VALID INEQUALITIES AND SUBADDITIVITY FOR INTEGER PROGRAMMING PROBLEMS

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1. Consider the integer programming problem (P_I)

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^t\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{0} \leqq \boldsymbol{x} \in \boldsymbol{Z}^n \end{array}$$

where $A \in \boldsymbol{R}^{m\times n}$, $\boldsymbol{b} \in \boldsymbol{R}^m$ and $\boldsymbol{c} \in \boldsymbol{R}^n$. Throughout this paper, $\boldsymbol{R}$ and $\boldsymbol{Z}$ denote the set of all real numbers and the set of all integers respectively.

Suppose that the set

$$X^I = \{\boldsymbol{x} \in \boldsymbol{R}^n\,;\, A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{0} \leqq \boldsymbol{x} \in \boldsymbol{Z}^n\}$$

of feasible solutions is bounded and not empty.

Let $\boldsymbol{\pi}$ be in $\boldsymbol{R}^n$ and π_0 be a real number. An inequality $\boldsymbol{\pi}^t\boldsymbol{x} \geqq \pi_0$ is called a valid inequality for X^I if it is satisfied by all $\boldsymbol{x} \in X^I$. We consider the associated linear programming problem (P_0)

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^t\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{0} \leqq \boldsymbol{x} \in \boldsymbol{R}^n. \end{array}$$

Let $\boldsymbol{x}^0$ be an optimal solution to the problem (P_0). A valid inequality for X^I, $\boldsymbol{\pi}^t\boldsymbol{x} \geqq \pi_0$, is called a cut if $\boldsymbol{\pi}^t\boldsymbol{x}^0 < \pi_0$.

We put

$$V = \{\boldsymbol{v} \in \boldsymbol{R}^m\,;\, \boldsymbol{v} = A\boldsymbol{x},\ \boldsymbol{0} \leqq \boldsymbol{x} \in \boldsymbol{Z}^n\}.$$

If $\boldsymbol{v}^1 = A\boldsymbol{x}^1 \in V$ and $\boldsymbol{v}^2 = A\boldsymbol{x}^2 \in V$, we have $\boldsymbol{v}^1 + \boldsymbol{v}^2 = A\,(\boldsymbol{x}^1 + \boldsymbol{x}^2) \in V$. Put $A = (a_{ij}) = (\boldsymbol{a}_1,\cdots,\ \boldsymbol{a}_n)$,

$$\boldsymbol{a}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ \vdots \\ \vdots \\ a_{mj} \end{pmatrix}, \text{ and } \boldsymbol{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} -j \qquad (1 \leqq j \leqq m),$$

then we have $A\boldsymbol{e}_j = \boldsymbol{a}_j$, therefore $\boldsymbol{a}_j \in V \quad (1 \leqq j \leqq m)$.

Let f be a real-valued function defined on V. f is said to be a subadditive function on V if

$$f(\boldsymbol{v}^1) + f(\boldsymbol{v}^2) \geqq f(\boldsymbol{v}^1 + \boldsymbol{v}^2)$$

for any $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ in V.

THEOREM 1. (*cf. Theorem 1.5 of [2] and Theorem 7.12 of [4]*) *Let f be a subadditive function on V satisfying $f(\boldsymbol{0}) = 0$. Then the inequality*

$$\sum_{j=1}^{n} f(\boldsymbol{a}_j)x_j \geqq f(\boldsymbol{b})$$

is a valid inequality for X^I.

Proof. First, we shall show that if f is subadditive, it holds

$$\sum_{j=1}^{l} f(\boldsymbol{v}_j)z_j \geqq f(\sum_{j=1}^{l} \boldsymbol{v}_j z_j) \qquad (1)$$

for any integer $l > 0$, any $\boldsymbol{v}_j \in V$ and any $z_j \in \boldsymbol{Z}$, $z_j \geqq 0$ $(1 \leqq j \leqq l)$.

