the above conditions can be rewritten as

$$x(0-) \in \mathfrak{C}_- \cap \ker O_- \quad \text{and} \quad x(0+) \in \mathfrak{C}_+ \cap \ker O_+ .$$

Invoking regularity of the matrix pairs $(E_\pm, A_\pm)$, a sufficient condition for observability of (6.4) is that $\mathfrak{C}_- \cap \ker O_- = \{0\}$, but the following simple example shows that this condition is not necessary.

Example 6.8. Consider $(C_-, E_-, A_-) = (0, 1, 0)$ and $(C_+, E_+, A_+) = (1, 1, 0)$ which reads as $\dot{x} \equiv 0$ with output $y \equiv 0$ on $(-\infty, 0)$ and $y \equiv x$ on $[0, \infty)$. Although $\mathfrak{C}_- \cap \ker O_- = \mathbb{R}$, the switched DAE is observable. $\triangleleft$

On the other hand, the condition $\mathfrak{C}_+ \cap \ker O_+ = \{0\}$ is not sufficient for observability, because in general $x(0+) = 0$ does not imply $x(0-) = 0$. A characterization of observability has to take into account the possible jumps from $x(0-)$ to $x(0+)$ as well as the induced impulses $x[0]$. Using the additional information $y[0] = 0$ and $y(0+) = 0$ we can find stronger sufficient conditions for observability. These and the above sufficient conditions can be summarized as follows:

1. In general, $x(0-) \in \mathfrak{C}_-$, so $\mathfrak{C}_- = \{0\}$ is sufficient for observability.
2. If $y^{(i)}(0-) = 0$ for all $i \in \mathbb{N}$, then $x(0-) \in \ker O_-$, so $\ker O_- = \{0\}$ is sufficient for observability.
3. If $y^{(i)}(0+) = 0$ for all $i \in \mathbb{N}$, then $x(0+) \in \ker O_+$ together with $x(0+) = \Pi_+ x(0-)$ implies that $x(0-) \in \Pi_+^{-1} \ker O_+$; hence, $\Pi_+^{-1} \ker O_+ = \{0\}$ is sufficient for observability.
4. If $y[0] = 0$, then Lemma 6.4 implies that $\Delta_0 x$ is in the null space of the matrix given by $\mathrm{col}(C_+ E_+^{\mathrm{imp}}, C_+(E_+^{\mathrm{imp}})^2, \cdots, C_+(E_+^{\mathrm{imp}})^{n_2-1})$ and Corollary 6.5 yields that $x(0-)$ is in the null space of $\mathrm{col}(C_+ E_+^{\mathrm{imp}}, C_+(E_+^{\mathrm{imp}})^2, \cdots, C_+(E_+^{\mathrm{imp}})^{n_2-1})(\Pi_+ - I)$; a sufficient condition for observability is therefore that the latter kernel is trivial.

Of course, the condition that the intersection of the above mentioned four "unobservable" subspaces for $x(0-)$ be trivial is another sufficient condition encompassing all four from above. Actually, it turns out that this condition is also necessary.

Theorem 6.9 (Characterization of observability)**.** *Consider the switched DAE* (6.1) *with the switching signal given by* (6.5)*. Use the notation* $O_\pm$ *as given by* (6.6)*, let* $O_+^- := O_+ \Pi_+$*,*

$$O_+^{\mathrm{imp}} := \mathrm{col}(C_+ E_+^{\mathrm{imp}}, C_+(E_+^{\mathrm{imp}})^2, \cdots, C_+(E_+^{\mathrm{imp}})^{n-1}),$$

where $E_+^{\mathrm{imp}} := \Pi_+^{\mathrm{imp}} E_+$*, and let* $O_+^{\mathrm{imp}-} := O_+^{\mathrm{imp}}(\Pi_+ - I)$*. Then* (6.1) *is observable if, and only if,*

$$\boxed{\{0\} = \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathrm{imp}-}.} \tag{6.7}$$

Proof. Because of Proposition 6.7, it suffices to consider (6.4) with zero output.
Sufficiency. Let $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ be a solution of the switched DAE (6.4) with $y \equiv 0$. In general $x(0-) \in \mathfrak{C}_-$; furthermore, $0 = y(0-) = \dot{y}(0-) = \ddot{y}(0-) = \cdots$ implies $x(0-) \in \ker O_-$. From $0 = y(0+) = \dot{y}(0+) = \ddot{y}(0+) = \cdots$, it follows that $x(0+) \in \ker O_+$, which together with

$x(0+) = \Pi_+ x(0-)$ yields $x(0-) \in \ker O_+^-$. Finally, $y[0] = 0$ implies $\Delta_0(x) \in \ker O_+^{\text{imp}}$ and since $\Delta_0(x) = x(0+) - x(0-) = (\Pi_+ - I)x(0-)$, it follows that $x(0-) \in \ker O_+^{\text{imp}-}$. Hence (6.7) yields $x(0-) = 0$, and regularity of the matrix pairs (E_-, A_-) and (E_+, A_+) implies $x \equiv 0$.

Necessity. Let $0 \neq x_0 \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-}$; then by regularity of the switched DAE (6.4) there exists a unique, non-trivial solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ of (6.4) with $x(0-) = x_0$. From $x(0-) \in \ker O_-$ it follows that $y^{(i)}(0-) = 0$ for all $i \in \mathbb{N}$, and hence by analyticity of y on $(-\infty, 0)$ it follows that $y \equiv 0$ on $(-\infty, 0)$. Corollary 6.5 and $x(0-) \in \ker O_+^{\text{imp}-}$ imply that $y[0] = 0$. Finally, $x(0-) \in \ker O_+^-$ implies that $x(0+) = \Pi_+ x(0-) \in \ker O_+$; hence, $y^{(i)}(0+) = 0$ for all $i \in \mathbb{N}$ and hence $y \equiv 0$ on $(0, \infty)$. Altogether this shows that there exists a nontrivial solution x with zero output, so the switched DAE (6.4) is not observable. □

The following corollary is an immediate consequence of Theorem 6.9 (in particular the necessity part of the proof).

Corollary 6.10. *The subspace* $\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-}$ *is the* unobservable subspace*, i.e. for every solution* x *of* (6.4) *it holds that* $x(0-) \in \mathcal{M}$ *if, and only if, the corresponding output is zero.* ◁

Remark 6.11. (The switched ODE special case) If the system in (6.1) is a switched ODE with $E_\pm = I_{n\times n}$, then $\mathfrak{C}_\pm = \mathbb{R}^n, \Pi_\pm = I_{n\times n}, \ker O_+^{\text{imp}-} = \mathbb{R}^n$, and the condition (6.7) reduces to

$$\{0\} = \ker O_- \cap \ker O_+ \,.$$

This result also appears in [52] as a sufficient condition for observability of switched ODEs. However, for the class of switching signals considered here, this condition is also shown to be necessary. ◁

Remark 6.12. (Order of subsystems important) The condition (6.7) is not symmetric; i.e., observability of the switched system (6.1) with the switching signal (6.5) does not, in general, imply observability of (6.1) with the reversed mode sequence. This is in stark contrast to results on switched ODEs, which are in general symmetric [52]. The underlying reason for this difference is the presence of jumps in the solutions of switched DAEs. Consider for example $(C_-, E_-, A_-) = (1, 0, 1)$ and $(C_+, E_+, A_+) = (0, 1, 0)$ which reads as $y \equiv x \equiv 0$ on $(-\infty, 0)$ and $\dot{x} = 0$ with $y \equiv 0$ on $[0, \infty)$. Hence the unique solution is given by $x \equiv 0$, which makes the switched DAE trivially observable. The converse switching signal, i.e. switching from (C_+, E_+, A_+) to (C_-, E_-, A_-), yields an unobservable switched DAE because the jump at zero "destroys" all information from the past. ◁

The utility of Theorem 6.9 is now demonstrated with the help of an example.

Example 6.13. Consider a switched DAE with two modes:

$$\Gamma_- : \begin{cases} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} u, \\ y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x, \end{cases}$$

$$\Gamma_+ : \begin{cases} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} u, \\ y = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} x. \end{cases}$$

Neither subsystem is observable in the classical sense. But we show that because of switching, it is possible to determine the exact value of the state trajectory. To write Γ_-, Γ_+ in quasi-Weierstraß form, we use the following transformation matrices which are obtained from the Wong sequences:

$$S_- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad T_- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$S_+ = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_+ = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The consistency, differential, and impulse projectors for each of these subsystems are:

$$\Pi_- = \Pi_-^{\text{diff}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \Pi_-^{\text{imp}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad O_- = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_+ = \Pi_+^{\text{diff}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \Pi_+^{\text{imp}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad O_+ = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and the subspaces indicated in Theorem 6.9 are:

$$\begin{aligned} \mathfrak{C}_- &= \operatorname{span}\{e_1, e_2, e_4\}, & \ker O_- &= \operatorname{span}\{e_1, e_2, e_3\}, \\ \ker O_+^- &= \operatorname{span}\{e_1, e_3, e_4\}, & \ker O_+^{\text{imp}-} &= \operatorname{span}\{e_2, e_3, e_4\}, \end{aligned}$$

where $e_i \in \mathbb{R}^4$, $i = 1, 2, 3, 4$, is the corresponding natural basis vector.

Clearly, $\mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-} = \{0\}$ and the switched system is observable according to Theorem 6.9. Note that each of the four subspaces $\mathfrak{C}_-$, $\ker O_-$, $\ker O_+^-$ and $\ker O_+^{\text{imp}-}$ is necessary to obtain a trivial intersection. If even one of them is not taken into account, then the intersection would be nontrivial. In fact, each of the subspaces restricts exactly one state variable. In view of Remark 6.12, note that the switched system with subsystem Γ_+ active on $(-\infty, 0)$ and Γ_- active on $[0, \infty)$, is not observable because (with the corresponding notation)

$$\{0\} \neq \mathfrak{C}_+ \cap \ker O_+ \cap \ker O_-^+ \cap \ker O_-^{\text{imp}+}.$$

As an illustration of constructing state trajectories from the knowledge of the output and the input, let us consider an input given by the following expression[2]:

$$u(t) = e^{2t} + \delta_{-1} + \delta_0,$$

and assume that the following output is produced by the switched system with $\sigma(t)$ specified

[2]Note that, for simplicity, we are misusing the notation by writing $u(t) = e^{2t} + \delta_{-1} + \delta_0$ because u is a piecewise-smooth distribution and therefore only the evaluations $u(t-)$, $u(t+)$, $u[t]$ are well defined. The correct way of writing would be to write $\hat{u}(t) = e^{2t}$ and $u = \hat{u}_{\mathbb{D}} + \delta_{-1} + \delta_0$.

in (6.5):

$$y(t) = \begin{cases} -1, & t \in (-\infty, -1), \\ 0, & t \in [-1, 0), \\ e^t + e^{2t} + \delta_0, & t \in [0, \infty). \end{cases}$$

The closed form solution for the state variables, parameterized by $a, b, c \in \mathbb{R}$, is given as follows:

$$x_1(t) = \begin{cases} e^{2t} - e^{t+1} + (a-1)e^t, & t \in (-\infty, -1), \\ (a-1)e^t + e^{2t}, & t \in [-1, 0), \\ 0, & t \in [0, \infty), \end{cases} \quad ; \quad x_3(t) = \begin{cases} -e^{2t} - \delta_{-1}, & t \in (-\infty, 0), \\ -e^t + e^{2t}, & t \in [0, \infty), \end{cases}$$

$$x_2(t) = \begin{cases} e^t b, & t \in (-\infty, 0), \\ e^t + e^{2t} + (b-1)e^t, & t \in [0, \infty), \end{cases} \quad ; \quad x_4(t) = \begin{cases} \frac{1}{2}e^{2t}c - 1, & t \in (-\infty, -1), \\ \frac{1}{2}e^{2t}c, & t \in [-1, 0), \\ -a\delta_0, & t \in [0, \infty). \end{cases}$$

First note that $x_3(0-) = -1$, which corresponds to the fact that in the homogeneous case the consistency space $\mathfrak{C}_-$ restricts $x_3(0-)$ to be zero. Since O_- restricts $x_4(0-)$, we would expect that $y(0-)$, $\dot{y}(0-)$, ..., determine $x_4(0-)$. In fact, $0 = y(0-) = x_4(0-)$. The space O_+^- restricts $x_2(0-)$, and hence by using the values for $y^{(i)}(0+)$, we are able to reconstruct $x_2(0-)$: $2 = y(0+) = x_2(0+) + x_4(0+) = 1 + b = 1 + x_2(0-)$, i.e. $x_2(0-) = 1$. Finally, $O_+^{\text{imp}-}$ restricts $x_1(0-)$; therefore, the information from the impulse of y at zero can be used to determine $x_1(0-)$: $\delta_0 = y[0] = x_2[0] + x_4[0] = -a\delta_0$, and hence $-1 = a = x_1(0-)$. Altogether, we were able to determine $x(0-)$ which together with the knowledge of u and the regularity of the matrix pairs $(E_\pm, A_\pm)$ makes it possible to uniquely reconstruct the whole state x. $\triangleleft$

6.2.1 Systems with Multiple Switching Instants

So far, we have studied switched DAEs with a single switching instant. For switched DAEs with more than two subsystems and multiple switchings, we build on the results of the previous section to obtain a characterization for the general case. We consider global solutions over the interval $(-\infty, \infty)$. If $t_0 = 0$ is the initial time, then over the interval $(-\infty, 0)$, the active subsystem is denoted by the matrices (E_0, A_0, C_0), and for $k \geq 1$, the triplet (E_k, A_k, C_k) represents the active subsystem over the interval $[t_{k-1}, t_k)$, so that t_k is the time instant at which transition occurs from mode $k-1$ to mode k. With $O_k^- := O_k\Pi_k$, and $O_k^{\text{imp}-} := O_k^{\text{imp}}(\Pi_k - I)$, we define $\mathcal{M}_k$ as:

$$\mathcal{M}_k := \mathfrak{C}_k \cap \ker O_k \cap \ker O_{k+1}^- \cap \ker O_{k+1}^{\text{imp}-}.$$

For a given $m \in \mathbb{N}$, define the following sequence of subspaces:

$$\mathcal{N}_m^m := \mathcal{M}_m \tag{6.8a}$$

$$\mathcal{N}_{k-1}^m := \mathcal{M}_{k-1} \cap \Pi_k^{-1}(\exp(-A_k^{\text{diff}}\tau_k)\mathcal{N}_k^m); \qquad 1 \le k \le m. \tag{6.8b}$$

Theorem 6.14 (Observability Characterization)**.** *Consider the switched system formed by family of DAEs* (6.1) *with switching signal σ comprising M switchings. For each positive integer $m < M$, define the sequence $\mathcal{N}_k^m$, for $0 \le k \le m$, according to* (6.8)*. The switched system is globally observable if, and only if, there exists an $m < M$ such that*

$$\mathcal{N}_0^m = \{0\}. \tag{6.9}$$

Proof. Sufficiency. We show that the identically zero output can only be produced by $x(\cdot) \equiv 0$. Fix m such that (6.9) holds. Assume that $y \equiv 0$ on $(-\infty, \infty)$; then according to Corollary 6.10, $x(t_m^-) \in \mathcal{M}_m = \mathcal{N}_m^m$. We next apply the inductive argument to show that $x(t_{k-1}^-) \in \mathcal{N}_{k-1}^m$ for $2 \le k \le m$. Assume that $x(t_k^-) \in \mathcal{N}_k^m$; then $x(t_{k-1}^+) \in \exp(-A_k^{\text{diff}}\tau_k)\mathcal{N}_k^m$. This implies that $x(t_{k-1}^-) \in \Pi_k^{-1}\exp(-A_k^{\text{diff}}\tau_k)\mathcal{N}_k^m$. Zero output on the interval (t_{k-2}, t_k) implies that $x(t_{k-1}^-) \in \mathcal{M}_{k-1}$ and thus $x(t_{k-1}^-) \in \mathcal{M}_{k-1} \cap \exp(-A_k^{\text{diff}}\tau_k)\mathcal{N}_k^m = \mathcal{N}_{k-1}^m$. As a result, $x(t_0^-) \in \mathcal{N}_0^m = \{0\}$, i.e., $x(t_0^-) = 0$; regularity of of the DAE (E_0, A_0, C_0) implies that $x(\cdot) \equiv 0$.

Necessity. Assume that $\mathcal{N}_0^m \neq \{0\}$ for each $m < M$. Since $\mathcal{N}_m^{m+1} \subseteq \mathcal{N}_m^m$ it follows that $\mathcal{N}_k^{m+1} \subseteq \mathcal{N}_k^m$ for $1 \le m < M$ and $0 \le k \le m$. Let $\mathcal{N}_k := \bigcap_{m \ge k} \mathcal{N}_k^m$, then, by finite dimensionality of $\mathbb{R}^n$,

$$\mathcal{N}_0 \neq \{0\}.$$

We will show that for all initial values $x^0 \in \mathcal{N}_0$ the unique solution $x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$ of the switched DAE

$$E_\sigma \dot{x} = A_\sigma x, \quad x(t_0-) = x^0$$

fulfills $y = C_\sigma x = 0$, which implies unobservability. To this end, we first show the following implication, $0 \leq k \leq m$:

$$x(t_k^-) \in \mathcal{N}_k \quad \Rightarrow \quad x(t_{k+1}^-) \in \mathcal{N}_{k+1}. \tag{6.10}$$

Assume $x(t_k^-) \in \mathcal{N}_k$. Since $x(t_k^+) = \Pi_{k+1} x(t_k^-)$ and $x(t_k^-) \in \mathcal{N}_k^m$ for any $m \geq k+1$, using Properties 2 and 3 in Appendix B, it follows that

$$\begin{aligned} x(t_{k+1}^-) &= e^{A_{k+1}^{\text{diff}} \tau_{k+1}} x(t_k^+) = e^{A_{k+1}^{\text{diff}} \tau_{k+1}} \Pi_{k+1} x(t_k^-) \\ &\in e^{A_{k+1}^{\text{diff}} \tau_{k+1}} \Pi_{k+1} \mathcal{N}_k^m \\ &\subseteq e^{A_{k+1}^{\text{diff}} \tau_{k+1}} \left(\Pi_{k+1} \mathcal{M}_k \cap e^{-A_{k+1}^{\text{diff}} \tau_{k+1}} \mathcal{N}_{k+1}^m \cap \mathfrak{C}_{k+1} \right) \subseteq \mathcal{N}_{k+1}^m . \end{aligned}$$

This conclusion is true for all $m \geq k+1$, so $x(t_{k+1}^-) \in \mathcal{N}_{k+1}$. The implication (6.10) is therefore shown and an inductive argument gives $x(t_k^-) \in \mathcal{N}_k$ for $0 \leq k < M$.

For any $0 \leq k < M$, $x(t_k^-) \in \mathcal{N}_k \subseteq \mathcal{M}_k \subseteq \ker O_k$ and $x(t) = e^{A_k^{\text{diff}}(t-t_k)} x(t_k)$, for all $t \in (t_{k-1}, t_k)$, $0 \leq k < M$ (by convention, $t_{-1} = -\infty$), which implies $y(t) = C_k x(t) = 0$ for all $t \in (t_{k-1}, t_k)$. Finally, $y[t_k] = 0$ because $x(t_k-) \in \mathcal{N}_k \subseteq \mathcal{M}_k \subseteq \ker O_k^{\text{imp}-}$. □

Example 6.15. Consider a switched DAE where the two modes

$$\Gamma_1 = \begin{cases} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \\ y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \end{cases}, \qquad \Gamma_2 = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \\ y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \end{cases}$$

are excited by the switching signal

$$\sigma(t) = \begin{cases} 1, & t \in (-\infty, 0), \\ 2, & t \in [0, \frac{\pi}{2}), \\ 1, & t \in [\frac{\pi}{2}, \infty). \end{cases}$$

Total number of switches is $M = 2$, and in the revised notation, (E_0, A_0, C_0) and (E_2, A_2, C_2) are specified by Γ_1, and Γ_2 determines (E_1, A_1, C_1); letting $\{e_1, e_2, e_3\}$ denote the natural basis vectors for $\mathbb{R}^3$, it can be verified that

$$\begin{aligned} &\mathfrak{C}_0 = \mathfrak{C}_2 = \text{span}\{e_2\}; \qquad &&\ker O_0 = \ker O_2 = \text{span}\{e_1, e_2\}; \qquad &&\ker O_1^- = \mathbb{R}^3; \\ &\mathfrak{C}_1 = \text{span}\{e_1, e_2\}; &&\ker O_1 = \text{span}\{e_1, e_2\}; &&\ker O_2^- = \mathbb{R}^3. \end{aligned}$$

Similarly, $\ker O_1^{\text{imp}-} = \mathbb{R}^3$ and $\ker O_2^{\text{imp}-} = \operatorname{span}\{e_2, e_3\}$. It follows that:

$$\mathcal{M}_0 = \mathfrak{C}_0 \cap \ker O_0 \cap \ker O_1^- \cap \ker O_1^{\text{imp}-} = \operatorname{span}\{e_2\},$$

and

$$\mathcal{N}_1^1 = \mathcal{M}_1 = \mathfrak{C}_1 \cap \ker O_1 \cap \ker O_2^- \cap \ker O_2^{\text{imp}-} = \operatorname{span}\{e_2\}.$$

Also, $A_1^{\text{diff}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so that $e^{A_1^{\text{diff}}\frac{\pi}{2}}\mathcal{M}_1 = \{e_1\}$, from which it is clear that,

$$\mathcal{N}_0^1 = \mathcal{M}_0 \cap \Pi_1^{-1}(e^{-A_1^{\text{diff}}\frac{\pi}{2}})\mathcal{M}_1^1 = \operatorname{span}\{e_2\} \cap \operatorname{span}\{e_1, e_3\} = \{0\}.$$

Hence the condition (6.9) holds. To see this explicitly, consider the closed form solution of the state trajectories, parameterized by a scalar a:

$$x_1(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ a \sin t, & t \in [0, \frac{\pi}{2}) \\ 0, & t \in [\frac{\pi}{2}, \infty) \end{cases} \quad ; \quad x_2(t) = \begin{cases} ae^{2t}, & t \in (-\infty, 0) \\ a \cos t, & t \in [0, \frac{\pi}{2}) \\ 0, & t \in [\frac{\pi}{2}, \infty) \end{cases} \quad ;$$

$$x_3(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ 0, & t \in [0, \frac{\pi}{2}) \\ -a\delta_{\frac{\pi}{2}}, & t \in [\frac{\pi}{2}, \infty). \end{cases}$$

For an identically zero output, the impulsive part of the output at second switching instant yields $a = x_2(0-) = 0$ and this makes $x(t) = 0, \forall\, t$. $\triangleleft$

As shown in the above example, even though the individual switchings between the subsystems do not make the system observable, a combination of multiple switches makes the system observable. However, as was the case with switched ODEs, these conditions depend on switching times and are not uniform over the class of switching signals. In the sequel, we give corollaries to Theorem 6.14 from which observability or unobservability of a switched system can be determined independently of the switching signal, and thus the results hold for all switching signals that have the same mode sequence.

Corollary 6.16 (Sufficient condition for observability of switched DAEs)**.** *Consider the family of DAEs* (6.1) *with a switching signal σ comprising M switchings. For each $m < M$,* *define the following sequence of subspaces:*

$$\begin{aligned}\overline{\mathcal{N}}_m^m &:= \mathcal{M}_m\\ \overline{\mathcal{N}}_{k-1}^m &:= \mathcal{M}_{k-1} \cap \Pi_k^{-1} \left\langle A_k^{\text{diff}} | \overline{\mathcal{N}}_k^m \right\rangle, \qquad k = m, \cdots, 1.\end{aligned}$$

The switched DAE (6.1) *is observable if there exists an m such that*

$$\overline{\mathcal{N}}_0^m = \{0\}.$$

Proof. It suffices to show that $\mathcal{N}_k^m \subseteq \overline{\mathcal{N}}_k^m$ for $0 \le k \le m$. First, note that $\mathcal{N}_m^m = \overline{\mathcal{N}}_m^m$. Assuming that $\mathcal{N}_k^m \subseteq \overline{\mathcal{N}}_k^m$ for $0 \le k \le m-1$, we claim that $\mathcal{N}_k^m \subseteq \overline{\mathcal{N}}_k^m$. Indeed, by Property 9 in Appendix B, and the recursion equation (6.8), we get

$$\begin{aligned}\mathcal{N}_{k-1}^m &= \mathcal{M}_{k-1} \cap \Pi_k^{-1} (\exp(-A_k^{\text{diff}} \tau_k) \mathcal{N}_k^m)\\ &\subseteq \mathcal{M}_{k-1} \cap \Pi_k^{-1} \left\langle A_k^{\text{diff}} | \overline{\mathcal{N}}_k^m \right\rangle = \overline{\mathcal{N}}_{k-1}^m,\end{aligned}$$

whence the desired result follows. □

The condition in Corollary 6.16 is not necessary as the following (ODE) example shows:

Example 6.17. Consider a switched DAE with mode sequence indexed as follows:

$$\begin{aligned}(E_0, A_0, c_0) &= \left(I, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, (1, -1) \right),\\ (E_1, A_1, c_1) &= \left(I, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, (0, 0) \right),\\ (E_2, A_2, c_2) &= \left(I, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (1, 0) \right).\end{aligned}$$

Easy calculations show that $\mathcal{M}_0 = \operatorname{span}\{(\begin{smallmatrix}1\\1\end{smallmatrix})\}$, $\mathcal{M}_1 = \operatorname{span}\{(\begin{smallmatrix}0\\1\end{smallmatrix})\}$ and $\overline{\mathcal{N}}_0^1 = \mathcal{M}_0$. Any solution x with zero output fulfills $x(t_0^-) = x(t_0^+) = (x_1^o, x_1^o)^\top$ for some $x_1^o \in \mathbb{R}$. Furthermore $x(t_1) = x(t_1^-) = \begin{bmatrix} 1 & \tau_1 \\ 0 & 1 \end{bmatrix} x(t_0^+) = (x_1^o(1+\tau_1), x_1^o)^\top \in \mathcal{M}_1$. Hence, either $x_1^o = 0$ or $\tau_1 = -1$. The latter is not possible, because we assumed that the switching times are in order, so $x_1^o = 0$ must hold. But this implies $x \equiv 0$, so all switched systems induced by the example are observable. ◁

The above example, however, satisfies the following necessary condition obtained as a corollary to Theorem 6.14

Corollary 6.18 (Necessary condition for observability of switched DAEs)**.** *Consider the family of DAEs* (6.1) *with a switching signal σ comprising M switchings. For each $m < M$, define the following sequence of subspaces:*

$$\begin{aligned} \underline{\mathcal{N}}_m^m &:= \mathcal{M}_m \\ \underline{\mathcal{N}}_{k-1}^m &:= \mathcal{M}_{k-1} \cap \Pi_k^{-1} \left\langle \underline{\mathcal{N}}_k^m | A_k^{\text{diff}} \right\rangle, \qquad k = m, \cdots, 1. \end{aligned}$$

The switched DAE (6.1) *is observable only if there exists an m such that*

$$\underline{\mathcal{N}}_0^m = \{0\}.$$

Proof. The proof proceeds similarly to Corollary 6.16, and we show that $\mathcal{N}_k^m \supseteq \underline{\mathcal{N}}_k^m$ for $0 \le k \le m$. Noting that $\mathcal{N}_m^m = \underline{\mathcal{N}}_m^m$, and assuming that $\mathcal{N}_k^m \supseteq \underline{\mathcal{N}}_k^m$ for $0 \le k \le m-1$, it follows, using Property 9 in Appendix B and the recursion equation (6.8), that

$$\begin{aligned} \mathcal{N}_{k-1}^m &= \mathcal{M}_{k-1} \cap \Pi_k^{-1} (\exp(-A_k^{\text{diff}} \tau_k) \mathcal{N}_k^m) \\ &\supseteq \mathcal{M}_{k-1} \cap \Pi_k^{-1} \left\langle \underline{\mathcal{N}}_k^m | A_k^{\text{diff}} \right\rangle = \underline{\mathcal{N}}_{k-1}^m, \end{aligned}$$

which proves the desired result. □

Once again, in order to show that there is enough gap between the necessary condition and the sufficient condition, consider the example where a system satisfies the necessary condition but not the sufficient condition and is unobservable.

Example 6.19. Consider a switched DAE with mode sequence indexed as follows:

$$\begin{aligned} (E_0, A_0, c_0) &= \left(I, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (0,0) \right), \\ (E_1, A_1, c_1) &= \left(I, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, (0,0) \right), \\ (E_2, A_2, c_2) &= \left(I, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (1,0) \right). \end{aligned}$$

Easy calculations show that $\mathcal{M}_0 = \mathbb{R}^2$, $\mathcal{M}_1 = \text{span}\{(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})\}$ and $\underline{\mathcal{N}}_0^1 = \{0\}$. Any solution x with zero output fulfills $x(t_0^-) = x(t_0^+) = (x_1^o, x_2^o)^\top$ for some $(x_1^o, x_2^o)^\top \in \mathbb{R}$, and $x(t_1) =$

$x(t_1^-) = \begin{bmatrix} \cos\tau_1 & \sin\tau_1 \\ -\sin\tau_1 & \cos\tau_1 \end{bmatrix} x(t_0^+) = (x_1^o \cos\tau_1 + x_2^o \sin\tau_1, x_2^o \cos\tau_1 - x_1^o \sin\tau_1)^\top \in \mathcal{M}_1$. Hence, $x_1^o \cos\tau_1 + x_2^o \sin\tau_1 = 0$ is the only constraint that specifies $x(t_0^-)$, which makes it impossible to recover $x(\cdot)$. $\triangleleft$

6.3 Determinability Conditions

In Theorem 6.9 and Theorem 6.14, we derived conditions which restrict $x(0-)$ to a single point. Since there are no switches over the interval $(-\infty, 0)$, and we have a unique solution over the interval $(-\infty, 0)$. Regularity assumption guarantees that the solution is also well defined over the interval $[0, \infty)$. It is possible that (6.9) does not hold for a given system but $x_{(t_k,\infty)}$ could still be determined uniquely from the knowledge of the output. This motivates the following definition.

Definition 6.20 (Determinability). *The switched DAE* (6.1) *is called* determinable *if for every pair of triplets* $(x_1, u_1, y_1), (x_2, u_2, y_2) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n+d_u+d_y}$ *which solve* (6.1)*, the implication* $(u_1, y_1) = (u_2, y_2) \Rightarrow {x_1}_{(t_i,\infty)} = {x_2}_{(t_i,\infty)}$ *holds for some* $t_i \geq 0$. $\triangleleft$

If we consider only the switching signal given by (6.5), then the Definition 6.20 essentially requires the state trajectory to be unique over the interval $(0, \infty)$. However, for a general switching signal, because no new information is accumulated in between the switching intervals, the definition of determinability requires the solution to be unique over an interval (t_k, ∞) for some t_k.

Just like for switched ODEs, the notion of determinability is, in general, weaker than observability. The reason is that, by regularity of the corresponding matrix pairs, knowledge of $x(t_k^-)$ yields knowledge of $x(t_k^+)$. In particular, observability implies determinability, but the converse is not true in general.

