VALID INEQUALITIES FOR MIXED INTEGER PROGRAMMING PROBLEMS

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Consider the mixed integer programming problem (P_M)

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^t\boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{0} \leq \boldsymbol{x} \in \boldsymbol{R}^n \\ & x_j \in \boldsymbol{Z} \quad (1 \leq j \leq n_1) \end{aligned}$$

where A is an $\boldsymbol{R}$-component $m \times n$ matrix, $\boldsymbol{b}$ is a vector in $\boldsymbol{R}^m$, $\boldsymbol{c}$ is a vector in $\boldsymbol{R}^n$ and n and n_1 are integers with $0 < n_1 < n$. $\boldsymbol{R}$ and $\boldsymbol{Z}$ denote the set of all real numbers and the set of all integers, respectively.

Let X be the set of all feasible solutions:

$$X = \left\{ \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \boldsymbol{R}^n ;\ A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq 0,\ x_j \in \boldsymbol{Z}\,(1 \leq j \leq n_1) \right\}.$$

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ be the column vectors of A: $A = (\boldsymbol{a}_1 \ \ldots \ \boldsymbol{a}_n)$, where $\boldsymbol{a}_j = (a_{1j} \ \ldots \ a_{nj})^t$. We denote by V the abelian group generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n_1}, \boldsymbol{b}$, i. e.,

$$V = \boldsymbol{a}_1\boldsymbol{Z} + \ldots + \boldsymbol{a}_{n_1}\boldsymbol{Z} + \boldsymbol{b}\boldsymbol{Z} \subset \boldsymbol{R}^n.$$

Let f be a real valued function defined on V, satisfying

$$f(\boldsymbol{v}_1) + f(\boldsymbol{v}_2) \geq f(\boldsymbol{v}_1 + \boldsymbol{v}_2)$$

for any $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, then f is called a subadditive function on V.

Let $\boldsymbol{\pi}$ be a vector in $\boldsymbol{R}^n$ and π_0 be a real number. An inequality $\boldsymbol{\pi}^t\boldsymbol{x} \geq \pi_0$. is called a valid inequality for X if every $\boldsymbol{x} \in X$ satisfies $\boldsymbol{\pi}^t\boldsymbol{x} \geq \pi_0$.

THEOREM 1. *Let f be a subadditive function on V satisfying $f(\boldsymbol{0}) = 0$. Then, for any $\boldsymbol{x} \in X$,*

$$\sum_{j=1}^{n_1} f(\boldsymbol{a}_j)x_j + f\Big(\sum_{j=n_1+1}^{n} \boldsymbol{a}_j x_j\Big) \geq f(\boldsymbol{b}).$$

Proof. For any $z_1, \ldots, z_{n_1} \in \boldsymbol{Z}$, we have

$$\sum_{j=1}^{n_1} f(\boldsymbol{a}_j) z_j \geq f(\sum_{j=1}^{n_1} \boldsymbol{a}_j z_j)$$

(See the proof of Theorem 1 in [4]).

For any $\boldsymbol{x} \in X$, we have $A\boldsymbol{x} = \boldsymbol{b}$, i. e.,

$$\sum_{j=1}^{n_1} \boldsymbol{a}_j x_j + \sum_{j=n_1+1}^{n} \boldsymbol{a}_j x_j = \boldsymbol{b},$$

so

$$f(\sum_{j=1}^{n_1} \boldsymbol{a}_j x_j + \sum_{j=n_1+1}^{n} \boldsymbol{a}_j x_j)$$

$$\leq f(\sum_{j=1}^{n_1} \boldsymbol{a}_j x_j) + f(\sum_{j=n_1+1}^{n} \boldsymbol{a}_j x_j)$$

$$\leq \sum_{j=1}^{n_1} f(\boldsymbol{a}_j) x_j + f(\sum_{j=n_1+1}^{n} \boldsymbol{a}_j x_j),$$

hence

$$\sum_{j=1}^{n_1} f(\boldsymbol{a}_j) x_j + f(\sum_{j=n_1+1}^{n} \boldsymbol{a}_j x_j) \geq f(\boldsymbol{b}).$$

For an integer $l (> 1)$, we denote by $\boldsymbol{Z}_l$ a complete residue system modulus l : $\boldsymbol{Z}_l = \{0, 1, \ldots, l\text{-}1\}$. We define the function $f_l : \boldsymbol{Z} \to \boldsymbol{Z}_l$ as follows: for any $a \in Z$, there exists the integer b such that $a \equiv b \pmod{l}$ and $b \in \boldsymbol{Z}_l$; we define $f_l(a) = b$. Then f_l is a subadditive function on $\boldsymbol{Z}$ and on any subset of $\boldsymbol{Z}$.

