A NOTE ON BULGARIAN SOLITAIRE

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Let n be any triangular number: $n=1+2+\cdots+k$, where k is a natural number.

Form a pile of n cards, then divide it into arbitrary piles with an arbitrary number of cards in each pile. Take one card from each pile and with them make a new pile. Keep repeating the procedure. It is conjectured that regardless of the initial state you will reach the consecutive state, i. e., $(1, 2, 3, \cdots, k)$ in finite steps: the game must end because the consecutive state cannot change. This game is called Bulgarian Solitaire.(1)

For example, in case $k=3$, $n=6$,

$$(1, 1, 4)\to(3, 3)\to(2, 2, 2)\to(1, 1, 1, 3)\to(2, 4)\to(1, 2, 3),$$

$$(6)\to(1, 5)\to(2, 4)\to(1, 2, 3),$$

and in case $k=4$, $n=10$,

$$(1, 1, 3, 5)\to(2, 4, 4)\to(1, 3, 3, 3)\to(2, 2, 2, 4)\to(1, 1, 1, 3, 4)\to$$
$$\to(2, 3, 5)\to(1, 2, 3, 4).$$

The above games end with the consecutive state in 5, 3 and 6 steps, respectively.

It is conjectured that for $n=1+2+\cdots+k$, any game must end in no more than $k(k-1)$ steps, and in 1982 Donald E. Knuth and his students of Stanford University comfirmed it for $k\leqq 10$ by computer.(2)

In this paper we shall show that the above conjecture cannot be

(1) M. Gardner, "Mathematical Games," *Scientific American*, Vol. 249, No. 2, 1983, 8-13.

(2) Gardner, *op. cit.*, 1983, 11.

made better in a sense, that is, we shall prove the following:

Let k be any natural number ($\geqq 3$). Put $n=1+2+\cdots+k$. The partition of n, $(1, 1, 2, 3, \cdots, k-2, k-1, k-1)$ reaches the consecutive state by Bulgarian operation in $k(k-1)$ steps.

The partition $(1, 1, 2, 3, \cdots, k-2, k-1, k-1)$ is called the top of the main trunk of Bulgarian tree by Gardner.(3)

Now we shall prove the above theorem for $k\geqq 6$; it is easily checked for $k\leqq 5$.

The initial state is $(1, 1, 2, 3, \cdots, k-2, k-1, k-1)$, so we have $(1, 2, 3, \cdots, k-2, k-2, k+1)$ after the lst step, $(1, 2, \cdots, k-3, k-3, k, k)$ after the 2nd step, $(1, 2, \cdots, k-4, k-4, k-1, k-1, k)$ after the 3rd step, and so on, $(1, 1, 4, 4, 5, \cdots, k)$ after the $(k-2)$ th step and $(3, 3, 4, 5, \cdots, k-1, k)$ after the $(k-1)$ th step. Hence we have $(2, 2, 3, 4, \cdots, k-2, k-1, k-1)$ after the kth step.

Let $2\leqq l\leqq k-3$. We shall show by induction on l that we have $(1, 2, \cdots, l-1, l+1, l+1, l+2, \cdots, k-1, k-1)$ after the lkth step.

If $l=2$, it is easily checked that we have $(1, 1, 3, 5, 5, 6, \cdots, k)$ after the $(2k-2)$ th step, so $(1, 3, 3, 4, 5, \cdots, k-2, k-1, k-1)$ after the $2k$th step.

If $l=3$, we have $(2, 2, 3, 4, \cdots, k-3, k-2, k-2, k)$ after the $(2k+1)$th step and $(1, 1, 3, 4, 6, 6, 7, \cdots, k)$ after the $(3k-2)$th step, so we have $(1, 2, 4, 4, 5, \cdots, k-1, k-1)$ after the $3k$th step.

Suppose, then, that $4\leqq l\leqq k-3$. By induction we may have $(1, 2, \cdots, l-2, l, l, l+1, \cdots, k-1, k-1)$ after the $(l-1)k$th step. Then we have $(1, 2, \cdots, l-3, l-1, l-1, l, \cdots, k-2, k-2, k)$ after the $((l-1)k+1)$ th

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(3) Gardner, *op. cit.*, 1983, 11.

step, and so on, $(1, 3, 3, 4, 5, \cdots, k-l+2, k-l+2, k-l+4, \cdots, k-1, k)$ after the $((l-1)k+l-3)$ th step. So we get $(1, 2, \cdots, k-l-2, k-l-2, k-l, \cdots, k-2, k, k)$ after the $((l-1)k+l+1)$th step. And we have $(1, 1, 3, 4, \cdots, l+1, l+3, l+3, l+4, \cdots, k)$ after the $((l-1)k+k-2)$ th step, so $(2, 3, \cdots, l, l+2, l+2, l+3, \cdots, k-1, k)$ after the $((l-1)k+k-1)$ th step, hence we obtain $(1, 2, \cdots, l-1, l+1, l+1, l+2, \cdots, k-1, k-1)$ after the $((l-1)k+k)=lk$th step.

Therefore, putting $l=k-3$, we have $(1, 2, \cdots, k-4, k-2, k-2, k-1, k-1)$ after the $(k-3)k$th step. So we get $(1, 2, \cdots, k-4, k-3, k-1, k-1, k-1)$ after the $(k-2)k$th step. Further we have $(1, 2, \cdots, k-4, k-2, k-2, k-2, k)$ after the $((k-2)k+1)$th step, $(1, 3, 3, 3, 5, \cdots, k)$ after the $((k-2)k+k-4)$th step, hence $(1, 2, \cdots, k-2, k-1, k)$ after the $(k-1)k$th step. This completes the proof.

We shall next show that for any triangular number $n=1+2+\cdots+k$, the partition (n) reaches the consecutive state in $(n-k)$th steps, where (n) is the next state of the partition $(\underbrace{1, 1, \cdots, 1}_{n})$ by Bulgarian operation.

Put $S_m=1+2+\cdots+m$ with $1 \leqq m \leqq k-1$. We shall show by induction on m the state after the S_mth step is $(1, 2, \cdots, m, n-S_m)$.

The state after the lst step is $(1, n-1)$ and the state after the 3rd step is $(1, 2, n-3)$, so the assertion holds for $m=1, 2$.

Suppose $3 \leqq m \leqq k-1$. By induction we may have $(1, 2, \cdots, m-1, n-S_{m-1})$ after the S_{m-1}th step. Then we have $(1, 2, \cdots, m-2, m, n-S_{m-1}-1)$ after the $(S_{m-1}+1)$th step, $(1, 3, 4, \cdots, m, n-S_m+2)$ after the $(S_{m-1}+m-2)$th step, so $(1, 2, \cdots m, n-S_m)$ after the $(S_{m-1}+m)=S_m$th step.

Hence, putting $m=k-1$, we have $(1, 2, \cdots, k)$ after the $(n-k)$th step because $S_{k-1}=n-k$.

For $t<S_{k-1}$, the state after the tth step cannot be consecutive con-

sidering $n-t>k$. Therefore for the first time we reach the consecutive state after the $(n-k)$th step.