As an illustration of a system which is determinable but not observable, we consider the following example:

Example 6.21. Let $(E_-, A_-, C_-) = \left(I_{2\times2}, I_{2\times2}, \begin{bmatrix} 1 & 0 \end{bmatrix}\right)$ be the subsystem prior to the switching, and $(E_+, A_+, C_+) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, I_{2\times2}, \begin{bmatrix} 0 & 1 \end{bmatrix}\right)$ be the subsystem after the switching. It can be checked that equation (6.7) does not hold, so $x(0-)$ cannot be determined from the output. But $x(0+) = \begin{bmatrix} y(0-) & 0 \end{bmatrix}^\top$ is completely specified by the output. Consequently, $x_{(0,\infty)}$ can be determined uniquely for this switched system. $\triangleleft$

Proposition 6.22 (Determinability of zero)**.** *The switched DAE* (6.1) *is determinable if, and only if,* $y \equiv 0$ *and* $u \equiv 0$ *implies* $x_{(t_k,\infty)} \equiv 0$ *for some* $t_k \geq 0$. ◁

Proof. The proof is analogous to the proof of Proposition 6.7. □

As was the case in observability, we first discuss the simple switching case and then generalize the result to multiple switchings case. The following result is derived from Theorem 6.9 and gives a characterization for systems that are determinable with switching signal (6.5).

Corollary 6.23. *Consider the switched DAE* (6.1) *with the switching signal given by* (6.5). *Then* (6.1) *is determinable if, and only if,*

$$\Pi_+(\mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-}) = \{0\}. \tag{6.11}$$

Proof. Because of Proposition 6.22 it suffices to consider (6.4) with zero output. Let $\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^{\text{imp}-} \cap \ker O_+$.

Sufficiency: Let $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ be a solution of the switched DAE (6.4) with $y \equiv 0$. According to Corollary 6.10, $x(0-) \in \mathcal{M}$. If (6.11) holds, then $x(0+) = \Pi_+ x(0-) = 0$. Regularity of each subsystem implies that $x_{(0,\infty)} \equiv 0$.

Necessity: If (6.11) does not hold, then there exists $0 \neq x_{0,+} \in \Pi_+(\mathcal{M})$. Choose $x_{0,-} \in \mathcal{M}$ with $x_{0,+} = \Pi_+ x_{0,-}$. By regularity of the switched DAE (6.4) there exists a unique, non-trivial solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ of (6.4) with $x(0-) = x_{0,-}$. Corollary 6.10 yields that $y \equiv 0$ and $x_{(0,\infty)} \not\equiv 0$ because $x(0+) = x_{0,+} \neq 0$. Hence the switched DAE (6.4) is not determinable. □

Remark 6.24. For subspaces $\mathcal{R}_1$, $\mathcal{R}_2$, and a linear map Π, $\Pi(\mathcal{R}_1 \cap \mathcal{R}_2) = \Pi(\mathcal{R}_1) \cap \Pi(\mathcal{R}_2)$ if, and only if

$$(\mathcal{R}_1 + \mathcal{R}_2) \cap \ker \Pi = \mathcal{R}_1 \cap \ker \Pi + \mathcal{R}_2 \cap \ker \Pi.$$

Using this and the fact that $\ker \Pi_+ \subseteq \ker O_+ \Pi_+$, the condition (6.11) can be simplified to

$$\Pi_+(\mathfrak{C}_- \cap \ker O_- \cap \ker O_+^{\text{imp}-}) \cap \ker O_+ = \{0\}. \qquad ◁$$

Next we consider the determinability of switched DAEs with multiple switchings. Let the subspace indicated in (6.11) be denoted as follows:

$$\mathcal{P}_k := \Pi_{k+1}(\mathfrak{C}_k \cap \ker O_k \cap \ker O_{k+1}^{\text{imp}-}) \cap \ker O_{k+1}\,.$$

The main result requires is developed using following sequence of subspaces:

$$\mathcal{Q}_0^0 = \mathcal{P}_0 \tag{6.12a}$$

$$\mathcal{Q}_0^k = \mathcal{P}_k \cap \Pi_{k+1}(\exp(A_{k+1}\tau_{k+1})\mathcal{Q}_0^{k-1}), \qquad k \geq 1. \tag{6.12b}$$

Theorem 6.25 (Determinability Characterization)**.** *Consider the switched system formed by family of DAEs* (6.1) *with switching signal σ comprising M switchings. For each positive integer $m < M$, define the sequence $\mathcal{Q}_0^m$, according to* (6.12)*. The switched system is determinable if, and only if, there exists an $m < M$ such that*

$$\mathcal{Q}_0^m = \{0\}. \tag{6.13}$$

Corollaries can be derived from this main result in which conditions for determinability are independent of switching times and depend only on the mode sequence.

Corollary 6.26 (Sufficient Condition for Determinability)**.** *Consider the family of DAEs* (6.1) *with a switching signal σ comprising M switchings. For each $m < M$, define the following sequence of subspaces:*

$$\overline{\mathcal{Q}}_0^0 := \mathcal{P}_0$$
$$\overline{\mathcal{Q}}_0^k := \mathcal{P}_k \cap \Pi_{k+1}\left\langle A_{k+1}^{\text{diff}} | \overline{\mathcal{Q}}_0^{k-1} \right\rangle, \qquad k = m, \cdots, 1.$$

The switched DAE (6.1) *is determinable if there exists an m such that*

$$\overline{\mathcal{Q}}_0^m = \{0\}. \qquad \triangleleft$$

Corollary 6.27 (Necessary Condition for Determinability)**.** *Consider the family of DAEs* (6.1) *with a switching signal σ comprising M switchings. For each $m < M$, define the following sequence of subspaces:*

$$\underline{\mathcal{Q}}_0^0 := \mathcal{P}_0$$
$$\underline{\mathcal{Q}}_0^k := \mathcal{P}_k \cap \Pi_{k+1}\left\langle \underline{\mathcal{Q}}_0^{k-1} | A_{k+1}^{\text{diff}} \right\rangle, \qquad k = m, \cdots, 1.$$

The switched DAE (6.1) *is determinable if there exists an m such that*

$$\underline{\mathcal{Q}}_0^m = \{0\}. \qquad \triangleleft$$

So far we have looked at the conditions that lead to observability/determinability of switched DAEs. It is natural to design the observers based on these conditions, and we intend to do so in the future.

Chapter 7

Application to Fault Detection in Electrical Networks

This chapter proposes a framework for fault detection and isolation (FDI) in electrical energy systems based on invertibility and observability techniques developed in the context of switched systems. In the absence of faults—the nominal mode of operation—the system behavior is described by one set of linear differential equations, or more in the case of systems with natural switching behavior, e.g., power electronics systems. Faults are categorized as *hard* and *soft*. A hard fault causes abrupt changes in the system structure, which results in an uncontrolled transition from the nominal mode of operation to a faulty mode governed by a different set of differential equations. A soft fault causes a continuous change over time of certain system structure parameters, which results in unknown additive disturbances to the set(s) of differential equations governing the system dynamics. In this setup, the dynamic behavior of an electrical energy system (with possible natural switching) can be described by a switched state-space model where each mode is driven by possibly known and unknown inputs. The problem of detection and isolation of hard faults is equivalent to uniquely recovering the switching signal associated with uncontrolled transitions caused by hard faults. The problem of detection and isolation of soft faults is equivalent to recovering the unknown additive disturbance caused by the fault. Uniquely recovering both switching signal and unknown inputs is the concern of the (left) invertibility problem in switched systems, and we are able to adopt theoretical results on that problem, developed earlier, to the present FDI setting. The application of the proposed framework to fault detection and isolation in switching electrical networks is illustrated with several examples. Also, to overcome the limitations of the invertibility framework that requires exact knowledge of the initial condition and derivates of the output, we then propose an alternate observer based approach for detection of soft faults, while assuming the switching signal is known.

7.1 Inversion Based Fault Detection and Isolation

Fault-tolerance (self-healing) may be defined as the ability of a system to adapt and compensate in a planned, systematic way to random component faults and keep delivering completely or partially the functionality for which it was designed [132]. Two main elements should be engineered into an electrical energy system to ensure fault-tolerance: (i) component redundancy, and (ii) fault detection and isolation mechanisms. Choosing the appropriate level of redundancy impacts other metrics, e.g., cost and weight. In this regard, the problem of optimal redundancy allocation has been addressed before [133]. The task of fault detection and isolation (FDI) is indispensable to ensure that component redundancy is managed appropriately. Failure to remove the faulty component from the system, even with sufficient redundant resources to tackle the fault, may entail further damage in other components and eventually bring the system down. A fault detection and isolation (FDI) system executes two actions: (i) detection makes a binary decision whether or not a fault has occurred, (ii) isolation determines the fault location, i.e., which component is faulty.

The literature in FDI is extensive [134, 135, 136, 137], and the methods used for the implementation of FDI can be broadly classified into three different categories: (i) model-based, which uses control-theoretic methods to design residual generators that can point to specific faults; (ii) artificial intelligence, which uses neural networks and fuzzy logic to develop expert systems that once trained can point to specific faults; and (iii) empirical and signal processing, which use spectral analysis to identify specific signatures of a certain fault. The following are a few references of each category application to FDI in electrical energy systems. The work in [138, 139, 140, 141, 142] is model-based, artificial intelligence methods are used in [143, 144, 145, 146], and empirical and signal processing methods in [147, 148].

The focus of this work is on model-based FDI methods, the foundations of which are built on control-theoretic concepts. Model-based methods include observer-based and parameter estimation approaches [149]. As the proposed work is closer to observer-based FDI, fundamental ideas behind this approach to FDI will be reviewed. Observer-based FDI was first introduced by Beard in [20] and further developed by Jones in [21]. The idea is based on using a Luenberger observer (see, e.g., [150]). In a non-switched linear system, it can be used to estimate the system states, given some output measurements and the inputs to the system. In the absence of faults, the state estimates obtained from the observer converge to the actual state values asymptotically. If a fault occurs, the state predicted with the Luenberger observer no longer converges to the true state of the system. By appropriately choosing the observer gain, the estimation error in the presence of a certain fault has certain geometrical characteristics that make the fault identifiable. The Beard-Jones approach is only applicable to deterministic linear time-invariant systems (LTI). The idea of using observers for FDI was extended to stochastic systems, where a Kalman filter approach was used to formulate the FDI problem [151]. This overcame the limitations of the Beard-Jones approach. There has been some work on FDI for nonlinear systems (see, e.g., [152]), where linearization around the system operating point, together with FDI techniques for linear systems, is used. The limitations of this approach are obvious. The Beard-Jones approach cannot handle systems with inherent switching behavior, e.g., power electronics systems. The goal of this research is to overcome these limitations by developing methods that apply to both non-switched and switched linear systems. To address this problem, the application of recent results in invertibility of switched systems will be investigated [65, 67].

In our framework, the system behavior in the absence of faults is described by a set of linear dynamical equations. Faults are categorized as hard, which cause an abrupt change in the system structure, and soft, which result in continuous variation of certain parameters of the system structure. When a hard fault occurs, the system trajectories follow a different set of linear dynamical equations and it is assumed that the dynamics of the system in the presence of such faults are known. If there is a finite number of hard faults under consideration, then we have a finite number of dynamical subsystems describing each possible system operational mode, including the nominal modes and all possible faulty modes. The occurrence of a hard fault results in the transition from a non-faulty mode to a faulty mode. The occurrence of soft faults will result in additional unknown forces driving the system dynamics. In this regard, the system can be thought of as a switched system where the switching signal and inputs are possibly unknown. In this setup, detecting and isolating a hard fault is equivalent to uniquely recovering the switching signal associated with the transition caused by the fault. Detection and isolation of soft faults is equivalent to recovering the unknown inputs that arise from the fault occurrence. To achieve this, we will use the notion of invertibility for switched systems. The problem of (left) invertibility of switched systems, introduced in Chapter 2, concerns the recovery of switching signal and input using the knowledge of the output and the initial state. The realization of FDI using invertibility can be summarized as follows. For hard faults, detection is equivalent to determining that the non-faulty mode can no longer produce the observed outputs, and isolation is equivalent to uniquely recovering the switching signal by identifying which faulty mode can produce the observed output. For soft faults, detection is equivalent to detecting the presence of an input disturbance, and isolation is equivalent to uniquely identifying the parameter change that causes this input disturbance to appear.

In the framework of non-switched systems, the problem of FDI using classical invertibility techniques has been studied before in [22]. In this work, the authors consider faults as additive unknown inputs to the system, and recover them as outputs of another dynamic system—the inverse system. Since the initial values of the state variables are not assumed to be known, the authors require the inverse system to be minimum-phase, which is possible for non-switched systems. Also, minimal realization is considered to save excessive computational effort. In this chapter, we also model soft faults as additive unknown inputs to the system. As the systems under consideration are switched systems, classical inversion techniques can no longer be used for detection of such faults and hence we use newly developed tools of invertibility for switched systems [65]. Also, we assume the initial conditions to be known and hence do not require stability of the inverse system. Furthermore, we do not consider using a minimal realization of the inverse system because the state variables of these minimal realizations for different modes might not coincide and therefore mode detection would not be possible.

The remainder of this chapter proceeds as follows. In Section 7.1.1, the system dynamical model, the notations, and the formulae that lead to the construction of an inverse switched system are presented. In Section 7.1.2, the notions introduced in Section 7.1.1 are used to construct the proposed FDI framework. Section 7.1.3 presents several case-studies that illustrate the ideas presented in Section 7.1.2, followed by some simulation results in Section 7.1.4. Concluding remarks are presented in Section 7.3.

7.1.1 Preliminaries: Inversion of Dynamical Systems

In the context of this work, it is assumed that, without loss of generality, the dynamic behavior of switching electrical systems can be described by a switched state-space model of the form

$$\Gamma : \begin{cases} \dot{x} = A_\sigma x + B_\sigma u + E_\sigma v, \\ y = C_\sigma x, \end{cases} \tag{7.1}$$

where $x \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^l$, and the switching signal takes values in the index set $\mathcal{P}$, that is, $\sigma : [0, \infty) \to \mathcal{P}$. The input u is assumed to be unknown, whereas the input v is assumed to be known.

For a fixed $p \in \mathcal{P}$ and known v, denote by $\Gamma_{p,x_0}(u)$ the trajectory of the corresponding subsystem with the initial state x_0 and the input u, and the corresponding output by

$\Gamma^{O}_{p,x_0}(u)$. Since switching signals are right-continuous, the outputs are also right-continuous, and whenever we take derivative of the output, we assume it is the right derivative.

Invertibility of Non-Switched Linear Systems

Consider affine linear systems of the form:

$$\dot{x} = Ax + Bu + Ev, \tag{7.2a}$$

$$y = Cx, \tag{7.2b}$$

where u is assumed to be unknown and v is assumed to be known.

As indicated in Chapter 2, the invertibility problem for linear systems is concerned with finding conditions for a linear time-invariant (LTI) system so that for a given initial state x_0 and known input v, the input-output map $\mathrm{H}_{x_0,v} : \mathcal{U} \to \mathcal{Y}$ is injective (left-invertibility) or surjective (right-invertibility), where $\mathcal{U}$ is the space of input functions u and $\mathcal{Y}$ is the corresponding output function space. The main computational tool for studying the problem in an algebraic setting is the *structure algorithm*, introduced in [75] and [88]. In the original formulation of this algorithm, it was assumed that all inputs were unknown. In this section, we tailor this formulation to the system (7.2), where some inputs are known, which is a more appropriate formulation for the FDI framework to be discussed in Section 7.1.2. Before proceeding with the formal formulation of the structure algorithm, we introduce an example that will help illustrating the main ideas behind it.

Example 7.1. Consider the circuit of Fig. 7.1, where it is assumed that the input voltage v_s is known, the load current i_{load} is unknown, and both the states i_L and v_C are measurable. By using the notation of (7.2), $x = [i_L\ v_C]^\top$, $u = i_{load}$, $v = v_s$, $y = [y_1\ y_2]^\top = [i_L\ v_C]^\top$, and

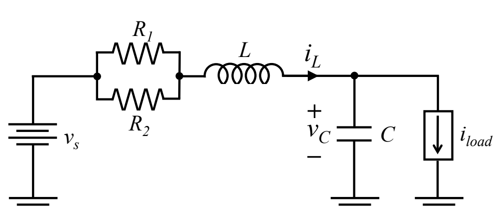


Figure 7.1: RLC circuit.

$$A = \begin{bmatrix} -\frac{R_1R_2}{(R_1+R_2)L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ -\frac{1}{C} \end{bmatrix},\ E = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{7.3}$$

The main idea behind the structure algorithm is that by differentiating the output, it is possible under certain conditions to invert the system, i.e., to find a one-to-one mapping between the output (and its derivatives) and the unknown input. In this example, it is straightforward to find such a mapping by differentiating the second output, $\dot{y}_2 = \begin{bmatrix} \frac{1}{C} & 0 \end{bmatrix} x - \frac{1}{C} i_{load}$, from which it follows that the unknown input i_{load} can be uniquely recovered from the output and its derivatives, $i_{load} = i_L - C\dot{y}_2$. Another important observation for solving the problem of inversion in switched affine linear systems, and later formalized in the form of the so-called *range theorem* [88], can be made regarding the outputs that can be generated by the system from all possible initial conditions. To explore the idea behind the generation of such an output set, consider the state-space description of a system without inputs which is not necessarily observable. For such a system, higher order derivatives of the output can be written as a linear combinations of its lower order derivatives. Similarly, for the system of Fig. 7.1, it is easy to verify that the outputs produced by this system satisfy the constraint $\ddot{y}_1 + \frac{1}{L}\dot{y}_2 + \frac{R_1R_2}{(R_1+R_2)L}\dot{y}_1 = \frac{1}{L}\dot{v}_s$. The construction is later formalized in this section. To determine how the outputs in this set are related to the state variables, consider the relation between outputs and states given by the observation equation $y = Cx$ and differentiate the first output as $\dot{y}_1 = \begin{bmatrix} -\frac{R_1R_2}{(R_1+R_2)L} & -\frac{1}{L} \end{bmatrix} x + \frac{1}{L} v_s$. It follows that the output and the state are related by the following functional relation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{d}{dt} & 0 \end{bmatrix} y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{R_1R_2}{(R_1+R_2)L} & -\frac{1}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} v_s. \tag{7.4}$$

Such relations are useful in mode identification of switched systems especially when the subsystems have the same output sets. ◁

The Structure Algorithm

We now proceed to formalize the ideas introduced in Example 7.1. The structure algoritm developed here is different from that given in Appendix A as we cover the more general case of partially known and partially unknown inputs. Consider the linear system (7.2).

Differentiate y to get $\dot{y} = C\dot{x} = CAx + CBu + CEv$. Let $q_1 = \text{rank}(CB)$; then there exists a nonsingular $l \times l$ matrix S_1 such that $D_1 := S_1 CB = \begin{bmatrix} \overline{D}_1 \\ 0 \end{bmatrix}$, where $\overline{D}_1$ has q_1 rows and rank q_1. Let $y_1 = S_1 \dot{y}$; $C_1 := S_1 CA$, and $E_1 := S_1 CE$. Thus, we have $y_1 = C_1 x + D_1 u + E_1 v_1$, where $v_1 = v$. Suppose that at step k, we have $y_k = C_k x + D_k u + E_k v_k$, where D_k has the form $\begin{bmatrix} \overline{D}_k \\ 0 \end{bmatrix}$; $\overline{D}_k$ has q_k rows and is full rank. Let the partition of C_k, E_k be $\begin{bmatrix} \overline{C}_k \\ \tilde{C}_k \end{bmatrix}$, $\begin{bmatrix} \overline{E}_k \\ \tilde{E}_k \end{bmatrix}$ respectively, where $\overline{C}_k$, $\overline{E}_k$ are the first q_k rows; y_k is partitioned as $\begin{bmatrix} \overline{y}_k \\ \tilde{y}_k \end{bmatrix}$, where $\overline{y}_k$ has the first q_k elements; and v_k has the form $\text{col}(v,\ \dot{v},\ \cdots,\ v^{(k-1)})$. If $q_k < l$, let M_k be the differential operator $M_k := \left[\begin{array}{c|c} I_{q_k} & 0 \\ \hline 0 & I_{l-q_k}(d/dt) \end{array}\right]$. Then $M_k y_k = \begin{bmatrix} \overline{C}_k \\ \tilde{C}_k A \end{bmatrix} x + \begin{bmatrix} \overline{D}_k \\ \tilde{C}_k B \end{bmatrix} u + \begin{bmatrix} \overline{E}_k & 0_{q_k \times r} \\ \tilde{C}_k E & \tilde{E}_k \end{bmatrix} v_{k+1}$, where $v_{k+1} := \text{col}(v,\ \dot{v},\ \cdots,\ v^{(k)})$. Let $q_{k+1} = \text{rank} \begin{bmatrix} \overline{D}_k \\ \tilde{C}_k B \end{bmatrix}$; then there exists a nonsingular $l \times l$ matrix S_{k+1} such that $D_{k+1} := S_{k+1} \begin{bmatrix} \overline{D}_k \\ \tilde{C}_k B \end{bmatrix} = \begin{bmatrix} \overline{D}_{k+1} \\ 0 \end{bmatrix}$, where D_{k+1} has q_{k+1} rows and rank q_{k+1}. Let $y_{k+1} := S_{k+1} M_k y_k$, $C_{k+1} := S_{k+1} \begin{bmatrix} \overline{C}_k \\ \tilde{C}_k A \end{bmatrix}$, $E_{k+1} := S_{k+1} \begin{bmatrix} \overline{E}_k & 0_{q_k \times r} \\ \tilde{C}_k E & \tilde{E}_k \end{bmatrix}$. Then $y_{k+1} = C_{k+1} x + D_{k+1} u + E_{k+1} v_{k+1}$ and we can repeat the procedure. Let $N_k := \prod_{i=0}^{k} S_{k-i} M_{k-i-1}$, $k = 1, 2, \cdots (M_{-1} := I; S_0 := I)$, $\overline{N}_k := [I_{q_k} \quad 0_{q_k \times (l-q_k)}] N_k$ and $\tilde{N}_k := [0_{(l-q_k) \times q_k} \quad I_{l-q_k}] N_k$. Then $y_k = N_k y$, $\overline{y}_k = \overline{N}_k y$, and $\tilde{y}_k = \tilde{N}_k y$. Using these notations, $y = y_0 = \tilde{y}_0 = \tilde{C}_0 x = Cx$, $E_0 = 0$ and $D_0 = 0$. Notice that since D_k has l rows and m columns, $q_k \leq \min\{l, m\}$ for all k and since $q_{k+1} \geq q_k$, using the Cayley-Hamilton theorem, it was shown in [88] that there exists a smallest integer $\alpha \leq n$ such that $q_k = q_\alpha$, $\forall k \geq \alpha$. If $q_\alpha = m$, the system is left-invertible and the inverse is

$$\Gamma^{-1} = \begin{cases} \overline{y}_\alpha & = \overline{N}_\alpha y, \\ \dot{z} & = (A - B\overline{D}_\alpha^{-1} \overline{C}_\alpha) z - B\overline{D}_\alpha^{-1} \overline{E}_\alpha v_\alpha + Ev + B\overline{D}_\alpha^{-1} \overline{y}_\alpha, \\ u & = -\overline{D}_\alpha^{-1} \overline{C}_\alpha z - \overline{D}_\alpha^{-1} \overline{E}_\alpha v_\alpha + \overline{D}_\alpha^{-1} \overline{y}_\alpha, \end{cases} \tag{7.5}$$

with the initial state $z(0) = x_0$.

The Range Theorem

From the structure algorithm, it can be seen that $\tilde{N}_k y = \tilde{y}_k = \tilde{C}_k x + \tilde{E}_k v_k$, for each k and hence,

$$\tilde{Y}_k = L_k x + J_{k-1} v_{k-1}, \tag{7.6}$$

where $\tilde{Y}_k = \begin{bmatrix} \tilde{y}_0 \\ \vdots \\ \tilde{y}_{k-1} \end{bmatrix} = \begin{bmatrix} \tilde{N}_0 \\ \vdots \\ \tilde{N}_{k-1} \end{bmatrix} y$, $L_k = \begin{bmatrix} \tilde{C}_0 \\ \vdots \\ \tilde{C}_{k-1} \end{bmatrix}$, and $J_k = \begin{bmatrix} & 0_{(l-q_0)\times kr} \\ \tilde{E}_1 & 0_{(l-q_1)\times(k-1)r} \\ \vdots & \vdots \\ \tilde{E}_k & 0_{(l-q_k)\times 0} \end{bmatrix}$.

Using the Cayley-Hamilton theorem, Silverman and Payne have shown in [88] that there exists a smallest number β, $\alpha \leq \beta \leq n$, such that $\operatorname{rank}(L_k) = \operatorname{rank}(L_\beta)$, $\forall k \geq \beta$. There also exists a number δ, $\beta \leq \delta \leq n$ such that $\tilde{C}_\delta = \sum_{i=0}^{\delta-1} P_i \big(\prod_{j=i+1}^{\delta} \tilde{R}_j\big) \tilde{C}_i$ for some matrices $\tilde{R}_j$ from the structure algorithm and some constant matrices P_i (see [88, p.205] for details). The number δ is not easily determined as α and β. The significance of α, β and δ is that they can be used to characterize the set of all outputs of a linear system as in the range theorem [88, Theorem 4.3]. We include the range theorem from [88] below in a modified form because of the presence of known input v. The proof, however, follows the same argument and is not repeated here. Define the differential operators $\mathbf{M}_1 := \left(\frac{d^\delta}{dt^\delta} - \sum_{i=0}^{\delta-1} P_i \frac{d^i}{dt^i}\right) \prod_{j=0}^{\alpha} \tilde{R}_j$ and $\mathbf{M}_2 := \sum_{j=0}^{\delta} \left(\prod_{l=j+1}^{\alpha} \tilde{R}_l\right) K_j \frac{d^{\delta-j}}{dt^{\delta-j}} - \sum_{j=0}^{\delta-1} \sum_{k=0}^{j} \left(\prod_{l=k+1}^{\alpha} \tilde{R}_l\right) K_j \frac{d^{j-k}}{dt^{j-k}}$ for some matrices K_i from the structure algorithm. The notation $|_t$ means "evaluating at t".

Theorem 7.2. *[88] A function $f : [t_0, T) \to \mathbb{R}^l$ is in the range of Γ_{x_0} if and only if*

(i) *f is such that $N_\delta f$ is defined and continuous;*

(ii) $\tilde{N}_k f|_{t_0} = \tilde{C}_k x_0 + \tilde{E}_k v_k|_{t_0}, \quad k = 0, \cdots, \delta - 1;$

(iii) $(\mathbf{M}_1 - \mathbf{M}_2 \overline{N}_\alpha) f(t) = \tilde{E}_\delta v_\delta(t) - \sum_{i=0}^{\delta-1} P_i \left(\prod_{j=0}^{\alpha} \tilde{R}_j\right) \tilde{E}_i v_i(t)$ *for all $t \in [t_0, T)$.* ◁

Compared to Theorem 4.1 in [88], condition *(ii)* in Theorem 7.2 (given above) has additional $\delta - \beta$ equations due to the presence of inhomogeneity v which, unlike [88], make the additional $\delta - \beta$ constraints linearly independent of the first β equations. Condition *(iii)* also gets modified.

With the help of extra notations, the range theorem is paraphrased in the following proposition for better understanding. Let $\mathbf{N} := \operatorname{col}(\tilde{N}_0, \ \cdots, \ \tilde{N}_{\delta-1})$, $L := L_\delta$, $J := J_{\delta-1}$, $\nu = v_{\delta-1}$, $\mathcal{C}^0$ be the class of continuous functions, and denote by $\widehat{\mathcal{Y}}$ the set of functions $f : \mathcal{D} \to \mathbb{R}^l$ for all $\mathcal{D} \subseteq [0, \infty)$ which satisfy *(i)* and *(iii)* of Theorem 7.2.

Proposition 7.3. *For a linear system* Γ*, using the structure algorithm on the system matrices, construct a set* $\widehat{\mathcal{Y}}$ *of functions, a differential operator* $\mathbf{N} : \widehat{\mathcal{Y}} \to \mathcal{C}^0$*, and the matrices* L*,* J*. There exists an input* $u \in \mathcal{C}^0$ *such that* $y = \Gamma^O_{x_0,v}(u)$ *if and only if* $y \in \widehat{\mathcal{Y}}$ *and* $\mathbf{N}y|_{t_0^+} = Lx_0 + J\nu|_{t_0^+}$. $\lhd$

For square invertible systems with $q_\alpha = l = m$, condition *(iii)* in Theorem 7.2 always holds and the set $\widehat{\mathcal{Y}}$ is simplified to the set of functions f for which $N_\delta f$ is defined and continuous. In particular, any $\mathcal{C}^n$ function will be in $\widehat{\mathcal{Y}}$. Also, note from the structure algorithm that regardless of what the unknown input is, the output, the state, and the known input are related by the equation $\mathbf{N}y|_t = Lx(t) + J\nu(t)$, for all $t \geq t_0$, not just at the initial time t_0. It is important to note that Proposition 7.3 provides the necessary and sufficient condition in terms of a differential operator $\mathbf{N}$, some matrices L, J and some set of functions $\widehat{\mathcal{Y}}$. Roughly speaking, the set $\widehat{\mathcal{Y}}$ characterizes continuous functions that can be generated by the system from all initial positions (the components of the output must be related to the system matrices A, B, C in some sense). This relation will be used later to identify the mode of operation in a switched system as the condition $\mathbf{N}y|_{t_0^+} = Lx_0 + J\nu|_{t_0^+}$ guarantees that the particular y can be generated starting from the particular initial state x_0 and v at time t_0. We evaluate $\mathbf{N}y$ and ν at t_0^+, to reflect that y, ν do not need to be defined for $t < t_0$. This is especially useful later when we consider switched systems where inputs and outputs can be piecewise right-continuous.

Switched Linear Systems

For switched linear systems (7.1), the map under consideration is a (switching signal $\times$ input)-output map $\mathrm{H}_{x_0,v} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$, where $\mathcal{S}$ is the space of switching signals. We are interested in knowing whether the preimage $(\sigma, u) = \mathrm{H}^{-1}_{x_0,v}(y)$ is unique. The issue of left-invertibility is of central importance in detection and isolation of hard faults because occurrence of a hard fault is the same as the transition of the system from nominal mode to a faulty mode. Thus, recovering the unknown switching signal is equivalent to identifying the hard fault in the system.

It was mentioned earlier that because of switch-singular pairs, if a subsystem is invertible, then it is possible that another subsystem might produce the same output starting from the same initial condition. This means that the pre-image of $\mathrm{H}_{x_0,v}$ at such (x_0, y) is not unique. We recall the simplified definition for the class of systems considered in this chapter.