Put $k = \sum_{j=1}^{l} z_j$, then $k \geqq 0$. We shall prove (1) by induction on k.

If $k = 0$, it holds by the asssumption of $f(\boldsymbol{0}) = 0$. If $k = 1$, (1) is trival. Let $r \geqq 2$. Suppose, as the induction hypothesis, that (1) is satisfied for $k = r-1$. Consider now $k = r$. We may assume $z_1 \geqq 1$. Then

$$f(\sum_{j=1}^{l} \boldsymbol{v}_j z_j) = f(\boldsymbol{v}_1 + \boldsymbol{v}_1 (z_1 - 1) + \sum_{j=2}^{l} \boldsymbol{v}_j z_j)$$

$$\leqq f(\boldsymbol{v}_1) + f(\boldsymbol{v}_1 (z_1 - 1) + \sum_{j=2}^{l} \boldsymbol{v}_j z_j)$$

$$\leqq f(\boldsymbol{v}_1) + f(\boldsymbol{v}_1)(z_1 - 1) + \sum_{j=2}^{l} f(\boldsymbol{v}_j) z_j$$

$$= \sum_{j=1}^{l} f(\boldsymbol{v}_j) z_j$$

by the induction hypothesis and by subadditivity. Thus, (1) is proven by induction.

Now, by the assumption of $X^I \neq \phi$ we have $\boldsymbol{b} \in V$. For any $\boldsymbol{x} = (x_j)_{1 \leqq j \leqq n}$ in X^I, we have

$$\sum_{j=1}^{n} \boldsymbol{a}_j x_j = \boldsymbol{b},$$

so we have by (1)

$$f(\boldsymbol{b}) = f(\sum_{j=1}^{n} \boldsymbol{a}_j x_j)$$

$$\leqq \sum_{j=1}^{n} f(\boldsymbol{a}_j) x_j$$

Hence it is a valid inequality for X^I.

2. For an integer m (> 1), $\boldsymbol{Z}_m$ denotes a complete residue system modulus m: $\boldsymbol{Z}_m = \{0, 1, \cdots, m-1\}$. We define the function f_m on $\boldsymbol{Z}$ to $\boldsymbol{Z}_m$ as follows: for any $n \in \boldsymbol{Z}$, there exists $a \in \boldsymbol{Z}_m$ such that $n \equiv a \pmod{m}$, so we define $f_m(n) = a$. For any a and $b \in \boldsymbol{Z}$, $a \equiv b \pmod{m}$ implies $b-a$ is divisible by m.

THEOREM 2. *The function f_m is subadditive on $\boldsymbol{Z}$ and on any* *subset U of* $\boldsymbol{Z}$.

Proof. We shall show for any n and $n' \in \boldsymbol{Z}$,

$$f_m(n) + f_m(n') \geqq f_m(n+n') \cdot \qquad (2)$$

If $f_m(n) + f_m(n') \leqq m-1$, (2) is satisfied with equality since we have $f_m(n) + f_m(n') = f_m(n+n') \pmod{m}$ and $0 \leqq f_m(n+n') \leqq m-1$. If $f_m(n) + f_m(n') \geqq m$, (2) is satisfied, since $0 \leqq f_m(n+n') \leqq m-1$.

EXAMPLE 1. We put

$$X^I = \{\boldsymbol{x} \in \boldsymbol{R}^4 ;\ 5x_1 + 3x_2 + 2x_3 + x_4 = 7,$$
$$0 \leqq x_j \in Z,\ 1 \leqq j \leqq 4\},$$

then we have

$$V = \{5x_1 + 3x_2 + 2x_3 + x_4 ;\ 0 \leqq x_j \in \boldsymbol{Z},\ 1 \leqq j \leqq 4\} \cdot$$

We have $V = \{x \in \boldsymbol{Z} ;\ x \geqq 0\}$, since $1 \in V$ and $0 \in V$. By Theorem 1 and Theorem 2, for every integer $m(>1)$, an inequality

$$f_m(5)x_1 + f_m(3)x_2 + f_m(2)x_3 + f_m(1)x_4 \geqq f_m(7)$$

is a valid inequality for X^I:

for $m = 2$, $\quad x_1 + x_2 + x_4 \geqq 1$,

for $m = 3$, $\quad 2x_1 + 2x_3 + x_4 \geqq 1$,

for $m = 4$, $\quad x_1 + 3x_2 + 2x_3 + x_4 \geqq 3$,

for $m = 5$, $\quad 3x_2 + 2x_3 + x_4 \geqq 2$.

