AN EXTENSION OF THE FLIPPING COINS PROBLEM

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A pile of M coins is arranged so that each coin is heads up. We take the top coin, flip it over and put it back on top of the pile. We then take the top two coins, and holding them together turn them upside down and put them back on top of the pile. We continue, taking in turn the top three, four and so on until the whole pile is turned upside down. We then commence at the top again flipping the top stack of one, two, three coins and continue in this fashion until the pile is again all heads up. How many flips are required? (Newman: The Flippin' Coins Problem, Mathematics Magazine, Vol. 54, No. 2, 51-59, 1981)

For $M=3$, the pile of coins will undergo these changes (brackets indicate the stack of coins flipped at each stage):

H]	T]	T]	T]	H]	H]	T]	H]	T]	H
H	H]	H]	T	T]	T]	H	H]	T]	H
H	H	H]	H	H	H]	T	T	T]	H

Thus the number of flips required for a pile of three coins is 9.

Newman proved that the number of flips for a pile of M coins is of the form Mk or $Mk-1$ (k is a natural number) and that this number is bounded by M^2 ($M>1$). This means that a pile of heads can occur only after a move that flips over the whole pile of M coins, or just prior to such a move. He proved these results by representing as binary decimals the state of the pile just after the pile flip,

i. e. Mth, $2M$th, $3M$th, etc., flip.

A coin has the two-fold state of the flip (head and tail). We now consider a thing that has the three-fold state of the flip like a signal (red, blue, yellow). We denote the three-fold state by 0, 1 and 2: if it is flipped over, the state a changes to the state $a+1$ (mod 3). A pile of M signals is arranged so that the state of each signal is 0. Then we consider the number of flips required for that the state of the signals is again all 0.

For example, the number of flips for $M=3$ is 9 as follows.

0	1	1	1	2	1	0	1	2	0
0	0	2	0	0	0	1	1	2	0
0	0	0	2	2	2	2	2	2	0

For $M=4$, it is shown in Table 1. We show the number of flips for M $(1 \leq M \leq 36)$ in Table 2.

NUMBER OF FLIPS				
	0	0	0	0
1	1	0	0	0
2	1	2	0	0
3	1	0	2	0
4	1	0	1	2
5	2	0	1	2
6	1	0	1	2
7	2	1	2	2
8	0	0	2	0
9	1	0	2	0
10	1	2	2	0
11	0	0	2	0
12	1	0	1	1
13	2	0	1	1
14	1	0	1	1
15	2	1	2	1
16	2	0	2	0

17	0	0	2	0
18	1	1	2	0
19	0	2	2	0
20	1	0	0	1
21	2	0	0	1
22	1	0	0	1
23	1	1	2	1
24	2	0	2	2
25	0	0	2	2
26	1	1	2	2
27	0	2	2	2
28	0	0	0	1
29	1	0	0	1
30	1	2	0	1
31	1	0	2	1
32	2	0	1	2
33	0	0	1	2
34	1	1	1	2
35	2	2	2	2
36	0	0	0	0

Table 1

M (NUMBER IN PILE)	NUMBER OF FLIPS
1	1
2	4
3	9
4	36
5	25
6	36
7	84
8	32
9	81
10	60
11	121
12	120
13	351

14	196
15	75
16	240
17	204
18	324
19	228
20	200
21	147
22	792
23	529
24	504
25	600
26	676
27	540
28	252
29	841
30	900
31	558
32	192
33	1089
34	2244
35	1225
36	324

Table 2

From an examination of Table 2, the following conjectures are made.

Conjecture 1. *The number of flips for a pile of M signals is of the form Mk (k is a natural number).*

Conjecture 2. *This number is bounded by* $3M^2$.

In this paper, we shall prove Conjecture 2, while Conjecture 1 is still an open question (Conjecture 1 is true for $M \leq 200$).

We consider the permutation of the positions of the signals after the first pile flip. For $M=3$, signals 1, 2 and 3 are in position 2, 3 and 1, respectively as shown in Table 3.

1]	1	2	3
2	2	1	1
3	3	3	2

Table 3

We denote by ψ_M the permutation of the positions of the signals 1, 2, ..., M after the pile flip, and by φ_M the inverse of ψ_M. For $M=3$, $\psi_3=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$ and $\varphi_3=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$. Newman showed that the order of the permutation φ_M is the order of the cycle containing M. Since the order of the cycle is bounded by M, the order of φ_M is so. We denote it by t, then the position of M signals is the same after t-th pile flip. Although the state of M signals after t-th pile flip may not be all 0, the state of M signals changes to all 0 after $3t$-th pile flip. $3t$-th pile flip means $3tM$-th flip. Since $t \leq M$, Conjecture 2 is proved.

FACOM M-180 II AD in Imformation Processing Center, Nagasaki University was used in order to obtain Table 1 and 2.