Definition 7.4. *Let $x_0 \in \mathbb{R}^n$ and $y \in \mathcal{C}^0$ on some time interval. The pair $(x_0,\ y)$ is a* switch-singular pair *of the two subsystems Γ_p, Γ_q if there exist u_1, u_2 such that $\Gamma^O_{p,x_0}(u_1) = \Gamma^O_{q,x_0}(u_2) = y$.* ◁

Based on the discussion in Chapter 3 (Section 3.1), one can arrive at the formula for checking if (x_0, y) is a switch-singular pair of Γ_p, Γ_q, utilizing the range theorem by Silverman and Payne (Theorem 7.2 in this Chapter). We will use our notations in Proposition 7.3. For the subsystem indexed by p, denote by $\mathbf{N}_p$, L_p, J_p, ν_p and $\widehat{\mathcal{Y}}_p$ the corresponding objects of interest as in Proposition 7.3. It follows from Definition 7.4 and Proposition 7.3 that (x_0, y) is a switch-singular pair if and only if $y \in \widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q$ and

$$\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y\big|_{t_0^+} = \begin{bmatrix} L_p \\ L_q \end{bmatrix} x_0 + \begin{bmatrix} J_p \nu_p|_{t_0^+} \\ J_q \nu_q|_{t_0^+} \end{bmatrix}, \tag{7.7}$$

where t_0 is the initial time of y. For a given (x_0, y), the condition (7.7) can be directly verified as all entities are known. One special case is $x_0 = 0, v \equiv 0, y \equiv 0$. It is obvious that with $u = 0$ and any switching signal, we always have $y \equiv 0$, *i.e.* $H_{0,0}(\sigma, u) = 0\ \forall \sigma$ regardless of the subsystem dynamics, and therefore the map $H_{x_0,v}$ is not one-to-one if the function $0 \in \mathcal{Y}$. So, whenever $\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y\big|_{t_0^+} = 0$, there exists an x_0 that forms switch-singular pair with such outputs.

Essentially, if a state and an output function (the time domain can be arbitrary) form a switch-singular pair, then there exist inputs for the two systems to produce that same output starting from that same initial state. Stated otherwise, if there are no switch-singular pairs between any of the subsystems, then the active subsystem is determined uniquely. From fault detection viewpoint, the absence of switch-singular pairs guarantees the detection and isolation of faults as demonstrated in Example 7.7 in Section 7.1.2.

Similar to Theorem 2.5, it has been shown in [65] that a switched system is invertible if and only if all subsystems are invertible and subsystem dynamics are such that there exist no switch-singular pairs among them. If these conditions are satisfied and the switched system is invertible, a switched inverse system can be constructed to recover the input u and switching signal σ from the knowledge of given x_0, y and v. For the switched inverse system, let $\overline{\mathcal{Y}}$ be the set of piecewise smooth functions such that, if $y \in \overline{\mathcal{Y}}$ and $y_{[t_0,t_0+\varepsilon)} \in \widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q$ for some $p \neq q$, $p, q \in \mathcal{P}$, $\varepsilon > 0$, then (7.7) does not hold. Define the *index inversion function*

$\overline{\Sigma}^{-1} : \mathbb{R}^n \times \overline{\mathcal{Y}} \to \mathcal{P}$ as:

$$\overline{\Sigma}^{-1}(x_0, y) = p \ : \ y_{[t_0, t_0+\varepsilon)} \in \widehat{\mathcal{Y}}_p \text{ and } \mathbf{N}_p y\big|_{t_0^+} = L_p x_0 + J_p \nu_p\big|_{t_0^+} \,, \tag{7.8}$$

where t_0 is the initial time of y, and $x_0 = x(t_0)$. The function $\overline{\Sigma}^{-1}$ is well-defined since p is unique by the fact that there are no switch-singular pairs. The existence of p is guaranteed if it is assumed that $y \in \overline{\mathcal{Y}}$ is an output from the modeled switched system.

7.1.2 Inversion-Based Fault Detection and Isolation

We categorize faults in electrical energy systems as *hard faults* and *soft faults*. Hard faults often result in an abrupt change of the system structure; examples include component open- and short-circuits or certain switching elements getting stuck in an open or closed position. We assume that the system configuration resulting from the hard faults can be modeled. Then in the context of the switched state-space description (7.1) discussed in the previous section, a hard fault can be thought of as an uncontrolled transition between two modes in $\mathcal{P}$. Soft faults, on the other hand, refer to a continuous variation—as opposed to an abrupt change—in certain parameters over the period of time; they may occur due to graceful degradation of the capacitance or the equivalent series resistance (ESR) of a capacitor. In the context of the switched state-space description (7.1), a soft fault can be thought of as an unknown additive disturbance and therefore can be naturally included in the vector of unknown inputs u.

The problem of detection and isolation of hard faults is equivalent to uniquely recovering the switching signal associated with the uncontrolled transition caused by the fault. The problem of detection and isolation of soft faults is equivalent to recovering the unknown additive disturbance caused by the fault. The theoretical concepts discussed in Section 7.1.1 are the foundations to develop the framework introduced in this section for detection and isolation of hard and soft faults in systems with inherent switching.

Generalized System Model

The state-space description given in (7.1) can be tailored to describe the non-faulty behavior of an electrical energy system with inherent switching and the faulty features described above. In this regard, the index set $\mathcal{P}$ of (7.1) is partitioned into two sets such that $\mathcal{P} = \mathcal{N} \cup \mathcal{F}$. The first set $\mathcal{N}$ contains the non-faulty modes among which transitions occur due to the possible inherent system switching, e.g., the different physical configurations of a buck or a boost converter dictated by the controlled position of the switches. The second set $\mathcal{F}$ contains the faulty modes that result from hard faults. Since the transitions from a mode in $\mathcal{N}$ to a mode in $\mathcal{F}$ are caused by a fault, these transitions are uncontrolled and, in general, unknown. The unknown system input u of (7.1) is also partitioned into two vectors as $u = [\mu \ \phi]^\top$. The first vector μ contains unknown inputs, e.g., a load in a buck converter modeled as a randomly varying current source, the measures of which are not available. The second vector ϕ contains unknown disturbances that arise due to soft faults.

Thus, formalizing the above ideas, system dynamic behavior under both non-faulty and faulty conditions can be described by a generalized switched state-space model of the form

$$\Gamma : \begin{cases} \dot{x} = A_\sigma x + B_\sigma u + E_\sigma v, \\ y = C_\sigma x, \end{cases} \tag{7.9}$$

where $x \in \mathbb{R}^n$, $u(t) = [\mu(t) \ \phi(t)]^\top \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^l$, $\sigma : [0, \infty) \to \mathcal{N} \cup \mathcal{F}$, and A_p, B_p, C_p, E_p with $p \in \mathcal{N} \cup \mathcal{F}$ defining the subsystems in (7.9).

Example 7.5. Consider again the circuit of Fig. 7.1 and assume the resistors R_1 and R_2 are subject to faults that can abruptly cause an open circuit across their terminals. Γ_1 represents the nominal mode of operation without any faults, Γ_2 describes the dynamics of the system when R_2 fails open, and Γ_3 corresponds to the case where R_1 fails open. Additionally, the capacitor is subject to graceful degradation, so its capacitance is given by $C(t) = C + \lambda_C(t)$, where C is the nominal capacitance, and the unknown function $\lambda_C(t) \leq 0$ captures the decrease in capacitance over time. Now, it is assumed that both the input voltage v_s and the load current i_{load} are known and both states i_L and v_c are measurable. In this scenario, following the notation of (7.9), $x = [i_L \ v_C]^\top$, $u(t) = \phi(t) = -\frac{1}{C+\lambda_C(t)}\left(\frac{\lambda_C(t)}{C}(i_L - i_{load}) + v_C \frac{d\lambda_C}{dt}\right)$, $v = [v_s \quad i_{load}]^\top$, and $\sigma : [0, \infty) \to \mathcal{P}$, where $\mathcal{P} = \{1, 2, 3\}$ and

$$\Gamma_1 \ : \ A_1 = \begin{bmatrix} -\frac{R_1 R_2}{(R_1+R_2)L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ E_1 = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & -\frac{1}{C} \end{bmatrix}, \ C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{7.10}$$

$$\Gamma_2 \ : \ A_2 = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ E_2 = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & -\frac{1}{C} \end{bmatrix}, \ C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{7.11}$$

$$\Gamma_3 \ : \ A_3 = \begin{bmatrix} -\frac{R_2}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \ B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ E_3 = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & -\frac{1}{C} \end{bmatrix}, \ C_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{7.12}$$

where A_1, B_1 describe the non-faulty circuit dynamics; A_2, B_2 describe the circuit dynamics after an open circuit in R_2; and A_3, B_3 describe the circuit dynamics after an open circuit in R_1. It is important to note that the disturbance $\phi(t)$ due to capacitor degradation in reality arises as a perturbation of the state-space representation matrices. However, as was shown in [21], and without loss of generality, it is always possible to rewrite this perturbation as an unknown additive disturbance as shown here. $\triangleleft$

It should be noted that, in the above example, ϕ denotes the degradation in the value of capacitance which is varying with time and is unknown. The capacitance can also be regarded as one of the model parameters, and in the case of Example 7.5 this parameter is time varying and the variation in this parameter is unknown. So, in general, uncertainties in model parameters may also be included as part of the unknown vector u in (7.9).

Fault Detection and Isolation

The first step in designing a fault detection and isolation system is to obtain the system generalized model (7.9), which includes non-faulty modes, faulty modes arising from hard faults and unknown disturbances caused by soft faults.

Once this generalized model (7.9) is obtained, the problem of fault detection and isolation is equivalent to finding $(\sigma,\ u)$ such that $\mathrm{H}_{x_0,v}(\sigma,\ u) = y$, where $\mathrm{H}_{x_0,v}$ is the input-output operator for a given initial state x_0 and a known input v, and y is the observed output. For (7.1), denote by $\mathrm{H}^{-1}_{x_0,v}(y)$ the preimage of a function y,

$$\mathrm{H}^{-1}_{x_0,v}(y) := \{(\sigma,\ u)\ :\ \mathrm{H}_{x_0,v}(\sigma,\ u) = y\}. \tag{7.13}$$

If the set in (7.13) reduces to a singleton, then the switched system is left-invertible[1], that is, there is a unique switching signal and input that generates the given output.

It is entirely possible that the preimage $\mathrm{H}^{-1}_{x_0,v}(y)$ is not unique. It happens because: (i) there is a subsystem that can produce the same output with more than one input u, or (ii) there is more than one subsystem that can produce the measured output. In the first case, there exist infinitely many inputs that can produce a given output on any compact interval with same initial and terminal state [65, Lemma 4] and therefore, *it is not possible to detect and isolate the occurrence of a particular soft fault.* In the second case, it is the existence of switch-singular pairs that prevents the mode identification. If such pairs exist among *the*

[1] An algebraic characterization of conditions under which a switched linear system is left-invertible appears in [65, Lemma 3].

faulty modes, then the occurrences of hard faults can still be detected but such faults cannot be isolated. In practice, in order to compute $\mathrm{H}^{-1}_{x_0,v}(y)$, it is necessary to (i) conduct mode identification, and (ii) recover the unknown input u.

Mode Detection or Hard Fault Detection and Isolation

The first step to obtain the inverse system is to conduct mode identification using the *index-inversion function* which is defined as follows:

$$\overline{\Sigma}^{-1}(x_0, y) = p \ : \ y_{[t_0,t_0+\varepsilon)} \in \widehat{\mathcal{Y}}_p \text{ and } \mathbf{N}_p y\big|_{t_0^+} = L_p x_0 + J_p \nu_p\big|_{t_0^+}, \ p \in \mathcal{N} \cup \mathcal{F}. \tag{7.14}$$

If the resulting mode p is unique and belongs to $\mathcal{N}$, then the occurrence of a hard fault is ruled out. However, if p belongs to $\mathcal{F}$, then a hard fault occurrence has been detected. If (7.14) results in more than one p that belongs to $\mathcal{F}$ but does not belong to $\mathcal{N}$, then a fault has occurred and it can be detected; however, it is not possible to isolate the fault as the faulty mode that is producing the observed output cannot be identified. In case (7.14) results in modes that belong to $\mathcal{N}$ and $\mathcal{F}$, then there is a switch-singular pair between a faulty and non-faulty mode, so one cannot conclude whether or not a fault has occurred.

It is true that the detection of hard faults depends upon the modeling of faulty modes and there may be cases where an unmodeled fault has occurred. In this case, the observed output of the system is not related to any of the modes, and the index-inversion function in (7.14) is ill-defined as the system is operating with unknown dynamics. So, in case the active mode cannot be identified, then the system has switched to a faulty mode that has not been modeled. Hence, the detection is still possible but one cannot identify where the fault has occurred.

Input Recovery or Soft Fault Detection and Isolation

Once the mode has been identified, and perhaps a hard fault has been detected and isolated, it is still necessary to check for the presence of soft faults, which can be accomplished by inverting the particular subsystem associated to the mode identified in (7.14), and then recovering u from the inverse of that particular subsystem. Practically, this can be achieved by running the following inverse switched system:

$$\Gamma_\sigma^{-1} = \begin{cases} \sigma(t) &= \overline{\Sigma}^{-1}(z(t), y_{[t,t+\varepsilon)}), \\ \dot{z} &= (A - B\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)} z - (B\overline{D}_\alpha^{-1}\overline{E}_\alpha)_{\sigma(t)} v_\alpha + Ev + (B\overline{D}_\alpha^{-1}\overline{N}_\alpha)_{\sigma(t)} y, \\ u &= (-\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)} z - (\overline{D}_\alpha^{-1}\overline{E}_\alpha)_{\sigma(t)} v_\alpha + (\overline{D}_\alpha^{-1}\overline{N}_\alpha)_{\sigma(t)} y, \end{cases}$$

with the initial condition $z(t_0) = x_0$. For each particular mode, the matrices $\overline{C}_\alpha$, $\overline{D}_\alpha$, $\overline{E}_\alpha$, $\overline{N}_\alpha$ are obtained similarly as explained in Section 7.1.1. The notation $(\cdot)_\sigma$ denotes the object in the parenthesis calculated for the subsystem with index $\sigma(t)$. The initial condition $\sigma(t_0)$ determines the initial active subsystem at the initial time t_0, from which time onwards, the active subsystem indexes and the input as well as the state are determined uniquely and simultaneously.

Remark 7.6. Ideally, for hard fault detection, one may only be interested in knowing whether any mode in $\mathcal{F}$ is active, and not necessarily the exact value of the switching signal at all times. But note that the value of state trajectory is required to determine the transition between modes, and the state trajectory can only be simulated if the exact value of the switching signal is known. However, to find relaxed conditions for hard fault detection without requiring the exact knowledge of state trajectories is a topic of ongoing research. Another similar direction of future work is to find less restrictive conditions which allow us to detect nonzero values of the unknown inputs induced by soft faults without necessarily recovering the soft fault exactly. $\triangleleft$

Example 7.7. Consider again the circuit in Fig. 7.1 with the same assumptions as in Example 7.5, which resulted in (7.10), (7.11), (7.12). In this case, mode detection is possible since

$$\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{d}{dt} & 0 \end{bmatrix}, \; J_1\nu_1 = J_2\nu_2 = J_3\nu_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_s \\ \dot{i}_{load} \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{R_1R_2}{(R_1+R_2)L} & -\frac{1}{L} \end{bmatrix}, \; L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{R_1}{L} & -\frac{1}{L} \end{bmatrix}, \; L_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{R_2}{L} & -\frac{1}{L} \end{bmatrix},$$

and $\widehat{\mathcal{Y}}_1 = \{y \,|\, \ddot{y}_1 + \frac{1}{L}\dot{y}_2 + \frac{R_1R_2}{(R_1+R_2)L}\dot{y}_1 = \frac{1}{L}\dot{v}_s\}, \widehat{\mathcal{Y}}_2 = \{y \,|\, \ddot{y}_1 + \frac{1}{L}\dot{y}_2 + \frac{R_1}{L}\dot{y}_1 = \frac{1}{L}\dot{v}_s\}, \widehat{\mathcal{Y}}_3 = \{y \,|\, \ddot{y}_1 + \frac{1}{L}\dot{y}_2 + \frac{R_2}{L}\dot{y}_1 = \frac{1}{L}\dot{v}_s\}$. If $y_1(t) = i_L(t) = 0$ for some t, then $\dot{y}_1(t) = -\frac{1}{L}y_2(t) + \frac{1}{L}v_s(t)$, so that for each $y_{[t,t+\varepsilon)}$ contained in $\widehat{\mathcal{Y}}_1$, $\widehat{\mathcal{Y}}_2$, or $\widehat{\mathcal{Y}}_3$, we must have $\dot{y}_1 = 0$, implying that $y_{[t,t+\varepsilon)} \in \cap_{i=1}^3 \widehat{\mathcal{Y}}_i$. Further, the expressions for $L_i, J_i, \mathbf{N}_i, i = 1,2,3$, suggest that the mode cannot be identified using the index inversion function in (7.14). Conversely, if $i_L \neq 0$ at any time, then the mode can always be recovered using (7.14). It follows that there are no switch-singular pairs as long as i_L is not identically zero. Therefore the occurrence of a fault in either R_1 or R_2 can be detected. However, if $R_1 = R_2$, even if a fault in either R_1 or R_2 can be detected, it cannot be isolated, i.e., we cannot determine whether the fault occurred in R_1 or R_2. Thus, in this example, isolation of hard faults is only possible if and only if $R_1 \neq R_2$. Inversion of the individual subsystems is also possible and $\phi(t)$ is given by

$$\phi(t) = \dot{y}_2 - \begin{bmatrix} \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \frac{1}{C} i_{load},$$

and therefore the detection of soft faults in the capacitor is also possible. It is important to note that $\phi(t) = -\frac{\lambda_C(t)}{C(C+\lambda_C(t))}(i_L - i_{load})$, and since everything is known except $\lambda_C(t)$, which happens to be proportional to the magnitude of the fault, recovering $\phi(t)$ gives a measure of the component degradation.

Assume now that apart from the measurements of both states, only v_s is known, while i_{load} is unknown, so that $v = v_s$, and $u = [i_{load} \; \phi]^\top$. In this case, mode identification is possible since

$$\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{d}{dt} & 0 \end{bmatrix}, \; J_1\nu_1 = J_2\nu_2 = J_3\nu_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} v_s,$$

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{R_1 R_2}{(R_1+R_2)L} & -\frac{1}{L} \end{bmatrix}, \; L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{R_1}{L} & -\frac{1}{L} \end{bmatrix}, \; L_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{R_2}{L} & -\frac{1}{L} \end{bmatrix},$$

and $\widehat{\mathcal{Y}}_1, \widehat{\mathcal{Y}}_2, \widehat{\mathcal{Y}}_3$ remain unchanged. However, inversion of the individual subsystems is not possible as, by taking derivatives of the outputs, we get

$$\begin{bmatrix} \dot{y}_2 \\ \ddot{y}_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{C} & 0 \\ \frac{R^2}{L^2} - \frac{1}{LC} & \frac{R}{L^2} \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -\frac{1}{C} & 1 \\ \frac{1}{LC} & -\frac{1}{L} \end{bmatrix} \begin{bmatrix} i_{load} \\ \phi \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{R}{L^2} \end{bmatrix} v_s,$$

and since the matrix multiplying $[i_{load} \quad \phi]^\top$ is not invertible, it is not possible to uniquely recover ϕ.

Finally, assume that apart from the measurements of both states, only i_{load} is known, while v_s is unknown. In this case, $\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3 = I_{2\times 2}$, $J_1 = J_2 = J_3 = 0_{2\times 2}$, $L_1 = L_2 = L_3 = I_{2\times 2}$. The operators in this case are of dimension lower than the previous two cases because the first derivative of each output is affected by a different unknown input. Since the system now has two unknown inputs and two outputs, $\widehat{\mathcal{Y}}_1$, $\widehat{\mathcal{Y}}_2$, $\widehat{\mathcal{Y}}_3$ consists of set of differentiable outputs. Thus, it is not possible to do mode identification, and therefore hard faults in resistors will go undetected. In terms of soft faults in the capacitor, it is still possible to detect them even if it is not possible to detect the mode as for all $p = 1, 2, 3$, it results that

$$\phi(t) = \dot{y}_2 - \begin{bmatrix} \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \frac{1}{C} i_{load}.$$

This is not true in general; i.e., the expression for inverting each subsystem is usually different, so to detect and isolate soft faults, it is necessary to identify the mode in which the system is operating. $\lhd$

7.1.3 Analytical Case-Studies

In this section, the application of the framework developed in Section 7.1.2 is illustrated by analyzing several power electronics circuits. Due to temperature variations, very high frequencies, and other variable conditions at which power electronics systems operate, the parameters of the components that comprise these systems may drift away over time from their nominal value. In this regard, we consider soft faults in capacitors and inductors and analyze how these faults affect typical power electronics circuits such as boost, buck, and boost-buck, and the conditions under which these faults can be detected and isolated in these circuits. We leave the illustration of hard fault detection to the simulation examples of Section 7.1.4. It is important to note that both hard and soft faults may cause performance degradation if not detected and accounted for. Although not discussed here, the system controller could be reconfigured to account for these variations so as to maintain a prescribed level of performance.

Boost Converter

Consider the boost converter of Fig. 7.2 where we assume both i_L and v_C can be measured, and the voltage v_s is perfectly known. Letting $x = [i_L \ v_C]^\top$, then the two modes of operation

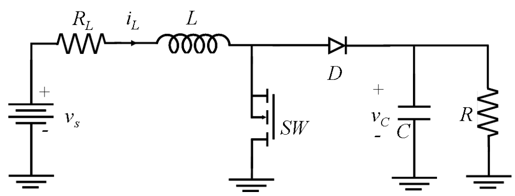


Figure 7.2: Boost converter.

of this converter are

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s, \\ y = x, \end{cases} \tag{7.15}$$

which corresponds to the case when SW is closed and D is open, and

$$\Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s, \\ y = x, \end{cases} \tag{7.16}$$

which corresponds to the case when SW is open, the diode D is conducting. We will assume that the switching signal is not available to the FDI system.

Capacitor Soft Faults

We can assume that as the capacitor degrades, its capacitance will decrease. Thus, without loss of generality, the capacitance of the capacitor can be described as $C(t) = C + \lambda_C(t)$, where C is the nominal capacitance, with $\lambda_C(t) \leq 0$ describing the fault magnitude. Then, the system dynamics can be described in a more general form to account for this fault as follows:

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & 0 \\ 0 & -\frac{1}{RC} + \rho_C(t) \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s, \\ y = x, \end{cases} \tag{7.17}$$

where $\rho_C(t) = \frac{1}{C+\lambda_C(t)}\left(\frac{\lambda_C(t)}{RC} - \frac{d\lambda_C(t)}{dt}\right)$, and

$$\Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & -\frac{1}{L} \\ \frac{1}{C} + \mu_C(t) & -\frac{1}{RC} + \rho_C(t) \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s, \\ y = x, \end{cases} \tag{7.18}$$

where $\mu_C(t) = \frac{-\lambda_C(t)}{C(C+\lambda_C(t))}$. Define ϕ_C as

$$\phi_C(t) := \begin{cases} \rho_C(t)v_C(t) & \text{if } \sigma(t) = 1, \\ \mu_C(t)i_L(t) + \rho_C(t)v_C(t) & \text{if } \sigma(t) = 2. \end{cases} \tag{7.19}$$

Then (7.17) and (7.18) can be rewritten as:

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \phi_C(t), \\ y = x, \end{cases} \tag{7.20}$$

$$\Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \phi_C(t), \\ y = x. \end{cases} \tag{7.21}$$

Note that $\phi_C(t)$ basically represents the unknown degradation in the value of the capacitor. We now use the tools from Section 7.1.2 to recover the switching signal $\sigma(t)$ and the unknown function $\phi_C(t)$. It is important to note that there might be cases in which the switching signal is available to the fault detection and isolation system, in which case, it would only be necessary to recover $\phi_C(t)$.

Mode Identification

Following the notation used in Section 7.1.2, it results that the operators $\mathbf{N_1}$, $\mathbf{N_2}$, J_1, J_2, L_1, L_2 are:

$$\mathbf{N}_1 = \mathbf{N}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{d}{dt} & 0 \end{bmatrix}, \; J_1\nu_1 = J_2\nu_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} v_s, \; L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{R_L}{L} & 0 \end{bmatrix}, \; L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{R_L}{L} & -\frac{1}{L} \end{bmatrix}.$$

Also, $\widehat{\mathcal{Y}}_1 = \{y \,|\, \dot{y}_1 + \frac{R_L}{L}y_1 = \frac{1}{L}v_s\}$, and $\widehat{\mathcal{Y}}_2 = \{y \,|\, \ddot{y}_1 + \frac{1}{L}\dot{y}_2 + \frac{R_L}{L}\dot{y}_1 = \frac{1}{L}\dot{v}_s\}$. The active mode can then be identified using (7.8), and the only possibility of switch-singular pair is if $-\frac{R_L}{L}i_L(t) = -\frac{R_L}{L}i_L - \frac{1}{L}v_C(t)$, or equivalently $v_C(t) = 0$. Thus, if the original system has switching when $v_C(t) = 0$, then it would not be observable in the output and in that case $\sigma(t)$ cannot be recovered uniquely[2]. However, this would mean that the voltage across the load becomes zero, which is not possible without a large variation on the capacitance; but before this occurs, since the capacitor degrades gracefully, the voltage $v_C(t)$ will remain greater than zero, and therefore the existence of a switch-singular pair is ruled out. Thus, other than the particular case described, the switching signal can be recovered using the following formula:

$$\sigma(t) = \begin{cases} 1 & \text{if } y_{[t,t+\varepsilon)} \in \widehat{\mathcal{Y}}_1 \text{ and } \mathbf{N}_1 y(t) = L_1 x(t) + J_1 v_s(t), \\ 2 & \text{if } y_{[t,t+\varepsilon)} \in \widehat{\mathcal{Y}}_2 \text{ and } \mathbf{N}_2 y(t) = L_2 x(t) + J_2 v_s(t). \end{cases} \tag{7.22}$$

Having recovered the switching signal $\sigma(t)$, one can activate the corresponding inverse subsystem to compute $\phi_C(t)$ and hence the change in the nominal value of the capacitor.

Unknown Input Recovery

In this case, both subsystems are invertible, so the detection and isolation of capacitor soft faults is possible. Applying the structure algorithm to Γ_1 and differentiating the output, we obtain:

$$\dot{y} \;=\; \begin{bmatrix} -\frac{R_L}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \phi_C(t),$$

from which it follows that $S_1 = I$ and therefore $D_1 = [0\ 1]^\top$. Furthermore, $q_1 = \text{rank}(D_1) = 1$, which is equal to the dimension of the input space and thus $q_\alpha = q_1$. Then following the notation used in Section 7.1.2, it results that $\overline{N}_\alpha = \begin{bmatrix} 0 & \frac{d}{dt} \end{bmatrix}$, $\overline{C}_\alpha = \begin{bmatrix} 0 & -\frac{1}{RC} \end{bmatrix}$, and $\overline{D}_\alpha = 1$; hence, the inverse system is described by

$$\Gamma_1^{-1} = \begin{cases} \overline{y}_\alpha &= \begin{bmatrix} 0 & \frac{d}{dt} \end{bmatrix} y, \\ \dot{z} &= \begin{bmatrix} -\frac{R_L}{L} & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \overline{y}_\alpha, \\ \phi_C &= \begin{bmatrix} 0 & \frac{1}{RC} \end{bmatrix} z + \overline{y}_\alpha\,. \end{cases} \tag{7.23}$$

[2] To recover non-unique switching signals in the presence of switch-singular pairs, see the algorithm proposed in [65]. A conceptually similar algorithm tailored for FDI framework appears in Section 7.1.4. It must be noted that these algorithms are non-causal and require the knowledge of future outputs to recover the value of switching signal in the past.

The similar procedure can be applied to Γ_2 to get dynamic representations for Γ_2^{-1}:

$$\Gamma_2^{-1} = \begin{cases} \overline{y}_\alpha &= \begin{bmatrix} 0 & \frac{d}{dt} \end{bmatrix} y, \\ \dot{z} &= \begin{bmatrix} -\frac{R_L}{L} & -\frac{1}{L} \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \overline{y}_\alpha, \\ \phi_C &= \begin{bmatrix} -\frac{1}{C} & \frac{1}{RC} \end{bmatrix} z + \overline{y}_\alpha \,. \end{cases} \tag{7.24}$$

The inverse switched system, comprising these inverse subsystems, produces ϕ_C as an output provided the initial condition is $z(t_0) = x(t_0)$,

Inductance Soft Faults

Following the same procedure as in the case of capacitor faults, the variation of the inductance over time can be described by $L(t) = L + \lambda_L(t)$, and the system dynamics is then given by

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \phi_L(t), \\ y = x, \end{cases} \tag{7.25}$$

$$\Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \phi_L(t), \\ y = x. \end{cases} \tag{7.26}$$

where

$$\phi_L(t) := \begin{cases} \frac{1}{L+\lambda_L(t)}\big((\frac{R_L}{L}\lambda_L(t) - \frac{d\lambda_L(t)}{dt})i_L - \frac{\lambda_L(t)}{L}v_s\big) & \text{if } \sigma(t) = 1, \\ \frac{1}{L+\lambda_L(t)}\big((\frac{R_L}{L}\lambda_L(t) - \frac{d\lambda_L(t)}{dt})i_L + \frac{\lambda_L(t)}{L}(v_C - v_s)\big) & \text{if } \sigma(t) = 2. \end{cases} \tag{7.27}$$

Mode Identification

Following the notion used in Section 7.1.2, it results that the operators $\mathbf{N_1}$, $\mathbf{N_2}$, J_1, J_2, L_1, L_2 are:

$$\mathbf{N}_1 = \mathbf{N}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{d}{dt} \end{bmatrix}, \; J_1 = J_2 = 0, \; L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{1}{RC} \end{bmatrix}, \; L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}.$$

Also, $\widehat{\mathcal{Y}}_1 = \{y \,|\, \dot{y}_2 + \frac{1}{RC} y_2 = 0\}$, and $\widehat{\mathcal{Y}}_2 = \{y \,|\, \ddot{y}_2 + \frac{1}{C}\dot{y}_1 + \frac{1}{RC}\dot{y}_2 = 0\}$. Note that $\widehat{\mathcal{Y}}_1 \subseteq \widehat{\mathcal{Y}}_2$, so the mode identification can only be carried out using (7.8) and the switch-singular pairs exist when $i_L = 0$, but this is only possible in discontinuous conduction mode (DCM), and before DCM is reached, i_L will remain greater than zero for certain amount of time, so it is possible to recover the unknown input and therefore identify the fault. Thus, other than this case, the switching signal can be recovered by using:

$$\sigma(t) = \begin{cases} 1 & \text{if } y_{[t,t+\varepsilon)} \in \widehat{\mathcal{Y}}_1 \text{ and } \mathbf{N}_1 y(t) = L_1 x(t), \\ 2 & \text{if } y_{[t,t+\varepsilon)} \in \widehat{\mathcal{Y}}_2 \text{ and } \mathbf{N}_2 y(t) = L_2 x(t). \end{cases} \tag{7.28}$$

Unknown Input Recovery

In this case, both subsystems are also invertible, so the detection and isolation of inductor soft faults is possible. The inverse subsystems are described by

$$\Gamma_1^{-1} = \begin{cases} \overline{y}_\alpha = \begin{bmatrix} \frac{d}{dt} & 0 \end{bmatrix} y, \\ \dot{z} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} z + \overline{y}_\alpha, \\ \phi_L = \begin{bmatrix} \frac{R_L}{L} & 0 \end{bmatrix} z - \frac{1}{L} v_s + \overline{y}_\alpha\,, \end{cases} \quad ; \quad \Gamma_2^{-1} = \begin{cases} \overline{y}_\alpha = \begin{bmatrix} \frac{d}{dt} & 0 \end{bmatrix} y, \\ \dot{z} = \begin{bmatrix} 0 & 0 \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} z + \overline{y}_\alpha, \\ \phi_L = \begin{bmatrix} \frac{R_L}{L} & \frac{1}{L} \end{bmatrix} z - \frac{1}{L} v_s + \overline{y}_\alpha\,. \end{cases}$$

Buck Converter

Consider the buck converter of Fig. 7.3 where we assume both i_L and v_C can be measured, and the voltage v_s is perfectly known. Let $x = [i_L \; v_C]^\top$. The case of capacitance soft-faults is similar to the boost converter case, so we omit the analysis and we focus on the more interesting case of inductor soft faults.