Let $t(>0)$ be an integer. For any j with $1 \leqq j \leqq t$ and any integer $m(>1)$, we define the function f_m^j on $\boldsymbol{Z}^t$ to $\boldsymbol{Z}_m$ as follows :

$$f_m^j((n_i)_{1 \leqq i \leqq t}) = f_m(n_j),$$

for any $(n_i)_{1 \leqq i \leqq t} \in \boldsymbol{Z}^t$.

THEOREM 3. *The function* f_m^j *is subadditive on* $\boldsymbol{Z}^t$ *and on any* *subset* V *of* $\boldsymbol{Z}^t$.

Proof. For any $(n_i) \in \boldsymbol{Z}^t$ and $(n_i') \in \boldsymbol{Z}^t$, we have

$$\begin{aligned} f_m^j((n_i)) + f_m^j((n_i')) &= f_m(n_j) + f_m(n_j') \\ &\geqq f_m(n_j + n_j') \\ &= f_m^j((n_i + n_i')) \\ &= f_m^j((n_i) + (n_i')) \end{aligned}$$

Therefore f_m^j is a subadditive function on $\boldsymbol{Z}^t$.

EXAMPLE 2. We put

$$\begin{aligned} X^I = \{x \in \boldsymbol{R}^5 ;\ & x_1 + 2x_2 + 2x_3 + 3x_4 + 5x_5 = 10, \\ & 3x_1 - 3x_2 + 2x_3 + 3x_4 + 2x_5 = 7, \\ & 0 \leqq x_j \in \boldsymbol{Z},\ 1 \leqq j \leqq 5\}. \end{aligned}$$

Put

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 & 5 \\ 3 & -3 & 2 & 3 & 2 \end{pmatrix} \text{ and } \quad \boldsymbol{b} = \begin{pmatrix} 10 \\ 7 \end{pmatrix},$$

then

$$X^I = \{x \in \boldsymbol{R}^5 ;\ A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{0} \leqq \boldsymbol{x} \in \boldsymbol{Z}^5\}.$$

Since $V = \{A\boldsymbol{x} ;\ \boldsymbol{0} \leqq \boldsymbol{x} \in \boldsymbol{Z}^5\}$, so $V \subset \boldsymbol{Z}^2$. For every integer $m(>1)$ and $j(=1,2)$, an inequality

$$f_m^j\begin{pmatrix}1\\3\end{pmatrix}x_1 + f_m^j\begin{pmatrix}2\\-3\end{pmatrix}x_2 + f_m^j\begin{pmatrix}2\\2\end{pmatrix}x_3 + f_m^j\begin{pmatrix}3\\3\end{pmatrix}x_4 + f_m^j\begin{pmatrix}5\\2\end{pmatrix}x_5 \geqq f_m^j\begin{pmatrix}10\\7\end{pmatrix}$$

is valid for X^I by Theorem 1 and Theorem 3 :

for $j=1$ and $m=3$, $\quad x_1 + 2x_2 + 2x_3 + 2x_5 \geqq 1$,

for $j=1$ and $m=4$, $\quad x_1 + 2x_2 + 2x_3 + 3x_4 + x_5 \geqq 2$,

for $j=2$ and $m=2$, $\quad x_1 + x_2 + x_4 \geqq 1$,

for $j=2$ and $m=3$, $\quad 2x_3 + 2x_5 \geqq 1$.