Further we define the function $p_i : \boldsymbol{Z}^t \to \boldsymbol{Z}$ as follows: for any $(\boldsymbol{a}_j)_{1 \leq j \leq t} \in \boldsymbol{Z}^t$, $p_i((\boldsymbol{a}_j)) = \boldsymbol{a}_i$. We put $f_l^i = f_l \circ p_i$; then f_l^i is a subadditive function on $\boldsymbol{Z}^t$ and on any subset of $\boldsymbol{Z}^t$.

Now, we consider the mixed integer programming problem (P_M). We assume all of the components of A and $\boldsymbol{b}$ are integers. This is actually equivalent to assuming the components rational.

THEOREM 2. *Let i be an integer with $1 \leq i \leq m$. If $a_{ij} \geq 0$ for any $j(n_1+1 \leq j \leq n)$, then*

$$\sum_{j=1}^{n_1} f_l(a_{ij})x_j + \sum_{j=n_1+1}^{n} a_{ij}x_j \geq f_l(b_i)$$

is a valid inequality for X.

Proof. We should notice that $V = a_1\boldsymbol{Z} + \cdots + a_{n_1}\boldsymbol{Z} + b\boldsymbol{Z} \subset \boldsymbol{Z}^n$. By Theorem 1, we have for any $\boldsymbol{x} \in X$,

$$\sum_{j=1}^{n_1} f_l^j(\boldsymbol{a}_j)x_j + f_l^i(\sum_{j=n_1+1}^{n} \boldsymbol{a}_j x_j) \geqq f_l^i(\boldsymbol{b}),$$

since f_l^i is a subadditive function on V. So it follows

$$\sum_{j=1}^{n_1} f_l(a_{ij})x_j + f_l(\sum_{j=n_1+1}^{n} a_{ij}x_j) \geqq f_l(b_i).$$

We have

$$f_l(\sum_{j=n_1+1}^{n} a_{ij}x_j) \leq \sum_{j=n_1+1}^{n} a_{ij}x_j$$

by our assumption. Therefore

$$\sum_{j=1}^{n_1} f_l(a_{ij})x_j + \sum_{j=n_1+1}^{n} a_{ij}x_j \geq f_l(b_i)$$

is a valid inequality for X.

EXAMPLE.

minimize $z = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5$

subject to $3x_1 + x_2 + 5x_3 - 2x_4 - 4x_5 = 15$

$5x_1 + x_2 - x_3 + 4x_4 - 5x_5 = 10$

$0 \leq x_j \in \boldsymbol{Z} \quad (1 \leqq j \leqq 4)$

$0 \leq x_5 \in \boldsymbol{R}$

The solution $x_1 = \frac{65}{28}$, $x_2 = 0$, $x_3 = \frac{45}{28}$, $x_4 = 0$, $x_5 = 0$ with $z = \frac{50}{7}$ is optimal for the associated linear programming problem without the integrality restrictions.

Put $A=\begin{pmatrix}3 & 1 & 5 & -2 & -4\\ 5 & 1 & -1 & 4 & -5\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}15\\ 10\end{pmatrix}$ and $\boldsymbol{x}=\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix}$.

We multiply both sides of the equation $A\boldsymbol{x}=\boldsymbol{b}$ on the left by the matrix $\begin{pmatrix}3 & 1\\ 5 & 1\end{pmatrix}^{-1}$. Then we have

$$-\frac{1}{2}\begin{pmatrix}-2 & 0 & 6 & -6 & 1\\ 0 & -2 & -28 & 22 & 5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix}=-\frac{1}{2}\begin{pmatrix}5\\ -45\end{pmatrix},$$

so

$$\begin{aligned}-2x_1 \quad +6x_3-6x_4+x_5 &= 5\\ -2x_2-28x_3+22x_4+5x_5 &= -45.\end{aligned}$$

The coefficients of x_5 are non-negative, so we obtain the following valid inequalities by Theorem 2:

for $i=1$ and $l=2$ $\quad x_5 \geq 1$

for $i=1$ and $l=3$ $\quad x_1+x_5 \geq 2$

for $i=1$ and $l=4$ $\quad 2x_1+2x_3+2x_4+x_5 \geq 1$

for $i=1$ and $l=6$ $\quad 4x_1+x_5 \geq 5$

for $i=2$ and $l=2$ $\quad 5x_5 \geq 1$

for $i=2$ and $l=4$ $\quad 2x_2+2x_4+5x_5 \geq 3$

for $i=2$ and $l=6$ $\quad 4x_2+2x_3+4x_4+5x_5 \geq 3$

for $i=2$ and $l=7$ $\quad 5x_2+x_4+5x_5 \geq 4$

The first, 5th, 6th and the last of these inequalities are cuts.

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