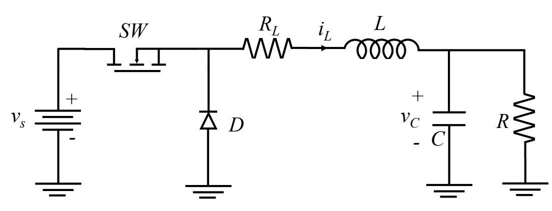


Figure 7.3: Buck converter.

Inductor Soft Faults

Assuming as before that the inductance variation over time can be described by $L(t) = L + \lambda_L(t)$, the converter dynamics are described by

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} \frac{-R_L}{L} & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \phi_L(t), \\ y = x, \end{cases} ; \quad \Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} \frac{-R_L}{L} & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \phi_L(t), \\ y = x, \end{cases} \tag{7.29}$$

where

$$\phi_L(t) = \begin{cases} \frac{1}{L+\lambda_L(t)}\big((\frac{R_L}{L}\lambda_L(t) - \frac{d\lambda_L(t)}{dt})i_L + \frac{\lambda_L(t)}{L}(v_C - v_s)\big) & \text{if } \sigma(t) = 1, \\ \frac{1}{L+\lambda_L(t)}\big((\frac{R_L}{L}\lambda_L(t) - \frac{d\lambda_L(t)}{dt})i_L + \frac{\lambda_L(t)}{L}v_C\big) & \text{if } \sigma(t) = 2. \end{cases}$$

So to recover the value of $\phi_L(t)$, one must first recover the switching signal using (7.14).

For the subsystems in (7.29), the operators involved in mode identification are $J_1 = J_2 = 0$, and

$$\mathbf{N}_1 = \mathbf{N}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{d}{dt} \end{bmatrix}, \; L_1 = L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix},$$

and $\widehat{\mathcal{Y}}_1 = \widehat{\mathcal{Y}}_2 = \{y \,|\, \ddot{y}_2 - \frac{1}{C}\dot{y}_1 + \frac{1}{RC}\dot{y}_2 = 0\}$. Since for any output produced by the system it is true that either $\mathbf{N}_1 y = L_1 x$ or $\mathbf{N}_2 y = L_2 x$, it follows that the equality in (7.7) always holds. In other words, every output produced by the switched system forms a switch-singular pair and the mode detection is not possible in this case. Since the recovery of the mode is the first step in the recovery of the unknown signal $\phi_L(t)$, it is not possible to detect the faults in the inductor using this approach. If the switching signal were already available to the FDI system, then we could bypass this problem and it would be possible to recover the unknown disturbance introduced by the inductance soft faults. The corresponding inverse subsystems would be described by

$$\Gamma_1^{-1} = \begin{cases} \overline{y}_\alpha &= \begin{bmatrix} \frac{d}{dt} & 0 \end{bmatrix} y, \\ \dot{z} &= \begin{bmatrix} 0 & 0 \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} z + \overline{y}_\alpha, \\ \phi_L &= \begin{bmatrix} \frac{R_L}{L} & \frac{1}{L} \end{bmatrix} z - \frac{1}{L}v_s + \overline{y}_\alpha\,, \end{cases} ; \quad \Gamma_2^{-1} = \begin{cases} \overline{y}_\alpha &= \begin{bmatrix} \frac{d}{dt} & 0 \end{bmatrix} y, \\ \dot{z} &= \begin{bmatrix} 0 & 0 \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} z + \overline{y}_\alpha, \\ \phi_L &= \begin{bmatrix} \frac{R_L}{L} & \frac{1}{L} \end{bmatrix} z + \overline{y}_\alpha\,. \end{cases} \tag{7.30}$$

Boost-Buck Converter

Consider a boost-buck converter given in Fig. 7.4, the dynamics of which are governed by

$$\Gamma_1 : \begin{cases} \dfrac{dx}{dt} = \begin{bmatrix} -\frac{R_{in}+r_s}{L_{in}} & \frac{r_s}{L_{in}} & -\frac{1}{L_{in}} \\ \frac{r_s}{L_{out}} & -\frac{(R+r_s)}{L_{out}} & 0 \\ \frac{1}{C} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{L_{in}} \\ 0 \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 \\ -\frac{1}{L_{out}} \\ 0 \end{bmatrix} v_{load}, \\ y = x, \end{cases}$$

and

$$\Gamma_2 : \begin{cases} \dfrac{dx}{dt} = \begin{bmatrix} -\frac{R_{in}}{L_{in}} & 0 & 0 \\ 0 & -\frac{R}{L_{out}} & -\frac{1}{L_{out}} \\ 0 & \frac{1}{C} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{L_{in}} \\ 0 \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 \\ -\frac{1}{L_{out}} \\ 0 \end{bmatrix} v_{load}, \\ y = x, \end{cases}$$

where $x = [i_{in}\ i_{out}\ v_C]^\top$. We assume that the switching signal and the load voltage v_{load} are both unknown and that measurements of all state variables are available. In the absence of faults, it is possible to recover both the switching signal and v_{load}. We show that in the presence of soft faults in L_{in} or C, it is still possible to recover the unknown input v_{load} as well as the input disturbance introduced by the corresponding soft fault. In contrast, in the presence of soft faults in L_{out}, it is not possible to recover both v_{load} and the input disturbance introduced by the fault. We just discuss the case of capacitor soft faults as the case of soft faults in L_{in} is very similar.

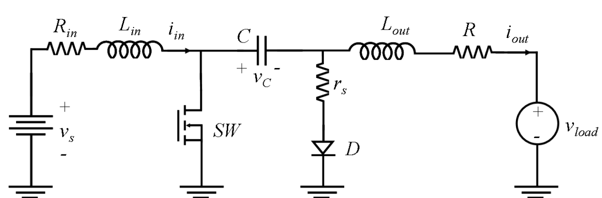


Figure 7.4: Boost-buck converter.

Capacitor Soft Faults

Introducing the faults as in previous examples, it follows that

$$\Gamma_1 : \begin{cases} \dfrac{dx}{dt} = \begin{bmatrix} -\frac{(R_{in}+r_s)}{L_{in}} & \frac{r_s}{L_{in}} & -\frac{1}{L_{in}} \\ \frac{r_s}{L_{out}} & -\frac{(R+r_s)}{L_{out}} & 0 \\ \frac{1}{C} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{L_{in}} \\ 0 \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 & 0 \\ -\frac{1}{L_{out}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{load} \\ \phi_C \end{bmatrix}, \\ y = x, \end{cases}$$

and

$$\Gamma_2 : \begin{cases} \dfrac{dx}{dt} = \begin{bmatrix} -\frac{R_{in}}{L_{in}} & 0 & 0 \\ 0 & -\frac{R}{L_{out}} & -\frac{1}{L_{out}} \\ 0 & \frac{1}{C} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{L_{in}} \\ 0 \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 & 0 \\ -\frac{1}{L_{out}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{load} \\ \phi_C \end{bmatrix}, \\ y = x, \end{cases}$$

where

$$\phi_C(t) = \begin{cases} \frac{1}{C+\lambda_C(t)}\left(\frac{\lambda_C(t)}{C} i_{in} - \frac{d\lambda_C(t)}{dt} v_C\right) & \text{if } \sigma(t) = 1, \\ \frac{1}{C+\lambda_C(t)}\left(\frac{\lambda_C(t)}{C} i_{out} - \frac{d\lambda_C(t)}{dt} v_C\right) & \text{if } \sigma(t) = 2. \end{cases}$$

Mode Identification

Applying the structure algorithm to each subsystem, we obtain: $\mathbf{N}_1 = \mathbf{N}_2 = \left[\begin{array}{ccc} & I_{3\times3} & \\ \hline \frac{d}{dt} & 0 & 0 \end{array}\right]$, $L_1 = \left[\begin{array}{ccc} & I_{3\times3} & \\ \hline -\frac{(R_{in}+r_s)}{L_{in}} & \frac{r_s}{L_{in}} & -\frac{1}{L_{in}} \end{array}\right]$, $L_2 = \left[\begin{array}{ccc} & I_{3\times3} & \\ \hline -\frac{R_{in}}{L_{in}} & 0 & 0 \end{array}\right]$, $J_1 = J_2 = \left[\begin{array}{c} 0_{3\times1} \\ \hline \frac{1}{L_{in}} \end{array}\right]$, and $\widehat{\mathcal{Y}}_1 = \{y \,|\, \ddot{y}_1 - \frac{r_s}{L_{in}}\dot{y}_2 + \frac{1}{L_{in}}\dot{y}_3 + \frac{(R_{in}+r_s)}{L_{in}}\dot{y}_1 = \frac{1}{L_{in}}\dot{v}_s\}$, $\widehat{\mathcal{Y}}_2 = \{y \,|\, \dot{y}_1 + \frac{R_{in}}{L_{in}} y_1 = \frac{1}{L_{in}} v_s\}$.
Thus, the switching signal can be recovered in exactly the same manner as in (7.22).

Unknown Input Recovery

In this case, there are two unknown inputs to be recovered, the load voltage v_{load} and the input disturbance introduced by the capacitor soft fault. For each mode, the corresponding inverse subsystems are:

$$
\Gamma_1^{-1}: \begin{cases} \dfrac{dz}{dt} = \begin{bmatrix} -\frac{(R_{in}+r_s)}{L_{in}} & \frac{r_s}{L_{in}} & -\frac{1}{L_{in}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} \frac{1}{L_{in}} \\ 0 \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{y}_2 \\ \dot{y}_3 \end{bmatrix}, \\ \begin{bmatrix} v_{load} \\ \phi_C \end{bmatrix} = \begin{bmatrix} -L_{out} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} + \begin{bmatrix} r_s & -(R+r_s) & 0 \\ \frac{1}{C} & 0 & 0 \end{bmatrix} z, \end{cases}
$$

and

$$
\Gamma_2^{-1}: \begin{cases} \dfrac{dz}{dt} = \begin{bmatrix} -\frac{R_{in}}{L_{in}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} \frac{1}{L_{in}} \\ 0 \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{y}_2 \\ \dot{y}_3 \end{bmatrix}, \\ \begin{bmatrix} v_{load} \\ \phi_C \end{bmatrix} = \begin{bmatrix} -L_{out} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} - \begin{bmatrix} 0 & R & 1 \\ 0 & \frac{1}{C} & 0 \end{bmatrix} z, \end{cases}
$$

with $z(t_0) = x(t_0)$.

7.1.4 Computer Implementation and Simulation Results

In this section, we provide an algorithm amenable for computer implementation that automates the necessary tasks of FDI discussed in previous sections. The effectiveness of the algorithm to detect and isolate both hard and soft faults was tested in a computer simulation. In this regard, MATLAB/Simulink were used as the platform to implement the algorithm. The boost converter case-study presented in Section 7.1.3 was simulated numerically using Simulink and PLECS [153], and random soft faults were injected in the simulation models to assess whether or not the algorithm implementation could successfully detect and isolate the corresponding injected faults. Additionally, a network of buck converters serving several loads was also simulated to test the effectiveness of the algorithm for detecting hard faults. The FDI algorithm was also successfully tested in a boost-buck converter, although the results are not included.

Algorithm for Automatic FDI

Based upon the concepts introduced in Section 7.1.2, we present Algorithm 4 that takes the parameters $x_0 \in \mathbb{R}^n$, measured output y (defined over a finite interval) and returns the set

$\mathcal{A}$, that contains the switching signal σ, the unknown inputs $u = [\mu \ \phi]^\top$, and the decision variables H_fault and S_fault that represent hard and soft faults respectively.

Algorithm 4: Fault Detection and Isolation in Switched Linear Systems

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begin $\texttt{FDI}(y_{[t_0,T)]}, x_0)$
- Let $\overline{\mathcal{P}} := \{p \in \mathcal{P} : y_{[t_0,t_0+\varepsilon)} \in \widehat{\mathcal{Y}}_p$ for some $\varepsilon > 0$ $\}$.
- Let $t^* := \min\{t \in [t_0, T) : y_{[t,t+\varepsilon)} \notin \widehat{\mathcal{Y}}_p$ for some $p \in \overline{\mathcal{P}},\ \varepsilon > 0\}$ otherwise $t^* = T$.
- Let $\mathcal{P}^* := \Sigma^{-1}(x_0,\, y_{[t_0,t^*)})$.
- **if** $\mathcal{P}^* \neq \emptyset$ **then**
  - Let $\mathcal{A} := \emptyset$
  - **foreach** $p \in \mathcal{P}^*$ **do**
    - **if** $p \in \mathcal{F}$ **then** H_fault$(p) = 1$
    - Let $u = \Gamma^{-1,O}_{p,x_0}(y_{[t_0,t^*)})$
    - **if** $\|\phi(i)\| > \delta$ **then** S_fault$(i) = 1$
    - $\mathcal{T} := \{t \in (t_0, t^*); (x(t), y_{[t,t^*)})$ is a switch-singular pair of Γ_p, Γ_q for some $q \neq p\}$.
    - **if** *$\mathcal{T}$ is a finite set* **then**
      - **foreach** $\tau \in \mathcal{T}$ **do**
        - let $\xi := \Gamma_p(u)(\tau)$
        - $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,\tau)},\ u_{[t_0,\tau)}, \text{H\_fault}, \text{S\_fault}) \oplus \texttt{FDI}(y_{[\tau,T)}, \xi)\}$
    - **else if** *$\mathcal{T} = \emptyset$ and $t^* < T$* **then**
      - let $\xi := \Gamma_p(u)(t^*)$
      - $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,t^*)},\ u_{[t_0,t^*)}, \text{H\_fault}, \text{S\_fault}) \oplus \texttt{FDI}(y_{[t^*,T)}, \xi)\}$
    - **else if** *$\mathcal{T} = \emptyset$ and $t^* = T$* **then**
      - $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,T)}, u_{[t_0,T)}, \text{H\_fault}, \text{S\_fault})\}$
    - **else**
      - $\mathcal{A} := \emptyset$
- **else**
  - $\mathcal{A} := \emptyset$

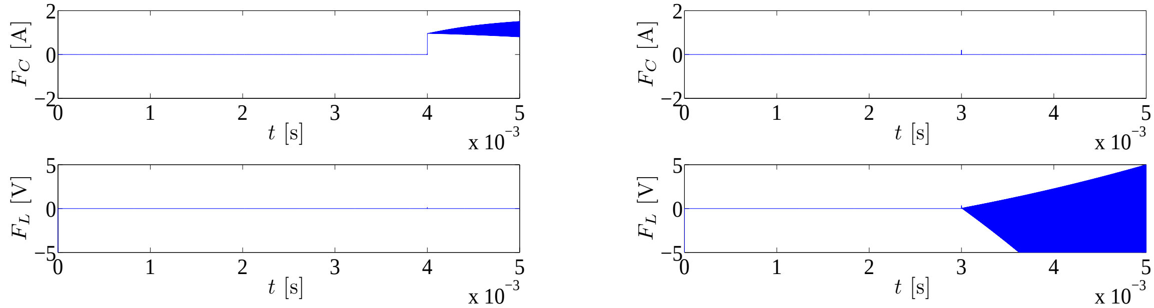
end

Since the occurrence of a hard fault is indicated by a switching signal taking values over a finite set, it is natural to associate a Boolean decision variable for its detection. A soft fault, on the other hand, is related to a real valued signal ϕ, and it may be undesirable to take any action when the value of ϕ is considerably smaller. Therefore, we declare a soft fault only when ϕ has crossed a certain threshold value, denoted by δ in the algorithm. Now, H_fault (respectively S_fault) is a vector and a value 1 in any of the entries indicates a hard (resp. soft) fault in the corresponding component. In case of multiple faults or switch-singular pairs, these decision vectors can have multiple 1's. In the algorithm, $\Sigma^{-1}(x_0, y)$ denotes the index-inversion map which returns the indices of the modes that can produce the given output and thus it may be multi-valued. If the returned set is empty, no subsystem is able to generate that y starting from x_0. The symbol $\oplus$ is used for concatenation and by convention $S \oplus \emptyset = \emptyset$ for any set S. Further, $\Gamma^{-1,O}_{p,x_0}$ indicates the output of the inverse subsystem p when initialized at x_0.

If the return is a non-empty set, the set must be finite and contains pairs of switching signals and inputs that generate the measured y starting from x_0. If the return is an empty set, it means that there is no switching signal and input that generate y. This may be the case if the system is operating in a configuration/faulty mode which has not been modeled or there is an infinite number of possible switching times. Also by our concatenation notation: if at any instant of time, the return of the procedure is an empty set, then that branch of the search will be empty because $\eta \oplus \emptyset = \emptyset$.

Boost Converter

We show simulation results for the boost converter case-study discussed in Section 7.1.3, the parameter values of which are given in Table 7.1. Note that although we are considering the converter that is operating in open-loop with a fixed duty ratio $D = 0.79$, the proposed FDI framework is independent of the type of control. Figure 7.5 shows the time evolution of the capacitor and inductor fault flags. For scaling purpose, we plot the capacitor fault flag $F_C = C\phi_C$, which is obtained by multiplying the nominal capacitance value and the input disturbance introduced by the capacitor soft fault (7.19). Similarly, the inductor fault flag $F_L = L\phi_L$ results from multiplying the nominal inductance value and the disturbance introduced by the inductor soft fault (7.27). Note that the fault flags F_C and F_L are indicators of soft faults and the actual error profile can be obtained by solving ODEs (7.19) and (7.27) for λ_C and λ_L respectively. Figure 7.6 shows the actual value of time varying capacitance $C(t) = C + \lambda_C(t)$, and the (recovered) value of $\lambda_C(t)$ obtained as a solution of (7.19). We choose to work with the fault flags instead of actual error profile because in some cases the underlying ODE solved to obtain λ_C or λ_L may be unstable and the small noise accumulated in the recovery of ϕ_C or ϕ_L may lead to inaccurate profiling of the error.



(a) Fault flag for a capacitor soft fault starting at $t = 4 \cdot 10^{-3}$s.

(b) Fault flag for an inductor soft fault starting at $t = 3 \cdot 10^{-3}$s.

Figure 7.5: Time evolution of boost converter capacitor fault flag F_C, and inductor fault flag, F_L.

Capacitor Soft Fault

In the simulation, the capacitor is described by a time-varying capacitance

$$C(t) = \begin{cases} C & \text{if } t < 4 \cdot 10^{-3} \text{ s}, \\ Ce^{-100(t-4\cdot 10^{-3})} & \text{if } t \geq 4 \cdot 10^{-3} \text{ s}. \end{cases} \tag{7.31}$$

Thus, this is equivalent to assuming that the capacitor remains fault-free up to $t = 4 \cdot 10^{-3}$ s and then it starts gracefully degrading, causing its capacitance to decrease. The FDI system captured this fault occurrence as can be seen in Fig. 7.5(a), where the capacitor fault flag F_C remains zero until $t = 4 \cdot 10^{-3}$ s and then it suddenly jumps indicating the presence of the soft fault. It is important to note that inductor fault flag F_L remains at zero, which

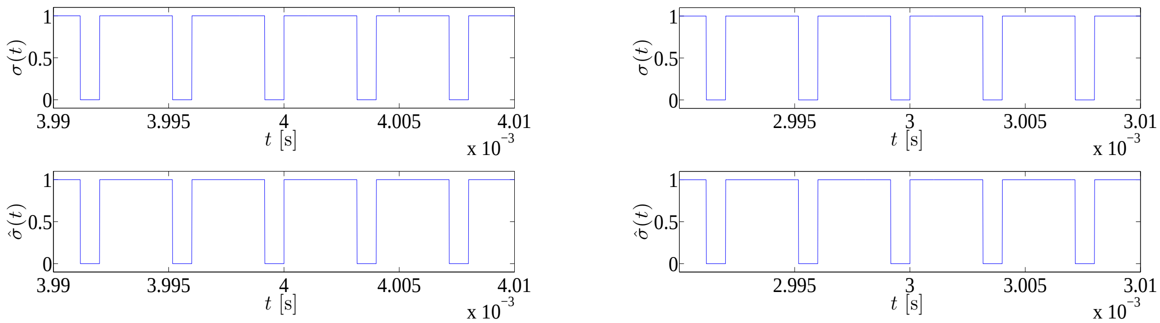
Table 7.1: Boost converter parameter values.

R_L [Ω]	L [H]	C [F]	R [Ω]	v_s [V]	f [kHz]	D
0.2	$5 \cdot 10^{-5}$	$2 \cdot 10^{-4}$	24	12	$200 \cdot 10^3$	0.79

(a) Actual value of capacitance.

(b) Recovered soft fault in capacitor.

Figure 7.6: Time varying capacitance and degradation from nominal value.



(a) $\sigma(t)$ and $\hat{\sigma}(t)$ around the capacitor soft fault occurrence.

(b) $\sigma(t)$ and $\hat{\sigma}(t)$ around the inductor soft fault occurrence.

Figure 7.7: Boost converter real switching signal $\sigma(t)$ and recovered version $\hat{\sigma}(t)$.

is consistent with the fact that no soft fault has occurred in the inductor. In Fig. 7.5(a), both F_C and F_L are curves oscillating at very high frequency, which is difficult to observe at a first glance due to the time scale used in the representation. Also, the degradation time constant chosen in the example is 0.01 s. In reality, degradation time constants are much larger; however, we chose this value to make the fault occurrence apparent in Fig. 7.5(a). This is by no means a limitation of the FDI system, which should be able to detect degrading faults with slower constant, but a limitation of the way the results are displayed.

Inductor Soft Fault

Similarly, to model inductor soft faults, the inductance is described by

$$L(t) = \begin{cases} L & \text{if } t < 3 \cdot 10^{-3} \text{ s}, \\ Le^{-200(t-3\cdot 10^{-3})} & \text{if } t \geq 3 \cdot 10^{-3} \text{ s}. \end{cases} \quad (7.32)$$

As shown in Fig. 7.5(b), the FDI system captured this fault and at time $t = 3 \cdot 10^{-3}$ s, the inductor fault flag F_L starts drifting from its previous zero value, indicating the presence of a soft fault in the inductor. As expected, the capacitor fault flag does not change after $3 \cdot 10^{-3}$ s.

Mode Detection

For completion, we show in Fig. 7.7 the real converter switching signal $\sigma(t)$, and the recovered switching signal $\hat{\sigma}(t)$ using the structure algorithm around the time of fault occurrence of both the capacitor and inductor. It is clear that the mode detection part, which is key for input recovery, works fine (consistent with the soft fault recovery results shown in Fig. 7.5).

DC Network

Consider the DC network of Fig. 7.8. The purpose of this system is to reliably provide DC power to three dispersed loads (described by resistors R_1, R_2 and R_3). Instead of using a single power supply, three distributed DC power supplies (buck converters) are used so as to ensure that a single fault (on the supply side) does not cause all the loads to lose power. The three power supplies and the three loads are connected through a network, where each transmission line linking two nodes is modeled as a resistor.

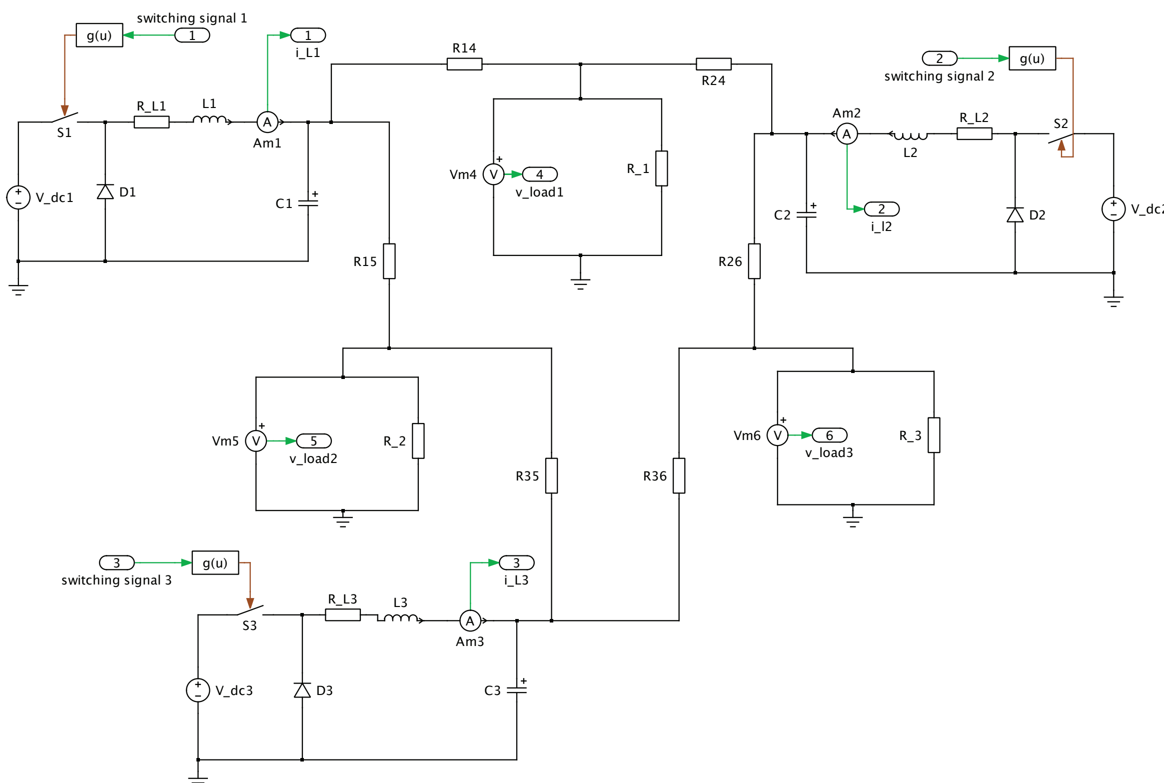


Figure 7.8: DC network implementation in PLECS.

We will focus on the problem of detecting hard and soft faults in this system. In particular, we will consider hard faults that cause an open circuit in a transmission line between two nodes. This also covers the case of short-circuits if the transmission elements are fused or they have some sort of relay protection. Soft faults considered include degradation of buck converter capacitors. We assume that the currents i_{L1}, i_{L2}, i_{L3} through the buck converter inductors are measured as well as the voltages v_{load_1}, v_{load_2}, v_{load_3} at the load buses. We assume the converters always work in a continuous conduction mode and therefore that there are eight possible nominal modes. For the parameter values given in Table 7.2, the resulting state-space description matrices are

$$A_p = 10^4 \cdot \begin{bmatrix} -0.83 & 0 & 0 & -8.33 & 0 & 0 \\ 0 & -0.83 & 0 & 0 & -8.33 & 0 \\ 0 & 0 & -0.83 & 0 & 0 & -8.33 \\ 0.07 & 0 & 0 & -6.72 & 3.30 & 3.31 \\ 0 & 0.06 & 0 & 3.30 & -6.72 & 3.30 \\ 0 & 0 & 0.07 & 3.31 & 3.31 & -6.71 \end{bmatrix}, \; E_p = 10^4 \cdot \begin{bmatrix} 8.33p_1 & 0 & 0 \\ 0 & 8.33p_2 & 0 \\ 0 & 0 & 8.33p_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.495 & 0.495 & 0 \\ 0 & 0 & 0 & 0.497 & 0 & 0.497 \\ 0 & 0 & 0 & 0 & 0.496 & 0.496 \end{bmatrix}, \qquad (7.33)$$

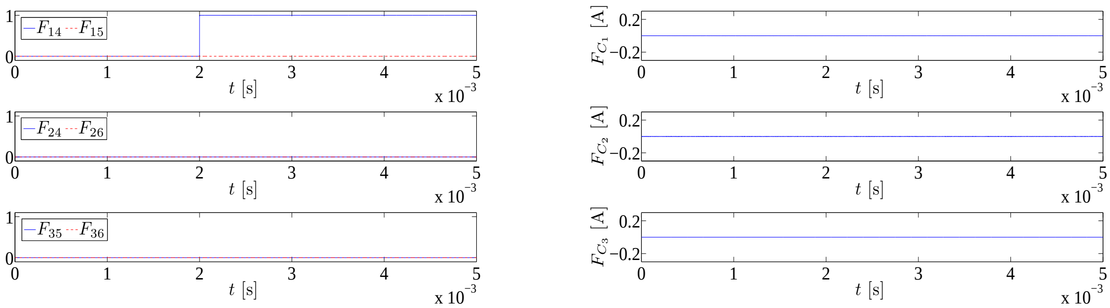
where $p_i \in \{0, 1\}$, with $i = 1, 2, 3$, and $p = \mathrm{dec}(p_1 p_2 p_3)$, where $\mathrm{dec}(\cdot)$ is the decimal representation of the binary number within the brackets.

Hard Faults in Transmission Lines

Consider a hard fault occurrence causing the transmission line linking buses 1 and 4 to open, which will result in eight new modes $p = 9 \dots 16$. For clarity of presentation, we just

Table 7.2: DC network parameter values, where $i = 1, 2, 3$.

L_i [H]	C_i [F]	V_i [V]	f [kHz]	D_i
$1.2 \cdot 10^{-5}$	$1.5 \cdot 10^{-3}$	12	$250 \cdot 10^3$	0.49
R_{L_i} [Ω]	R_1 [Ω]	R_2 [Ω]	R_3 [Ω]	$R_{14(5)}, R_{24(6)}, R_{35(6)}$ [Ω]
0.1	0.5	0.8	0.6	0.01



(a) Transmission line fault flags $F_{14}, F_{15}, F_{24}, F_{26}, F_{35}, F_{36}$.

(b) Buck converter capacitor fault flags $F_{C_1}, F_{C_2}, F_{C_3}$.