3. Consider the integer programming problem (P_I)

$$\begin{aligned} &\text{minimize} && z = \boldsymbol{c}^t \boldsymbol{x} \\ &\text{subject to} && A\boldsymbol{x} = \boldsymbol{b} \\ & && \boldsymbol{0} \leqq \boldsymbol{x} \in \boldsymbol{Z}^n \end{aligned}$$

where $A \in \boldsymbol{R}^{m\times n}$, $\boldsymbol{b} \in \boldsymbol{R}^m$ and $\boldsymbol{c} \in \boldsymbol{R}^n$. We assume A is an integer matrix and $\boldsymbol{b}$ and $\boldsymbol{c}$ are integer vectors (they may be rational ; indeed they satisfy the assumption by multiplying by an integer). Suppose that $m < n$ and rank $A = m$. Consider the associated linear programming problem (P_0)

$$\begin{aligned} &\text{minimize} && z = \boldsymbol{c}^t \boldsymbol{x} \\ &\text{subject to} && A\boldsymbol{x} = \boldsymbol{b} \\ & && \boldsymbol{0} \leqq \boldsymbol{x} \in \boldsymbol{R}^n. \end{aligned}$$

Let B be an optimal basis matrix to the problem (P_0), so (P_I) can be written

$$\begin{aligned} &\text{minimize} && z = z_B + \bar{\boldsymbol{c}}_N^t \boldsymbol{x}_N \\ &\text{subject to} && \boldsymbol{x}_B + \bar{N}\boldsymbol{x}_N = \bar{\boldsymbol{b}} \\ & && \boldsymbol{0} \leqq \boldsymbol{x}_B \in \boldsymbol{Z}^m,\ \boldsymbol{0} \leqq \boldsymbol{x}_N \in \boldsymbol{Z}^{n-m}. \end{aligned}$$

Put $l = n - m$. We assume $\boldsymbol{x}_N = (x_1, \cdots, x_l)^t$ and $\boldsymbol{x}_B = (x_{l+1}, \cdots, x_n)^t$ without loss of generality. Then the problem (P_I) can be written

$$\begin{aligned} &\text{minimize} && z = z_B + \sum_{j=1}^{l} \bar{c}_j\, x_j \\ &\text{subject to} && x_{l+i} + \sum_{j=1}^{l} \bar{a}_{ij}\, x_j = \bar{b}_i \quad (1 \leqq i \leqq m) \\ & && 0 \leqq x_j \in \boldsymbol{Z} \qquad (1 \leqq j \leqq n)\ . \end{aligned}$$

If $\boldsymbol{b}$ is an integer vector, then $\bar{\boldsymbol{x}} = (\bar{\boldsymbol{x}}_B, \bar{\boldsymbol{x}}_N) = (\bar{\boldsymbol{b}}, \boldsymbol{0})$ is in X^I, so $\bar{\boldsymbol{x}}$ is an optimal solution to the problem (P_1). Therefore we assume $\bar{\boldsymbol{b}}$ is not an integer vector. Then there exists i ($1 \leqq i \leqq m$) such that b_i is not an integer : consider the constraint

$$x_{l+i} + \sum_{j=1}^{l} \bar{a}_{ij}x_j = \bar{b}_i , \qquad (3)$$

where $\bar{a}_{ij}$ and $\bar{b}_i$ are rational numbers.

We define $f(a) = a - [a]$ for any $a \in \boldsymbol{R}$, where [ ] is a Gauss' symbol, i. e., $[a]$ denotes the maximal integer not greater than a. Then we have $0 \leqq f(a) < 1$. For any a and $b \in$ R, we write $a \equiv b$ (mod 1) when $b - a$ is an integer. Then we have

$$\sum_{j=1}^{l} f(\bar{a}_{ij})x_j \equiv f(\bar{b}_i) \quad (\text{mod } 1)$$

by the constraint (3), so we have

$$\sum_{j=1}^{l} f(\bar{a}_{ij})x_j \geqq f(\bar{b}_i) ; \qquad (4)$$

this inequality is called a Gomory cut because it does not hold when $x_j = 0\ (1 \leqq j \leqq l)$.