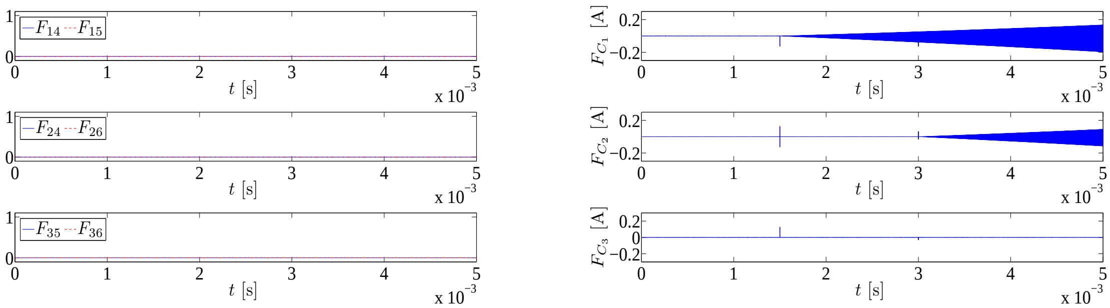
Figure 7.9: Time evolution of DC network hard fault flags in transmission lines and soft fault flags in buck converter capacitors for a hard fault in transmission line R_{14} starting at $t = 2 \cdot 10^{-3}$.

provide the elements of the matrices in (7.33) that will get modified as a result of this fault. $A_9(4,4) = \cdots = A_{16}(4,4) = -3.35 \cdot 10^4$; $A_9(4,5) = \cdots = A_{16}(4,5) = 0$; $A_9(5,5) = \cdots = A_{16}(5,5) = -3.49 \cdot 10^4$; $A_9(5,4) = \cdots = A_{16}(5,4) = 0$; $C_9(4,4) = \cdots = C_{16}(4,4) = 0$; $C_9(4,5) = \cdots = C_{16}(4,5) = 0.98$.

In the simulation environment, this fault was injected at $t = 2 \cdot 10^{-3}$. The output of the FDI system is displayed in Fig. 7.9. As can be seen in Fig. 7.9(a), at the time of fault occurrence, $t = 2 \cdot 10^{-3}$, the flag F_{14}, which indicates a hard fault in transmission line R_{14}, changes from zero to one, whereas the flags for the remaining transmission elements remain at zero, as expected; i.e., there is no false alarm. Figure 7.9(b) shows the fault flags for soft faults in the buck converter capacitors; as expected, they remain at zero even after the hard fault occurrence, so there is no false alarm in this case either.

Soft Faults in Buck Converter Capacitors

In the simulation environment, these faults will be modeled in a similar fashion as in (7.31) for the boost converter numerical example. To illustrate the ability of the FDI system to detect and isolate two faults, we assume capacitor C_1 starts degrading at $t = 1.5 \cdot 10^{-3}$ and capacitor C_2 starts degrading at $t = 3 \cdot 10^{-3}$. The degradation is exponential as in (7.31), with a rate of 100 s^{-1}. The FDI system captured both fault occurrences as can be seen in Fig. 7.10(b), where the fault flag F_{C_1} corresponding to capacitor C_1 remains at zero until $t = 1.5 \cdot 10^{-3}$ and then starts increasing. The other two fault flags remain at zero



(a) Transmission line fault flags $F_{14}, F_{15}, F_{24}, F_{26}, F_{35}, F_{36}$.

(b) Buck converter capacitor fault flags $F_{C_1}, F_{C_2}, F_{C_3}$.

Figure 7.10: Time evolution of DC network hard fault flags in transmission lines and soft fault flags in buck converter capacitors for a soft fault in capacitor C_1 starting at $t = 1.5 \cdot 10^{-3}$ and a soft fault in capacitor C_2 starting at $t = 3 \cdot 10^{-3}$.

until $t = 2 \cdot 10^{-3}$, when F_{C_2} starts increasing, which means capacitor C_2 starts gracefully degrading. Flag F_{C_3} remains at zero at all times, which indicates capacitor C_3 does not degrade in the simulation period considered. It is important to note that the flags for hard faults displayed in Fig. 7.10(a) remain at zero, indicating no hard faults occurrences.

7.2 Fault Detection using Observers

This section proposes a novel framework for model based fault detection in switching electrical systems based on a geometric approach and the design of reduced order asymptotic observers. We introduce the concept of conditioned invariant subspaces and generalized unobservable subspaces in switched linear systems, which are utilized in deriving a sufficient condition to solve the problem of fault detection. Based on these conditions, reduced order observers are designed for the switched system without assuming observability of the individual modes. For a certain class of switching signals, it is shown that the state estimates obtained from the proposed observer converge to the observable component of the actual state asymptotically. The outputs of these observers are designed to be sensitive to a particular fault, thus acting as residual signals for detection and isolation of faults.

In our earlier work on fault detection in electrical systems, we used the invertibility techniques to recover the hard faults and soft faults. In detecting the soft faults, the mapping between the output and the corresponding fault was assumed to be invertible, which allowed us to exactly reconstruct the fault using the derivatives of the output and the initial condition of the state variable. However, in practice, the initial condition may not be available and the computation of derivatives of the output is not always feasible due to noise in the measurements. In order to relax these two assumptions, this section proposes an observer based technique for detecting faults in electrical systems. Because the initial condition is not assumed to be known, an asymptotic observer is first designed for the switched system to estimate the actual state. We no longer require the derivatives of the output because we do not aim to reconstruct the fault; rather, we are just interested in knowing whether the fault has some nonzero value. The observers we thus design do not require knowledge of the initial condition, and using the output feedback, the state estimates generated by these observers converge to the actual state. If we now look at the difference between the actual output and the estimated output, then we require certain linear combination of this output estimation error to be sensitive to the faults in the system. This way the faults could be detected by monitoring the nonzero values of the output estimation error.

Earlier similar approaches on fault detection involve modeling the faults in actuators and sensors as unknown functions of time [154, 155]. In order to detect a particular fault, the goal is to generate a residual signal corresponding to each fault such that the residual is non-zero if and only if the corresponding fault has occurred. Such residuals are generated as the output of an auxiliary dynamical system. The residual generator for each fault must be designed in such a way that the output of this auxiliary system is nonzero for one fault only and identically zero for the remaining ones. Thus, we require the mapping between the output of the residual generator and its corresponding fault to be invertible or input observable. Moreover, with the initial condition unknown, it is desirable that the resulting transients from the state estimation die down quickly; this can be achieved by requiring the states affected by that particular fault to be observable.

However, this problem has mainly been studied for linear time-invariant systems only. With switching being an inherent component of power electronic systems, it becomes vital to study the problem of fault detection for a richer class of systems that comprises switching between a family of dynamical subsystems. Using our recent work on observability of switched linear systems [102] as an additional motivation, this section investigates the problem of fault detection in switched systems whose subsystems are comprised of linear dynamics. The objective is to first formalize the conditions on subsystem dynamics under which the problem can be solved and then design the residual generators for each fault by using the idea of constructing asymptotic observers for switched systems.

The geometric conditions proposed for the solution of fault detection and filter design are inherently based on the concept of controlled-invariant and conditioned-invariant subspaces. These notions were introduced by Basile and Maro, along with some applications in dynamical systems [156, 157]. Several others, such as [158], studied the applications of these invariant subspaces in the design of compensators and regulators. Willems [159, 160] then generalized these notions to give various versions of the disturbance decoupling problem using feedback. Later, this notion of invariant subspaces was extended to a certain class of parameter varying subsystems in [161], and their application was studied in fault detection problems [162]. Very recently, the concept of invariant subspaces has found its application in the disturbance decoupling problem for switched linear systems [163, 164].

Using similar ideology, we extend these fundamental ideas for fault detection in switched systems by proposing a solution based on the concept of conditioned-invariant subspaces and designing asymptotic observers which act as residual generators.

7.2.1 Preliminaries

Non-Switched Systems

Let us first review the basic ideas adopted in the literature on fault detection of linear time-invariant non-switched systems. Although we will generalize these concepts to switched systems, the extension is non-trivial and is developed based on the notions of observability and invertibility of switched systems from our work in Chapter 2 and Chapter 4.

Consider a non-switched linear time-invariant system subjected to two faults $m_1(t)$ and $m_2(t)$:

$$\dot{x} = Ax + Bu + J_1 m_1 + J_2 m_2, \tag{7.34}$$

$$y = Cx. \tag{7.35}$$

The state estimator for such a system is designed as:

$$\dot{\hat{x}} = (A + LC)\hat{x} + Bu - Ly, \tag{7.36}$$

$$\hat{y} = C\hat{x}. \tag{7.37}$$

In the absence of faults, the error $\tilde{y}(t) := \hat{y}(t) - y(t)$ will converge to zero if the designed observer is stable. However, when a fault occurs, say $m_1(t) \neq 0$, then the observer no longer estimates the actual state of the plant; as a result, the error $\tilde{y}(t)$ grows with time and by assigning an appropriate threshold, faults can be detected. A rather interesting problem is whether we can use the properties of the innovation $\tilde{y}$ to isolate the faulty component in the system. Beard [20] and Jones [21] suggested that it is possible to confine $\tilde{y}(t)$ to a fixed direction by appropriately choosing the feedback gain matrix L. Looking more into the directional properties of the innovation $\tilde{y}$, define the following two linear transformations:

$$r_1(t) = H^1\tilde{y}(t) = H^1(y(t) - \hat{y}(t)),$$
$$r_2(t) = H^2\tilde{y}(t) = H^2(y(t) - \hat{y}(t)).$$

For fault detection the purpose is to find L, H^1, H^2 such that the failure of the first actuator shows up in $r_1(t)$ but has no effect on $r_2(t)$, and the failure of the second actuator shows up in $r_2(t)$ and has no effect on $r_1(t)$. Clearly, if the growth of $\tilde{y}(t)$ is constrained to independent subspaces, then H^1 and H^2 can simply be taken as projection matrices onto these independent subspaces, as done by Beard and Jones. However, a more general approach, adopted by Massoumnia [155], is to design the matrices H^i, along with the matrix L, as a part of the design process. Proceeding further, the error dynamics for the state estimation error $\tilde{x}(t) := \hat{x}(t) - x(t)$ are given by:

$$\dot{\tilde{x}} = (A + LC)\tilde{x} - J_1 m_1 - J_2 m_2, \tag{7.38}$$
$$r_1 = H^1 C\tilde{x}, \qquad r_2 = H^2 C\tilde{x}. \tag{7.39}$$

Now, for nonzero $m_2(t)$ not to effect $r_1(t)$, the transfer function between m_2 and r_1 should be zero, that is, the image of J_2 should be contained in the unobservable subspace of the pair $(H^1C, A + LC)$. Also, for a nonzero $m_1(t)$ to show up in $r_1(t)$, the image of J_1 should not intersect the unobservable subspace of $(H^1C, A+LC)$. Similar reasoning holds for $r_2(t)$ and the unobservable subspace of $(H^2C, A + LC)$. Thus, the goal of fault detection is to design L, H^1, and H^2 that achieve this objective. On the other hand, instead of looking for these matrices, the problem can be formulated in terms of existence of certain subspaces $\mathcal{W}^1$ and $\mathcal{W}^2$ that contain the images of J_2 and J_1 respectively and that can be assigned as unobservable subspaces of $(H^1C, A+LC)$ and $(H^2C, A+LC)$ respectively for some H^1, H^2, and L. If such subspaces $\mathcal{W}^1$ and $\mathcal{W}^2$ exist and can be computed directly from the system data, then we can also compute the required H^1, H^2, and L to solve the fault detection problem; see [155] for details. This is the essence of the geometric approach that we shall generalize to switched systems in our work. The subspaces $\mathcal{W}^1$ and $\mathcal{W}^2$ are termed as unobservability subspaces in classical literature. We will first develop similar objects for switched systems and design the residual generators using these concepts.

Switched Systems

We consider the problem of fault detection in switched linear systems represented by the following set of equations:

$$\dot{x} = A_\sigma x + B_\sigma u, \tag{7.40a}$$

$$y = C_\sigma x, \tag{7.40b}$$

where we assume that the switching signal σ takes values in the finite index set $\{1, \dots, p\}$. Due to the actuator and sensor faults, the dynamics of (7.40) get modified as follows:

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) + \sum_{i=1}^{k} J_i m_i(t) \tag{7.41a}$$

$$y(t) = C_{\sigma(t)} x(t), \tag{7.41b}$$

where $m_i(t)$ denotes the magnitude of i-th fault, at time instant t, acting in the direction given by J_i. Each J_i is assumed to be a non-zero vector in $\mathbb{R}^n$ and its range space is denoted by $\mathcal{J}_i$.

Problem Formulation

In order to detect a particular fault, our goal is to generate a residual signal $r_i(t)$ corresponding to each fault $m_i(t)$ such that $r_i(t) \neq 0$ if and only if $m_i(t) \neq 0$. Each residual signal is generated as an output of an auxiliary dynamical system. This is achieved by first determining the observable part of the switched system, which we denote by z. The goal is then to estimate the observable part of the system and generate residues as a function of this observable part. In the absence of fault, z is estimated perfectly and the corresponding r_i stays near zero. In the presence of fault, state estimation error in z must manifest in r_i. Since the system under consideration involves switches, it is not a trivial matter to define how the observable space is defined and how to construct asymptotic observers for these observable components. Furthermore, only certain types of structural constraints would allow for the state estimation error to appear in the residues. To better explain our ideas, we assume that the system is subjected to two faults only, i.e., $m_1(t)$ and $m_2(t)$. For definiteness, we design the residual generating auxiliary dynamical system with output $r_1(t)$ which is sensitive to $m_1(t)$ and is not affected by $m_2(t)$. The same technique could be reciprocated to design fault detection mechanism for $m_2(t)$. In case of multiple faults, see Remark 7.17 on how these techniques could be generalized.

In the sequel, we first introduce some new geometric tools for a switched system and derive conditions on geometric structure of the switched system that allow for the solution to fault detection problem to exist. Later we use them in the construction of asymptotic observers and residual generators.

7.2.2 Geometric Tool Set

Notations: Let $\oplus$ denote the external direct sum of two subspaces, so that we use $\mathcal{V}^{\oplus p}$ as shorthand for $\mathcal{V} \oplus \cdots \oplus \mathcal{V}$. Also, let $A_{1\cdots p}$ denote the matrix $[A_1, A_2, \ldots, A_p]$. The largest A-invariant subspace contained in a subspace $\mathcal{V}$ is denoted by $\langle \mathcal{V} | A \rangle$, whereas the $\langle A | \mathcal{V} \rangle$ denotes the smallest A-invariant subspace containing the subspace $\mathcal{V}$. With a matrix A, $\mathcal{R}(A)$ denotes the column space (range space) of A. The sum of two subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ is defined as $\mathcal{V}_1 + \mathcal{V}_2 := \{v_1 + v_2 : v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2\}$. For a possibly non-invertible matrix A, the pre-image of a subspace $\mathcal{V}$ under A is given by $A^{-1}\mathcal{V} = \{x : Ax \in \mathcal{V}\}$. Let $\ker A := A^{-1}\{0\}$; then it is seen that $A^{-1} \ker C = \ker(CA)$ for a matrix C. For convenience of notation, let $A^{-\top}\mathcal{V} := (A^\top)^{-1}\mathcal{V}$ where $A^\top$ is the transpose of A, and it is understood that $A_2^{-1}A_1^{-1}\mathcal{V} = A_2^{-1}(A_1^{-1}\mathcal{V})$. Also, we denote the products of matrices A_i as $\prod_{i=j}^{k} A_i := A_j A_{j+1} \cdots A_k$ when $j < k$, and $\prod_{i=j}^{k} A_i := A_j A_{j-1} \cdots A_k$ when $j > k$. The notation $\mathrm{col}(A_1, \ldots, A_k)$ means the vertical stack of matrices $A_1, \cdots, A_k$, that is, $[A_1^\top, \ldots, A_k^\top]^\top$.

Conditioned-Invariant Subspaces

The concept of conditioned-invariant subspace was introduced in [156] for non-switched linear time invariant systems. Some of the properties of these subspaces appear in Appendix B. The basic utility of conditioned-invariant subspace $\mathcal{V}$ is that if $x(t_0)/\mathcal{V}$ is known, then there exists a dynamic observer that generates $x(t)/\mathcal{V}$ at each time t. In other words, if the output feedback gain matrix L is chosen such that $\mathcal{V}$ is invariant under the resulting closed loop dynamics, i.e., $(A + LC)\mathcal{V} \subseteq \mathcal{V}$, then it is possible to construct a reduced order subsystem with state space $\mathbb{R}^n/\mathcal{V}$ that generates $x(t)/\mathcal{V}$ as output. Motivated by this idea, we now extend the concept of conditioned-invariant subspace to switched systems.

Definition 7.8 (Conditioned-invariant subspace)**.** *A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is said to be conditioned-invariant if there exist maps L_i such that*

$$(A_i + L_iC_i)\mathcal{V} \subseteq \mathcal{V}, \qquad \forall\, i = 1, \dots, p. \tag{7.42}$$

Proposition 7.9. *A subspace $\mathcal{V}$ is conditioned-invariant if, and only if,*

$$A_{1\cdots p}(\ker C_1 \oplus \ker C_2 \oplus \cdots \oplus \ker C_p \cap \mathcal{V}^{\oplus p}) \subseteq \mathcal{V}. \tag{7.43}$$

Proof. (Sufficiency). If (7.43) holds, then

$$A_1(\ker C_1 \cap \mathcal{V}) + \cdots + A_p(\ker C_p \cap \mathcal{V}) \subseteq \mathcal{V},$$

and it follows that for each $i = 1, \dots, p$,

$$A_i(\ker C_i \cap \mathcal{V}) \subseteq \mathcal{V}.$$

From Proposition B.2 in Appendix B, it follows that for each $i = 1, \dots, p$ there exists L_i such that $(A_i + L_iC_i)\mathcal{V} \subseteq \mathcal{V}$.

(Necessity). Conversely, if there exist L_i satisfying (7.42), then, using Proposition B.2, $A_i(\ker C_i \cap \mathcal{V}) \subseteq \mathcal{V}$ for each $i = 1, \dots, p$, whence the desired result follows. □

Let $\mathcal{V}(\mathcal{J})$ denote a conditioned-invariant subspace that contains the subspace $\mathcal{J}$, and let $\mathcal{V}^*(\mathcal{J}) := \inf \mathcal{V}(\mathcal{J})$; then a recursive algorithm for computing $\mathcal{V}^*(\mathcal{J})$ using Proposition 7.9 is given below:

$$\mathcal{V}_0 = \mathcal{J} \tag{7.44a}$$

$$\mathcal{V}_{k+1} = \mathcal{J} + A_{1\cdots p}(\ker C_1 \oplus \ker C_2 \oplus \cdots \oplus \ker C_p \cap \mathcal{V}_k^{\oplus p}). \tag{7.44b}$$

It is easy to see that $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots$, and since $\mathcal{V}_k$ is contained in $\mathbb{R}^n$ for each k, there exists a k^* such that $\mathcal{V}_{k^*+1} = \mathcal{V}_{k^*}$ and we let $\mathcal{V}^* = \mathcal{V}_{k^*}$.

Unobservable Subspaces

While introducing the conditioned-invariant subspaces, it was assumed that the initial condition is known modulo certain subspace in order to maintain the same information about the state for all times using an observer. However, if the initial condition is absolutely unknown, a desired property of the observer is that it generates an *estimate* that converges to the actual state with time. This motivates the introduction of unobservable subspaces. Roughly speaking, for a non-switched LTI system, a subspace $\mathcal{W}$ is called unobservable if there exists a stable observer whose output $\hat{x}/\mathcal{W}$ converges to $x/\mathcal{W}$. Precisely speaking, for a fixed mode q, if $\mathcal{W}_q$ is an unobservable subspace, then there exist matrices H_q and L_q such that [23]:

$$\mathcal{W}_q = \langle \ker H_q C_q | A_q + L_q C_q \rangle .$$

It turns out that H_q that satisfies the above equality is picked such that $\ker H_q C_q = \ker C_q + \mathcal{W}_q$. A particular object of interest is the smallest unobservable subspace of the i-th subsystem containing a subspace $\mathcal{J}$ which is denoted by $\mathcal{W}_i^*(\mathcal{J})$, and is computed as the limit of the following sequence:

$$\mathcal{W}_{i,0} = \mathbb{R}^n \tag{7.45a}$$

$$\mathcal{W}_{i,k+1} = \mathcal{V}^*(\mathcal{J}) + (A_i^{-1}\mathcal{W}_{i,k} \cap \ker C_i). \tag{7.45b}$$

Unobservable subspaces are an important tool for state estimation as one can always design a stable state estimator for the state vector modulo the unobservable subspace. Our goal is now to define the unobservable subspace of the switched system such that a stable estimator could be designed for its orthogonal complement.

Definition 7.10 (Unobservable Subspace)**.** *A subspace $\mathcal{W}$ is called unobservable if there exist maps $L_i : \mathcal{Y} \mapsto \mathbb{R}^n$, and $H_i : \mathcal{Y} \mapsto \mathcal{Y}$ such that*

$$\mathcal{W} = \left\langle \cap_{j=1}^{p} \ker H_j C_j | A_i + L_i C_i \right\rangle , \quad \forall\, i = 1, \ldots, p. \tag{7.46}$$

From the definition, it is clear that every unobservability subspace is also a conditioned-invariant subspace.

Proposition 7.11. *If $\mathcal{V}$ is a conditioned-invariant subspace such that $\mathcal{V} \subset \mathcal{W}_q^*$, for some q, then there exists a matrix $L_q : \mathbb{R}^{d_y} \to \mathbb{R}^n$ that satisfies:*

$$(A_q + L_q C_q)\mathcal{V} \subset \mathcal{V} \qquad \textit{and} \qquad (A_q + L_q C_q)\mathcal{W}_q^* \subseteq \mathcal{W}_q^*; \tag{7.47}$$

that is, the output feedback matrix L_q renders both $\mathcal{V}$ and $\mathcal{W}_q^*$ invariant.

Proof. Let $w_1, w_2, \ldots, w_{k_1}$ be the basis for $\mathcal{W}_q^*$, and let $w_1, w_2, \ldots, w_{k_2}$, $k_2 \leq k_1$ be the basis for $\ker C_q \cap \mathcal{W}_q^*$. Since $\mathcal{W}_q^*$ is also conditioned invariant for mode i, it holds that $A_q(\ker C_q \cap \mathcal{W}_q^*) \subseteq \mathcal{W}_q^*$. Pick the gain L_q such that $L_q C_q w_j = -A_q w_j$, for $j = k_2+1, \ldots, k_1$. Then, for $j = 1, \ldots, k_2$, we get $(A_q + L_q C_q) w_j = A_q w_j \in \ker C_q \cap \mathcal{W}_q^* \subseteq \mathcal{W}_q^*$. And for $j = k_2+1, \ldots, k_1$, we get $(A_q + L_q C_q) = 0 \in \mathcal{W}_q^*$.

Also, let $v_1, v_2, \ldots, v_{l_1}$ be the basis for $\mathcal{V}$, and let $v_1, v_2, \ldots, v_{l_2}$, $l_2 \leq l_1$ be the basis for $\ker C_q \cap \mathcal{V}$. Since $\mathcal{V} \subseteq \mathcal{W}_q^*$, then $\ker C_q \cap \mathcal{V} \subseteq \mathcal{W}_q^* \cap \ker C_q$, and it follows that $\operatorname{span}\{v_{l_1+1}, \ldots, v_{l_2}\} \subseteq \operatorname{span}\{w_{k_1+1}, \ldots, w_{k_2}\}$. Hence, $(A_q + L_q C_q) v_j = 0$ for $j = l_2+1, \ldots, l_1$. Also, because $\mathcal{V}$ is conditioned invariant, $A_q(\ker C_q \cap \mathcal{V}) \subseteq \mathcal{V}$, which gives $(A_q + L_q C_q) v_j = A v_j$ for $j = 1, \ldots, l_2$. Therefore, $(A_q + L_q C_q)\mathcal{V} \subseteq \mathcal{V}$. □

For the switched system, we are interested in finding $\mathcal{W}(\mathcal{J})$ which denotes an unobservable subspace that satisfies (7.46) and contains the subspace $\mathcal{J}$. Towards this end, we first find a conditioned-invariant subspace that contains $\mathcal{J}$ using (7.44) and denote it by $\mathcal{V}^*(\mathcal{J})$; then, for each mode q, we find the smallest unobservable subspace using (7.45) that contains $\mathcal{V}^*(\mathcal{J})$ and denote it simply by $\mathcal{W}_q^*(\mathcal{J})$; and for each of these subspaces we choose L_q that satisfies (7.47). Let $\mathcal{W}^*(\mathcal{J}) := \inf \mathcal{W}(\mathcal{J})$; then the following algorithm is used to compute $\mathcal{W}^*(\mathcal{J})$:

$$\mathcal{W}_0 = \mathcal{W}_1^*(\mathcal{J}) \cap \mathcal{W}_2^*(\mathcal{J}) \cap \cdots \cap \mathcal{W}_p^*(\mathcal{J}) \tag{7.48a}$$

$$\mathcal{W}_{k+1} = \langle \mathcal{W}_k | A_1 + L_1 C_1 \rangle \cap \langle \mathcal{W}_k | A_2 + L_2 C_2 \rangle \cap \cdots \cap \langle \mathcal{W}_k | A_p + L_p C_p \rangle \,. \tag{7.48b}$$

The sequence $\mathcal{W}_k$ is such that $\mathcal{W}_0 \supseteq \mathcal{W}_1 \supseteq \cdots$, and there exists a k^* such that $\mathcal{W}_{k^*+1} = \mathcal{W}_{k^*}$ and we let $\mathcal{W}^*(\mathcal{J}) = \mathcal{W}_{k^*}$. Also note that $\mathcal{V}^*(\mathcal{J})$ is contained in $\mathcal{W}_k$ for each $k \geq 0$, so that $\mathcal{J} \subseteq \mathcal{V}^*(\mathcal{J}) \subseteq \mathcal{W}_{k^*}$, and it also follows by construction that $\mathcal{W}^*(\mathcal{J})$ is $(A_q + L_q C_q)$-invariant, for each q. Next we show that for a certain class of switching signals, it is possible to design a stable estimator for the reduced state space $\mathbb{R}^n / \mathcal{W}^*(\mathcal{J})$ even though not every subsystem is observable in that space.

Synopsis

As already mentioned, we are going to work with two faults only and design a filter which detects the fault $m_1(t)$ and is unaffected by $m_2(t)$. The basic idea is to first compute the smallest unobservability subspace that contains the image of J_2 using (7.48). For brevity, we denote the resulting subspace by $\mathcal{W}^*$. We then construct an observer that estimates the state modulo $\mathcal{W}^*$, so that $\mathbb{R}^n/\mathcal{W}^*$ is the observable space and if $P : \mathbb{R}^n \to \mathbb{R}^n/\mathcal{W}^*$ is the canonical

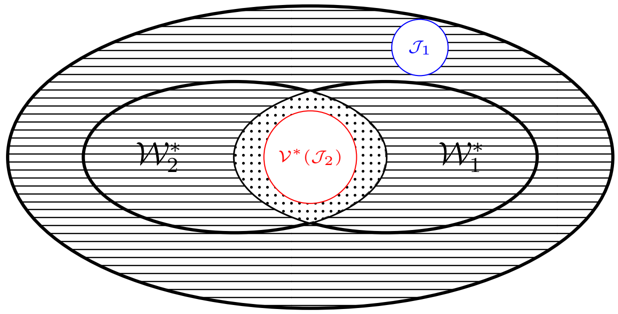


Figure 7.11: We seek a residual generator which is sensitive to faults along $\mathcal{J}_1$ and insensitive to faults along $\mathcal{J}_2$. With $\mathcal{W}_1^*$ and $\mathcal{W}_2^*$ denoting the unobservable subspaces of individual subsystems that contain $\mathcal{V}^*(\mathcal{J}_2)$, we find the output feedback matrices for individual subsystems which limit the propagation of faults along $\mathcal{J}_2$ to a common unobservable space (denoted by dots) of the system for a certain class of switching signals. The remaining striped region is the observable subspace containing $\mathcal{J}_1$ for which a switching observer is designed. The output of this observer is the residual error signal which depends on the fault m_1.

projection on to this observable space, then the estimator is constructed for $z := Px$. We then pick the residual signal $r_1(t)$ to be a linear combination of the difference between the estimated output $C\hat{z}(t)$ and the measured output $y(t)$. If there are no faults, then this difference converges to zero and in the presence of any fault this error has significantly large value. Under the added constraint that the image of J_1 does not intersect $\mathcal{W}^*$, it is shown that $r_1(t)$ is only sensitive to $m_1(t)$ and remains unaffected by nonzero values of $m_2(t)$. See Fig. 7.11 for the graphic illustration of the strategy adopted for fault detection.

In the next section, we give the construction of the observer that estimates the observable component $z = Px$, followed by geometric conditions for fault detection in Section 7.2.4.

7.2.3 Observer Design

Our goal is to design an observer that estimates $z = Px$, where $P : \mathbb{R}^n \to \mathbb{R}^n/\mathcal{W}^*$ is the canonical projection onto the observable space, so that $\ker P = \mathcal{W}^*$. Recall that we have fixed $\mathcal{W}^*$ to be the unobservable subspace that contains $\mathcal{J}_2$ and computed through (7.48). In the process, the gain matrices L_q are picked such that (7.47) holds, where $\mathcal{V}^*$ is taken to be the smallest conditioned-invariant subspace containing $\mathcal{J}_2$. Also, in the sequel, the smallest unobservable subspace of mode q that contains $\mathcal{V}^*$ will be denoted simply by $\mathcal{W}_q^*$. It is seen that $\mathcal{V}^* \subseteq \mathcal{W}^* \subseteq \mathcal{W}_q^*$, for each $q \in \{1, \dots p\}$, and the gain L_q renders $\mathcal{V}^*$, $\mathcal{W}^*$ and $\mathcal{W}_q^*$ invariant. Next, for each $q \in \{1, \dots, p\}$, let us define $E_q := -PL_q$ and the matrix F_q as the map induced by $(A_q + L_q C_q)$ on the factor space $\mathbb{R}^n/\mathcal{W}^*$, that is, $F_q P = P(A_q + L_q C_q)$; also, let $G_q := PB_q$.

In order to design an asymptotic observer, we first define a set of switching signals $\mathcal{S}$ consisting of all periodic switching signals; that is, if $\sigma \in \mathcal{S}$, then

$$\sigma(t_{k-1}) = (k \mod p) + 1, \qquad \forall\, k \geq 1,$$

so that σ is a periodic signal taking values in the set $\{1, \dots, p\}$; and from now onwards all the entities associated with subscript, or superscript, k refer to the corresponding entity for mode $(k \mod p) + 1$. Furthermore, it is assumed that there exist a time T_D such that, for each $\sigma \in \mathcal{S}$,

$$t_k - t_{k-1} \leq T_D, \qquad\qquad \forall\, k \geq 1.$$

It is shown in Theorem 7.13 that if the actual system is executed for certain $\sigma \in \mathcal{S}$, then the observer design given below generates the desired converging state estimate.

Before proceeding with the construction of the observer, we recall a useful result from the literature [44], which will be used in the proof of Theorem 7.13 and specifies the integer N in (7.50).

Lemma 7.12. *Let $d_z := n - \dim \mathcal{W}^*$, and consider the insertion maps defined by the matrices $W_k : \mathcal{W}_k^*/\mathcal{W}^* \to \mathbb{R}^n/\mathcal{W}^*$ with orthonormal column vectors which form the basis of $\mathcal{W}_k^*/\mathcal{W}^*$. For $\sigma \in \mathcal{S}$, there exists a positive number $N \leq \sum_{i=0}^{d_z-1} p(pd_z)^i - 1$, such that for almost all values of $t_1, t_2, \dots, t_N$, the following holds:*

$$\mathcal{R}(W_k)^\perp + \mathcal{R}(\Psi_{k-1}^q W_{k-1})^\perp + \cdots + \mathcal{R}(\Psi_{k-N}^q W_{k-N})^\perp = \mathbb{R}^n/\mathcal{W}^*, \tag{7.49}$$

where Ψ_i^j denotes the flow matrix from t_i to t_j, that is, $\Psi_i^j := e^{F_j \tau_j} e^{F_{j-1}\tau_{j-1}} \cdots e^{F_{i+1}\tau_{i+1}}$. $\lhd$

Lemma 7.12 suggests that there exists an upper bound on the integer N such that (7.49) holds. For our observer design proposed, we fix N to be the smallest positive integer that satisfies (7.49).

The observer we now propose is a hybrid dynamical system of the form:

$$\dot{\hat{z}}(t) = F_k \hat{z}(t) + G_k u(t) + E_k y, \qquad t \in [t_{k-1}, t_k), \tag{7.50a}$$

$$\hat{z}(t_k) = \hat{z}(t_k^-) - \xi_k(t_k^-), \quad k \geq 1, \tag{7.50b}$$

$$\xi_k(t_k^-) = \begin{cases} \mathcal{O}_k(y_{[t_{k-N-1}, t_k)}, u_{[t_{k-N-1}, t_k)}), k > N, \\ 0, \qquad\qquad\qquad\qquad 1 \leq k \leq N, \end{cases} \tag{7.50c}$$

with an arbitrary initial state $\hat{z}(t_0) \in \mathbb{R}^{d_z}$, $d_z := n - \dim \mathcal{W}^*$. The hybrid observer proposed in (7.50) consists of certain dynamics for the observable component of the system, whose estimate is denoted by $\hat{z}$. Equation (7.50a) denotes the evolution of $\hat{z}$ on the reduced space $\mathbb{R}^n/\mathcal{W}^*$, where the matrices F_k, G_k and E_k are chosen in such a manner that the subspace $\mathcal{W}^*$ is invariant under the resulting output feedback. Equation (7.50b) introduces an error correction term ξ_k at the switching instants t_k which is computed through the operator $\mathcal{O}_k$. The basic idea is to design the operator $\mathcal{O}_k$ so that the parts of information available for z from each subsystem are gathered and processed in such a manner that $\hat{z}(t) \to Px(t)$. It will turn out that the operator $\mathcal{O}_k$ includes dynamic observers for partial states at each mode, and some inversion algorithm logic. In the sequel, we develop the structure of the operator $\mathcal{O}_k$ and based on that, a procedure for implementation of hybrid observer is outlined in Algorithm 5. It is then shown in Theorem 7.13 that the state estimates computed according to the parameter bounds in Algorithm 5 indeed converge to the actual state of the system.

For the construction of observer, let us introduce the error term $\tilde{z} := \hat{z} - Px$; then the error dynamics due to (7.40a) and (7.50) are described by

$$
\begin{aligned}
\dot{\tilde{z}}(t) &= \dot{\hat{z}}(t) - P\dot{x}(t) = F_k\hat{z}(t) - P(A_k + L_kC_k)x(t) \\
&= F_k\tilde{z}(t), \quad t \neq t_k, & (7.51\text{a}) \\
\tilde{z}(t_k) &= \tilde{z}(t_k^-) - \xi_k(t_k^-). & (7.51\text{b})
\end{aligned}
$$

For each $q \in \{1, \ldots, p\}$, we pick H_q such that $\mathcal{W}_q^* = \langle \ker H_qC_q | A_q + L_qC_q \rangle$, so that the space $\mathbb{R}^n/\mathcal{W}_q^*$ is observable under mode q if we choose the output for that mode to be $H_qy(t) = H_qC_qx(t)$. This motivates us to introduce the output estimation error $\tilde{y}(t) = \overline{R}_q\hat{z}(t) - H_qy(t)$, where the matrix $\overline{R}_q$ is chosen such that $\overline{R}_qP = H_qC_q$ which is possible because $\ker P = \mathcal{W}^* \subseteq \ker H_qC_q = \ker C_q + \mathcal{W}_q^*$ (see Appendix B). It is observed that

$$
\tilde{y}(t) = \overline{R}_q\hat{z}(t) - H_qy(t) = \overline{R}_q\hat{z}(t) - H_qC_qx(t) = \overline{R}_q\hat{z}(t) - \overline{R}_qPx(t) = \overline{R}_q\tilde{z}(t). \qquad (7.52)
$$

Denote by $\overline{w}^q$ the observable component of the error dynamics of mode q, and let $P_q : \mathbb{R}^n/\mathcal{W}^* \to \mathbb{R}^n/\mathcal{W}_q^*$ be a matrix with orthonormal row vectors such that $\overline{w}^q = P_q\tilde{z}$, so that P_q is the canonical projection. Define S_q as the map induced by F_q on $\mathbb{R}^n/\mathcal{W}_q^*$, i.e., $S_qP_q = P_qF_q$. Also, define R_q such that $\overline{R}_q = R_qP_q$, (e.g. let $R_q = \overline{R}_qP_q^\top$, where $P_q^\top$ is the right inverse of P_q because P_q is an orthogonal matrix with full row-rank). Clearly, (S_q, R_q) is an observable pair. Thus, for the interval $[t_{k-1}, t_k)$, we obtain

$$\dot{\overline{w}}^k = P_k\dot{\tilde{z}} = P_kF_k\tilde{z} = S_kP_k\tilde{z} = S_k\overline{w}^k, \qquad \overline{w}^k(t_{k-1}) = P_k\tilde{z}(t_{k-1}), \tag{7.53a}$$

$$\tilde{y} = \overline{R}_k\tilde{z} = R_kP_k\tilde{z} = R_k\overline{w}^k. \tag{7.53b}$$

Since $\overline{w}^k$ is observable over the interval $[t_{k-1}, t_k)$, a standard Luenberger observer, whose role is to estimate $\overline{w}^k(t_k^-)$ at the end of the interval, is designed as:

$$\dot{\hat{\overline{w}}}^k = S_k\hat{\overline{w}}^k + K_k(\tilde{y} - R_k\hat{\overline{w}}^k), \quad t \in [t_{k-1}, t_k), \tag{7.54a}$$

$$\hat{\overline{w}}^k(t_{k-1}) = 0, \tag{7.54b}$$

where K_k is a matrix such that $(S_k - K_kR_k)$ is Hurwitz. Note that we have fixed the initial condition of the estimator to be zero for each interval.

Let us denote the vector $[\tau_{i+1}, \cdots, \tau_j]$ simply by $\tau_{\{i+1,j\}}$ (where $j > i$), which will be often dropped when used as an argument for succinct presentation. With $j > i$, define the state-transport matrix

$$\Psi_i^j(\tau_{\{i+1,j\}}) := e^{F_j\tau_j}e^{F_{j-1}\tau_{j-1}}\cdots e^{F_{i+1}\tau_{i+1}}, \tag{7.55}$$

and for convenience $\Psi_i^i := I$. We now consider the insertion maps $W_i : \mathcal{W}_i^*/\mathcal{W}^* \to \mathbb{R}^n/\mathcal{W}^*$ with orthonormal column vectors which form the basis of $\mathcal{W}_i^*/\mathcal{W}^*$, and then define a matrix $\Theta_i^k(\tau_{\{i+1,k\}})$ whose columns form a basis of the subspace $\mathcal{R}(\Psi_i^k(\tau_{\{i+1,k\}})W_i)^\perp$; that is,

$$\mathcal{R}(\Theta_i^k(\tau_{\{i+1,k\}})) = \mathcal{R}(\Psi_i^k(\tau_{\{i+1,k\}})W_i)^\perp, \quad i = k-N, \cdots, k.$$

By construction, each column of Θ_i^k is orthogonal to the subspace $\mathcal{W}_i^*$ that has been transported from t_i^- to t_k^- along the error dynamics (7.51a). This matrix Θ_i^k will be used for filtering out the unobservable component in the state estimate obtained from the mode i after being transported to the time t_k^-. As a convention, we take Θ_i^k to be a null matrix whenever $\mathcal{R}(\Psi_i^k(\tau_{\{i+1,k\}})W_i)^\perp = \{0\}$. From Lemma 7.12, it follows that the matrix

$$\Theta_k := [\Theta_k^k \vdots \cdots \vdots \Theta_{k-N}^k] \tag{7.56}$$

has rank n. Equivalently, $\Theta_k^\top$ has n independent columns and is left-invertible, so that

$(\Theta_k^\top)^\dagger = (\Theta_k \Theta_k^\top)^{-1}\Theta_k$, where $\dagger$ denotes the left-pseudo-inverse. Introduce the notation

$$\begin{aligned} \xi^-_{\{k-N,k-1\}} &:= \mathrm{col}(\xi_{k-N}(t^-_{k-N}), \dots, \xi_{k-1}(t^-_{k-1})), \\ \hat{\overline{w}}^-_{\{k-N,k\}} &:= \mathrm{col}(\hat{\overline{w}}^{k-N}(t^-_{k-N}), \dots, \hat{\overline{w}}^{k}(t^-_{k})), \end{aligned} \tag{7.57}$$

and define the vector Ξ_k as follows:

$$\Xi_k(\hat{\overline{w}}^-_{\{k-N,k\}}, \xi^-_{\{k-N,k-1\}}) := \begin{bmatrix} \Theta_k^{k\top} \Psi_k^k P_k^\top \hat{\overline{w}}^k(t_k^-) \\ \Theta_{k-1}^{k\top} \left(\Psi_{k-1}^k P_{k-1}^\top \hat{\overline{w}}^{k-1}(t^-_{k-1}) - \Psi^k_{k-1}\xi_{k-1}(t^-_{k-1}) \right) \\ \vdots \\ \Theta_{k-N}^{k\top} \left(\Psi_{k-N}^k P_{k-N}^\top \hat{\overline{w}}^{k-N}(t^-_{k-N}) - \sum_{l=k-N}^{k-1} \Psi^k_l \xi_l(t^-_l) \right) \end{bmatrix}.$$

We then compute $\xi_k(t_k^-)$ in (7.50c) as:

$$\xi_k(t_k^-) = (\Theta_k^\top)^\dagger \Xi_k(\hat{\overline{w}}^-_{\{k-N,k\}}, \xi^-_{\{k-N,k-1\}}), \tag{7.58}$$

which corresponds to the operator $\mathcal{O}_k$. Finally, as the last piece of notation, we define the matrices M_j^k, $j = k-N, \cdots, k$, as follows:

$$[M_k^k, M_{k-1}^k, \cdots, M_{k-N}^k] := (\Theta_k^\top)^\dagger \times \mathrm{blockdiag}\left(\Theta_k^{k\top}\Psi_k^k, \Theta_{k-1}^{k\top}\Psi_{k-1}^k, \cdots, \Theta_{k-N}^{k\top}\Psi_{k-N}^k \right). \tag{7.59}$$

Each M_j^k ($j = k-N, \cdots, k$) is a d_z by d_z matrix whose argument is $\tau_{\{k-N+1,k\}}$ in general (due to the inversion of $\Theta_k^\top$), while the argument of both Θ_j^k and Ψ_j^k is $\tau_{\{j+1,k\}}$.

With all the quantities defined so far, the proposed observer (7.50) is implemented according to Algorithm 5.

The following theorem shows that the implementation in Algorithm 5 guarantees the convergence of the estimation error to zero.

Theorem 7.13. *Assume that, for each q in $\{1, \dots, p\}$, L_q is chosen such that $\|F_q\| \le a$; then* (7.49) *holds for almost all $\sigma \in \mathcal{S}$, and for such $\sigma \in \mathcal{S}$ the observer given by* (7.50) *and implemeted according to Algorithm 5 generates a converging estimate for $z = Px$, $|\hat{z} - z| \to 0$, where x evolves according to system* (7.41)*, and $P : \mathbb{R}^n \to \mathbb{R}^n/\mathcal{W}^*$ is the canonical projection.*

Proof. Using (7.51), it follows for $\sigma \in \mathcal{S}$ that the estimation error $\tilde{z}(t)$ for the interval $[t_k, t_{k+1})$ is bounded by

$$|\tilde{z}(t)| = |e^{F_{k+1}(t-t_k)}\tilde{z}(t_k)| \le e^{a(t-t_k)}|\tilde{z}(t_k)|,$$

Algorithm 5: Implementation of hybrid observer

Input : σ, u, v, y.

Initialization: Run (7.50) for $t \in [t_0, t_{N+1})$ with some $\hat{z}(t_0)$.

foreach $k \geq N+1$ **do**

  for $j = k - N$ **to** k **do**

    Compute the injection gain K_j such that

$$\|M_j^k(\tau_{\{k-N+1,k\}})P_j^\top e^{(S_j - K_j R_j)\tau_j} P_j\| \leq c \tag{7.60}$$

    where the constant c is chosen so that

$$0 < c < \frac{1}{N+1}. \tag{7.61}$$

    Obtain $\hat{\overline{w}}^j(t_j^-)$ by running the individual observer (7.54) for the j-th mode.

  Compute $\xi_k(t_k^-)$ from (7.58), as an implementation of (7.50c).

  Compute $\hat{z}(t_k)$ using (7.50b) and run (7.50a) over the interval $[t_k, t_{k+1})$.

---

where a is a constant such that $\|F_k\| \leq a$ for all $F_k \in \{F_1, \ldots, F_p\}$, and thus,

$$|\tilde{z}(t)| \leq e^{aT_D} |\tilde{z}(t_k)|.$$

Therefore, if $|\tilde{z}(t_k)| \to 0$ as $k \to \infty$, then we achieve that

$$\lim_{t \to \infty} |\tilde{z}(t)| = 0. \tag{7.62}$$

The remainder of the proof shows that $|\tilde{z}(t_k)| \to 0$ as $k \to \infty$ under the conditions stated in the theorem statement.

Note that, if $w^q := W_q^\top \tilde{z}$ denotes the unobservable component of $\tilde{z}$ under mode q, then $\tilde{z}(t_k^-)$ can be written as

$$\tilde{z}(t_k^-) = \begin{bmatrix} P_k \\ W_k^\top \end{bmatrix}^{-1} \begin{bmatrix} \overline{w}^k(t_k^-) \\ w^k(t_k^-) \end{bmatrix} = P_k^\top \overline{w}^k(t_k^-) + W_k w^k(t_k^-). \tag{7.63}$$

The matrix $\Psi_i^j(\tau_{\{i+1,j\}})$, defined in (7.55), transports $\tilde{z}(t_i^-)$ to $\tilde{z}(t_j^-)$ along (7.51a) by

$$\tilde{z}(t_j^-) = \Psi_i^j(\tau_{\{i+1,j\}})\tilde{z}(t_i^-) - \sum_{l=i}^{j-1} \Psi_l^j(\tau_{\{l+1,j\}})\xi_l(t_l^-). \tag{7.64}$$

We now have the following series of equivalent expressions for $\tilde{z}(t_k^-)$:

$$
\begin{aligned}
\tilde{z}(t_k^-) &= P_k^\top \overline{w}^k(t_k^-) + W_k w^k(t_k^-) \\
&= \Psi_{k-1}^k P_{k-1}^\top \overline{w}^{k-1}(t_{k-1}^-) + \Psi_{k-1}^k W_{k-1} w^{k-1}(t_{k-1}^-) - \Psi_{k-1}^k \xi_{k-1}(t_{k-1}^-) \\
&= \Psi_{k-2}^k P_{k-2}^\top \overline{w}^{k-2}(t_{k-2}^-) + \Psi_{k-2}^k W_{k-2} w^{k-2}(t_{k-2}^-) - \Psi_{k-2}^k \xi_{k-2}(t_{k-2}^-) - \Psi_{k-1}^k \xi_{k-1}(t_{k-1}^-) \\
&\;\;\vdots \\
&= \Psi_{k-N}^k P_{k-N}^\top \overline{w}^{k-N}(t_{k-N}^-) + \Psi_{k-N}^k W_{k-N} w^{k-N}(t_{k-N}^-) - \sum_{l=k-N}^{k-1} \Psi_l^k \xi_l(t_l^-).
\end{aligned}
\tag{7.65}
$$

To appreciate the implication of this equivalence, we first note that for each $k-N \leq i \leq k$, the term $\Psi_i^k P_i^\top \overline{w}^i(t_i^-)$ transports the observable information of the i-th mode from the interval $[t_{i-1}, t_i)$ to the time instant t_k^-. This observable information is corrupted by the unknown term $w^i(t_i^-)$, but since the information is being accumulated at t_k^- from modes $i = k-N, \cdots, k$, the idea is to combine the partial information from each mode to recover $\tilde{z}(t_k^-)$. This is done by making use of the Lemma 7.12. This lemma shows that the matrix Θ_k defined in (7.56) has rank $n - d_w$, and is left-invertible. Keeping in mind that the range space of each Θ_i^k is orthogonal to $\mathcal{R}(\Psi_i^k W^i)$, each equality in (7.65) leads to the following relation:

$$
\Theta_i^{k\top} \tilde{z}(t_k^-) = \Theta_i^{k\top} \left(\Psi_i^k P_i^\top \overline{w}^i(t_i^-) - \sum_{l=i}^{k-1} \Psi_l^k \xi_l(t_l^-) \right), \tag{7.66}
$$

for $i = k-N, \cdots, k$. Stacking (7.66) from $i = k$ to $i = k-N$, and employing the left-inverse of $\Theta_k^\top$, we obtain that

$$
\tilde{z}(t_k^-) = (\Theta_k^\top)^\dagger \Xi_k(\overline{w}_{\{k-N,k\}}^-, \xi_{\{k-N,k-1\}}^-), \tag{7.67}
$$

where $\overline{w}_{\{k-N,k\}}^-$ is defined similarly as in (7.57). It is seen from (7.67) that, if we are able to estimate $\overline{w}_{\{k-N,k\}}^-$ without error, then the plant state $z(t_k^-)$ is exactly recovered by (7.67) because $z(t_k^-) = \hat{z}(t_k^-) - \tilde{z}(t_k^-)$ and both entities on the right side of the equation are known. However, since this is not the case, $\overline{w}_{\{k-N,k\}}^-$ has been replaced with its estimate $\hat{\overline{w}}_{\{k-N,k\}}^-$ in (7.58), and $\xi_k(t_k^-)$ is set as an estimate of $\tilde{z}(t_k^-)$ there.

Thanks to the linearity of Ξ_k in its arguments, it is noted that

$$\begin{aligned}\tilde{z}(t_k) &= \tilde{z}(t_k^-) - \xi_k(t_k^-) \\ &= (\Theta_k^\top)^\dagger \Big(\Xi_k(\overline{w}_{\{k-N,k\}}^-, \xi_{\{k-N,k-1\}}^-) - \Xi_k(\hat{\overline{w}}_{\{k-N,k\}}^-, \xi_{\{k-N,k-1\}}^-) \Big) \\ &= -(\Theta_k^\top)^\dagger \Xi_k(\tilde{\overline{w}}_{\{k-N,k\}}^-, 0),\end{aligned} \tag{7.68}$$

where $\tilde{\overline{w}}_{\{k-N,k\}}^- := \hat{\overline{w}}_{\{k-N,k\}}^- - \overline{w}_{\{k-N,k\}}^- = \mathrm{col}(\tilde{\overline{w}}^{k-N}(t_{k-N}^-), \ldots, \tilde{\overline{w}}^k(t_k^-))$. It follows from (7.53) and (7.54) that

$$\tilde{\overline{w}}^i(t_{i-1}) = \hat{\overline{w}}^i(t_{i-1}) - \overline{w}^i(t_{i-1}) = 0 - P_i \tilde{z}(t_{i-1}),$$

and

$$\tilde{\overline{w}}^i(t_i^-) = e^{(S_i - K_i R_i)\tau_i} \tilde{\overline{w}}^i(t_{i-1}) = -e^{(S_i - K_i R_i)\tau_i} P_i \tilde{z}(t_{i-1}).$$

Plugging this expression in (7.68), and using the definition of M_j^k ($j = k-N, \ldots, k$) from (7.59), we get

$$\tilde{z}(t_k) = \sum_{j=k-N}^{k} M_j^k(\tau_{\{k-N+1,k\}}) P_j^\top e^{(S_j - K_j R_j)\tau_j} P_j \tilde{z}(t_{j-1}).$$

Then, from the selection of gains K_j's satisfying (7.60), we have that

$$|\tilde{z}(t_k)| \leq \sum_{j=k-N}^{k} c|\tilde{z}(t_{j-1})|. \tag{7.69}$$

Finally, the statement of the following lemma aids us in the completion of the proof of Theorem 7.13. Applying Lemma 7.14 to (7.69), we see that $|\tilde{z}(t_k)| \to 0$ as $k \to \infty$, whence the desired result follows. □

Lemma 7.14. *A sequence* $\{a_i\}$ *satisfying*

$$|a_i| \leq c(|a_{i-1}| + |a_{i-2}| + \cdots + |a_{i-N-1}|), \qquad i > N,$$

with $0 \leq c < 1/(N+1)$ *converges to zero:* $\lim_{i\to\infty} a_i = 0$.

Proof. Let $c = \alpha/(N+1)$ with $0 < \alpha < 1$. Then it is obvious that, for $i > N$,

$$|a_i| \leq \frac{\alpha}{N+1} \sum_{k=i-N-1}^{i-1} |a_k| \leq \alpha \max_{i-N-1 \leq k \leq i-1} |a_k|. \tag{7.70}$$

Similarly, it follows that

$$
\begin{aligned}
|a_{i+1}| &\leq \alpha \max_{i-N\leq k\leq i} |a_k| \\
&\leq \alpha \max \left\{ |a_{i-N-1}|, \max_{i-N\leq k\leq i-1} |a_k|, |a_i| \right\} \\
&\leq \alpha \max_{i-N-1\leq k\leq i-1} |a_k|,
\end{aligned}
$$

where the last inequality follows from (7.70). By induction, this leads to

$$
\max_{i\leq k\leq i+N} |a_k| \leq \alpha \max_{i-N-1\leq k\leq i-1} |a_k|.
$$

That is, the maximum value of the sequence $\{a_i\}$ over the length of window $N+1$ is strictly decreasing and converging to zero, which proves the desired result. □

7.2.4 Conditions for Fault Detection

With our novel definitions for unobservability subspaces for switched systems and the observer design proposed in the previous section, we can extend the ideas from [155] at an abstract level to develop a similar condition for the solution of fault detection problem. It is shown that the observer (7.50) along with the residual signal

$$
r_1(t) := \overline{R}_k \hat{z}(t) - H_k y(t) \tag{7.71}
$$

serves as a filter for detecting the fault $m_1(t)$ under certain structural conditions. In the sequel, we first discuss a sufficient condition for exact fault detection so that $r_1(t)$ is sensitive to all real valued functions $m_1(t)$, followed by a weaker condition for generic fault detection so that the residual $r_1(t)$ is sensitive to almost every real-valued function $m_1(t)$.

Exact Fault Detection

The condition given in the following theorem guarantees that (7.50) and (7.71) generate a system that detects every possible fault $m_1(t)$.

Theorem 7.15. *Let $\mathcal{W}_q^*(\mathcal{J}_2)$ denote the smallest unobservable subspace of mode q containing* *the conditioned-invariant subspace $\mathcal{V}^*(\mathcal{J}_2)$; then the fault detection problem is solvable if*

$$\mathcal{W}_q^*(\mathcal{J}_2) \cap \mathcal{J}_1 = \{0\}, \qquad \textit{for all} \quad q = 1, \ldots, p, \tag{7.72}$$

i.e., the fault signature $\mathcal{J}_1$ intersects trivially with the smallest unobservable subspace containing $\mathcal{V}^*(\mathcal{J}_2)$ for each mode.

Proof. Consider the observer given in (7.50). If, due to the presence of faults, the system dynamics follow the equations given in (7.41), then instead of (7.51) the error dynamics $\tilde{z} := \hat{z} - Px$ are given by:

$$\begin{aligned} \dot{\tilde{z}} &= F_k \hat{z} - P(A_k - L_k C_k)x - PJ_1 m_1(t) - PJ_2 m_2(t), \\ &= F_k \tilde{z} - PJ_1 m_1(t), \end{aligned} \tag{7.73}$$

where the second equality follows from the fact that the range of J_2 is contained in $\mathcal{W}^*(\mathcal{J}_2)$ and P is an orthogonal projection on $\mathbb{R}^n/\mathcal{W}^*(\mathcal{J}_2)$, so that $PJ_2 = 0$. Also, $\mathcal{W}^*(\mathcal{J}_2) \subseteq \mathcal{W}_k^*(\mathcal{J}_2)$ for each k, and because of (7.72), it follows that $PJ_1 \in \mathcal{J}_1$. The residual signal is rewritten as:

$$r_1(t) = \overline{R}_k \hat{z}(t) - H_k y(t) = \tilde{y}(t) = \overline{R}_k \tilde{z}(t), \tag{7.74}$$

and we claim that $r_1(t)$ is sensitive to $m_1(t)$. Assume for the moment that $\left\langle \ker \overline{R}_q | F_q \right\rangle \subseteq \mathcal{W}_q^*$; then $\left\langle \ker \overline{R}_q | F_q \right\rangle \cap \mathcal{J}_1 = \{0\}$, which in turn implies that the largest (F_q, J_1)-invariant subspace contained in $\ker \overline{R}_q$ is $\{0\}$ (see Appendix B). Hence, for each mode q, the mapping $m_1 \mapsto r_1$ is invertible, which is the desired result.

It remains to show that $\left\langle \ker \overline{R}_q | F_q \right\rangle \subseteq \mathcal{W}_q^*$. This follows from the following set of inclusions:

$$\begin{aligned} \left\langle \ker \overline{R}_q | F_q \right\rangle &= \ker \overline{R}_q \cap \ker \overline{R}_q F_q \cap \cdots \cap \ker \overline{R}_q F_q^{n-1} \\ &= P(\ker H_q C_q) \cap P(\ker H_q C_q (A_q + L_q C_q)) \cap \cdots \cap P(\ker H_q C_q (A_q + L_q C_q)^{n-1}) \\ &\subseteq \ker H_q C_q \cap \ker H_q C_q (A_q + L_q C_q) \cap \cdots \cap \ker H_q C_q (A_q + L_q C_q)^{n-1} \\ &= \langle \ker H_q C_q | A_q + L_q C_q \rangle = \mathcal{W}_q^*. \end{aligned}$$

Since $\tilde{z}(t) \to 0$ when $m_1(t) \equiv 0$ from our observer design, the condition (7.72) in theorem statement indeed guarantees that nonzero values of $m_1(\cdot)$ generate nonzero values $r_1(\cdot)$. $\square$

Generic Fault Detection

The condition given in Theorem 7.15 for exact fault detection basically requires each subsystem to be invertible with respect to the designed output r_1. This way r_1 can detect the presence of all possible non-zero real-valued functions $m_1(\cdot)$. However, in general the faults that can hide themselves from the output are very few and are of specific form. If those kinds of functions are excluded from consideration, then weaker conditions can be derived for detecting generic faults. Towards this end, a weaker condition is proposed in the following theorem and in the proof we discuss the kinds of faults that are excluded for generic fault detection.

Theorem 7.16. *Consider the system* (7.41) *for which the observer* (7.50) *is designed. The set of faults m_1, which is detectable through the residual signal r_1 in* (7.71)*, forms a dense subset in the space of all real-valued measurable functions over* $[t_0, \infty)$ *if*

$$\mathcal{W}^*(\mathcal{J}_2) \cap \mathcal{J}_1 = \{0\}. \tag{7.75}$$

Sketch of Proof. For the system of equations given by (7.73) and (7.74), we need to show that the set of functions $m_1(\cdot)$ for which $r_1(\cdot) \equiv 0$ is nowhere dense. We consider two cases:

Case 1: There exists a mode q^* such that $\mathcal{W}^*_{q^*}(\mathcal{J}_2) \cap \mathcal{J}_1 = \{0\}$. In that case, if a non-zero $m_1(\cdot)$ generates identically zero $r_1(\cdot)$ and mode q^* is activated during the interval $[t_k, t_{k+1})$, then it must be true that $m_{1_{[t_k,t_{k+1})}} \equiv 0$. Furthermore, m_1 must bring $\tilde{z}(t_k)$ to zero because the set of states that generate $r_{1_{[t_k,t_{k+1})}} \equiv 0$ have trivial intersection with the set of states that generate zero r_1 under other modes. Clearly, such inputs form a nowhere dense set.

Case 2: $\cap_{q=1}^{p} \mathcal{W}^*_q(\mathcal{J}_2) \cap \mathcal{J}_1 = \{0\}$. In this case, if a dense set of inputs produces identically zero r_1, then it must be true that for each mode $q \in \{1, \dots, p\}$, the smallest $(A_q, \mathcal{J}_1)$ invariant subspace is contained in $\cap_{q=1}^{p} \mathcal{W}^*_q(\mathcal{J}_2)$. In particular, for each q, there is an A_q-invariant subspace $\cap_{q=1}^{p} \mathcal{W}^*_q(\mathcal{J}_2)$ whose intersection with $\mathcal{J}_1$ is nontrivial. This is a contradiction to (7.75).

Since $\mathcal{J}_1$ is assumed to have dimension one, only one of the two cases can hold, and we showed that fault detection is possible for generic faults in the above two cases. □

Remark 7.17. In case there are more than two faults, say $m_1, m_2, \dots, m_k$, then one can apply the same techniques to construct residual signals that are sensitive to one fault and remain unaffected by other faults. For example, to design a fault detection filter for m_1, we replace $\mathcal{J}_2$ by $\sum_{i=2}^{k} \mathcal{J}_k$ and look at the unobservable subspaces containing $\sum_{i=2}^{k} \mathcal{J}_k$. Thus, the condition (7.72) is replaced by

$$\mathcal{W}_q^*(\mathcal{J}_2) \cap \mathcal{J}_1 = \{0\}, \qquad \text{for all} \quad q = 1, \dots, p, \tag{7.76}$$

and (7.75) is replaced by

$$\mathcal{W}^*(\mathcal{J}_2) \cap \mathcal{J}_1 = \{0\}. \tag{7.77}$$

Similar changes are made in the construction of the observer as well. ◁

7.2.5 Case-Studies

Example 7.18. Consider the boost converter of Fig. 7.12 where we assume both i_L and v_C can be measured, and the voltage v_s is perfectly known. Letting $x = [i_L \; v_C]^\top$, then the two modes of operation of this converter are

$$\Gamma_1 : \left\{ \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s, \right. \tag{7.78}$$

which corresponds to the case when S_1 is closed and D is open, and

$$\Gamma_2 : \left\{ \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s, \right. \tag{7.79}$$

which corresponds to the case when S_1 is open, and the diode D is conducting. We will assume that the switching signal is not available to the FDI system.