Now there exists an integer $D(>1)$ such that $D\bar{a}_{ij} \in \boldsymbol{Z}$ $(1 \leqq j \leqq l)$ and $D\bar{b}_i \in \boldsymbol{Z}$, so the constraint (3) can be written

$$Dx_{l+i} + \sum_{j=1}^{l} D\bar{a}_{ij}x_j = D\bar{b}_i .$$

We obtain by Theorem 2 a valid inequality

$$\sum_{j=1}^{l} f_D(D\bar{a}_{ij})x_j \geqq f_D(D\bar{b}_i).$$

This inequality is the same as a Gomory cut (4), since putting $\bar{a}_{ij} = \frac{D_a}{D}$ and $\bar{b}_i = \frac{D_b}{D}$ $(D_a, D_b \in \boldsymbol{Z})$, we have

$$f(\bar{a}_{ij}) = \frac{f_D(D_a)}{D} = \frac{f_D(D\bar{a}_{ij})}{D} ,$$

$$f(\bar{b}_i) = \frac{f_D(D_b)}{D} = \frac{f_D(D\bar{b}_i)}{D} .$$

EXAMPLE 3. Consider the integer programming problem

$$\text{minimize} \quad z = x_1 + x_2 + x_3 + 2x_4 + 2x_5$$

$$\text{subject to} \quad x_1 + 2x_2 + x_3 + x_4 + 5x_5 = 10$$

$$3x_1 - 3x_2 + 2x_3 - 3x_4 + 3x_5 = 5$$

$$0 \leqq x_j \in \boldsymbol{Z}, \quad 1 \leqq j \leqq 5$$

The optimal linear programming tableau is

$$\text{minimize} \quad z = \frac{5}{3} x_3 + \frac{5}{3} x_4 + \frac{5}{9} x_5$$

$$\text{subject to} \quad x_1 + \frac{7}{9} x_3 - \frac{1}{3} x_4 + \frac{7}{3} x_5 = \frac{40}{9}$$

$$x_2 + \frac{1}{9} x_3 + \frac{2}{3} x_4 + \frac{4}{3} x_5 = \frac{25}{9}$$

or

$$\text{subject to} \quad 9x_1 + 7x_3 - 3x_4 + 21x_5 = 40$$

$$9x_2 + x_3 + 6x_4 + 12x_5 = 25$$

(see Example 1 of [2]).

By Theorem 3 we have valid inequalities as follows :

for $j = 1$ and $m = 3$, $x_3 \geqq 1$,

for $j = 1$ and $m = 7$, $2x_1 + 4x_2 \geqq 5$,

for $j = 1$ and $m = 9$, $7x_3 + 6x_4 + 3x_5 \geqq 4$,

for $j = 2$ and $m = 2$, $x_2 + x_3 \geqq 1$,

for $j = 2$ and $m = 3$, $x_3 \geqq 1$,

for $j = 2$ and $m = 4$, $x_2 + x_3 + 2x_4 \geqq 1$,

for $j = 2$ and $m = 6$, $3x_2 + x_3 \geqq 1$,

for $j = 2$ and $m = 9$, $x_3 + 6x_4 + 3x_5 \geqq 7$.

The third and the last inequalities of these are Gomory cuts.

References

[1] B. E. Gillett, Introduction to Operations Research, McGraw-Hill, 1976.

[2] R. E. Gomory and E. L. Johnson, Some continuous functions related to corner polyhedra I, Mathematical Programming 3 (1972), 23-85.

[3] R. E. Gomory and E.L. Johnson, Some continuous functions related to corner polyhedra II, Mathematical Programming 3 (1972), 359-389.

[4] H. Konno, Seisu-keikaku-ho (in Japanese), Sangyo-Tosho, 1981.