We can assume that as the capacitor degrades, its capacitance will decrease. Thus, without loss of generality, the capacitance of the capacitor can be described as $C(t) = C + \lambda_C(t)$, where C is the nominal capacitance, with $\lambda_C(t)$ describing the fault magnitude. Similarly, the inductance of the inductor can be described as $L(t) = L + \lambda_L(t)$, where L is the nominal

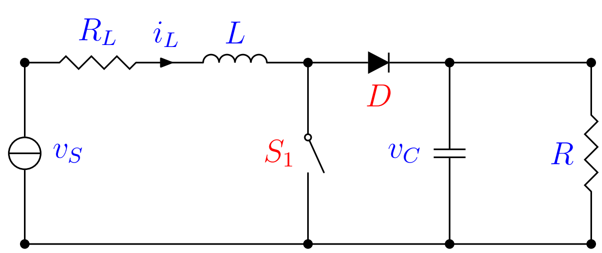


Figure 7.12: A boost converter.

inductance, and $\lambda_L(t)$ describes the fault magnitude. Then, the system dynamics can be described in a more general form to account for these faults as follows:

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \phi_C(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \phi_L(t), \end{cases} \tag{7.80}$$

$$\Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} -\frac{R_L}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \phi_C(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \phi_L(t), \end{cases} \tag{7.81}$$

where $\phi_C(t)$, $\phi_L(t)$ basically represent the unknown degradation in the value of the capacitor and inductor, respectively, and have the following expressions (see [165] for details):

$$\phi_C(t) := \begin{cases} \frac{1}{C+\lambda_C(t)}\big(\frac{\lambda_C(t)}{RC} - \frac{d\lambda_C(t)}{dt}\big) v_C(t) & \text{if } \sigma(t) = 1, \\ \frac{1}{C+\lambda_C(t)}\Big(\frac{-\lambda_C(t)}{C} i_L(t) + \big(\frac{\lambda_C(t)}{RC} - \frac{d\lambda_C(t)}{dt}\big) v_C(t)\Big) & \text{if } \sigma(t) = 2, \end{cases}$$

and

$$\phi_L(t) := \begin{cases} \frac{1}{L+\lambda_L(t)}\big((\frac{R_L}{L}\lambda_L(t) - \frac{d\lambda_L(t)}{dt}) i_L - \frac{\lambda_L(t)}{L} v_s\big) & \text{if } \sigma(t) = 1, \\ \frac{1}{L+\lambda_L(t)}\big((\frac{R_L}{L}\lambda_L(t) - \frac{d\lambda_L(t)}{dt}) i_L + \frac{\lambda_L(t)}{L}(v_C - v_s)\big) & \text{if } \sigma(t) = 2. \end{cases}$$

Let $J_1 = [1\ 0]^\top$ be the signature flag for inductor fault and $J_2 = [0\ 1]^\top$ correspond to the capacitor fault. To better illustrate the theory, we consider different cases of observation matrices.

Case 1: First consider the simple case, where $C_1 = C_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. We show that it is possible to detect faults both in the inductor and capacitor. Note that $\mathcal{W}_1^*(\mathcal{J}_1) = \mathcal{W}_2^*(\mathcal{J}_1) = \mathcal{J}_1$, and $\mathcal{W}_1^*(\mathcal{J}_2) = \mathcal{W}_2^*(\mathcal{J}_2) = \mathcal{J}_2$. It now follows from the conditioned-invariance subspace algorithm that:

$$\mathcal{V}^*(\mathcal{J}_2) = \mathcal{W}^*(\mathcal{J}_2) = \mathcal{J}_2,$$

and therefore,

$$\mathcal{J}_1 \cap \mathcal{W}^*(\mathcal{J}_2) = \{0\}.$$

Thus, it is possible to construct residual generator that is sensitive to J_1, i.e., faults in the inductor. Also, it can be verified that

$$\mathcal{V}^*(\mathcal{J}_1) = \mathcal{W}^*(\mathcal{J}_1) = \mathcal{J}_1,$$

so that

$$\mathcal{J}_2 \cap \mathcal{W}^*(\mathcal{J}_1) = \{0\},$$

and we can construct another residual generator that responds to faults corresponding to J_2, i.e., the capacitor. The construction of residual generator now follows:

For r_1, pick H_1^1 such that $\ker H_1^1 C_1 = \ker C_1 + \mathcal{W}^*(\mathcal{J}_2)$, and H_2^1 such that $\ker H_2^1 C_2 = \ker C_2 + \mathcal{W}^*(\mathcal{J}_2)$. One may pick $H_1^1 = H_2^1 = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$.

For r_2, pick H_1^2 such that $\ker H_1^2 C_1 = \ker C_1 + \mathcal{W}^*(\mathcal{J}_1)$, and H_2^2 such that $\ker H_2^2 C_2 = \ker C_2 + \mathcal{W}^*(\mathcal{J}_1)$. One may pick $H_1^2 = H_2^2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

Case 2: Next, we consider the case where $C_1 = C_2 = [0\ 1]$. In this case, it is not possible to detect a fault either in the inductor or the capacitor. Note that $\mathcal{V}^*(\mathcal{J}_1) = \mathbb{R}^2$, and $\mathcal{V}^*(J_2) = \mathcal{J}_2$. Also, $\mathcal{W}_1^*(\mathbb{R}^2) = \mathcal{W}_1^*(\mathcal{J}_2) = \mathcal{W}_2^*(\mathcal{J}_1) = \mathcal{W}_2^*(\mathcal{J}_2) = \mathbb{R}^2$.

Since $\mathcal{W}^*(\mathcal{J}_2) = \mathbb{R}^2$, we have $\mathcal{J}_1 \cap \mathcal{W}^*(J_2) = \mathcal{J}_1 \neq \{0\}$, so fault detection in the inductor is not possible.

Also, $\mathcal{W}^*(\mathcal{V}^*(\mathcal{J}_1)) = \mathbb{R}^2$, so that $\mathcal{J}_2 \cap \mathcal{W}^*(\mathcal{V}^*(\mathcal{J}_1)) = \mathcal{J}_2 \neq \{0\}$, and hence the faults occurring in the capacitor cannot be detected either.

Case 3: Now consider the case where $C_1 = [1\ 0]$, $C_2 = [0\ 1]$; we show that it is possible to detect faults in the inductor only. To see this, we compute $\mathcal{V}^*(\mathcal{J}_1) = \mathbb{R}^2$, and $\mathcal{W}^*(\mathbb{R}^2) = \mathbb{R}^2$, so that $\mathcal{J}_2 \cap \mathcal{W}^*(\mathcal{V}^*(\mathcal{J}_1)) = \mathcal{J}_2 \neq \{0\}$ which implies that the fault in the capacitor cannot be detected. However, $\mathcal{V}^*(\mathcal{J}_2) = \mathcal{J}_2$, and $\mathcal{W}^*(\mathcal{J}_2) = \mathcal{J}_2$ which in turn gives $\mathcal{J}_1 \cap \mathcal{W}^*(\mathcal{V}^*(\mathcal{J}_2)) = \{0\}$. Hence, the faults in the inductor that appear in the system along $\mathcal{J}_1$ can be detected. If we change the observation matrices to be $C_1 = [0\ 1]$, $C_2 = [1\ 0]$, then, using similar computation, it can be shown that the faults in the capacitor could be detected but the ones in the inductor cannot be detected.

The last case is of particular interest because it suggests that by processing information from one sensor at a time, we can detect a fault either in the capacitor or the inductor. The simultaneous processing of multiple measurements requires more computational power, and by processing each measurement one at a time we can lighten the computational burden while achieving the goal of fault detection. Moreover, if we combine the two subcases in Case 3, it is seen that we only need to run a 1-dimensional observer and this is again a much better option in practice. ◁

Example 7.19. As another case-study, we consider a network of boost converters. A Simulink schematic of such a circuit is given in Fig. 7.13. The network comprises three boost converters, and since each of them can operate in two modes, there are $2^3 = 8$ modes of operation in total.

If we measure the current and voltage for each boost converter in the network, then it is seen that the soft faults in each inductor and capacitor can be detected generically.

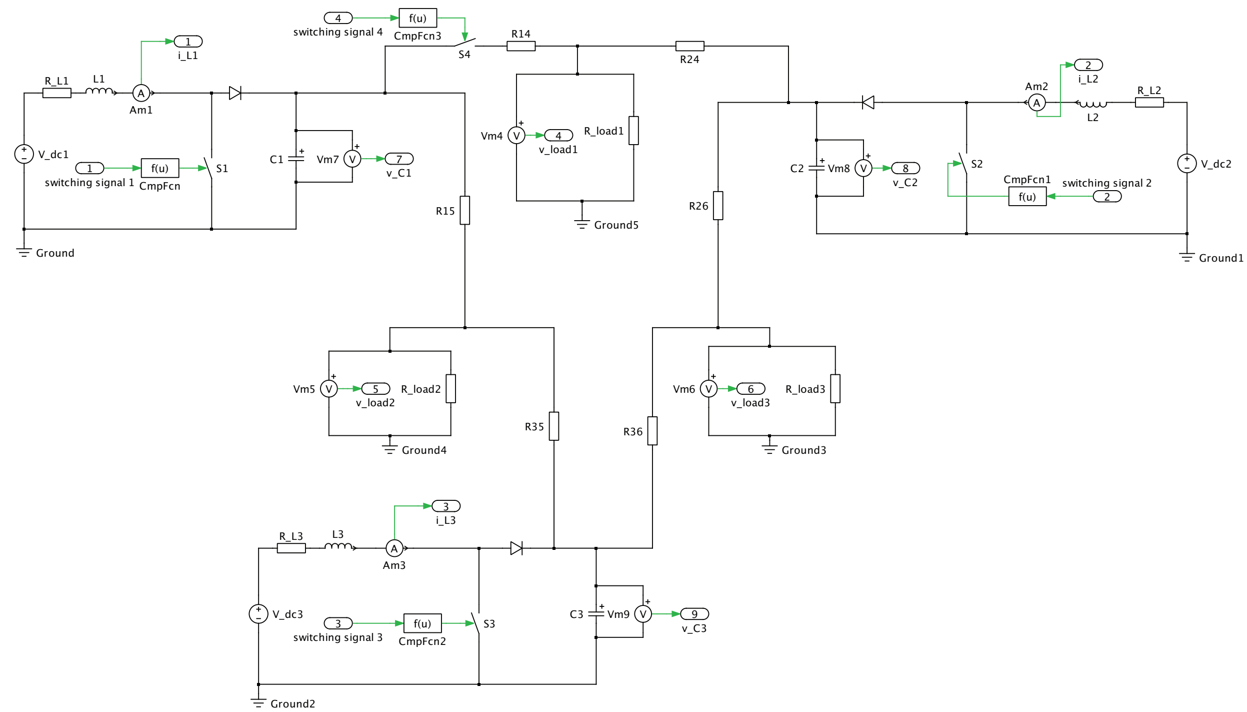


Figure 7.13: A network of boost converters.

For example, if m_1 denotes the faults in the inductor of the first boost converter, then $J_1 := [1\ 0\ 0\ 0\ 0\ 0]^\top$. Similarly, for other faults, the vector J_i has only one nonzero entry corresponding to the fault it represents. It is verified that,

$$\mathcal{W}^* \left(\sum_{i=2}^{5} \mathcal{J}_i \right) \cap \mathcal{J}_1 = \{0\},$$

so that an observer could be designed with a residual generator sensitive to m_1 only. In general for this system, for each $k \in \{1, \ldots, 6\}$, we have

$$\mathcal{W}^* \left(\sum_{i=1, i \neq k}^{6} \mathcal{J}_i \right) \cap \mathcal{J}_k = \{0\},$$

where $\mathcal{J}_k$ denotes the direction of fault either in the capacitor or the inductor of one of the constituent boost converters. $\triangleleft$

7.3 Comparison

The method of the inversion-based FDI relies on differentiation of the outputs, so the robustness against output measurement noise is imperative. Also, the need for accurate fault impact models could potentially hinder the effectiveness of the method. The inversion-based FDI method is naturally suited for a distributed implementation, which can be accomplished by appropriately breaking the system into smaller interconnected systems. Unlike observer-based methods, where a bank of filters is required (each sensitive to a particular fault), with the inversion-based method, it is unnecessary to have each faulty system model running concurrently. To illustrate, using the system of Fig. 7.8, the system could be broken into three interconnected subsystems, each roughly composed of a source, a load and the lines linking these components. Each subsystem will have additional inputs (possibly unknown) resulting from the interconnection with other subsystems. In this scenario, it is possible to undertake FDI locally using the inversion approach. This would also allow the recovery of those possibly unknown inputs introduced when partitioning the system. Additionally, if the individual subsystems are invertible, these unknown inputs can also be uniquely recovered. This would be of interest if such unknown inputs cannot be measured for reconfiguration strategies in the system operation.

Even though the invertibility techniques help detect hard faults when the system configuration changes, and yield greater information about the soft faults as they can be recovered exactly, it does require the knowledge of initial condition and derivatives of the output. Also, in most applications, the recovery of exact magnitude of faults is not desirable and only their occurrence needs to be detected. To overcome these shortcomings, the observability techniques may be preferred at times. Investigating this application in detail would surely provide more grounds to compare the two techniques.

It is natural to study the problem of fault detection and isolation for nonlinear switched systems as has been done in [166] for actuator faults. The framework proposed in this chapter can be extended to nonlinear systems in a conceptually similar manner using the results on invertibility and observability of nonlinear switched systems. In this regard, preliminary work based on the utilization of the results in [67] to FDI in nonlinear systems is illustrated in [167], with specific application to detecting faults in transmission lines of electric power systems.

Chapter 8

Conclusions and Future Work

8.1 Future Work: Synthesis Problems

The most significant aspect of the information-extracting structural properties, in addition to being theoretically rich, is their utility in solving some of the prominent design problems. Some of the earlier work on switched systems – relating to the properties of stability [28, 11, 27, 12], observability [102], and invertibility [65, 105] – addresses the problems from an analytical perspective, and in this chapter we propose problems aiming towards the transition from the analysis to system design. The observability property allows for state estimation from the measured outputs by designing appropriately an auxiliary dynamical system, called *observer*; and the concepts related to invertibility of switched systems are utilized in designing switching signals and control inputs for generating the desired output trajectories. We have already addressed the design of observers in Chapters 4 and 5, so in this chapter we address the design problems based on invertibility tools. Studying the design problems in the context of switched systems has not received much attention in the literature and this chapter develops the framework for exploring some system design problems with switching dynamics. It is noted that this chapter only provides a preliminary approach to the development of these problems and their solution requires much further research.

From the viewpoint of system design and synthesis, system inversion tools could be utilized in designing stabilizing state feedback laws, and generating inputs for exact output generation and asymptotic output tracking. For nonswitched systems, it has been shown that the dynamics of the inverse system are the same as dynamics of the actual plant when the output is constrained to zero (also called *zero dynamics*). If, using the inversion techniques, the input is now chosen to maintain the zero output, and the zero dynamics are assumed to be stable, then such inputs act as stabilizing state feedback laws for systems with stable zero dynamics (also called *minimum-phase* systems) [62, 61]. If the minimum-phase assumption is not imposed, then usually the input constructed from the inverse system may be unbounded in its norm. Thus, it becomes relevant to investigate how the bounded outputs can be produced from bounded inputs for the class of nonminimum-phase systems [64]. Another application of invertibility techniques could be seen in asymptotic output tracking. The invertibility problem setup requires the initial condition to be known; even though this knowledge produces the desired output exactly, a more practical setup demands the initial condition to be unknown. In such cases, it is useful to find methods that result in tracking the desired output because the exact output generation is no longer possible. However, with nonswitched systems, only a restricted set of outputs can be tracked as the prescribed output trajectory is required to be sufficiently smooth. Classical system dynamics are not able track outputs which exhibit discontinuities, or that are only piecewise smooth. Having already developed invertibility tools in the context of switched systems, it would now be interesting to investigate how one can design algorithms to track these richer and more general classes of outputs. Also, in the process, it may be possible to discover new stabilization techniques for switched systems. Thus, studying such problems in the context of switched systems provides a richer and more useful framework to solve the aforementioned design problems, and below we look ahead to our methodology towards developing the solution of these problems.

8.1.1 Exact Output Generation

Towards the end of Chapter 2, the problem of output generation was discussed which involved computing the input $u(\cdot)$ and the switching signal $\sigma(\cdot)$ that reproduce a desired output function $y_d(\cdot)$. In general, the computed input $u(\cdot)$ need not be bounded even when the given function $y_d(\cdot)$ is bounded. From the applications standpoint, it is desirable to have the norm of the input u (generated by the inverse system) to be bounded by some non-decreasing function of the norm of the output and its derivatives, a problem which we call *bounded output generation.*

Before we introduce the formal problem statement and related concepts, let us redefine the class of systems considered for bounded output generation problem:

$$\dot{x} = f_\sigma(x) + \sum_{i=1}^{m} g_{i,\sigma}(x)u_i, \tag{8.1a}$$

$$y = h_\sigma(x). \tag{8.1b}$$

The following definitions are fundamental to the development in this section.

Definition 8.1 (Class $\mathcal{K}, \mathcal{K}_\infty, \mathcal{KL}$ functions)**.** *A function $\chi : [0,\infty) \mapsto [0,\infty)$ is said to be of* class $\mathcal{K}$ *if it is continuous, strictly increasing, and $\chi(0) = 0$. If χ is also bounded, then it is said to be of* class $\mathcal{K}_\infty$*. A function $\beta : [0,\infty) \times [0,\infty) \mapsto [0,\infty)$ is said to be of* class $\mathcal{KL}$ *if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \in [0,\infty)$ and $\beta(r,t) \to 0$ as $t \to \infty$ for each fixed $r \in [0,\infty)$.* $\lhd$

The problem of bounded output generation can now be formally stated as follows.

Problem 8.1 (Bounded Output Generation)**.** *Assume that x_0 and $y_d(\cdot)$ are given such that the following holds for some $p \in \mathcal{P}$:*

$$\hat{y}_d(t_0) = Z_p\big(x_0, \dot{\tilde{y}}_{d_1}(t_0), \cdots, \tilde{y}_{d_k}^{(k)}(t_0)\big), \qquad \forall k = 0, 1, \cdots, \alpha_p - 1, \tag{8.2}$$

where $\hat{y}_d$ and Z are defined in (2.21) *and* (2.20)*, respectively. Find the switching signal $\sigma(\cdot)$ and the input $u(\cdot)$ that reproduce $y_d(\cdot)$ exactly as the output of the system* (8.1)*, i.e.,*

$$y = \mathrm{H}_{x_0}(\sigma, u) = y_d$$

and

$$|u(t)| \leq \beta(|x(t_0)|, t - t_0) + \chi\left(\left|Y^N(t)\right|\right), \quad \textit{for each } t \geq t_0, \tag{8.3}$$

for some $\beta \in \mathcal{KL}$, and $\chi \in \mathcal{K}_\infty$. $\lhd$

The condition (8.2) merely states that the initial value of the state and the desired output must be related in such a manner that at least one subsystem can reproduce the desired output trajectory exactly. With the implicit assumption that y_d is reproducible by the systems (8.1), we can then construct the switching signal $\sigma(\cdot)$ using the index-inversion function and the output generation algorithm in Chapter 2. The additional requirement on the size of the input in (8.3) does not always hold, and this section will focus on how this property of bounded input can be achieved.

Since we use the inverse system to compute the input u that reproduces the output, the stability of the inverse system is critical for bounded output generation. The full order inverse subsystems obtained by the structure algorithm are not stable in general. Classically, the constraints for the bounded output generation problem are imposed on the so-called *reduced order* inverse system, which may have stable or unstable dynamics. In the former case, where the state of the reduced order inverse system is stable (with zero inputs), the systems go by the name of *minimum-phase systems* in the classic control literature. In dealing with nonlinear systems, however, the minimum-phase property – which can also be called the stability of the system with output constrained to zero – may not be enough. The inverse system may be stable with zero output (which acts as input to the inverse system) but unstable under other outputs of bounded magnitude. Consider the following example as a motivation:

Example 8.2. Consider the following system with its inverse:

$$\Gamma : \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_3 x_1^2 + u, \\ \dot{x}_3 = -x_3 + x_3 x_1, \\ y = x_1, \end{cases} \quad \Rightarrow \quad \Gamma^{-1} : \begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = \ddot{y}, \\ \dot{z}_3 = -z_3 + z_3 z_1, \\ u = z_3 z_1^2 + \ddot{y}. \end{cases}$$

The zero-dynamics obtained by setting $y \equiv 0$ are:

$$\dot{x}_3 = -x_3,$$

which are clearly asymptotically stable and hence the system is minimum-phase. Suppose that the output to be tracked is given by $y_d \equiv 2$; then $\dot{y} \equiv \ddot{y} \equiv 0$. If the initial condition is chosen such that $z_1(0) > 1$, $z_2(0) = 0$, and $z_3(0) \neq 0$, then $z_3(t) \to \infty$ and consequently $u(t) \to \infty$ as $t \to \infty$, even though the output $y(t)$ and its derivatives remain bounded for all $t \geq 0$. $\triangleleft$

In the above example, even though the inverse system is stable under zero output (which is actually acting as an input to the inverse system), the state may not be bounded in response to the bounded output. So the example suggests that the bounded output generation problem requires some property stronger than the minimum-phase. The notion of *input-to-state stability*, introduced in [168], captures this phenomenon.

Definition 8.3 (Input-to-State Stability (ISS))**.** *The system* (8.1) *is called* input-to-state stable *with respect to the input u if for some functions $\chi \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for every initial state $x(t_0)$, and every input u, the corresponding solution satisfies the inequality*

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \chi(\|u_{[t_0,t]}\|_\infty), \tag{8.4}$$

for all $t \geq t_0 \geq 0$. $\triangleleft$

The ISS property basically requires the state of the system to be bounded by a suitable function of the input, modulo a decaying term depending on initial conditions. For the bounded output generation problem, we basically require the inverse system to be ISS with respect to the output y and its derivatives, as they are acting as the inputs to this system. This guarantees that bounded outputs result in bounded states and outputs converging to zero generate states converging to zero. If we now assume invertibility, then u produced by the inverse system also remains bounded because it is merely a bounded function of the state and the output. We will make this argument precise shortly.

The above remarks suggest that we need to consider a stronger variant of the minimum-phase property for the output generation problem. Note that the system considered in Example 8.2 is actually in its *normal form* [18], so it is easy to compute its zero dynamics which may not be the case in general. In order to bypass the step of computing the normal forms and deriving conditions in terms of original system coordinates, we consider the dual notion of ISS called *output-to-state stability* (OSS) introduced in [169]. It is shown in [170] that the class of *weakly uniform OSS* (see Definition 8.5) systems includes the systems in global normal form with ISS inverse dynamics. This result allows us to derive conditions for the bounded output generation problem in terms of the OSS property of the actual subsystems (rather than their inverses). As ISS can be seen as the robust version of asymptotic stability with zero inputs, OSS can be regarded as the robust version of detectability for nonlinear systems. Its definition basically requires the state of the system to be bounded by a suitable function of the output plus a decaying term depending on initial conditions.

Definition 8.4 (Uniform Output-to-State Stability (OSS))**.** *The system* (8.1) *is called* uniformly output-to-state stable *with respect to the output y if for some functions $\chi \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for every initial state $x(t_0)$, and every input u, the corresponding solution satisfies the inequality*

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \chi(\|y_{[t_0,t]}\|_\infty), \tag{8.5}$$

for all t in the domain of the corresponding solution. ◁

The OSS property, in general, requires the states to be bounded for all kinds of output; in particular, it includes outputs that may have unbounded derivatives. This property is stronger than what is required for bounded output generation as the inverse system is run by output and its derivatives. Thus, we only require the system to be OSS with respect to some derivative of the output which motivates us to define the following weaker notion.

Definition 8.5 (Weak Uniform OSS)**.** *The system* (8.1) *is called* weakly uniformly output-to-state stable of order N *with respect to output y and its derivatives if there exists a positive* *integer N, a class $\mathcal{K}_\infty$ function χ, and a class $\mathcal{KL}$ function β, such that for every initial state $x(0)$, and every input $u \in \mathcal{C}^{n-1}$, the corresponding solution satisfies the inequality*

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \chi\left(\left\|Y^N_{[t_0,t]}\right\|_\infty\right), \tag{8.6}$$

for all t in the domain of the corresponding solution, and $Y^N := \operatorname{col}\left(y, \dot{y}, \cdots, y^{(N)}\right)$. ◁

It is clear that if $N = 0$ in Definition 8.5, then we recover the OSS property of Definition 8.4. Also, if the system is weakly uniformly OSS of order N, then it is also weakly uniformly OSS of order $N + 1$; however, the converse is not true in general.

Based on the OSS results developed in [171] and [169], we can attribute a Lyapunov function to weakly uniformly OSS systems. A system is weakly uniformly OSS of order N if there exists a smooth, positive definite, radially unbounded function $V : \mathbb{R}^n \mapsto \mathbb{R}$ that satisfies

$$\frac{\partial V}{\partial x}\left(f(x) + \sum_{i=1}^{m} g_i(x)u_i\right) \leq -\chi_1(|x|) + \chi_2(|Y^N|), \tag{8.7}$$

for some functions $\chi_1, \chi_2 \in \mathcal{K}_\infty$.

Our primary goal is to arrive at conditions that guarantee weak uniform OSS property of system (8.1). Very recently, the results dealing with OSS of switched systems have been published in [36], and we use this result to give a sufficient condition for the switched system (8.1) to be weakly uniformly OSS.

In order to deal with the derivatives of the output and use existing results on OSS, we associate an auxiliary output function $H_p^{\alpha_p}$ with each mode p, where α_p is the relative order of the system defined in Chapter 2. The function $H_p^{\alpha_p}$ is defined from the following recursion:

$$\begin{aligned} H_p^0(x) &:= h_p(x), \\ H_p^k(x, u, \dot{u}, \cdots, u^{(k-1)}) &:= \frac{\partial H_p^{k-1}}{\partial x}\left(f_p(x) + \sum_{i=1}^{m} g_{i,p}u_i\right) + \sum_{i=1}^{k-2} \frac{\partial H_p^{k-1}}{\partial u^(i)} u^{(i+1)}, \quad 1 \leq k \leq \alpha_p. \end{aligned} \tag{8.8}$$

For SISO case, this auxiliary function comes out to be:

$$H_p^{\alpha_p} = \operatorname{col}\left(h_p(x), L_{f_p}h_p(x), \cdots, L_{f_p}^{\alpha_p - 1}h_p(x), L_{f_p}^{\alpha_p}h_p(x) + L_{g_p}L_{f_p}^{\alpha_p-1}h_p(x)u\right),$$

where α_p now denotes the relative degree of the subsystem p.

With this definition of auxiliary output function, we arrive at the following result.

Theorem 8.6. *Consider the switched system* (8.1). *Assume that there exist functions* $\varphi_1, \varphi_2, \chi \in \mathcal{K}_\infty$*, smooth functions* $V_p : \mathbb{R}^n \mapsto \mathbb{R}$*, constants* $\lambda_s > 0$*,* $\mu \geq 1$ *such that for all* $x \in \mathbb{R}^n$ *and all* $p, q \in \mathcal{P}$ *the following holds:*

$$\varphi_1(|x|) \leq V_p(x) \leq \varphi(|x_2|) \tag{8.9}$$

$$|x| > \chi(|H_p^{\alpha_p}(x, u, \dot{u}, \cdots, u^{(\alpha_p - 1)})|) \Rightarrow \frac{\partial V_p}{\partial x}\left(f_p(x) + \sum_{i=1}^{m} g_{p,i}(x) u_i\right) \leq -\lambda_s V_p(x) \quad \forall\, u \tag{8.10}$$

$$V_p(x) \leq \mu V_q(x) \tag{8.11}$$

If σ *is the switching signal with average dwell time*

$$\tau_a \geq \frac{\ln \mu}{\lambda_s}, \tag{8.12}$$

then the switched systems is weakly uniformly OSS of order k*, where* $k = \max_{p \in \mathcal{P}} \alpha_p$*.* ◁

The result is built upon Theorem 1 in [36] and the proof proceeds in exactly the similar manner.

Next, we claim that the weak uniform OSS property in conjunction with the left-invertibility of the switched system leads to the solution of bounded output generation problem as the inputs generated by the inverse system are then bounded. To see this, recall the expression for the input u obtained from the structure algorithm (2.19), where u is expressed as a function of state and the derivatives of the output, i.e.,

$$u = \Upsilon(x, \dot{y}, \ddot{y}, \ldots, y^{\alpha_p - 1}). \tag{8.13}$$

We can find a class $\mathcal{K}_\infty$ function $\tilde{\chi}$ such that, at each time instant, we get the following bounds:

$$|\Upsilon(x, \dot{y}, \ddot{y}, \ldots, y^{\alpha_p - 1})| \leq \tilde{\chi}(|(x, \dot{y}, \ddot{y}, \ldots, y^{\alpha_p - 1})|) \tag{8.14a}$$

$$\leq \tilde{\chi}(2|x|) + \tilde{\chi}(2|Y^{\alpha_p - 1}|). \tag{8.14b}$$

The result of Theorem 8.6 implies the existence of a class $\mathcal{KL}$ function $\hat{\beta}(\cdot, \cdot)$ and a class $\mathcal{K}_\infty$ function $\hat{\chi}$ such that

$$|x(t)| \leq \hat{\beta}(x_0, t - t_0) + \hat{\chi}(\|Y^k\|_{[t_0, t]}). \tag{8.15}$$

Combining (8.14b) and (8.15), we get

$$\begin{aligned}|u(t)| &\le \tilde{\chi}\left(2\hat{\beta}(|x_0|, t-t_0) + \hat{\chi}(\|Y^k\|_{[t_0,t]})\right) + \tilde{\chi}(2|Y^{\alpha_p-1}(t)|)\\ &\le \tilde{\chi}\left(4\hat{\beta}(|x_0|, t-t_0)) + \tilde{\chi}(2\hat{\psi}(\|Y^k\|_{[t_0,t]})\right) + \tilde{\chi}(2|Y^{\alpha_p-1}(t)|)\\ &=: \beta(|x_0|, t-t_0) + \chi(|Y^k(t)|),\end{aligned}$$

where $\beta(s_1, s_2) = \tilde{\chi}(4\hat{\beta}(s_1, s_2))$, and $\chi(s) = \tilde{\chi}(2\hat{\chi}(s) + \tilde{\chi}(s))$.

This idea of bounding the norm of the state with the output norm using OSS concept and then using the left-invertibility to bound the size of the input was initially employed in the study of *output-input-stability* of nonswitched nonlinear systems in [170].

8.1.2 Asymptotic Output Tracking

In Chapter 2, Section 2.5 and this chapter's Section 8.1.1, we discussed the problem of output generation where we construct a switching signal and an input function that reproduce *exactly* a prescribed output function $y_d(\cdot)$. As we have seen, this is possible when certain components of the state of the system are *fixed* at time $t = 0$ based on the value of the desired output $y_d(\cdot)$ and its derivatives at that time instant. Moreover, this method implicitly assumes that similar constraints are satisfied by the state during its evolution whenever the desired output or its derivatives are not smooth, so that the exact output generation is possible after switching to another subsystem. However, presetting the initial state to a prescribed value is not a common practice and, in addition, the initial state may be different from the desired one because of unknown perturbations. For practical considerations, we are now interested in the problem of producing an output that, irrespectively of what the initial state of the system is, *converges asymptotically* to the prescribed reference function $y_d(\cdot)$. This problem is called *output tracking.* Similar to the previous section, the term *bounded output tracking* refers to finding bounded inputs for bounded output functions. It turns out that this requirement can be naturally embedded in the original problem statement.

Problem 8.2 (Output Tracking)**.** *For a given reference function $y_d(\cdot)$ and an arbitrary initial state $x(t_0)$, find $\sigma(\cdot)$ and $u(\cdot)$ such that the solution of the differential equation* (8.1a) *satisfies*

$$y(t) = h_{\sigma(t)}(x(t)) \to y_d(t) \quad \textit{as} \quad t \to \infty,$$

and

$$|u(t)| \leq \beta(|x(t_0)|, t-t_0) + \chi\left(\left|Y^N(t)\right|\right), \quad \textit{for each } t \geq t_0,$$

for some $\beta \in \mathcal{KL}$*, and* $\chi \in \mathcal{K}_\infty$. ◁

The problem of output tracking with switched nonlinear systems is an ongoing work and this section only provides the motivation and basic approach we intend to adopt for the solution of this problem. To motivate further discussion, consider the following example:

Example 8.7. (Tracking with x_0 unknown) Consider two linear systems with equal relative degrees in their normal forms:

$$\Gamma_1 : \begin{cases} \dot{x}_1 = x_1 + x_2 + u, \\ \dot{x}_2 = -x_2 + x_1, \\ \;y = x_1, \end{cases} \quad \Rightarrow \quad \Gamma_1^{-1} : \begin{cases} \dot{x}_1 = \dot{y}, \\ \dot{x}_2 = -x_2 + y, \\ \;u = -x_1 - x_2 + \dot{y}. \end{cases}$$

$$\Gamma_2 : \begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = x_2 - x_1 + u, \\ \;y = x_2, \end{cases} \quad \Rightarrow \quad \Gamma_2^{-1} : \begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = \dot{y}, \\ \;u = x_1 - x_2 + \dot{y}. \end{cases}$$

Note that if the subsystems Γ_1 and Γ_2 are driven by the inputs generated by their corresponding inverses, then the closed-loop systems are:

$$\Gamma_1 : \begin{cases} \dot{x}_1 = \dot{y}_d, \\ \dot{x}_2 = -x_2 + x_1, \\ \;y = x_1, \end{cases} \quad ; \quad \Gamma_2 : \begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = \dot{y}_d, \\ \;y = x_2. \end{cases}$$

If $e_y := y - y_d$, then $\dot{e}_y \equiv 0$ which in turn gives $e_y(t) \equiv e_y(t_0)$. Thus, if the initial conditions are not chosen in some particular manner that gives $e_y(t) = 0$, the input generated by the inverse system will not be able to reduce the error between the desired output and the system output.

Assume for a moment that the switching signal is known. If the input u is modified by introducing some damping terms, so that u takes the form

$$u(t) = \begin{cases} -x_1 - x_2 + \dot{y}_d - a_1(y - y_d) & \text{if } \sigma(t) = 1, \\ x_1 - x_2 + \dot{y}_d - a_2(y - \dot{y}_d) & \text{if } \sigma(t) = 2, \end{cases} \tag{8.16}$$

where $a_1, a_2 > 0$, then the closed-loop dynamics are:

$$\Gamma_1 : \begin{cases} \dot{x}_1 = \dot{y}_d - a_1(y - y_d), \\ \dot{x}_2 = -x_2 + x_1, \\ \; y = x_1, \end{cases} \quad ; \quad \Gamma_2 : \begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = \dot{y}_d - a_2(y - y_d), \\ \; y = x_2. \end{cases}$$

This time the error dynamics are:

$$\dot{e}_y(t) = \begin{cases} -a_1 e_y(t) & \text{if } \sigma(t) = 1, \\ -a_2 e_y(t) & \text{if } \sigma(t) = 2. \end{cases}$$

So the error dynamics switch between two stable modes and the error between the desired output and system output asymptotically converges to zero. Since a_1, a_2 are design parameters, the stability may be obtained in the general case without imposing the slow switching constraint.

In case the switching signal is not known, it would be interesting to design a switching law that makes the error dynamics asymptotically stable. $\triangleleft$

In SISO nonswitched linear systems, every time the output is differentiated a pole is introduced at the origin in the inverse system. So a full-order inverse of a system with relative degree r has exactly r poles at the origin. From the structure of the inverse system in normal form, it is clear that these poles are actually controllable. Thus, modifying the input by including the damping terms enables us to move these poles in the left half-plane to stabilize the system. The zero dynamics, on the other hand, are not controllable; that is why we assume them to be stable.

In the general MIMO case, it is natural to ask: To what extent can the full-order inverse system be stabilized? The early work on system inversion, e.g. [75, 76, 78, 79], was concerned with the existence of inverses and algorithms for constructing inverses which were not necessarily of minimal dimensions, although Silverman [75] did show that inverses of lower order than that of the given system could be constructed. For the linear case, Bengtsson [172] showed that the poles of a minimal order inverse of a system are unique and common to all inverses of the system. In fact, by analogy with SISO systems, he defined the invariant zeros of a MIMO system as the poles of its minimal order inverse.[1] Since we are dealing with full order inverses, an interesting question is whether we can construct these inverses

[1] If the minimal order inverse system is stable, then the actual system is minimum-phase.

such that the poles which are not the poles of a minimal order inverse can be specified arbitrarily. Building on the work of [172], Patel [173] showed that whenever a system has a stable minimal order inverse, then we can always construct inverses of higher order which are also stable, and whose responses can be specified to some extent by assigning a subset of the poles. This approach is also generalized for nonlinear case in applications such as output tracking problems [18, 63] and feedback stabilization [62, 174].

Thus, with minimum-phase assumption, it is possible to introduce additional damping terms in the control law which stabilize the inverse of each subsystem, and without any switching this leads to asymptotic output tracking. Whenever the prescribed output involves discontinuities, it is natural to have switching in order to have better tracking performance, and because of switching, care must be taken on two fronts:

- Since the initial condition is no longer assumed to be known, we cannot use the output generation algorithm scheme of section 2.5 to compute the switching signal. So we have to come up with a new switching strategy in order to minimize the output tracking error.

- Even though the dynamics of the output tracking error are Hurwitz for each subsystem, they are of different dimension and heavily coordinate dependent.

Addressing these two questions is of primary concern in the study of output tracking when the subsystems are assumed to be minimum-phase as we look for well-defined class of outputs for which switching laws could be designed to achieve asymptotic tracking. An extension of this problem is to consider the switched systems that comprise some minimum-phase and some nonminimum-phase systems. In short, we are interested in solving the following two sub-cases to Problem 8.2.

Sub-problem 8.2–A (Output Tracking with all Subsystems Minimum-phase)**.** *Assume that each subsystem of the switched system* (8.1) *is left-invertible and minimum-phase. Determine the class of outputs* $\mathcal{Y}$*, so that for a given reference signal* $y_d(\cdot) \in \mathcal{Y}$ *and an arbitrary initial state* $x(t_0)$*, one can find a switching signal* $\sigma(\cdot)$ *and an input* $u(\cdot)$ *such that the solution of the differential equation* (8.1a) *satisfies*

$$y(t) = h_{\sigma(t)}(x(t)) \to y_d(t) \quad \text{as} \quad t \to \infty,$$

and

$$|u(t)| \leq \beta(|x(t_0)|, t - t_0) + \chi\left(\left|Y^N(t)\right|\right), \quad \text{for each } t \geq t_0,$$

for some $\beta \in \mathcal{KL}$, *and* $\chi \in \mathcal{K}_\infty$. ◁

Sub-problem 8.2–B (Output Tracking with some Subsystems Minimum-phase)**.** *Assume that there are some subsystems in the family of subsystems considered in* (8.1) *that are left-invertible and minimum-phase. Determine the class of outputs* $\mathcal{Y}$*, so that for a given reference signal* $y_d(\cdot) \in \mathcal{Y}$ *and an arbitrary initial state* $x(t_0)$*, one can find a switching signal* $\sigma(\cdot)$ *and an input* $u(\cdot)$ *such that the solution of the differential equation* (8.1a) *satisfies*

$$y(t) = h_{\sigma(t)}(x(t)) \to y_d(t) \quad \text{as} \quad t \to \infty,$$

and

$$|u(t)| \leq \beta(|x(t_0)|, t - t_0) + \chi\left(\left|Y^N(t)\right|\right), \quad \text{for each } t \geq t_0,$$

for some $\beta \in \mathcal{KL}$, *and* $\chi \in \mathcal{K}_\infty$. ◁

The inherent idea behind in solving Sub-problem 8.2–B is to design the switching law so that the activation time for minimum-phase subsystems is large enough compared to the total activation time of nonminimum-phase subsystems; see [41, 42], and [36] for exposition of this idea in stabilization and IOSS framework, respectively.

The minimum-phase assumption is not necessary for bounded output tracking problems. In nonswitched nonlinear systems, Devasia et al. [64] have shown that non-causal methods can be employed to generate bounded outputs with bounded inputs with nonminimum-phase systems. Their method essentially drives the state at time t_0 to a stable manifold, and from that point onwards the output keeps the state in that manifold. This method of presetting the initial condition using noncausal methods seems infeasible for switched systems and impossible when the switching signal is not known. Thus, an interesting direction of work is to determine whether there exist any feasible methods for solving the output tracking problem with switched systems when *none* of the constituent subsystems is required to be minimum-phase.

Sub-problem 8.2–C (Output Tracking without Minimum-phase Subystems)**.** *Assume that each subsystem of the switched system* (8.1) *is only left-invertible. Determine a class of outputs* $\mathcal{Y}$*; and for each* $y_d(\cdot) \in \mathcal{Y}$*, find a switching signal* $\sigma(\cdot)$*, and an input* u *such that*

$$y(t) = h_{\sigma(t)}(x(t)) \to y_d(t) \quad \text{as} \quad t \to \infty,$$

and

$$|u(t)| \leq \beta(|x(t_0)|, t - t_0) + \chi\left(\left|Y^N(t)\right|\right), \quad \text{for each } t \geq t_0,$$

for some $\beta \in \mathcal{KL}$, *and* $\chi \in \mathcal{K}_\infty$. ◁

Note that the Sub-problem 8.2–C essentially requires us to seek a switching signal that would stabilize a family of unstable systems. The work of [39, 40] suggests that one possible way of achieving this objective is to look for a stable convex combination among the family of unstable systems; and more recently the averaging methods for stability of switched systems in [43] have been developed to solve this problem. Thus, it is safe to say that not much work has been published concerning the stability of switched systems when all subsystems are unstable. Nevertheless, preliminary research on Sub-problem 8.2–C shows that it is possible to achieve bounded output tracking for certain classes of switching signals even when all the subsystems are nonminimum-phase. Studying this issue in more depth is a topic of future work.

8.2 Concluding Remarks

The thesis addressed the problems of invertibility and observability in switched systems. The major portion of our work so far has focused on analyzing the structure of the systems that reveal information about these properties, and with the exception of designing the observers for switched systems with ODEs, these properties have not been studied from a design perspective. Therefore, with the design of control systems as our central objective for the future work, we conclude this document by quickly reviewing the work done while highlighting the directions that can be pursued.

Invertibility of Switched Nonlinear Systems

We started off with invertibility of nonlinear switched systems. A necessary and sufficient condition for the invertibility of switched systems was given which required the invertibility of individual subsystems and the nonexistence of switch-singular pairs. The concept of switch-singular pairs introduced in [65] for linear systems was extended to nonlinear systems. We then developed the formula for checking if (x_0, y) is a switch-singular pair of two subsystems. In the case of single-input single-output bilinear systems, the formula for checking the existence of a switch-singular pair reduces to checking the rank of some matrices. Assuming that all the subsystems are invertible, we developed an algorithm to recover the input and switching signal that generate a desired output trajectory from given initial state.

The work may be extended further by developing conditions for checking the existence of switch-singular pairs which are more constructive, as it is in general not feasible to verify (2.25) for every output and state. Another research direction is to approach the problem geometrically and investigate characterizations equivalent to nonexistence of switch-singular pairs to obtain geometric criteria for left-invertibility of switched systems. The geometric criteria for invertibility may provide more useful conditions for system design because they do not require us to look at the derivatives of the output. In doing so, one may look for similarities among the subsystems which exhibit switch-singular pairs.

Invertibility with Uncertainties

The problem of robust invertibility dealt with the reconstruction of a switching signal in the presence of disturbances in the output and uncertain initial conditions. The idea of the gap between the subspaces was used to arrive at a conservative but easy-to-implement algorithm for switching signal recovery. In the case of minimum-phase subsystems, we showed that the switching signal can be recovered for all times under the dwell-time assumption. The ongoing work is focused on developing better algorithms and obtaining tighter bounds for reconstructing the switching signal. Relaxing the dwell-time assumption to average dwell-time is also being considered.

Asymptotic Output Tracking is an interesting problem that was proposed at the beginning of this chapter. Because of its practical utility and the theoretic concepts involved, we will prioritize this problem in our future work. We presented a basic approach to the solution of this problem. The first step is to work under the minimum-phase assumption and classify the class of signals which can be tracked asymptotically with the help of a suitable switching law. We can then build on this approach for a more general class of switched systems whose subsystems are not necessarily minimum-phase, but the tracking may still be possible by appropriate switching.

Observability and Observers

We presented conditions for observability of switched linear and nonlinear systems with state jumps, and asymptotic observers were designed based on these conditions. For switched DAEs, these characterizations are formulated in terms of consistency projectors and the newly introduced differential and impulse projectors which are obtained by utilizing the so-called Wong sequences.

Observers for DAEs is a topic that has not received much attention in the literature. For linear systems, based on the conditions developed, an observer was constructed that combines the partial information obtained from each mode at some time instant to get an estimate of the state vector. In case of DAEs, the major concern is to incorporate the knowledge from the impulsive part of the output. This suggests that different designs for observers are required for this class of systems. Building observers for such systems based on the conditions already developed is an ongoing work.

Semi-global observability of switched nonlinear systems is a possible extension of the results presented in Chapter 5. The conditions for observability of switched nonlinear systems given in Chapter 5 are applicable locally because we assumed the existence of certain functions whose gradients span a certain codistribution over a set $\mathcal{X}^o$. If the state trajectories are contained in $\mathcal{X}^o$, then it was shown that the system is observable. However, in a more practical setup, we want to determine the observability of the system over a *given* set $\mathcal{X}$. It may be the case that a particular codistribution of interest is nonsingular at each $x \in \mathcal{X}$, and for semi-global extension of our results, we are interested in knowing whether we can determine a set of functions with linearly independent 1-forms that span the codistribution under consideration. It is seen that this is not always possible. To determine conditions for semi-global extension of our observability results is a task of future work.

Appendix A

Structure Algorithm for Linear Systems

The structure algorithm

Consider the dynamics of a non-switched linear systems, which are written as

$$\Gamma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du. \end{cases} \tag{A.1}$$

The main computational tool for studying the problem in algebraic setting is the *structure algorithm*, introduced by [75] and [88]. We discuss this algorithm briefly, and the reader is referred to [88] for further technical details and proofs. Consider the linear system (2). Let n be the state dimension, m the input dimension, and l the output dimension. Let $q_0 = \text{rank}(D)$; there exists a nonsingular $l \times l$ matrix S_0 such that $D_0 := S_0 D = \begin{bmatrix} \overline{D}_0 \\ 0 \end{bmatrix}$, where $\overline{D}_0$ has q_0 rows and rank q_0. Let $y_0 = S_0 y$ and $C_0 := S_0 C$. Thus, we have $y_0 = C_0 x + D_0 u$. Suppose that at step k, we have $y_k = C_k x + D_k u$, where D_k has the form $\begin{bmatrix} \overline{D}_k \\ 0 \end{bmatrix}$; $\overline{D}_k$ has q_k rows and is full rank. Let the partition of C_k be $\begin{bmatrix} \overline{C}_k \\ \tilde{C}_k \end{bmatrix}$, where $\overline{C}_k$ is the first q_k rows, and the partition of y_k be $\begin{bmatrix} \overline{y}_k \\ \tilde{y}_k \end{bmatrix}$, where $\overline{y}_k$ is the first q_k elements. If $q_k < l$, let M_k be the differential operator $M_k := \left[\begin{array}{c|c} I_{q_k} & 0 \\ \hline 0 & I_{l-q_k}(d/dt) \end{array}\right]$. Then $M_k y_k = \begin{bmatrix} \overline{C}_k \\ \tilde{C}_k A \end{bmatrix} x + \begin{bmatrix} \overline{D}_k \\ \tilde{C}_k B \end{bmatrix} u$. Let $q_{k+1} = \text{rank} \begin{bmatrix} \overline{D}_k \\ \tilde{C}_k B \end{bmatrix}$; then there exists a nonsingular matrix $l \times l$ matrix S_{k+1} such that $D_{k+1} := S_{k+1} \begin{bmatrix} \overline{D}_k \\ \tilde{C}_k B \end{bmatrix} = \begin{bmatrix} \overline{D}_{k+1} \\ 0 \end{bmatrix}$, where D_{k+1} has q_{k+1} rows and rank

q_{k+1}. Let $y_{k+1} := S_{k+1}M_k y_k$, $C_{k+1} := S_{k+1}\begin{bmatrix} \overline{C}_k \\ \tilde{C}_k A \end{bmatrix}$. Then $y_{k+1} = C_{k+1}x + D_{k+1}u$ and we can repeat the procedure. Let $N_k := \prod_{i=0}^{k} S_{k-i}M_{k-i-1}$, $k = 1, 2, \cdots (M_{-1} := I; S_0 = I)$, $\overline{N}_k := [I_{q_k} \quad 0_{q_k \times (l-q_k)}]$ and $\tilde{N}_k := [0_{(l-q_k)\times q_k} \quad I_{l-q_k}]N_k$. Then $y_k = N_k y$, $\overline{y}_k = \overline{N}_k y$, and $\tilde{y}_k = \tilde{N}_k y$. Using these notations, $y = y_0 = \tilde{y}_0 = \tilde{C}_0 x = Cx$; $S_0 = I$ and $D_0 = 0$. Notice that since D_k has l rows and m columns, $q_k \leq \min\{l, m\}$ for all k and since $q_{k+1} \geq q_k$, using the Cayley-Hamilton theorem, it was shown in [88] that there exists a smallest integer $\alpha \leq n$ such that $q_k = q_\alpha$, $\forall k \geq \alpha$.

If $q_\alpha = m$, the system is left-invertible and an inverse is

$$\Gamma^{-1} = \begin{cases} \overline{y}_\alpha & = \overline{N}_\alpha y, \\ \dot{x} & = (A - B\overline{D}_\alpha^{-1}\overline{C}_\alpha)x + B\overline{D}_\alpha^{-1}\overline{y}_\alpha\,, \\ u & = -\overline{D}_\alpha^{-1}\overline{C}_\alpha x + \overline{D}_\alpha^{-1}\overline{y}_\alpha\,, \end{cases} \tag{A.2}$$

with the initial state x_0.

From the structure algorithm, it can be seen that $\tilde{y}_k = \tilde{C}_k x$, $\forall k$ and hence,

$$\begin{bmatrix} \tilde{N}_0 \\ \vdots \\ \tilde{N}_k \end{bmatrix} y = \begin{pmatrix} \tilde{y}_0 \\ \vdots \\ \tilde{y}_k \end{pmatrix} = \begin{bmatrix} \tilde{C}_0 \\ \vdots \\ \tilde{C}_k \end{bmatrix} x =: L_k x, \quad \forall k. \tag{A.3}$$

Using the Cayley-Hamilton theorem, Silverman and Payne have shown in [88] that there exists a smallest number β, $\alpha \leq \beta \leq n$, such that $\mathrm{rank}(L_k) = \mathrm{rank}(L_\beta)$, $\forall k \geq \beta$. There also exists a number δ, $\beta \leq \delta \leq n$ such that $\tilde{C}_\delta = \sum_{i=0}^{\delta-1} P_i \big(\prod_{j=i+1}^{\delta} \tilde{R}_j\big)\tilde{C}_i$ for some matrices $\tilde{R}_j$ from the structure algorithm and some constant matrices P_i (see [88, p.205] for details).

Derivation of operators W_p and V_p

In the construction of inverse system, there are two operators acting on the output y that require a closer look. The first one is $\overline{N}_\alpha$, which appears in the dynamical equations of the inverse system; and the other one is $\mathbf{N} := \mathrm{col}(\widetilde{N}_0, \widetilde{N}_1, \cdots, \widetilde{N}_{\beta-1})$ that appears in the statement of Proposition 3.2. Both of them are differential operators acting on the output y. Below we seek a simpler representation so that $\mathbf{N}y$ and $\overline{N}_\alpha y$ can be written as a matrix (with real coefficients) times a vector (comprising of output and its derivatives).

From the structure algorithm, we have

$$M_0 y_0 = M_0 S_0 y = \begin{bmatrix} \overline{S}_0 \\ 0 \end{bmatrix} y + \frac{d}{dt} \begin{bmatrix} 0 \\ \widetilde{S}_0 \end{bmatrix} y$$
$$=: K_{0,0}\, y + \frac{d}{dt} K_{0,1}\, y. \tag{A.4}$$

In general, let

$$M_i y_i = K_{i,0} y + \frac{d}{dt}\big(K_{i,1} y + \cdots \frac{d}{dt}\big(K_{i,i} y + K_{i,i+1} y\big)\big). \tag{A.5}$$

Then in view of $M_{i+1} y_{i+1} = \begin{bmatrix} \overline{S}_{i+1} \\ 0 \end{bmatrix} M_i y_i + \frac{d}{dt} \begin{bmatrix} 0 \\ \widetilde{S}_{i+1} \end{bmatrix} M_i y_i$, we have the $l \times l$ matrices $K_{i,j}$ defined recursively as follows:

$$K_{i+1,j} = \begin{bmatrix} \overline{S}_{i+1} \\ 0 \end{bmatrix} K_{i,j} + \begin{bmatrix} 0 \\ \widetilde{S}_{i+1} \end{bmatrix} K_{i,j-1}, \quad 0 \le j \le i+2, i \ge 0, \tag{A.6}$$

where $K_{i,-1} = 0 \; \forall\, i$ by convention and $K_{0,0}$, $K_{0,1}$ are initialized in (A.4). Using the notation in (A.5), in view of $\widetilde{N}_k y = \widetilde{S}_k M_{i-1} y_{i-1}$ and $\overline{N}_k y = \overline{S}_k M_{i-1} y_{i-1}$, then

$$\mathbf{N} y =: \widetilde{G}_0 y + \frac{d}{dt}\big(\widetilde{G}_1 y + \cdots + \frac{d}{dt}\big(\widetilde{G}_{\beta-2} y + \frac{d}{dt}\widetilde{G}_{\beta-1} y\big)\big), \tag{A.7}$$
$$\overline{N}_\alpha y =: \overline{G}_0 y + \frac{d}{dt}\big(\overline{G}_1 y + \cdots + \frac{d}{dt}\big(\overline{G}_{\alpha-1} y + \frac{d}{dt}\overline{G}_\alpha y\big)\big), \tag{A.8}$$

where $\widetilde{G}_i := \begin{bmatrix} \widetilde{S}_0 K_{-1,i} \\ \widetilde{S}_1 K_{0,i} \\ \vdots \\ \widetilde{S}_{\beta-1} K_{\beta-2,i} \end{bmatrix}$, $\overline{G}_j := \overline{S}_\alpha K_{\alpha-1,j}$, $0 \le i \le \beta - 1$, $0 \le j \le \alpha$, $K_{-1,0} = I$, and $K_{j,k} = 0 \; \forall k \ge j+2, \forall\, j$. Next, introduce the notations: $W := \begin{bmatrix} \widetilde{G}_0 & \cdots & \widetilde{G}_{\beta-1} & \widetilde{G}_\beta & \cdots & \widetilde{G}_n \end{bmatrix}$ and $V := \begin{bmatrix} \overline{G}_0 & \cdots & \overline{G}_\alpha & \overline{G}_{\alpha+1} & \cdots & \overline{G}_n \end{bmatrix}$, where $\widetilde{G}_i = 0$ for $\beta \le i \le n$ and $\overline{G}_i = 0$ for $\alpha + 1 \le i \le n$. Then for a subsystem Γ_p, we have

$$\mathbf{N}_p y = W_p Y^n \qquad \text{and} \qquad \overline{N}_{\alpha_p} y = V_p Y^n. \tag{A.9}$$

Appendix B

Review: Geometric Control Theory

Geometric tools have been employed for the control of dynamical systems and in this appendix we review some of the basic notions and identities that have been used in the draft.

Some Useful Identities

Let $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}$ be any linear subspaces, A be a (not necessarily invertible) $n \times n$ matrix, and B, C, X be matrices of suitable dimension. For a matrix B, $\mathcal{R}(B)$ denotes the column space (range space) of B. The pre-image of $\mathcal{V}$ through A is given by $A^{-1}\mathcal{V} = \{x : Ax \in \mathcal{V}\}$. The following properties can be found in the literature such as [175], or developed with little effort.

1. $A\mathcal{R}(B) = \mathcal{R}(AB)$ and $A^{-1} \ker B = \ker(BA)$.
2. $A^{-1}A\mathcal{V} = \mathcal{V} + \ker A$, and $AA^{-1}\mathcal{V} = \mathcal{V} \cap \mathcal{R}(A)$.
3. $A^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2) = A^{-1}\mathcal{V}_1 \cap A^{-1}\mathcal{V}_2$, and $A(\mathcal{V}_1 \cap \mathcal{V}_2) \subseteq A\mathcal{V}_1 \cap A\mathcal{V}_2$ (with equality if and only if $(\mathcal{V}_1 + \mathcal{V}_2) \cap \ker A = \mathcal{V}_1 \cap \ker A + \mathcal{V}_2 \cap \ker A$, which holds, in particular, for any invertible A).
4. $A\mathcal{V}_1 + A\mathcal{V}_2 = A(\mathcal{V}_1 + \mathcal{V}_2)$, and $A^{-1}\mathcal{V}_1 + A^{-1}\mathcal{V}_2 \subseteq A^{-1}(\mathcal{V}_1 + \mathcal{V}_2)$ (with equality if and only if $(\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{R}(A) = \mathcal{V}_1 \cap \mathcal{R}(A) + \mathcal{V}_2 \cap \mathcal{R}(A)$, which holds, in particular, for any invertible A).
5. $(\ker A)^{\perp} = \mathcal{R}(A^{\top})$.
6. $(A^{\top}\mathcal{V})^{\perp} = A^{-1}\mathcal{V}^{\perp}$ and $(A^{-1}\mathcal{V})^{\perp} = A^{\top}\mathcal{V}^{\perp}$.
7. $\langle A|\mathcal{V}\rangle = \mathcal{V} + A\mathcal{V} + A^2\mathcal{V} + \cdots + A^{n-1}\mathcal{V}$ and $\langle \mathcal{V}|A\rangle = \mathcal{V} \cap A^{-1}\mathcal{V} \cap A^{-2}\mathcal{V} \cap \cdots \cap A^{-(n-1)}\mathcal{V}$.
8. $\langle \mathcal{V}_1 \cap \mathcal{V}_2|A\rangle = \langle \mathcal{V}_1|A\rangle \cap \langle \mathcal{V}_2|A\rangle$ and $\langle A|\mathcal{V}_1 \cap \mathcal{V}_2\rangle \subset \langle A|\mathcal{V}_1\rangle \cap \langle A|\mathcal{V}_2\rangle$.

9. $e^{At}\mathcal{V} \subseteq \langle A|\mathcal{V}\rangle$ and $\langle \mathcal{V}|A\rangle \subseteq e^{At}\mathcal{V}$ for any t.

10. $\langle A|\mathcal{V}\rangle^{\perp} = \left\langle \mathcal{V}^{\perp}|A^{\top}\right\rangle$.

    Next, we introduce the matrix $G := \text{col}(C, CA, \ldots, CA^{n-1})$.

11. $e^{At}\ker G = \ker G$ and $e^{A^{\top}t}\mathcal{R}(G^{\top}) = \mathcal{R}(G^{\top})$ for all t.

12. $\langle \ker G|A\rangle = \ker G$ and $\left\langle A^{\top}|\mathcal{R}(G^{\top})\right\rangle = \mathcal{R}(G^{\top})$.

13. The equation $BX = C$ can be solved for X, if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(B)$.

14. The equation $XB = C$ can be solved for X, if and only if $\ker B \subseteq \ker C$.

15. Let $\mathcal{W}$ be A-invariant and $P : \mathbb{R}^n \to \mathbb{R}^n/\mathcal{W}$ be the canonical projection, then there exists a unique map $\overline{A} : \mathbb{R}^n \to \mathbb{R}^n/\mathcal{W}$ such that $\overline{A}P = PA$. We say that the map $\overline{A}$ is induced by A on the factor space $\mathbb{R}^n/\mathcal{W}$.

16. Let $W : \mathcal{W} \to \mathbb{R}^n$ be an insertion map; then there exists a unique map $A|\mathcal{W}$ such that $AW = W(A|\mathcal{W})$, and $A|\mathcal{W}$ is the restriction of A to $\mathcal{W}$.

In the sequel we introduce some classical notion from geometric based linear multivariable control for the class of following linear time-invariant systems:

$$\dot{x} = Ax + Bu, \tag{B.1a}$$

$$y = Cx. \tag{B.1b}$$

Conditioned-Invariant Subspaces

Definition B.1. *[176, Chapter 5] A subspace $\mathcal{V}$ of $\mathbb{R}^n$ is called conditioned invariant if there exists an observer for $x/\mathcal{V}$.*

Proposition B.2. *[176, Chapter 5] The following statements are equivalent:*

1. *$\mathcal{V}$ is conditioned-invariant,*

2. *$A(\ker C \cap \mathcal{V}) \subseteq \mathcal{V}$,*

3. *there exists a matrix L such that $(A + LC)\mathcal{V} \subseteq \mathcal{V}$.*

Controlled-Invariant Subspaces

Definition B.3. *[176, Chapter 4] A subspace $\mathcal{W} \subseteq \mathbb{R}^n$ is called controlled invariant if for any $x_0 \in \mathcal{W}$, there exists an input function u such that the solution of $\dot{x} = Ax + Bu$ satisfies $x(t) \in \mathcal{W}$, for all $t \geq 0$.*

Proposition B.4. *[176, Chapter 4] Consider the system $\dot{x} = Ax + Bu$. Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. The following statements are equivalent:*

1. *$\mathcal{W}$ is controlled invariant,*
2. *$A\mathcal{W} \subseteq \mathcal{W} + \mathcal{R}(B)$,*
3. *there exists a linear map F such that $(A + BF)\mathcal{W} \subseteq \mathcal{W}$.*

Invertibility Conditions

Theorem B.5. *Consider the dynamical system* (B.1)*, and let $\mathcal{W}(\ker C)$ denote the largest controlled-invariant subspace contained in $\ker C$. Then the system is left-invertible if and only if*

$$\mathcal{W}(\ker C) \cap \mathcal{R}(B) = \{0\}.$$

A related notion to that of invertibility is that of input observability [76]. System (B.1) is said to be input observable if and only if

$$\langle \ker C | A \rangle \cap \mathcal{R}(B) = \{0\}.$$

Note that every A-invariant subspace is controlled-invariant, so that $\langle \ker C | A \rangle \subseteq \mathcal{W}(\ker C)$. Thus, left-invertibility implies input-observability. However, the converse is not true in general.

Theorem B.6. *If B is monic and $\dim \mathcal{R}(B) = 1$, then* (B.1) *is left-invertible if and only if*

$$\langle \ker C | A \rangle \cap \mathcal{R}(B) = \{0\}.$$

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